

On Non-Perturbative Unitarity in Gravitational Scattering

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We argue that the tree-level graviton-scalar scattering in the Regge limit is unitarized by non-perturbative effects within General Relativity. At Planckian energy the back reaction of the incoming graviton on the background geometry produces a non-perturbative plane wave which, in turn, softens the UV-behavior. Our amplitude then interpolates between the perturbative graviton-scalar scattering at low energy and scattering on a classical plane wave in the Regge limit.

Keywords: QFT, Gravity

I. INTRODUCTION

It is well known that perturbative scattering amplitudes involving gravitons violate the unitarity bound at Planckian energy even at tree-level. For instance, the scattering of a graviton and a massless scalar field is given by [1]

$$A_{tree} = -\frac{i\kappa^2 su}{4t} \quad (1)$$

where s, t, u are the usual Mandelstam variables and κ is the dimensionful gravitational coupling. This state of affairs has given rise to an extensive activity in searching for a UV-completion of General Relativity (GR). String theory is one such complete theory whose legacy rests partly on the fact that it predicts an amplitude that is perturbatively unitary.

On the other hand, one may question whether the assumption of asymptotic in- and out-states on which (1) is built holds for gravitons of Planckian energy since Gravity is a non-linear theory whose coupling strength increases with energy. One argument in favor of it is that a single graviton can always be boosted to an inertial frame where its energy is small. However, for a two body scattering with large COM energy there is no boost for which both particles have small energy. Thus back-reaction will have to be taken into account for at least one in-going particle. This idea is not new. It was explored already many years ago by 't Hooft [2] and others [3] who replaced an ingoing scalar of transplanckian energy by a gravitational shock wave. The starting point on which we base our argument for a non-perturbative unitarization of (1) is similar to this although the details are somewhat different. We perform a Lorentz boost such that the energy of the incoming scalar is small while the incoming

graviton has transplanckian energy so that back reaction on geometry has to be taken into account. Luckily, an exact solution to Einstein's equation, accounting for the complete back reaction on geometry is available in the form of a pp-wave [4, 5] (c.f. [6–10]). As a result, the non-perturbative generalization of (1) in the large s but small t (or Regge) limit can be reduced to a perturbative calculation on top of a pp-wave as illustrated in Fig. 1.

II. PERTURBATIVE LIMIT

To see how this comes about let us first recover the perturbative amplitude (1) for $(h_{(1)}, \phi_{(2)} \rightarrow h_{(3)}, \phi_{(4)})$ in position space. Without restricting the generality we make the following momentum assignments

$$\begin{aligned} p_{(1)} &\equiv p = (p_u, 0, 0, 0), & p_{(2)} &\equiv k = (0, k_v, 0, 0), \\ p_{(3)} &\equiv q = (q_u, q_v, q_1, q_2), & p_{(4)} &\equiv l = (l_u, l_v, l_1, l_2). \end{aligned}$$

In position space the t -channel diagram can then be calculated as follows: We first solve for the internal graviton \tilde{h} around Minkowski background, η , through

$$\begin{aligned} 0 &= G_{\mu\nu}(\eta + \lambda h_{(1)} + \lambda h_{(3)} + \lambda^2 \tilde{h}) \\ &= \lambda^2 \frac{\delta G_{\mu\nu}}{\delta g} \Big|_{\eta}(\tilde{h}) + \lambda^2 \frac{\delta^2 G_{\mu\nu}}{\delta g^2} \Big|_{\eta}(h_{(1)}, h_{(3)}), \end{aligned} \quad (2)$$

where $G_{\mu\nu}$ is the Einstein tensor and we assume that $h_{(1)}$ and $h_{(3)}$ satisfy the linearized Einstein equation

$$\frac{\delta G_{\mu\nu}}{\delta g} \Big|_{\eta}(h_{(1)}) = \frac{\delta G_{\mu\nu}}{\delta g} \Big|_{\eta}(h_{(3)}) = 0. \quad (3)$$

Here, λ is a dimensionless parameter whose sole purpose is to keep track of the order in perturbation in h . Next, we solve for the outgoing scalar field $\tilde{\phi}$ with the help of the Ansatz $\phi = \phi_{(2)} + \lambda^2 \tilde{\phi}$,

$$0 = \square_{\eta + \lambda^2 \tilde{h}} \phi = \lambda^2 \square_{\eta} \tilde{\phi} + \square_{\lambda^2 \tilde{h}} \phi_{(2)}, \quad (4)$$

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where \square_g stands for scalar wave operator in the background metric g . We note that (2) fixes \tilde{h} only up to a solution of the homogeneous equation. The latter reproduces 3-particle (1 graviton) scattering amplitude upon substitution into (4). Note also that in (2) we can replace $h_{(1)}$ by a wave packet since the equation for \tilde{h} is linear in $h_{(1)}$.

Equivalently, we can treat $h_{(1)}$ as a background field and solve \tilde{h} as a linearized fluctuation around that background. Setting $h_{(1)} \equiv H$ for later convenience, the linearized Einstein equation reads

$$0 = \frac{\delta G_{\mu\nu}}{\delta g} \Big|_{\eta+\lambda H} (h). \quad (5)$$

With the Ansatz [11] $h = \lambda h_{(3)} + \lambda^2 \tilde{h}$, we expand the background once again and get

$$0 = \lambda^2 \frac{\delta G_{\mu\nu}}{\delta g} \Big|_{\eta} (\tilde{h}) + \lambda^2 \frac{\delta^2 G_{\mu\nu}}{\delta g^2} \Big|_{\eta} (H, h_{(3)}), \quad (6)$$

which is in agreement with (2).

In what follows we will work with the latter form since it is suitable to accommodate non-linear effects for the incoming graviton, H . Indeed, suppose that H has momentum p of order M_{Pl} . Then H cannot be treated as a perturbation of Minkowski space-time and back reaction on the geometry has to be taken into account. This can be done by replacing H by a pp-wave. In Einstein-Rosen coordinates [5] the pp-wave metric reads

$$ds^2 = 2dUdV - \gamma_{ij}(U)dy^i dy^j, \quad (i, j = 1, 2), \quad (7)$$

with $\gamma_{ij} \sim (\delta_{ij} + h_{(1)ij})$ in the perturbative limit (here and in what follows we absorb λ in h). However, in the non-perturbative regime, Brinkmann coordinates [4] are more convenient, with

$$ds^2 = 2dudv - H_{ab}(u)x^a x^b du^2 - dx_a^2, \quad (a, b = 1, 2) \quad (8)$$

which is an exact solution, $G_{\mu\nu}(\eta + H) = 0$. Then (6) is the correct generalization of (2) provided $h_{(3)}$ has small momentum which is compatible with the Regge limit, $t \ll M_{Pl}^2$. In Brinkmann coordinates, for $h_{(3)}$ transverse, traceless with asymptotic polarization vector ϵ_a , the linear solution for \tilde{h} on the pp-wave takes the form [12]

$$\tilde{h}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{i}{q_v} \epsilon^a \Sigma_{ab} \epsilon^b + \epsilon_u^2 & \epsilon_u \epsilon_1 & \epsilon_u \epsilon_2 \\ 0 & \epsilon_u \epsilon_1 & \epsilon_1 \epsilon_1 & \epsilon_1 \epsilon_2 \\ 0 & \epsilon_u \epsilon_2 & \epsilon_2 \epsilon_1 & \epsilon_2 \epsilon_2 \end{pmatrix} \Phi(x) \quad (9)$$

where

$$\Phi(x) = \frac{1}{\sqrt{|E|}} e^{iq_v(v + \frac{\Sigma_{ab} x^a x^b}{2}) + iq_i E_a^i x^a + i \frac{q_i q_j}{q_v} F^{ij}} \quad (10)$$

is the solution of the scalar wave equation and

$$\Sigma_{ab} = \dot{E}_a^i E_{bi}, \quad F^{ij} = \int_{-\infty}^u \gamma^{ij}(s) ds. \quad (11)$$

Here Σ_{ab} is the deformation tensor for the Vierbein E_a^i , subject to

$$\ddot{E}_{ai} = H_{ab}(u) E_i^b, \quad \text{with} \quad \lim_{u \rightarrow \infty} E_a^i(u) = \delta_a^i, \quad (12)$$

and $\gamma_{ij} = E_{ia} E_a^j$. Finally, the longitudinal polarization ϵ_u is given by

$$\epsilon_u = \epsilon^a \left[\frac{q_j}{q_v} E_a^j + \Sigma_{ab} x^b \right]. \quad (13)$$

In order to disentangle the disconnected 1-graviton contribution, we then substitute (9) into (4)

$$\square_\eta \tilde{\phi} = \partial_v \left[\epsilon_u^2 - \frac{i}{q_v} \epsilon^a \epsilon^b \Sigma_{ab} \right] \Phi \partial_v \phi_{(2)} \\ - \partial_v \epsilon_u \epsilon^a \Phi \partial_a \phi_{(2)} - \partial_a \epsilon_u \epsilon^a \Phi \partial_v \phi_{(2)} + \partial_a \epsilon^a \epsilon^b \Phi \partial_b \phi_{(2)} \quad (14)$$

with $\phi_{(2)}$ is the incoming scalar field. We can then make connection to perturbation theory around Minkowski metric by integrating $\square_\eta \tilde{\phi}$ against $\phi_{(4)}$,

$$\int_{R^4} \phi_{(4)} \square_\eta \tilde{\phi} = \lim_{u \rightarrow \infty} \int_{\Sigma_u} \left(\phi_{(4)} \partial_v \tilde{\phi} - \tilde{\phi} \partial_v \phi_{(4)} \right) \\ - \lim_{u \rightarrow -\infty} \int_{\Sigma_u} \left(\phi_{(4)} \partial_v \phi_{(2)} - \phi_{(2)} \partial_v \phi_{(4)} \right) \quad (15)$$

where we have used that $\tilde{\phi} \rightarrow \phi_{(2)}$ as $u \rightarrow -\infty$. The second term on the r.h.s. then subtracts the disconnected contribution to the scattering. At 0^{th} order in H this gives

$$-(2\pi)^4 \delta^4(k + q + l) (\epsilon^\mu k_\mu)^2. \quad (16)$$

This reproduces the familiar 1-graviton scattering. This is expected since \tilde{h} reduces to $h_{(3)}$ in (2) for vanishing H .

Let us now consider the first order in H . We first choose a polarization for H by setting

$$E_a^i = \delta_a^i + \hat{\sigma}_a^i e^{ip_\mu x^\mu}, \quad (17)$$

where $\hat{\sigma}_a^i$ is one of the Pauli matrices. The contributions, linear in H , come from the expansion of ϵ_u and Σ_{ab} in (14). To continue we note that the deformation in ϵ_u simply takes account of the fact that the transversality condition of $h_{(3)}$ depends on H in the coordinate basis, so that (13) still holds. The contribution at first order in H to the connected four point function is then

$$- (\epsilon^a \epsilon^b \hat{\sigma}_{ab}) p_u \frac{k_v^2}{q_v} \quad (18)$$

where we used that $\Sigma_{ab} = ip_u \hat{\sigma}_{ab}$. In terms of the Mandelstam variables this can be written as

$$- (\epsilon^a \epsilon^b \hat{\sigma}_{ab}) \frac{s^2}{t} \quad (19)$$

which is the t -channel contribution of the scalar-graviton into scalar-graviton scattering amplitude.

Next we replace $\phi_{(2)}$ by the linearized approximation of the scalar solution $\Phi_{(2)}$ in the pp-wave background,

$$\Phi_{(2)}(x) = \frac{e^{ik_v \left[v + \frac{\Sigma_{ab} x^a x^b}{2} \right]}}{\sqrt{|E|}} \sim \left[1 + \frac{i \Sigma_{ab} x^a x^b}{2} \right] e^{ik_v v}.$$

This should, in addition, account for the the s and u -channel contribution. Indeed, solving for the scalar wave equation in the pp-background before and after interacting with \tilde{h} takes into account the interaction with H . Furthermore, this should take into account the contact term. Indeed, the last term in (14) gives an new contribution

$$-(\epsilon^a \epsilon^b \hat{\sigma}_{ab}) p_u k_v = -(\epsilon^a \epsilon^b \hat{\sigma}_{ab}) s. \quad (20)$$

Adding this to (19) we get

$$(2\pi)^4 \delta^4(p+k+q+l) (\epsilon^a \epsilon^b \hat{\sigma}_{ab}) \frac{su}{t} \quad (21)$$

which is the correct perturbative limit including all channels as well as the contact interaction.

III. NON-PERTURBATIVE CALCULATION

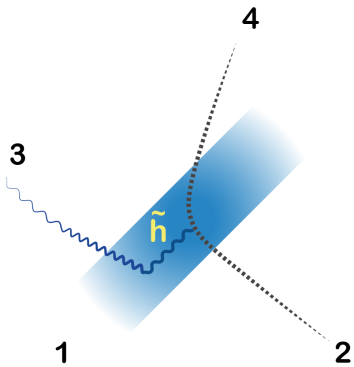


FIG. 1. Non-perturbative four particle scattering process where the ingoing graviton is replaced by a pp-wave (shaded strip).

In order to take the complete back-reaction of the incoming graviton into account we make the substitution $\eta \rightarrow \eta + H$ and insert the exact solutions $\Phi_{(2)}$ and $\Phi_{(4)}$ for the scalar fields together with the internal graviton $\tilde{h}_{\mu\nu}$ on the pp-wave into (14). Then integrating (14) against $\Phi_{(4)}$ we end up with (ignoring the one graviton contribution (16)).

$$\begin{aligned} A &= \int \Phi_{(4)} \partial_\mu \tilde{h}^{\mu\nu} \partial_\nu \Phi_{(2)} \\ &= \int \Phi_{(4)} \left(\frac{i}{q_v} \epsilon^a \epsilon^b \Sigma_{ab} k_v^2 + i k_v \epsilon^a \epsilon^b \Sigma_{ab} \right) \Phi_{(2)}, \quad (22) \end{aligned}$$

where the terms in the bracket come from the first line in (14) and the integral at $u \rightarrow -\infty$ in (15) subtracts the disconnected contribution, with $h_{(3)}$ now disconnected in the plane wave background. This integral can be further simplified following [12],

$$\begin{aligned} A &= -(2\pi)^2 \delta(k_v + q_v + l_v) \int du \frac{(\epsilon^a \epsilon^b \Sigma_{ab}) \left(\frac{k_v^2}{q_v} + k_v \right)}{|E| |E_{(2)}|^{\frac{1}{2}} \sqrt{|l_v B|}} \\ &\times \exp \left[i \left(- \frac{J_a (B^{-1})^{ab} J_b}{2l_v} + \frac{q_i q_j F^{ij}}{q_v} + \frac{l_i l_j}{l_v} F^{ij} \right) \right], \quad (23) \end{aligned}$$

where $E_{(2)}$ again solves (12) but with *ingoing* boundary condition, $\lim_{u \rightarrow -\infty} (E_{(2)})_a^i(u) = \delta_a^i$, and

$$J_a = (l_i + q_i) E_a^i, \quad B_{ab} = \Sigma_{ab}^{(2)} - \Sigma_{ab}. \quad (24)$$

To continue we note that E_a^i are functions of $z = p_u u$ only while Σ_{ab} , and B are of the form $p_u \Sigma(z)$, $p_u B(z)$ respectively [13]. This allows us to extract the p_u dependence as

$$\begin{aligned} A &= (2\pi)^2 \delta(s+u+t) \frac{s}{t} \int dz \frac{(\epsilon^a \epsilon^b \Sigma_{ab})}{|E| |E_{(2)}|^{\frac{1}{2}} \sqrt{|B|}} \frac{1}{\sqrt{|B|}} \\ &\times \exp \left[i \left(- \frac{J_a (B^{-1})^{ab} J_b}{u} + 2 \left(\frac{l_i l_j}{u} + \frac{q_i q_j}{t} \right) F^{ij} \right) \right], \quad (25) \end{aligned}$$

where the extra factor of $1/p_u$ multiplying F^{ij} in the last term is due to the change of measure $ds = \frac{dz}{p_u}$ in (11). In addition we used that

$$\delta(s+u+t) = \delta((k_v + q_v + l_v) \cdot p_u) = \frac{\delta(k_v + q_v + l_v)}{p_u} \quad (26)$$

where $2q_v p_u = t$, $2k_v p_u = s$. Using (26), the form (25) of the 4-point amplitude makes the unitarity of the amplitude at large s manifest. On the other hand for small s (25) reduces to the perturbative amplitude (21). It is not hard to see that the terms in (9) containing ϵ_u and ϵ_a will similarly give a non-perturbative deformation of (16) preserving the unitarity of the latter. We would like to stress, however, that we do not expect that the total cross-section for the $(\phi(k), h_{\mu\nu}(p) \rightarrow \phi(l), h_{\mu\nu}(q))$ scattering is unitary since, due to the absence of momentum conservation, the integral over the outgoing momenta is not constrained. However, this feature is expected for a scattering on an external potential.

In order to complete the argument that the amplitude is bounded we need to convince ourselves that the integral in (25) is finite. As mentioned before, all the steps performed in obtaining (25) are equally valid when replacing $h_{(1)}$ by a wave packet which is the more realistic set-up. Let us then consider the particular case when the pp-wave is a sandwich wave [6], that is, it vanishes for $|u| > u_0$ where $u_0 > 0$. A generic feature of such pp-waves is the focusing of geodesics [6, 7] which implies, in particular, that $|\gamma_{ij}|$ vanishes at some point $u > u_0$.

Consequently the amplitude of (10) will be singular at this point and so will $\Sigma_{ab}(z)$ and $\mathbf{A}(z)$. We can further simplify to the case where the pp-wave is delta function supported in u with linear polarization. In this case we have

$$H_{ab}(u) \propto \delta(u) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (27)$$

It is a simple matter to show (e.g. [7]) that E_2^a has a simple zero (and thus γ_{22} has a double zero) while $E_1^a > 0$. Therefore, the zero of $|E_{(2)}(z)|$ and the pole $\mathbf{B}(z)$ cancel against each other so that we are left with a simple pole coming from $\Sigma_{ab}(z)$. Thus, the integral exists in the sense of distributions and is bounded in the CoM energy, s (and also in u).

The remaining terms in which lead to the 3-point amplitude (16) in the perturbative limit will also receive non-perturbative contributions upon replacing $\phi_{(2)}$ and $\phi_{(4)}$ by the exact solution in the pp-wave background. It is not hard to see that that these are bounded in the large s limit.

IV. BACK REACTION

So far we have ignored the back-reaction of the scattered particles on the pp-wave. On the other hand, due to the focusing of the geodesics we expect that the energy-momentum density of the matter (and gravitons) will diverge at the focusing points [7]. That this is indeed the case can be seen by recalling the scalar field $\Phi_2(x)$ in the pp-wave background given in (10). From this we see that the amplitude of $\Phi_2(x)$ grows like $|\gamma_{ij}|^{-\frac{1}{4}} \sim \frac{1}{\sqrt{u}}$ near the focusing point which we set to $u = 0$ for convenience. The dominant contribution to the stress tensor thus comes from

$$T_{uu} \sim \partial_u \Phi_2 \partial_u \Phi_2 \sim \frac{1}{u^3}. \quad (28)$$

Thus back reaction is important near the focusing point and (27) should be modified accordingly. In order to obtain a self-consistent solution let us define $\mathcal{E} := |E| = \sqrt{|\gamma_{ij}|}$. Then we have [7]

$$\frac{\ddot{\mathcal{E}}}{\mathcal{E}} = -\frac{1}{2}R_{uu} - \frac{1}{8}\text{tr}((\gamma^{-1}\dot{\gamma})^2) + \frac{1}{16}\text{tr}((\gamma^{-1}\dot{\gamma}))^2 \quad (29)$$

with

$$R_{uu} = T_{uu} \sim (\partial_u \frac{1}{\sqrt{\mathcal{E}}})^2 \sim \frac{1}{4} \frac{\dot{\mathcal{E}}^2}{\mathcal{E}^3}. \quad (30)$$

Near the focusing point, where \mathcal{E} vanishes, the curvature term dominates so that near $u = 0$,

$$\frac{\ddot{\mathcal{E}}}{\mathcal{E}} \sim -\frac{\dot{\mathcal{E}}^2}{\mathcal{E}^3} \quad (31)$$

which has a first integral, $\dot{\mathcal{E}} = e^{\frac{1}{\mathcal{E}}}$. Let us then consider z as a function of \mathcal{E} , that is,

$$\frac{dz}{d\mathcal{E}} = e^{-\frac{1}{\mathcal{E}}}. \quad (32)$$

Upon substitution into (25) focusing on the pre-exponential factor near the zero-locus of \mathcal{E} , as before, we find

$$\int dz(\dots) \sim \int \frac{\sqrt{\dot{\mathcal{E}}}}{\mathcal{E}} dz = \int \frac{e^{-\frac{1}{2\mathcal{E}}}}{\mathcal{E}} d\mathcal{E} \quad (33)$$

which is finite near the focusing point.

Before we close this section we should perhaps comment on the justification of (30) which we claimed to be the dominating term near the focusing point. This is apparent when expressing $\Phi_{(2)}$ in Einstein-Rosen coordinates

$$\Phi_{(2)}(x) = \frac{e^{ik_v V}}{\sqrt{|E|}}.$$

On the other hand, the actual calculations are done in Brinkmann coordinates where the phase of the scalar field (10) oscillates rapidly near $u = 0$. So one might argue that a more singular contribution to T_{uu} comes from differentiating the phase. However, this is clearly an artifact of the choice of Brinkmann coordinates. Indeed the coordinate transformation

$$\begin{aligned} U &= u, \\ V &= v + \frac{1}{2} \dot{E}^i{}_a E_{bi} x^a x^b = v + \frac{1}{2} \Sigma_{ab} x^a x^b, \\ y^i &= E^i{}_a x^a, \end{aligned} \quad (34)$$

is singular at the focusing point away from $x = 0$. In Rosen coordinates this rapid oscillation is simply expressed by noticing that for $x \neq 0$, $u \rightarrow 0$ maps into $V \sim \frac{x^2}{u} \rightarrow \infty$ which is not a singular point. Another way to see that the phase does not dominate in Brinkmann coordinates either is to consider a wave packet of compact support in the v -direction which will cut-off the wave function, at fixed v in the x -direction as required by causality. On the other hand the pre-exponential factor accounting for (30) is not a coordinate artifact. It simply reflects the focusing of the geodesics which is a geometric property of plane waves [6].

V. DISCUSSION

We have shown that the perturbative 4-particle amplitude evolves in the large s , small t limit into a non-perturbative expression involving a macroscopic non-linear plane wave, which is manifestly unitarity at the expense of smearing out the momentum conservation constraint. This picture is intuitively satisfactory, since due to backreaction, we expect that an energetic graviton sources a growing number of soft gravitons, eventually approaching a classical solution. Earlier approaches

based on related ideas were proposed by 't Hooft and others [2, 3] used a gravitational shock wave to represent an energetic scalar particle (in the geometrical optics approximation) and studied the propagation of a scalar field in that background. One might suggest that this setting should be related to ours via a boosted reference frame in which the incoming scalar $\phi_{(2)}$ is energetic while the graviton $h_{(1)}$ is perturbative. However, since the shock wave approximation [2] works only for point particle sources or superposition thereof [3], the matching is not clear. In fact the scenario of [2] does not have simple perturbative limit. On the other hand, at the calculational level some of our formulae are essentially identical to those in [12] but the physical motivation and interpretation is quite different.

Finally, we should emphasize that our result does not yet allow us to conclude that GR unitarizes itself completely since large momentum transfer, where new physics usually arises, is not covered by our analysis. Other approaches which focus on the large t limit instead

can be found in [14] for instance, where it is argued that black holes may unitarize the cross section in the large t limit. We have nothing new to say about that regime apart, perhaps, that in order to set up an experiment involving gravitons with large momentum transfer at least one of the ingoing gravitons must have energy of the order of M_{Pl} in which case our analysis becomes relevant. The same comment applies, of course, to scattering at transplanckian energies in string theory [15].

ACKNOWLEDGMENTS

The authors would like to thank C. Gomez, Tomas Prochazka for discussions and, in particular, S. Mukhanov for substantial input. This work has been supported by the Excellence Cluster ‘Origins: From the Origin of the Universe to the First Building Blocks of Life’.

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- [1] F. A. Berends and R. Gastmans, *Nucl. Phys.* **B88**, 99 (1975).
- [2] G. 't Hooft, *Phys. Lett.* **B198**, 61 (1987).
- [3] V. Ferrari, P. Pendenza, and G. Veneziano, *Gen. Rel. Grav.* **20**, 1185 (1988).
- [4] Brinkmann, H.W. *Math. Ann.* **94** (1925).
- [5] A. Einstein and N. Rosen, *J. Franklin Inst.* **223**, 43 (1937).
- [6] R. Penrose, *Reviews of Modern Physics* **37** (1965).
- [7] J. Garriga and E. Verdaguer, *Phys. Rev.* **D43**, 391 (1991).
- [8] J. Griffiths, *Oxford mathematical monographs*. Clarendon Press (1991).
- [9] M. Blau, “Plane waves and Penrose limits,” (2011).
- [10] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers, and E. Herlt, *Exact solutions of Einstein's field equations*, Cambridge Monographs on Mathematical Physics (Cambridge Univ. Press, Cambridge, 2003).
- [11] The choice of the linear contribution is fixed by the initial condition to have an ingoing graviton $h = \lambda h_{(3)}$.
- [12] T. Adamo, E. Casali, L. Mason, and S. Nekovar, *Class. Quant. Grav.* **35**, 015004 (2018), arXiv:1706.08925 [hep-th].
- [13] It should be clear from the context that when u is one of the Brinkmann coordinate and when it is one of the Mandelstam variables.
- [14] G. Dvali and C. Gomez, (2010), arXiv:1005.3497 [hep-th].
- [15] G. Dvali, C. Gomez, R. S. Isermann, D. Lust, and S. Stieberger, *Nucl. Phys.* **B893**, 187 (2015), arXiv:1409.7405 [hep-th].