The existence of extreme events is a fascinating phenomenon in natural and social sciences. They appear whenever the probability distribution has a ‘heavy tail’, differing very much from the equilibrium one. Examples are ‘rogue waves’ in the ocean [1] and their analogues in nonlinear optics [2, 3], Lévy flights [4], and other numerous examples in physics [5, 6], biology [7], Earth science [8], etc. [9]. Most famous are the statistics of income and wealth [10] with a power-law (Pareto) probability distribution [11] describing social inequality and responsible for the renowned ‘80/20 rule’. The power laws can be however very different; for some outstanding cases the power exponents are less than 2 leading to indefinite mean values, to say nothing of higher moments. Here we present the first evidence of such probability distributions of photon numbers using nonlinear effects pumped by parametrically amplified vacuum noise, known as bright squeezed vacuum (BSV). We observe a Pareto distribution with power exponent 1.3 when BSV pumps supercontinuum generation, and other heavy-tailed distributions for the optical harmonics generated from BSV. Unlike in other fields, we can flexibly control the Pareto exponent by changing the experimental parameters. Besides photonic applications such as ghost imaging, this extremely fluctuating light is also interesting for quantum thermodynamics as a resource to produce more efficiently non-equilibrium states by single-photon subtraction [12], which we demonstrate in experiment.

‘Common sense’ or, rather, the central limit theorem tells us that the probability distributions of random physical values tend to be Gaussian. The more surprising are deviations from this tendency, from the behaviour of wave height in the ocean to the statistics of solar flares and to the distribution of city populations. Following oceanology, the term ‘rogue waves’ denotes events whose magnitude considerably exceeds the ones expected from Gaussian statistics. A distribution with a high probability of such events is said to have a ‘heavy tail’.

An example of extreme heavy-tailed distribution is the Pareto one, scaling as the power law with an exponent $1 + k$: $P(x) \propto x^{-(1+k)}$. For any $k$, certain statistical moments of this distribution do not converge. At $k < 1$, even the mean value diverges. A power-law distribution can appear due to the exponential amplification of an initially broad distribution [9]. In economics, this happens due to the very non-uniform behaviour of different people [10], causing very ‘unfair’ distribution of the wealth: top 1% own almost the same as the rest 99% [13]. As we show below, a similar phenomenon occurs in nonlinear optics, ‘wealthiest’ 1% of pulses contain 3 times more energy than the rest combined (Fig. 3c), when the exponential rate of frequency conversion is governed by the amplified vacuum noise.

Vacuum noise, also called zero-point vacuum fluctuations, originates from the non-commutativity of photon creation and annihilation operators [14]. It causes spontaneous transitions in atoms as well as spontaneous parametric down-conversion (PDC) and four-wave mixing (FWM). In other words, PDC and FWM ‘visualize’ the vacuum noise by parametrically amplifying it. Because parametric amplification is accompanied by squeezing, the quantum state of light produced through PDC and FWM is called squeezed vacuum. This state manifests superbunching [15, 16], quadrature and photon-number squeezing [17, 19].

Through PDC from strong picosecond pulses, we produce squeezed vacuum that is bright enough to pump nonlinear effects such as the generation of optical harmonics or supercontinuum. The spectral width of the resulting bright squeezed vacuum (BSV) exceeds 35 THz for pumping at 800 nm (Fig. 1a).

Within this spectral band, the vacuum noise is exponentially amplified from its initial ‘brightness’ equivalent to one photon per mode to $N = \sinh^2(G)$, where the parametric gain $G$ scales with the pump laser field amplitude. Figure 1 shows this exponential amplification, reaching $15.3 \pm 0.5$. As a result, the mean energy per pulse for the whole band is up to 10 µJ. Around this mean, the photon number has a very broad distribution (Fig. 1c). It is broader than not only a Poissonian distribution (blue solid line) typical for a shot noise limited laser, but also an exponential one (dashed black line), typical for light with thermal statistics.

We further use BSV to pump optical harmonic generation (Fig. 2a). Panels b,c of Fig. 2 display the probabil-
Continuum is generated (Fig. 3a). The spectrum after the fibre (Fig. 3c), is almost unchanged (grey contour) if BSV is weak (energy per pulse 0.2 nJ), but it is considerably broadened (pink contour) if the input energy per pulse is 40 nJ.

The experimentally obtained probability distribution of photon number per pulse at wavelength 770 nm under pumping with 48 nJ pulses is displayed in Fig. 3c (red points). It has a Pareto scaling with \( k = 0.49 \pm 0.02 \), found from the fit of complementary cumulative distribution function (CCDF) for more accurate determination; see Methods and the Extended Data. The shape of the probability distribution can be well explained assuming that the ‘blue’ side of the supercontinuum is free of Raman processes and is generated through high-gain FWM. The number of photons in the supercontinuum scales then as an exponential function of the number of photons in BSV, which has a broad distribution (Fig. 1).

Calculation based on the probability theory (see Methods for more details).

Indeed, the right-top insets of Fig. 2b, c show the time traces of photon numbers per pulse normalized to their mean values. We see ‘extreme events’: a SH pulse and a TH pulse exceeding their mean values more than 200 and 650 times, respectively. Using the analogy of ‘rogue waves’ in the ocean, this would correspond to a wave of about a kilometer height. This extremely ‘heavy-tailed’ behaviour is because the number of photons in an optical harmonic pulse scales as the power function of the pump number of photons, which in our case already has a very broad distribution. In the case of supercontinuum generation, this tendency is even stronger because the dependence on the pump is exponential.

To obtain supercontinuum, we use BSV centered at 800 nm. The initially 80-nm broad spectrum of BSV is filtered to 10 nm (see Methods) before launching it into the standard single-mode fused silica fibre, where supercontinuum is generated (Fig. 3a). The spectrum after the fibre (Fig. 3c) is almost unchanged (grey contour) if BSV is weak (energy per pulse 0.2 nJ), but it is considerably broadened (pink contour) if the input energy per pulse is 40 nJ.
ods) shows that the supercontinuum will have a Pareto photon-number probability distribution with the exponent $k$ depending on the bandwidth and the mean photon number of BSV. To test this, we measure the probability distributions for the supercontinuum with the BSV energy per pulse reduced to 30 nJ. For bandwidths 10 and 3 nm we obtain $k = 0.64 \pm 0.02$ and $k = 0.31 \pm 0.02$, respectively. The corresponding graphs can be found in the Extended Data section. Thus, the variation of BSV power and bandwidth are versatile instruments to control the index of the Pareto distribution for supercontinuum.

The power-law probability distribution shown in Fig. 3 has very unusual features. With $k < 1$, the mean number of photons per pulse is not defined and depends on the time of observation; it makes our distribution much different from the ones reported earlier [20, 21]. This fractal-like behavior is typical for Pareto distributions where the mean values do not converge [22]. Similar to the ‘coastline paradox’ [23], where the coastline appears the longer, the better one measures, the mean photon number per pulse will be the higher, the more data are used to determine it, $(N) \propto s^{1/k-1}$, where $s$ is the dataset size [9]. In a real experiment the distribution gets always truncated through some physical mechanisms. In our case, this occurs for photon numbers per pulse above $10^6$ due to the detector saturation.

The power-law behaviour is only present within certain spectral bands of the supercontinuum. It occurs on the wings of the spectra and gets suppressed in the middle (Fig. 3f). As the pulse energy increases, the spectrum broadens and the fluctuations get stronger but they also move further from the initial central wavelength. The reason is that at near-centre frequencies, the supercontinuum is so bright that it itself generates new sidebands through FWM. But because FWM is equivalent to two-photon loss, it leads to the depletion of intensity fluctuations [24]. The same tendency is visible in the second-order normalized correlation function $g^{(2)}(\lambda, \lambda') = \langle N(\lambda)N'(\lambda') \rangle / \langle N(\lambda) \rangle \langle N(\lambda') \rangle$. In the right-bottom panel, we see $g^{(2)}$ values as high as 170, which considerably exceeds the strongest superbunching reported to date [25, 26]. Similarly to the mean, the measured value depends on the time of observation and is just the lower boundary of superbunching.

Superbunching has interesting consequences for photon subtraction experiments [27] (Fig. 4a), used for example to test quantum thermodynamics [28]. In such an experiment, a quantum state of light $|\psi\rangle$ is fed to a beamsplitter, after which a single-photon detector can register a reflected photon. Provided that the detector registers a single photon, the state of light after the beamsplitter is a photon-subtracted state, $|\mu\rangle \propto \hat{a}^\dagger |\psi\rangle$, where $\hat{a}$ is the photon annihilation operator. Counterintuitively, the photon-subtracted state has the mean photon number increased by a factor $g^{(2)}$ compared to the initial

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**FIG. 3. Supercontinuum pumped by squeezed vacuum.** a, the experimental setup: after BSV is generated in a BBO crystal, the pump is cut off by a dichroic mirror (DM) and BSV is filtered by a bandpass filter (BP) and coupled into a single-mode fibre (SMF); further, the supercontinuum is analyzed either by a CCD camera after a spectrometer or by a photodetector (PD) after a monochromator. b, the resulting average spectrum under pumping with 0.2 nJ pulses (grey) and 40 nJ pulses (pink). c, the probability distribution of photon number per pulse (points), for supercontinuum pumped with 48 nJ pulses (2 nJ pulses (grey) and 40 nJ pulses (pink)). d-f, spectral distribution of photon-number fluctuations: single-pulse spectra of supercontinuum for the input energy per pulse 24 nJ (d) and 40 nJ (f) and the corresponding distributions of the normalized second-order correlation function (e and g, respectively).

The inset shows a time trace for pulse height normalized to the mean. d-g, spectral distribution of photon-number fluctuations: single-pulse spectra of supercontinuum for the input energy per pulse 24 nJ (d) and 40 nJ (f) and the corresponding distributions of the normalized second-order correlation function (e and g, respectively).
state \[12, 29\]. The photon subtraction process admits a direct interpretation similar to Maxwell’s demon experiment \[28\]. Figure 4 shows the result of photon subtraction from the state of supercontinuum, after its frequency filtering with a monochromator at wavelength 780 nm. The mean photon number is increased up to 140 times as a result of photon subtraction, depending on the bandwidth and the energy per pulse of the BSV pumping the supercontinuum generation. This drastic increase in the mean photon number shows that the supercontinuum can be brought out of equilibrium by the subtraction process much more efficiently than thermal or BSV light. As a result, much more work could be, in principle, extracted, which makes the supercontinuum a useful resource for quantum thermodynamics \[12\].

![FIG. 4. Photon subtraction from supercontinuum.](image)

In conclusion, we have demonstrated heavy-tailed distributions of photon numbers for nonlinear effects generated from bright squeezed vacuum. In particular, supercontinuum generation leads to an extremal distribution of photon numbers for nonlinear effects and higher moments. The mechanism behind this phenomenon is the exponential dependence of the frequency conversion efficiency on the photon number of the pump, which has already a very broad distribution. The subtraction of a single photon from the supercontinuum increases the mean number of photons by more than two orders of magnitude, which can be further exploited to extract large amount of thermodynamical work. 

In addition, light with strong fluctuations in the photon number can be useful in ghost imaging \[30, 31\] where the contrast of the image is given by the bunching parameter \[32, 33\]. In particular, recently time-domain ghost imaging has been demonstrated with incoherent supercontinuum, although without high photon-number fluctuations \[34\]. The use of supercontinuum with Pareto photon distribution will drastically increase the contrast.

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METHODS

Generation of BSV. For the experiments with optical harmonics, BSV is generated in a 10 mm beta barium borate (BBO) crystal through type-I collinear frequency-degenerate PDC [35] pumped by 1.6 ps pulses of regeneratively amplified Ti:sapphire laser at 800 nm with a 5 kHz repetition rate and energy per pulse up to 0.5 mJ (Fig. 2b). To reduce the effect of spatial walk-off [36], the pump beam is focused into the crystal with a cylindrical lens. The resulting BSV spectrum is centered at 1600 nm (Fig. 1b). For the supercontinuum generation, BSV at 800 nm is used, generated through type-I collinear degenerate PDC in two 3 mm BBO crystals [37] from the frequency doubled radiation of the same laser (wavelength 400 nm, energy per pulse up to 0.2 mJ). After cutting off BSV with a dichroic mirror, BSV is filtered spatially and spectrally: with a slit and 4f monochromator (Fig. 3a) to a few-mode case, respectively. In the latter case the fibre itself provides single-mode spatial filtering.

Generation of optical harmonics. SH and TH are generated by tightly focusing BSV on the surface of a 1 mm slab of lithium niobate crystal, with the z crystal axis in the plane of the slab and the BSV polarized along z. Under this condition, the largest components of second and third nonlinear susceptibilities are used, providing a high efficiency even though phase matching is not satisfied [35] (Fig. 2b). The photon number distribution of BSV is measured by a charge-integrating detector based on an infrared p-i-n diode, providing a linear response up to 10^6 photons per pulse and noise equivalent to 270 photons per pulse [18].

After cutting off BSV with a short-pass filter, each harmonic is additionally filtered with bandpass filters at 800 nm (SH) or 532 nm (TH) to block the other one.

Supercontinuum is generated in a 5 m single-mode fused silica fibre and further, to achieve single-mode detection, spectrally filtered with a monochromator whose resolution is 1 nm (Fig. 3b). The photon-number probability distribution is measured by a visible charge-integrating detector (the same as in the case of optical harmonics). The single-pulse spectra are recorded by a CCD camera at the output of a spectrometer (Fig. 3f) and further processed to measure the second-order normalized correlation function g(2)(λ, λ′) shown in Fig. 3g.

Probability distribution for optical harmonics. In the absence of pump depletion, the number of photons in the nth harmonic is

\[ N_{nw} = KN_{w}^{n}, \]

where \( K \) depends on the conversion efficiency and \( N_{w} \) is the number of photons in the fundamental radiation.

The probability distribution \( P_{nw}(N_{nw}) \) for the harmonic radiation can be obtained from the one for the fundamental radiation \( P_{\omega}(N_{\omega}) \) as \( P_{nw}(N_{nw})dN_{nw} = P_{\omega}(N_{\omega})dN_{\omega} \):

\[ P_{nw}(N_{nw}) = \frac{P_{\omega} \left( \sqrt{N_{nw}/K} \right)}{n \sqrt{K} N_{nw}^{1-1/n}}. \]

For BSV, the envelope of the photon-number distribution is [38, 39]

\[ P_{B}(N_B) = \frac{1}{\sqrt{2\pi(N_B)^{3/2}}}e^{-\frac{N_B}{2(N_B)^{3/2}}}. \]

Then, for its \( n^{th} \) harmonic we get

\[ P_{nw}(N_{nw}) = \frac{2^{\gamma(2n-1)!}}{n\sqrt{2\pi \gamma(N_{nw})^{1/n}}} e^{-\frac{1}{2} \gamma(N_{nw})^{1/n}}. \]

This is a heavy-tailed generalized Gamma distribution but all its moments are finite.

Probability distribution for supercontinuum is obtained by assuming that the latter is generated through FWM from BSV. The number of photons in supercontinuum is then \( N_{SC} = \sinh^{2}(\kappa N_{B}) \), where \( N_{B} \) is the BSV photon number and \( \kappa \) characterizes the interaction strength.

Similar to the harmonics case, from the BSV photon-number distribution \( P_{B}(N_B) \) we obtain the supercontinuum photon-number distribution:

\[ P_{SC}(N) = \frac{e^{-\frac{\text{arcsinh}N}{2\kappa(N_B)}}}{\sqrt{8\pi\kappa(N_B)(1+N)\text{arcsinh}\sqrt{N}}}, \]

where \( \langle N_B \rangle \) is the BSV mean photon number.

For large \( N \), Eq. (5) has asymptotic scaling typical for the Pareto distribution [11],

\[ P(N) \propto \frac{1}{N^{1+k}}, \]

with the tail exponent (Pareto index) \( k > 0 \) (see Extended Data, Fig. 5). The latter tends to zero at \( \kappa(N_B) \gg 1 \).

Analysis of the CCDF. We characterize the tails of probability distributions through their complementary cumulative distribution functions (CCDF) [40],

\[ \tilde{C}(N) = \int_{N}^{\infty} P(N')dN', \]

whose analysis is possible even if the moments diverge.

The tail index of a distribution is defined as

\[ \alpha = \lim_{N \to \infty} \frac{H(N)}{N}, \]
where $H(N) = -\log \tilde{C}(N)$ is called the hazard function. If $\alpha = \text{const}$, the distribution decays exponentially. For diverging $\alpha$ the tail decays faster than exponential, for $\alpha = 0$ slower. The latter means a heavy-tailed distribution \[\text{11}\].

For the $n$th optical harmonic of BSV, the CCDF is

\[ \tilde{C}_{n\omega}(N_{n\omega}) = \text{Erfc} \left[ \frac{2^{1/2} \sqrt{(2n-1)!!}}{\sqrt{2}} \frac{N_{n\omega}}{N_{B}} \right], \tag{9} \]

where $\text{Erfc}(x)$ is the complementary error function. CCDFs for SH and TH, together with the experimental data, are presented in Extended Data, Fig. 7. From these dependences and the corresponding $H(N)/N$ dependences (Extended Data, Fig. 8), one can see that $\alpha$ tends fast to zero, therefore the distributions are heavy-tailed.

In the case of supercontinuum generated from BSV, the CCDF is

\[ \tilde{C}_{SC}(N) = \text{Erfc} \left[ \frac{\arcsinh \sqrt{N}}{2 \kappa(N_B)} \right]. \tag{10} \]

Similarly to the case of harmonics, $\alpha$ tends to zero, the distribution is heavy-tailed. However, it exhibits a faster tendency to zero than for any harmonic (Extended Data, Fig. 8).

The CCDF \[\text{10}\] is of the form $N^{-k}L(N)$, where $k$ is the tail exponent and $L(N)$ is a slowly varying function (i.e. $\lim_{N \to \infty} \frac{L(tN)}{L(N)} = 1$, for any $t > 1$ \[\text{11}\]). Therefore distribution \[\text{5}\] belongs to regularly varying distributions with a finite tail exponent. It is tail equivalent to the Pareto distribution with the same $k$. The tail exponent tends to $(4\kappa(N_B))^{-1}$. For a sufficiently strong pump, $k$ goes below unity and then, all moments are indefinite. It makes \[\text{5}\] very different from the other heavy-tailed distributions.

The tail exponents linearly increase with the number of modes. Indeed, BSV with $M$ modes has

\[ P_{B,M}(N_B) = \frac{N_B^{M/2-1}}{\Gamma(M/2)} \left( \frac{M}{2(N_B)} \right)^{M/2} e^{-\frac{M N_B}{2(N_B)}}, \tag{11} \]

where $\Gamma(x)$ is the gamma function. The corresponding CCDF for supercontinuum is

\[ \tilde{C}_{SC,M}(N) = \frac{\Gamma \left( \frac{M}{2}, \frac{\arcsinh \sqrt{N}}{2 \kappa(N_B)/M} \right)}{\Gamma(M/2)}, \tag{12} \]

where $\Gamma(s, x)$ is the upper incomplete Gamma function. The latter leads to

\[ k = \frac{M}{4\kappa(N_B)}. \tag{13} \]

Remarkably, the distribution \[\text{11}\] for $M = 2$ becomes exactly the exponential one. As a result, Eq. \[\text{13}\] immediately tells us that with a BSV pump, $k < 1$ is achieved with twice lower mean photon number than with a thermal pump.

**Experimental tail exponents** $k$, are derived from the fits of the linear part of CCDFs in the log-log scale. The error of the fit $\Delta k_e = 0.02$ is mainly caused by imprecise estimation of the starting and ending points of this range. The other method, based on the calculation of the maximum likelihood estimator \[\text{9}\], provides similar exponents in our case. An example of CCDF with both fits and the resulting exponents are presented in Extended Data, Fig. 7 and Table 4.

The theoretical exponents $k$, estimated from Eq. \[\text{13}\], are somewhat smaller than the experimental ones. We get the mean gain value $(\kappa(N_B) = 4 \pm 0.5$ and $2.5 \pm 0.5$ for $P = 48$ and $30$ nJ, respectively) from the nonlinear dependence, similar to the one for BSV (Fig. \[\text{1b}\]). The number $M$ of modes ($M = 5 \pm 1$ and $2 \pm 0.2$ for $\Delta \lambda_B = 10$ and $3$ nm, respectively) is obtained from the $g^{(2)}$ value \[\text{10}\] for extremely weak BSV ($P = 3$ pJ) after the fibre. The uncertainties in both measurements result in the relative error $\delta k = 25\%$.

**Taking detection noise into account.** The dark noise of charge-integrating photodetectors is well described by a Gaussian distribution with zero mean,

\[ P(N) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{N^2}{2\sigma^2}}, \tag{14} \]

with the standard deviation $\sigma$. Because this noise is independent from the fluctuations of the detected light, it just adds to the photon-number noise.

The total probability distribution is given by the convolution of the photon-number and dark noise probability distributions. The convolution of Eq. \[\text{14}\] with Eq. \[\text{4}\], with no fitting parameters, perfectly coincides with the experimental histograms for the generated harmonics (Fig. 2).

**Photon subtraction** experiment is performed using two avalanche photodiodes in a Hanbury Brown Twiss (HBT) configuration. After spectral filtering of the supercontinuum down to a 1 nm bandwidth, the output of the monochromator is attenuated with a neutral density filter in order to obtain on average much less than one photon ($\langle N_{\psi} \rangle \ll 1$) before the beamsplitter of the HBT setup. The mean number of photons after measurement of a single photon in one arm ($N_{\psi'}$) is measured with the second avalanche photodiode.
EXTENDED DATA

FIG. 5. Calculated photon-number distribution for FWM from BSV with \( \kappa(N_B) = 0.5 \) (black line) and 2.5 (red line). Multimode pumping changes the tail exponent \( k \); as an example, the distribution for \( \kappa(N_B) = 2.5 \) and \( M = 5 \) modes (blue line) demonstrates nearly the same scaling as single-mode pumping with \( \kappa(N_B) = 0.5 \) (black line).

FIG. 6. Photon-number histograms for the supercontinuum pumped by 30 nJ pulses of BSV with bandwidth \( \Delta \lambda_B = 10 \text{ nm} \) (red points) and \( \Delta \lambda_B = 3 \text{ nm} \) (black points). The Pareto fits are shown by blue lines. The peak below \( 1 \ldots 3 \times 10^4 \) photons follows the exponential distribution; we attribute it to the thermalization effect in supercontinuum [41].

<table>
<thead>
<tr>
<th></th>
<th>( P ), ( \Delta \lambda_B )</th>
<th>( k_e )</th>
<th>( k_t )</th>
<th>( \langle N \rangle ), photons/pulse</th>
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<td>Fig. 3</td>
<td>10 nm</td>
<td>0.49</td>
<td>0.31</td>
<td>( 1.18 \times 10^4 )</td>
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<td>Fig. 3</td>
<td>30 nm</td>
<td>0.64</td>
<td>0.5</td>
<td>( 6.5 \times 10^3 )</td>
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<td>Fig. 6</td>
<td>30 nJ, 3 nm</td>
<td>0.30</td>
<td>0.2</td>
<td>( 5.6 \times 10^3 )</td>
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TABLE I. Characteristics of the supercontinuum photon-number distributions from Figs. 3 and 6. \( P \) and \( \Delta \lambda_B \) are BSV energy per pulse and bandwidth, respectively; \( k_e \) and \( k_t \) are experimental and theoretical tail exponents. Despite \( k \) being less than 1, the mean number of photons \( \langle N \rangle \) exists for the measured distributions due to truncation.

FIG. 7. (a) CCDFs for the data on optical harmonics from Fig. 2b,c: 2\( \omega \) (black points) and 3\( \omega \) (red points). The corresponding theoretical distributions [9] for \( n = 2, 3 \) are shown by gray and pink lines respectively. \( \langle N_{n\omega} \rangle \) are taken from the experimental data. (b) CCDF for the data on the supercontinuum from Fig. 3 (red points), its fit (blue line) with Eq. (6) leading to \( k_e = 0.49 \), and Eq. (6) with \( k = 0.53 \) (black line) obtained from the maximum likelihood estimator [9].

FIG. 8. Experimental (points) and theoretical (lines) \( H(N)/N \) for second (black) and third (red) harmonics from Fig. 2b,c and supercontinuum (blue) from Fig. 3. For the latter, the theoretical values are calculated from a fit with the Pareto distribution (Fig. 7b).