HIDDEN SYMMETRIES AND DECAY FOR THE VLASOV EQUATION ON THE KERR SPACETIME

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ABSTRACT. This paper proves the existence of a bounded energy and integrated energy decay for solutions of the massless Vlasov equation in the exterior of a very slowly rotating Kerr spacetime. This combines methods previously developed to prove similar results for the wave equation on the exterior of a very slowly rotating Kerr spacetime with recent work applying the vector-field method to the relativistic Vlasov equation.

1. Introduction

In this paper we prove the existence of a bounded energy, and an integrated energy decay estimate for solutions of the massless Vlasov equation in the exterior of a very slowly rotating Kerr spacetime.

For parameters a, M, with $|a| \leq M$, the exterior region of the Kerr spacetime is represented in Boyer-Lindquist coordinates (t, r, θ, ϕ) by $\mathbb{R} \times (r_+, \infty) \times S^2$ with the Lorentzian metric

$$g = -\left(1 - \frac{2Mr}{\Sigma}\right) \mathrm{d}t^2 - \frac{4Mar\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\phi + \frac{\Pi\sin^2\theta}{\Sigma}\mathrm{d}\phi^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2, \tag{1.1}$$

where $r_+ = M + \sqrt{M^2 - a^2}$, and

$$\Delta = r^2 - 2Mr + a^2, \qquad \Sigma = r^2 + a^2\cos^2\theta, \qquad \Pi = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta.$$

For $|a| \leq M$, the Kerr spacetimes contain a black hole and are stationary and axisymmetric, that is to say ∂_t and ∂_ϕ are Killing vector fields. Although the exterior is extendible as an analytic manifold, it is globally hyperbolic and foliated by surfaces of constant t, Σ_t, which are Cauchy surfaces.

The Vlasov equation governs the evolution of massive or massless particles which do not self-interact [14]. The particles are represented by a distribution function on phase space, which evolves under the geodesic flow, so it is constant along geodesics. In the context of kinetic theory, the equation is known as the collisionless Boltzmann equation.

Let $(\mathcal{M}, g)$ be a time oriented Lorentzian manifold of dimension $1 + 3$, with timelike vector field T_+. For the case of massless Vlasov, the distribution function is a non-negative function defined on the bundle of future light cones $\mathcal{C}^+$,

$$\mathcal{C}^+ = \bigcup_{x \in \mathcal{M}} \mathcal{C}^+_x,$$

$$\mathcal{C}^+_x = \{(x, v) : v \in T_p\mathcal{M},\ g(v, v) = 0, g(v, T_+) < 0\}.$$

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For $m > 0$, the set $g(v,v) = -m^2$, $g(v,T_+) < 0$ is sometimes called the mass shell, and $\mathcal{C}^+$ is its analogue for the massless case considered here. The Vlasov equation is

$$\mathcal{X} f = 0, \tag{1.2}$$

where $\mathcal{X}$ is the geodesic spray, the vector field on $T\mathcal{M}$ which generates the geodesic flow. The geodesic spray is the Lagrangian vector field of $L = \frac{1}{2}g(v,v)$ [1, §3.7]. We have that $\mathcal{X}L = 0$, in particular $\mathcal{X}$ is tangent to $\mathcal{C}^+$. In case the distribution function f is a function $f : \mathcal{C}^+ \to \mathbb{R}$, we shall refer to equation (1.2) as the massless Vlasov equation.

A local coordinate system (x^a) on $\mathcal{M}$ induces natural coordinates (x^a, v^a) on $T\mathcal{M}$, where $v^a = dx^a(v)$. The coordinate form of $\mathcal{X}$ is

$$\mathcal{X} = v^a \left(\frac{\partial}{\partial x^a} - v^b \Gamma^c{}_{ab} \frac{\partial}{\partial v^c} \right), \tag{1.3}$$

where $\Gamma^c{}_{ab}$ is the Christoffel symbol of the metric g_{ab}.

In the Kerr exterior, it is convenient to use the Boyer-Lindquist coordinates $(x^a) = (t, r, \theta, \phi)$ and the corresponding natural coordinates (x^a, v^a). On $\mathcal{C}^+$, we locally use coordinates $(t, r, \theta, \phi, v^r, v^\theta, v^\phi)$, and treat the quantities $v^t, v_t, v_r, v_\theta, v_\phi$ as functions of these. To facilitate the presentation of our main result, we introduce

$$E_{\text{model},3}[f](t) = \int_{\Sigma_t} \int_{\mathcal{C}^+_x} \left(\frac{(r^2+a^2)^2}{\Delta} v_t^2 + \Delta v_r^2 + v_\theta^2 + \frac{1}{\sin^2\theta} v_\phi^2 \right) |f|_2 \mathrm{d}^3 v \mathrm{d}^3 x,$$

where

$$|f|_2 = \left| M^2 v_t^2 + v_\theta^2 + \frac{1}{\sin^2\theta} v_\phi^2 \right|^2 f, \quad \mathrm{d}^3 x = \sin\theta \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi, \quad \mathrm{d}^3 v = \frac{1}{|v_t|} r^2 \sin\theta \mathrm{d}v^r \mathrm{d}v^\theta \mathrm{d}v^\phi.$$

The term $|f|_2$ should be understood as a strengthening of the f by two factors of $M^2 v_t + v_\phi^2 + \sin^{-2}\theta v_\theta^2$. As explained in Section 3.2, these two factors arise from strengthening the energy by two second-order multiplication symmetries of the Vlasov equation. The volume forms d^3x and d^3v are given here because they have simple coordinate expressions, although they are not the naturally induced volume forms defined on Σ_t and $\mathcal{C}^+_x$, which are used in the rest of this paper and introduced in Sections 2.1 and 2.2.

Our main results are

Theorem 1 (Uniformly bounded energy). *There are positive constants C and $\bar\epsilon$ such that if $M > 0$, $|a| \le \bar\epsilon M$, and $f : \mathcal{C}^+ \to [0,\infty)$ is a smooth solution of the Vlasov equation (1.2) in the exterior of the Kerr spacetime with parameters (M,a), then, for all t in $\mathbb{R}$,*

$$E_{\mathit{model,3}}[f](t) \le C E_{\mathit{model,3}}[f](0). \tag{1.4}$$

Theorem 2 (Morawetz estimate). *There are positive constants C, $\bar\epsilon$, and $\bar r$ and a function $\mathbf{1}_{r\not\simeq 3M}$ which is identically 1 for $|r-3M| \ge \bar r$ and zero otherwise such that if $M > 0$, $|a| \le \bar\epsilon M$, and $f : \mathcal{C}^+ \to [0,\infty)$ is a smooth solution of the Vlasov equation (1.2) in the exterior of the Kerr spacetime with parameters (M,a), then,*

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{\Sigma_t} \int_{\mathcal{C}^+_x} \left(\left(M \frac{\Delta^2}{(r^2+a^2)^2} \right) v_r^2 + \mathbf{1}_{r\not\simeq 3M} \frac{1}{r} \left(M^2 v_t^2 + v_\theta^2 + \frac{1}{\sin^2\theta} v_\phi^2 \right) \right) |f|_2 \mathrm{d}^3 v \mathrm{d}^4 x, \\ \le C E_{\mathit{model,3}}[f](0), \end{aligned} \tag{1.5}$$

where $\mathrm{d}^4x = \mathrm{d}\bar{t}\mathrm{d}^3x$.

More precisely,

$$\int_{-\infty}^{\infty}\int_{\Sigma_t}\int_{\mathcal{C}_x^+} M\frac{\Delta^2}{(r^2+a^2)^2}v_r^2|f|_2 + r^5\tilde{\mathcal{R}}'\tilde{\mathcal{R}}'\mathcal{L}f\,\mathrm{d}\mu_{\mathcal{C}_x^+}\mathrm{d}\mu_g \leq CE_{model,3}[f](0), \tag{1.6}$$

where $\tilde{\mathcal{R}}'$ *is given in equation* (3.8) *and where* $\mathrm{d}\mu_{\mathcal{C}_x^+}$ *and* $\mathrm{d}\mu_g$ *are the natural volume forms on* $\mathcal{C}_x^+$ *and* $\mathcal{M}$ *by the metric* g.

The main innovation in this paper is to combine the vector field technique introduced in [13] for proving dispersive estimate for the relativistic Vlasov equation with earlier work on dispersion of fields outside a Kerr black hole, in particular the method of [2], see also [3].

The method used in [2] is a generalization of the vector-field method, which relies on the stress-energy tensor and spacetime symmetries to construct momenta appropriate for the proof of energy estimates and integrated energy estimates. The proof of the non-linear stability of Minkowski space [9] is an important application of this method. The vector-field method was recently applied to prove dispersive estimates for the relativistic Vlasov equation as part of a proof of non-linear stability for the massless and massive Vlasov-Nordström systems on Minkowski space [13] (see also [17] for the nonrelativistic Vlasov equation). Previous stability results for Minkowski space as a solution of the Einstein-Vlasov system include the massive [15] and massless Vlasov cases [10] in spherical symmetry and, recently, the massless case without symmetry [19].

Energy bounds and Morawetz estimates, analogous to Theorems 1 and 2 respectively, have already been proved for the wave equation outside a very slowly rotating Kerr black hole [11, 18, 2]. Strictly speaking, an energy bound should be an integral over spacelike hypersurfaces of an integrand that is quadratic in v, rather than of order 6, as appears in $E_{\mathrm{model},3}[f]$, but we will consistently ignore this distinction. Away from (an open set about) $r = 3M$, the horizon at $r = r_+$, and null infinity at $r \to \infty$, the integrand in the Morawetz estimate is a bounded multiple of the integrand appearing in the energy; however, the integral is over all of space-time, instead of a single spacelike hypersurface. Thus, the Morawetz estimate implies that the local energy in a fixed r region (away from $r = r_+$ and $r \to \infty$ and sufficiently far from $r = 3M$) is integrable in time. Hence, on average, it must decay in time. Thus, Morawetz estimates are also called integrated local energy decay estimates. Energy bounds and Morawetz estimates are a useful tool in proving pointwise estimates, for instance of the form $\sup_{r\in(r_+,R],(\theta,\phi)\in S^2}|\psi(t,r,\theta,\phi)| \lesssim t^{-p}$ for some p. For the wave equation in the subextremal range $|a| < M$, the entire argument from energy estimates and Morawetz bounds to pointwise bounds can be found in [12].

In the Kerr spacetime, there are null geodesics that can orbit at fixed r, and these are the primary obstacle in proving Morawetz estimates. Furthermore, for $|a| > 0$, the vector field ∂_t ceases to be timelike near $r = r_+$, which prevents the existence of a conserved, positive energy. Since both the wave equation and massless Vlasov equation admit solutions that approximate null geodesics for arbitrary lengths of time, any Morawetz estimate must degenerate on such solutions. On the orbiting null geodesics, the factor $\tilde{\mathcal{R}}'$ vanishes, providing sufficient degeneracy; for $|a| \ll M$, the roots of $\tilde{\mathcal{R}}'$ are all near $r = 3M$, which is why $\tilde{\mathcal{R}}'^2$ in equation (1.6) can be replaced by $\mathbf{1}_{r\not\approx 3M}(M^2v_t^2v_\theta^2 + \sin^{-2}\theta v_\phi^2)^2$

in equation (1.5). Our analysis is dependent on the fact that $\tilde{\mathcal{R}}'$ can be expanded purely in terms of r dependent factors and constants of motion along the null geodesics. This is a consequence of the remarkable observation of Carter that, in addition to the geodesic constants of motion arising from the metric and the two Killing vectors, there is a fourth constant of motion, called a hidden symmetry [8].

Steady states for the massive Vlasov equation in the exterior of a fixed Schwarzschild space-time (where $a = 0$ and representing the exterior of a star or black hole) have been constructed and used to study accretion disks [16]. The existence of these steady states implies that no Morawetz estimate, analagous to Theorem 2, can hold for the massive Vlasov equation outside a Schwarzschild black hole.

The formation of black holes for the massive Einstein-Vlasov system has been studied in [4, 6]. For the coupled Einstein-massless Vlasov system, there are spherically symmetric steady states [5]. The existence of such solutions suggests, in contrast to the results in this paper, that there are spherically symmetric solutions of the Einstein-massless Vlasov system which have a nonzero, static configuration of massless Vlasov matter outside a Schwarzschild-like black hole. However such solutions seem to require a large Vlasov field and cannot be small perturbations of the Schwarzschild solution.

The paper is organized as follows. Section 2 contains an introduction to the geometry of the Kerr spacetime, and in particular a discussion on the multiplication symmetries of the field in Section 2.1, and a presentation of the properties of the stress-energy tensor of the Vlasov equation in Section 2.2. Section 3 contains the proof of the Morawetz estimates. The relevant energies are defined in Section 3.1; Section 3.2 introduces the vector field used to perform the estimates; relevant bulk terms are estimated in Section 3.3; and the proof is concluded in Section 3.4.

2. Preliminaries

Throughout, the indices $a, b, c, \dots$ denote integers in $\{0, 1, 2, 3\}$. Underlined indices $\underline{a}_1, \dots, \underline{a}_k$ are used exclusively to parametrize the set of symmetries used for the calculation, as explained in Section 2.1. The Einstein summation convention is used throughout the paper.

2.1. The Kerr geometry.

For $M > 0$ and $|a| \leq M$, the exterior region of the Kerr space-time is $(t, r, \omega) \in \mathbb{R} \times (r_+, \infty) \times S^2$, where $r_+ = M + \sqrt{M^2 - a^2}$ is the larger of the two roots of $\Delta = 0$, together with the metric given in equation (1.1). Typically, we will parameterise S^2 be spherical coordinates θ, ϕ. Although this exterior can be extended as a smooth Lorentzian manifold, it is globally hyperbolic, with the surfaces of constant t providing a foliation by Cauchy hypersurfaces. Thus, there is a well-posed initial-value-problem for many PDEs, including the Vlasov equation, with initial data posed, for example, on the hypersurface $t = 0$. For $M > 0$ and $|a| \leq M$, the exterior region of the Kerr space-time describes the exterior region of a rotating black hole.

The vector field $T_\perp = \partial_t + \omega_\perp \partial_\phi$ with $\omega_\perp = 2aMr/\Pi$ is orthogonal to the surfaces of constant t, and hence to $\partial_r, \partial_\theta, \partial_\phi$. This vector field is not normalised, and, instead, $g(T_\perp, T_\perp) = -\Delta\Sigma/\Pi$. The rotation speed of the black hole is $\omega_{\mathcal{H}} = a/(r_+^2 + a^2)$. Independently of θ, one has $\omega_{\mathcal{H}} = \lim_{r \to r_+} \omega_\perp$.

Many calculations are simplified by working only with the following form of the inverse Kerr metric:

$$\Sigma g^{ab} = \Delta \partial_r^a \partial_r^b + \frac{1}{\Delta}\mathcal{R}^{ab}, \tag{2.1}$$

where

$$\begin{aligned}
\Delta &= r^2 - 2Mr + a^2,\\
\Sigma &= \Omega^{-2} = r^2 + a^2\cos^2\theta,\\
\mathcal{R}^{ab} &= -(r^2+a^2)^2\partial_t^a\partial_t^b - 4aMr\partial_t^{(a}\partial_\phi^{b)} + (\Delta - a^2)\partial_\phi^a\partial_\phi^b + \Delta Q^{ab},\\
Q^{ab} &= \partial_\theta^a\partial_\theta^b + \cot^2\theta\partial_\phi^a\partial_\phi^b + a^2\sin^2\theta\partial_t^a\partial_t^b.
\end{aligned}$$

This form of the expression allows us to avoid having to work with $\Pi = (r^2+a^2)^2 - a^2\Delta\sin\theta$, except when working with $T_\perp = \partial_t + (2aMr/\Pi)\partial_\phi$. In fact, except in the volume form and inside Q^{ab}, this notation typically allows us to avoid all θ dependent factors. It will later be useful to use a conformal factor Ω defined by

$$\Omega^{-2} = \Sigma.$$

The volume form on the Kerr exterior in Boyer-Lindquist coordinates is

$$\sqrt{|g|}\mathrm{d}t\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = \Sigma\sin\theta\mathrm{d}t\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi.$$

Since Σ is uniformly equivalent to r^2, integrals with respect to this volume form are equivalent to those with respect to $\mathrm{d}^3x\mathrm{d}t$.

In the Kerr spacetime, ∂_t and ∂_ϕ are Killing vectors and Q^{ab} is a conformal Killing tensor. We use the following notation

$$e = v_a\partial_t^a, \qquad l_z = v_a\partial_\phi^a, \qquad q = v_av_bQ^{ab}.$$

A basis for the multiplicative factors that give symmetries for the null Vlasov equation is

$$\mathbb{S} = \bigcup_{n=0}^{\infty}\mathbb{S}_n, \qquad \mathbb{S}_n = \{e^{n_t}l_z^{n_\phi}q^{n_q} : n_t + n_\phi + 2n_q = n\}.$$

Of particular importance in this analysis is

$$\mathbb{S}_2 = \{e^2, el_z, l_z^2, q\} = \{S_{\underline{a}}\}_{\underline{a}}.$$

Since each element of $\mathbb{S}_2$ is the contraction of a 2-tensor with v_av_b, thus, we introduce $S_{\underline{a}}^{ab}$ such that

$$S_{\underline{a}} = S_{\underline{a}}^{ab}v_av_b.$$

The quantity $\mathcal{R}^{ab}v_av_b$ plays a crucial role in our analysis. It can be written as a linear combination of the $S_{\underline{a}}$, with coefficients that are polynomial in r, M, a. To simplify a lot of the calculations, we use the notation $\mathcal{R}^{\underline{a}}$ to denote these coefficients and use the Einstein summation convention in the $\underline{a}$ indices. We also use the notation $\mathcal{R}$ to denote $\mathcal{R}^{ab}v_av_b$. Thus, we have four expressions for the following quantity

$$\mathcal{R} = \mathcal{R}^{ab}v_av_b = \mathcal{R}^{\underline{a}}S_{\underline{a}} = \mathcal{R}^{\underline{a}}S_{\underline{a}}^{ab}v_av_b. \tag{2.2}$$

For other quantities that are linear combinations of the $S_{\underline{a}}$ (possibly with coefficients that are polynomial or rational functions of r, M, a), we also use the Einstein summation convention in the $\underline{a}$ variables to expand the quantity. In addition to $\mathcal{R}$ and derivatives of rescalings of it, we also use

$$\mathcal{L} = M^2 e^2 + l_z^2 + q,$$

which can be expanded as

$$\mathcal{L}^{\underline{a}} S_{\underline{a}} = \mathcal{L}^{\underline{a}} S_{\underline{a}}^{ab} v_a v_b. \tag{2.3}$$

One crucial way in which $\mathcal{R}$ appears in the analysis of whether null geodesics fall into the black hole, asymptote to an orbiting null geodesic, or escape to infinity. Equation (2.1) can be contracted with $v_\alpha v_\beta$ to derive an ODE for $\mathrm{d}r/\mathrm{d}\lambda$. One consequence of this is that a null geodesic has a turning point, where $\mathrm{d}r/\mathrm{d}\lambda$ vanishes, only when $\mathcal{R} = 0$. Furthermore, by a standard dynamical systems analysis of one-dimensional systems, there can only be a trajectory remaining at fixed r when $\mathcal{R} = 0 = \partial_r \mathcal{R}$.

Following [2], in the remainder of the paper, it is useful to consider double-indexed collections of vector fields $X^{a\underline{ab}}$. These are sometimes called 2-symmetry-strengthened vector fields. The same notation is also used for stress-energy tensors, cf. Section 2.2.

2.2. **The stress-energy tensor and symmetries of the Vlasov equation.** Throughout this subsection, let $(\mathcal{M}, g)$ be a globally hyperbolic, Lorentzian manifold of dimension $3+1$. Consider the vector bundle $\mathcal{V} = T\mathcal{M}$, and consider $\mathcal{C}^+$.

The volume element on $\mathcal{C}_x^+$ induced from the volume element $\mathrm{d}\mu_{T_x\mathcal{M}} = (-\det g)^{1/2} \mathrm{d}v^0 \wedge \ldots \wedge \mathrm{d}v^3$ is given by the Gelfand-Leray form [7, Chapter 7] of $\mathrm{d}\mu_{T_x\mathcal{M}}$ with respect to $-L = -\frac{1}{2} g(v, v)$, restricted to $\mathcal{C}_x^+$. That is $\mathrm{d}\mu_{T_x\mathcal{M}} = \mathrm{d}L \wedge \mathrm{d}\mu_{\mathcal{C}_x^+}$, where $\mathrm{d}L$ is the exterior derivative of L calculated on $T_x\mathcal{M}$. In local coordinates (x^a) with a taking values $0, \ldots, 3$, this takes the (not unique) form

$$\mathrm{d}\mu_{\mathcal{C}_x^+} = \sqrt{|g|} \frac{\mathrm{d}v^1 \wedge \mathrm{d}v^2 \wedge \mathrm{d}v^3}{(-v_0)}.$$

This can also be found as an appropriate limit of $i_X (\mathrm{d}\mu_{T_x\mathcal{M}})$ with $X^a = (-g(v, v))^{-1} v^a$ on the hypersurface $\{v : g(v, v) = -m^2, \text{ and } v \text{ future directed}\}$ as $m \to 0^+$.

In the particular case of the Kerr spacetime, recall $\sqrt{|g|} = \Sigma \sin\theta$ is uniformly equivalent to $r^2 \sin\theta$. For r large, v_0 is negative on $\mathcal{C}_x^+$. For r near r_+, where ∂_t ceases to be timelike, the situation is more complicated. If $v_0 < 0$, then (r, θ, ϕ) have the standard orientation, and if $v_0 > 0$, then (v^r, v^θ, v^ϕ) have the reverse orientation. $v_0 = 0$ only occurs on a set of codimension 1. Therefore, when integrating $\mathrm{d}\mu_{\mathcal{C}_x^+}$ over $\mathcal{C}_x^+$ using the orientation induced by (v^r, v^θ, v^ϕ), one always has $|v_0|^{-1}$ in the denominator. As a consequence, integrals with respect to $\mathrm{d}\mu_{\mathcal{C}_x^+}$ on $\mathcal{C}_x^+$ with the orientation induced by a future-directed normal are uniformally equivalent to integrals over $\mathcal{C}_x^+$ computed in the (v^r, v^θ, v^ϕ) coordinates using the measure $\mathrm{d}^3 v$.

Recall the following (see [4]):

Definition 3. *The Vlasov stress-energy tensor is defined to be*

$$\mathrm{T}_{ab}[f]_x = \int_{\mathcal{C}_x^+} f(x, v) v_a v_b \mathrm{d}\mu_{\mathcal{C}_x^+}.$$

For the remainder of this paper, the term "stress-energy tensor" will refer to the Vlasov stress-energy tensor. The Vlasov stress-energy tensor is symmetric, traceless, and divergence-free for the massless Vlasov equation. If f is non-negative, the stress-energy tensor satisfies the dominant energy condition.

Killing tensors play a crucial role in understanding the symmetries of the Vlasov equation. Recall $K_{a_1\dots a_n}$ is a conformal Killing tensor if, for some $n \in \mathbb{N}$, there is a tensor field $p_{a_1\dots a_{n-1}}$ such that

$$K_{a_1\dots a_n} = K_{(a_1\dots a_n)},$$
$$\nabla_{(b}K_{a_1\dots a_n)} = g_{(ba_1}p_{a_2\dots a_n)}.$$

If K is a conformal Killing tensor, then there are several relevant and well-known consequences. $K_{a_1\dots a_n}\dot{\gamma}^{a_1}\dots\dot{\gamma}^{a_n}$ is constant along any null geodesic γ. On $T\mathcal{M}$, the function $(x, v) \mapsto K_{a_1\dots a_n}v^{a_1}\dots v^{a_n}$ is a solution of the Vlasov equation (1.2). Hence, its restriction to $\mathcal{C}^+$ satisfies the massless Vlasov equation. This is a well-known property of the Vlasov fields, which has already been exploited in [13, Section 2.8]. From this the following follows by direct calculation.

Lemma 4. *If $K_{a_1\dots a_n}$ is a conformal Killing tensor, then the map*

$$f(x, v) \mapsto K_{a_1\dots a_n}v^{a_1}\dots v^{a_n}f(x, v)$$

is a symmetry of the null Vlasov equation, in the sense that if $f(x, v)$ is a solution of the Vlasov equation, then so is $K_{a_1\dots a_n}v^{a_1}\dots v^{a_n}f(x, v)$.

Lemma 5. *Let $n \in \mathbb{N}$, and $\{S_{\underline{a}}\}_{\underline{a}}$ be a collection of symmetries for the Vlasov equation of the form $(K_{\underline{a}})_{a_1\dots a_{N(\underline{a})}}v^{a_1}\dots v^{a_{N(\underline{a})}}$. The stress-energy tensor defined by*

$$\mathrm{T}_{ab\underline{a}_1\dots\underline{a}_n}[f]_x = \int_{\mathcal{C}^+_x} S_{\underline{a}_n}\dots S_{\underline{a}_1}f(x, v)v_a v_b \mathrm{d}\mu_{\mathcal{C}^+_x}$$

satisfies

1. *(symmetry)* $\mathrm{T}_{ab\underline{a}_1\dots\underline{a}_n}[f] = \mathrm{T}_{ba\underline{a}_1\dots\underline{a}_n}[f]$,
2. *(trace-free)* $\mathrm{T}^a{}_{a\underline{a}_1\dots\underline{a}_n}[f] = 0$, *and,*
3. *(divergence-free) if f is a solution of the Vlasov equation,* $\nabla^a\mathrm{T}_{ab\underline{a}_1\dots\underline{a}_n}[f] = 0$.
4. *(dominant energy condition) Furthermore, suppose Y^a is a future-directed causal vector, and suppose that $X^{a\underline{a}_1\cdots\underline{a}_n}$ is such that for any set of real numbers $\{\sigma_{\underline{a}}\}_{\underline{a}}$, the vector $X^{a\underline{a}_1\cdots\underline{a}_n}\sigma_{\underline{a}_1}\dots\sigma_{\underline{a}_n}$ is a future-directed causal vector. In this case, if f is nonnegative, then $\mathrm{T}_{ab\underline{a}_1\dots\underline{a}_n}[f]X^{a\underline{a}_1\cdots\underline{a}_n}Y^b \geq 0$.*

Proof. For any sequence of values for $\underline{a}_1, \dots, \underline{a}_n$, consider the sequence of concomitants defined by $\mathrm{T}_{ab\underline{a}_1\dots\underline{a}_k}[b] = \mathrm{T}_{ab\underline{a}_1\dots\underline{a}_{k-1}}[S_{\underline{a}_k}b]$. Since T_{ab} is symmetric and is trace-free, the $\mathrm{T}_{ab\underline{a}_1\dots\underline{a}_k}$ have the same property by induction. Similarly, since each $S_{\underline{a}_k}$ is a symmetry, the $\mathrm{T}_{ab\underline{a}_1\dots\underline{a}_k}$ is divergence-free for the Vlasov equation by induction.

Suppose the dominant energy condition fails for $\mathrm{T}_{ab\underline{a}_1\dots\underline{a}_n}$. Thus, there is some smooth $f : \mathcal{C}^+ \to [0, \infty)$, an $x \in \mathcal{M}$, and $X^{a\underline{a}_1\cdots\underline{a}_n}$ and Y^a as in the statement of the theorem such that

$$\int_{\mathcal{C}^+_x} S_{\underline{a}_n}\dots S_{\underline{a}_1}f(x, v)v_a v_b X^{a\underline{a}_1\cdots\underline{a}_n}Y^b \mathrm{d}\mu_{\mathcal{C}^+_x} < 0.$$

Thus, there is a $w \in T_x\mathcal{M}$ such that $S_{\underline{a}_n} \dots S_{\underline{a}_1} f(x,w) w_a w_b X^{a\underline{a}_1\cdots\underline{a}_n} Y^b < 0$. Let $\sigma_{\underline{a}}$ be the value of $S_{\underline{a}}$ at (x,w). (Since the $S_{\underline{a}}$ are assumed to be multiplicative symmetry operators depending on (x,v), this is possible.) Since the $S_{\underline{a}}$ are polynomial in the v^a, they are continuous in $\mathcal{C}_x^+$. Thus, there is an open neighbourhood W of w in $\mathcal{C}_x^+$ such that $S_{\underline{a}_n} \dots S_{\underline{a}_1} f(x,v) v_a v_b X^{a\underline{a}_1\cdots\underline{a}_n} Y^b < S_{\underline{a}_n} \dots S_{\underline{a}_1} f(x,w) w_a w_b X^{a\underline{a}_1\cdots\underline{a}_n} Y^b/2 < 0$. Let χ be a smooth function on $\mathcal{C}_x^+$ that is one on an open neighbourhood W' of w and that is supported in W. Thus, $f\chi$ is a non-negative function on $\mathcal{C}_x^+$ and

$$
\begin{aligned}
0 &> \int_{\mathcal{C}_x^+} (\chi(v) f(x,v)) v_a v_b (X^{a\underline{a}_1\cdots\underline{a}_n} \sigma_{\underline{a}_n} \dots \sigma_{\underline{a}_1}) Y^b \mathrm{d}\mu_{\mathcal{C}_x^+} \\
&> \left(\int_{\mathcal{C}_x^+} (\chi(v) f(x,v)) v_a v_b \mathrm{d}\mu_{\mathcal{C}_x^+} \right) (X^{a\underline{a}_1\cdots\underline{a}_n} \sigma_{\underline{a}_n} \dots \sigma_{\underline{a}_1}) Y^b
\end{aligned} \tag{2.4}
$$

Since $X^{a\underline{a}_1\cdots\underline{a}_n}\sigma_{\underline{a}_1}\dots\sigma_{\underline{a}_n}$ and Y^a are timelike and future-directed vector fields, and T_{ab} satisfies the dominant energy condition, it follows that the final term in inequality (2.4) must be nonnegative, which contradicts inequality (2.4). Thus, by contradiction, $\mathrm{T}_{ab\underline{a}_1\cdots\underline{a}_n}$ must satisfy the dominant energy condition. □

One concludes this section by the standard conservation of energies for Vlasov fields. Let$\{S_{\underline{a}}\}_{\underline{a}}$ be a collection of symmetries, $X^{a\underline{a}_1\cdots\underline{a}_k}$ a collection of vector fields, and Σ be a spacelike hypersurface. The energy of f with respect to the vector X on the hypersurface Σ is

$$
E_X[f](\Sigma) = \int_\Sigma \mathrm{T}_{ab\underline{a}_1\dots\underline{a}_k}[f] X^{a\underline{a}_1\cdots\underline{a}_k} \mathrm{d}\nu_\Sigma^b,
$$

Let now Σ_1, Σ_2 be hypersurfaces and R be an open set such that $\partial R = \Sigma_2 - \Sigma_1$. The following lemma states the conservation of energies of Vlasov fields:

Lemma 6. *Let Ω be a positive function on $\mathcal{M}$, and $q^{\underline{a}_1\cdots\underline{a}_k}$ be a collection of functions on $\mathcal{M}$. The following identity holds:*

$$
E_X[f](\Sigma_2) - E_X[f](\Sigma_1) = \int_R \Pi_{X,\Omega,q}[f](R) \mathrm{d}\mu_g,
$$

where

$$
\Pi_{X,\Omega,q}[f] = -\frac{1}{2}\Omega^2 \mathrm{T}_{ab\underline{a}_1\dots\underline{a}_k}[f] \mathrm{Lie}_{X^{\underline{a}_1\cdots\underline{a}_k}} (\Omega^{-2} g^{ab}) + \mathrm{T}_{ab\underline{a}_1\dots\underline{a}_k} g^{ab}[f] q^{\underline{a}_1\cdots\underline{a}_k}.
$$

Remark 7. *Later in Section 3, to simplify the notations, $\Pi_{X,\Omega,q}[f]$ is sometimes denoted Π_X, since Ω and q are clear from context.*

Proof. The proof is a straightforward consequence of the fact that $\mathrm{T}_{ab\underline{a}_1\dots\underline{a}_k}[f]$ is traceless and divergence free. □

3. The bounded-energy estimate

The essential parts of the proof are to construct 2-symmetry-strengthened vector fields $\mathbf{T}_\chi$ and $\mathbf{A}$ such that

$$E_{\mathbf{T}_\chi} \geq 0, \tag{3.1a}$$

$$\Pi_{\mathbf{A}} \geq 0, \tag{3.1b}$$

$$\Pi_{\mathbf{T}_\chi} \lesssim \frac{|a|}{M}\Pi_{\mathbf{A}}, \tag{3.1c}$$

$$E_{\mathbf{T}_\chi} \gtrsim |E_{\mathbf{A}}|. \tag{3.1d}$$

A simple bootstrap argument then shows, for sufficiently small $|a|/M$, that $E_{\mathbf{T}_\chi}$ is uniformly bounded by its initial value and that the spacetime integral of $\Pi_{\mathbf{A}}$ is bounded by a multiple of $E_{\mathbf{T}_\chi}$ at any time.

Of the properties above, the first property (3.1a) is ensured by taking $\mathbf{T}_\chi$ to be future-directed and causal. The second property (3.1b) is ensured finding $\mathbf{A}$ (together with a conformal factor and a collection of auxiliary function q) such that (suppressing symmetry indices for simplicity)

$$\Omega^{-2}\Pi_{\mathbf{A},\Omega,q} = \left(-\frac{1}{2}\mathrm{Lie}_{\mathbf{A}}(\Omega^{-2}g^{ab}) - q\Omega^{-2}g^{ab}\right)\mathrm{T}_{ab}$$

is non-negative. The remaining two properties (3.1c)-(3.1d) are quantitative, allowing one term to be dominated, rather than qualitative, merely requiring a term to be signed, and so they are more complicated. However, if $\mathbf{T}_\chi$ were Killing, then the associated bulk term would vanish, and the third condition (3.1c) would hold trivially; since the Kerr exterior has no globally Killing, causal vector, we instead construct an approximately Killing vector field, with $|a|/M$ being a measure of the failure of $\mathbf{T}_\chi$ to be Killing, so that the third condition (3.1c) holds. The fourth condition (3.1d) holds from the dominant energy condition, as long as $\mathbf{A}$ can be chosen to have a length bounded by the length of $\mathbf{T}_\chi$.

Ideally, one would construct $\mathbf{A}$ and $\mathbf{T}_\chi$ that are vector fields, but, following [2], we take them to be 2-symmetry-strengthened vector fields. The Kerr spacetime has orbiting null geodesics, which we define to be ones which neither are absorbed through the event horizon nor escape to null infinity. The projection of such geodesics to Σ_t fills an open set in the Kerr spacetime, but not its tangent space. Because of the presence of orbiting null geodesics in an open set, it is not possible to find a vector field $\mathbf{A}$ such that $\left(\mathrm{Lie}_{\mathbf{A}}(\Omega^{-2}g^{ab}) - q\Omega^{-2}g^{ab}\right)\mathrm{T}_{ab}$ is non-negative; however, although [2] doesn't use the terminology introduced in this paper, it introduced 2-symmetry-strengthened vector fields that, with the energies and bulk terms for the wave equation, satisfy the four conditions (3.1). Most of the rest of this paper consists of constructing these 2-symmetry-strengthended vector fields and demonstrating they have the desired properties. In [2], it was important to work with 2-symmetry-strengthened vector fields so that the quadratic stress-energy tensor for the wave equation could be written as a bilinear quantity. In this paper, it is again convenient to work with 2-symmetry-strengthened vector fields, so that we can more easily define the notion of a causal 2-symmetry-strengthened vector field.

The calculations in this paper are significantly simpler than in [2]. Both papers rely on properties of null geodesics and on the fact that, for a geodesic γ with the energies and bulk terms defined by $E_X[\gamma] = \dot\gamma^a X_a$ and $\Pi_X[\gamma] = \nabla_{(a}X_{b)}\dot\gamma^a\dot\gamma^b$, the estimates (3.1a)-(3.1d) are valid. The calculations in this paper are relatively quick, since the behaviour of null geodesics completely determines the behaviour of solutions to the Vlasov equation. In contrast, solutions of the wave equation are only accurately modelled by null geodesics in the high-frequency limit. To treat the wave equation in [2], a significant amount of additional work is required to show that a similar method can be used uniformly without a frequency decomposition.

3.1. **The blended energy.** In this subsection, we construct a causal 2-symmetry-strengthened vector field.

Definition 8. *Let*

$$\begin{aligned} T_\perp &= \left(\partial_t + \frac{2aMr}{\Pi}\partial_\phi\right)^a = (\partial_t + \omega_\perp\partial_\phi)^a\,,\\ T_\chi^a &= (\partial_t + \chi\omega_{\mathcal{H}}\partial_\phi)^a,\\ \mathbf{T}_\chi^{a\underline{ab}} &= T_\chi^a\delta^{\underline{ab}}, \end{aligned}$$

where $\omega_{\mathcal{H}} = a/(r_+^2+a^2)$ is the rotation speed of the horizon, $\chi = \chi(r)$ is a function that is 1 for $r < r_\chi$, smoothly decreasing on $r \in [r_\chi, r_\chi + M]$, and identically 0 for $r > r_\chi + M$, and where r_χ is chosen sufficiently large. For simplicity, we take $r_\chi = 10M$.

Lemma 9. *There is a positive constant $\bar\epsilon$ such that if $|a| < \bar\epsilon M$, $t \in \mathbb{R}$, and $f : \mathcal{C}^+ \to [0,\infty)$ is continuous, then*

$$E_{T_\perp}[f](\Sigma_t) \simeq \int_{\Sigma_t}\int_{\mathcal{C}_x^+}\left(\frac{(r^2+a^2)^2}{\Delta}v_t^2 + \Delta v_r^2 + Q^{ab}v_a v_b\right) f\mathrm{d}\mu_{\mathcal{C}_x^+}\mathrm{d}\mu_{\Sigma_t}, \tag{3.2}$$

$$\simeq \int_{\Sigma_t}\int_{\mathcal{C}_x^+}\left(\frac{(r^2+a^2)^2}{\Delta}v_t^2 + \Delta v_r^2 + v_\theta^2 + \frac{1}{\sin^2\theta}v_\phi^2\right) f\mathrm{d}\mu_{\mathcal{C}_x^+}\mathrm{d}\mu_{\Sigma_t}, \tag{3.3}$$

$$E_{T_\perp}[f](\Sigma_t) \simeq E_{T_\chi}[f](\Sigma_t), \tag{3.4}$$

$$\simeq E_{model,3}. \tag{3.5}$$

Furthermore, if f is a C^1 solution of the Vlasov equation, then

$$\sqrt{\det g}\left|-\frac{1}{2}\Omega^2\mathrm{T}[f]_{ab}\mathrm{Lie}_{T_\chi}(\Omega^{-2}g^{ab})\right| = \Delta|\partial_r\chi||v_r||v_\phi|\sin\theta. \tag{3.6}$$

Proof. This proof follows the argument of the proof for Lemma 3.1 of [2].

Let $\omega_\perp$ denote $2aMr/\Pi$. Since the normal satisfies $\mathrm{d}\nu_{\Sigma_t}^a = T_\perp^a(\Pi/\Delta)\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$, and since $-g_{ab}T_\perp^aT_\perp^b = \Delta\Sigma/\Pi$, the $T_\perp$ energy is

$$E_{T_\perp} = \int_{\Sigma_t}\mathrm{T}_{ab}T_\perp^aT_\perp^b\frac{\Pi}{\Delta}\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = \int_{\Sigma_t}\int_{\mathcal{C}_x^+} f\left(\frac{\Pi}{\Delta}(T_\perp^a v_a)^2 + \frac{1}{2}\Sigma g^{ab}v_a v_b\right)\mathrm{d}\mu_{\mathcal{C}_x^+}\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi.$$

The integrand can be expanded as

$$\frac{\Pi}{\Delta}(T_\perp^a v_a)^2 + \frac{1}{2}\Sigma g^{ab} v_a v_b = \frac{1}{2}\left(\Delta(v_r)^2 + \frac{(r^2+a^2)^2}{\Delta}(T_\perp^a v_a)^2 + Q^{ab} v_a v_b + v_\phi^2\right) \tag{3.7a}$$

$$- \frac{1}{2\Delta}\left(4aMr - 2\omega_\perp(r^2+a^2)^2\right) v_t v_\phi \tag{3.7b}$$

$$+ \frac{1}{2\Delta}\left(-a^2 + (r^2+a^2)^2\omega_\perp^2\right) v_\phi^2 - a^2\sin^2\theta (T_\perp^a v_a)^2. \tag{3.7c}$$

Since the coefficients $4aMr - 2\omega_\perp(r^2+a^2)^2$ and $-a^2 + (r^2+a^2)^2\omega_\perp^2$ vanish at $r = r_+$, are bounded by factors that go uniformly to 0 on bounded sets as $a \to 0$, and grow as $r \to \infty$ no faster than r and a constant respectively, for $|a|$ sufficiently small, the terms in lines (3.7b)-(3.7c) are dominated by those on the right-hand side of line (3.7a). Thus, the terms on the left and right side of line (3.7a) are equivalent. This proves estimate (3.2). Estimate (3.3) follows from the equivalence

$$(T_\perp^a v_a)^2 + Q^{ab} v_a v_b \simeq (T_\perp^a v_a)^2 + v_\theta^2 + \frac{1}{\sin^2\theta} v_\phi^2.$$

The T_χ energy can be estimated using the fact that $T_\perp - T_\chi = (\omega_\perp - \chi\omega_{\mathcal{H}})\partial_\phi$ is orthogonal to $T_\perp$, so

$$E_{T_\perp} - E_{T_\chi} = \int_{\Sigma_t} (\omega_\perp - \chi\omega_{\mathcal{H}}) v_\phi (T_\perp^a v_a) \frac{\Pi}{\Delta} \mathrm{d}\mu_{\Sigma_t}.$$

The coefficient $\omega_\perp - \chi\omega_{\mathcal{H}}$ vanishes linearly at $r = r_+$, is bounded by a function that goes to zero uniformly as $a \to 0$, and goes to zero as $r \to \infty$ like r^{-4}, so, by a simple Cauchy-Schwarz estimate, one finds $|E_{T_\perp} - E_{T_\chi}| \lesssim |a| E_{T_\perp}$, and $E_{T_\perp} \simeq E_{T_\chi}$. Finally, $E_{T_\perp}$ and $E_{\text{model},3}$ are equivalent, since, in considering the integration on the cone, $\sqrt{\det g} = \Sigma\sin\theta$ is uniformly equivalent to $r^2\sin\theta$ for a sufficiently small.

The contraction of the stress-energy tensor with the Lie derivative can be calculated directly from $\mathrm{Lie}_{T_\chi}(\Omega^{-2}g^{ab}) = -2\Delta\partial_r^{(a}\partial_\phi^{b)}$. □

Corollary 10. *There is a positive constant $\bar\epsilon$ such that if $|a| \le \bar\epsilon$, $t \in \mathbb{R}$, and $f : \mathcal{C}^+ \to [0,\infty)$ is continuous, then*

$$E_{\mathbf{T}_\chi}[f](\Sigma(t)) \simeq E_{model,3}[f](t).$$

Proof. This follows from applying estimates (3.3) and (3.4), substituting $(M^2 v_t^2 + v_\theta^2 + \csc^2\theta v_\phi^2)^2 f$ for f, recognising $E_{\mathbf{T}_\chi}[f]$ as being obtained from $E_{T_\chi}[f]$ substituting $(M^2 v_t^2 + q + l_z^2)^2 f$ for f, and observing the uniform equivalence of $(M^2 v_t^2 + q + l_z^2)$ and $(M^2 v_t^2 + v_\theta^2 + \csc^2\theta v_\phi^2)$. □

3.2. **Set-up for radial vector fields.** In this subsection, we define a radial 2-symmetry-strengthened vector field, $\mathbf{A}$, in terms of unspecified scalar functions, which will be chosen in the following subsection. The main result of this subsection is that the bulk term, $\Pi_{\mathbf{A}}$, can be written as a sum of two terms, with the second involving a square and the first involving a second derivative. A square is always non-negative. One should expect that the second-derivative term will be non-negative on orbits, since the orbits are known to be unstable. In the following subsection, the scalar functions are chosen so that this second-derivative term is non-negative everywhere, not just on the orbits.

Definition 11. *If z and w are smooth functions of r and the parameters M and a, the Morawetz vector field and the reduced scalar functions are defined to be*

$$\mathbf{A}^{a\underline{ab}} = -zw\mathcal{L}^{(\underline{a}}\tilde{\mathcal{R}}'^{\underline{b})}\partial_r^a,$$
$$q^{\underline{ab}} = \frac{1}{2}(\partial_r z)w\mathcal{L}^{(\underline{a}}\tilde{\mathcal{R}}'^{\underline{b})},$$

where

$$\tilde{\mathcal{R}}'^{\underline{a}} = \partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right),$$
$$\mathcal{L} = \mathcal{L}^{\underline{a}}S_{\underline{a}} = M^2e^2 + l_z^2 + q,$$

and $\mathcal{R}^{\underline{a}}$ and $\mathcal{L}$ are defined in equations (2.2) and (2.3). We also introduce

$$\tilde{\tilde{\mathcal{R}}}'' = \partial_r\left(w\frac{z^{1/2}}{\Delta^{1/2}}\tilde{\mathcal{R}}'\right).$$

The following lemma is a trivial observation in the current context. This is in contrast with the situation for the wave equation where the rearrangement of the symmetry indices required some calculation and introduced additional terms at the initial and final time, which had to be dominated by the energies.

Lemma 12 (Rearrangements)**.**

$$\mathcal{L}^{(\underline{a}}\mathcal{R}^{\underline{b})}S_{\underline{a}}S_{\underline{b}} = \mathcal{L}^{(\underline{a}}\mathcal{R}^{\underline{b})}S_{\underline{a}}^{ab}S_{\underline{b}}^{cd}v_av_bv_cv_d$$
$$= \mathcal{L}^{\underline{a}}\mathcal{R}^{\underline{b}}S_{\underline{a}}^{ab}S_{\underline{b}}^{cd}v_av_bv_cv_d.$$

Proof. Apply the definition $S_{\underline{a}} = S_{\underline{a}}^{ab}v_av_b$ and similarly in $\underline{b}$, and then observe that the contraction in $abcd$ is against four copies of v, so that it is automatically symmetric in $\underline{ab}$. □

Lemma 13. *With $\mathbf{A}$ and q as above, one finds*

$$\Pi_{\mathbf{A},\Omega,q} = \mathcal{L}^{\underline{a}}\left(-z^{1/2}\Delta^{3/2}\partial_r\left(w\frac{z^{1/2}}{\Delta^{1/2}}\tilde{\mathcal{R}}'^{\underline{b}}\right)\partial_r^e\partial_r^g + \frac{1}{2}w\tilde{\mathcal{R}}'^{\underline{b}}\tilde{\mathcal{R}}'^{\underline{c}}S_{\underline{c}}^{eg}\right)S_{\underline{a}}^{ab}S_{\underline{b}}^{cd}v_av_bv_cv_dv_ev_gf.$$

Proof. Recall

$$\Omega^{-2}g^{eg} = \Delta\partial_r^e\partial_r^g + \frac{1}{\Delta}\mathcal{R}^{eg},$$
$$\Omega^{-2}\Pi_{\mathbf{A},\Omega,q} = -\frac{1}{2}\mathrm{Lie}_{\mathbf{A}^{\underline{ab}}}(\Omega^{-2}g^{ab}) + \Omega^{-2}q^{\underline{ab}}g^{ab}.$$

Thus,

$$\Omega^{-2}\Pi_{\mathbf{A},\Omega,q} = \Bigg(\frac{1}{2}\left(\mathcal{L}^{(\underline{a}} z w \tilde{\mathcal{R}}'^{\underline{b})} \partial_r \Delta - 2\Delta \mathcal{L}^{(\underline{a}} \partial_r \left(z w \tilde{\mathcal{R}}'^{\underline{b})} \right) \right) \partial_r^e \partial_r^g + \frac{1}{2}\mathcal{L}^{(\underline{a}} z w \tilde{\mathcal{R}}'^{\underline{b})} \partial_r \left(\frac{\mathcal{R}^{eg}}{\Delta} \right)$$

$$+ \frac{1}{2}(\partial_r z) w \mathcal{L}^{(\underline{a}} \tilde{\mathcal{R}}'^{\underline{b})} \partial_r^e \partial_r^g + \frac{1}{2}(\partial_r z) w \mathcal{L}^{(\underline{a}} \tilde{\mathcal{R}}'^{\underline{b})} \frac{1}{\Delta} \mathcal{R}^{eg} \Bigg)$$

$$S_{\underline{a}}^{ab} S_{\underline{b}}^{cd} v_a v_b v_c v_d v_e v_g$$

$$= \mathcal{L}^{\underline{a}} \Bigg(- z^{1/2} \Delta^{3/2} \partial_r \left(w \frac{z^{1/2}}{\Delta^{1/2}} \tilde{\mathcal{R}}'^{\underline{b}} \right) \partial_r^e \partial_r^g + \frac{1}{2} w \tilde{\mathcal{R}}'^{\underline{b}} \partial_r \left(\frac{z}{\Delta} \mathcal{R}^{eg} \right) \Bigg)$$

$$S_{\underline{a}}^{ab} S_{\underline{b}}^{cd} v_a v_b v_c v_d v_e v_g .$$

Substituting $\tilde{\mathcal{R}}'^{eg} = \partial_r \left(\frac{z}{\Delta} \mathcal{R}^{eg} \right)$ gives the desired result. □

3.3. **Choosing the weights.** In this subsection, we choose the weights z and w, so that $\Pi_{\mathbf{A},\Omega,q}$ is non-negative for all r. The choices are the same as those appearing for the wave equation in [2].

Here, we recall how the weight functions z and w are chosen, following the explanation in remark 3.8 of [2]. The goal in choosing the various weight functions is to obtain nonnegativity for the two terms in $\Pi_{\mathbf{A},\Omega,q}$, namely $-z^{1/2}\Delta^{3/2}\partial_r \left(w \frac{z^{1/2}}{\Delta^{1/2}} \tilde{\mathcal{R}}' \right) v_r^2$ and $\frac{1}{2} w \tilde{\mathcal{R}}' \tilde{\mathcal{R}}'$. For $|a| \ll M$, the orbiting null geodesics are near $r = 3M$. On orbiting null geodesics, $\tilde{\mathcal{R}}'(r; M, a; e, l_z, q)$ vanishes and $-\tilde{\tilde{\mathcal{R}}}''(r; M, a; e, l_z, q)$ is positive. Thus, the desired non-negativity holds on orbiting null geodesics regardless of the choice of z and w. The functions z and w are chosen so that the non-negativity extends to all other null geodesics. These functions can be chosen so that $-\tilde{\tilde{\mathcal{R}}}''(r; M, a; e, l_z, q)$ remains positive everywhere and so that $\tilde{\mathcal{R}}'(r; M, a; e, l_z, q)$ vanishes only in a neighbourhood of $r = 3M$.

We have chosen the weights so that the following properties hold:

(1) The definition of $\tilde{\mathcal{R}}'$ in equation (3.8) is made so that $w\tilde{\mathcal{R}}'\partial_r \left(\frac{z}{\Delta} \mathcal{R} \right)$ takes the form $w\tilde{\mathcal{R}}'^2$ in Lemma 13.
(2) $M^2\epsilon_{e^2}^2$ is the coefficient of e^2 in $\tilde{\mathcal{R}}'(r; M, a; e, l_z, q)$ and $\tilde{\tilde{\mathcal{R}}}''(r; M, a; e, l_z, q)$. Note that $M^2\epsilon_{e^2}^2$ plays the same role as $\epsilon_{\partial_t^2}^2$ in [2], where the differential symmetry operator ∂_t^2 for the wave equation plays the role of the multiplicative symmetry e^2 for the Vlasov equation. The use of a dimensionless parameter, ϵ_{e^2}, in this paper clarifies that the small parameter $|a|/M$ can be chosen uniformly in M to close the bootstrap argument.
(3) z_1 is such that, if z_2 had been equal to 1, which corresponds to $\epsilon_{e^2} = 0$, then the coefficient of M^2e^2 in $\tilde{\mathcal{R}}'(r; M, a; e, l_z, q)$ would be zero.
(4) z_2 is such that, if $\epsilon_{e^2} > 0$, then the coefficient of $M^2\epsilon_{e^2}e^2$ in $\tilde{\mathcal{R}}'(r; M, a; e, l_z, q)$ is non-negative and a perturbation (in ϵ_{e^2}) of the coefficient of q.
(5) w_1 is such that, if z_2 and w_2 had both been equal to 1, then the coefficient of Mel_z in $\tilde{\tilde{\mathcal{R}}}''(r; M, a; e, l_z, q)$ would vanish.

(6) w_2 is such that
   (a) $\tilde{\tilde{\mathcal{R}}}''(r; M, a; e, l_z, q)$ is positive everywhere, and
   (b) $(zw\tilde{\mathcal{R}}'(r; M, a; e, l_z, q))^2 g(\partial_r, \partial_r) \lesssim (M^2e^2 + l_z^2 + q)^2 g(T_\chi, T_\chi)$.

   In particular, from the dominant energy condition, condition 6b allows us to show that $E_{\mathbf{A}} \lesssim E_{\mathbf{T}_\chi}$. Once the form $w_2 = Cr^{-1}$ was chosen, the factor of $C = 1/2$ was chosen so that, when $a = 0$ and $\epsilon_{e^2} = 0$, the coefficient of $l_z^2 + q$ in $\tilde{\tilde{\mathcal{R}}}''$ is equal to 1.

The factors $\tilde{\mathcal{R}}'$, z_1, z_2, and z_1 are uniquely defined by the above properties. In contrast, the factor w_2 is both overdetermined, since we have chosen it to satisfy two conditions that are not a priori obviously compatible, and underdetermined, since it so happens that there are many functions that allow these two conditions to be satisfied.

Definition 14. *Given a positive value for the parameter ϵ_{e^2}, we use the following weights to define the Morawetz vector field,*

$$z = z_1 z_2, \qquad w = w_1 w_2,$$
$$z_1 = \frac{\Delta}{(r^2+a^2)^2}, \qquad w_1 = \frac{(r^2+a^2)^4}{3r^2 - a^2},$$
$$z_2 = 1 - M^2\epsilon_{e^2}\frac{\Delta}{(r^2+a^2)^2}, \qquad w_2 = \frac{1}{2r}.$$

The reason for these choices is explained in Remark 3.8 of [2].

In the following lemma, big-O notation is used in the r variable. The notation $f = O(r^{-l})$ means that f is independent of v_r, e, l_z, and q and that there is a constant C such that for positive M and sufficiently small $|a|/M$, uniformly in $r > r_+$, there is the bound $|f(r, M, a)| \le Cr^{-l}$. The notation $f = g + hO(r^{-l})$ denotes that there is a function $k = k(r, M, a)$ such that $k = O(r^{-l})$. The notation $f = g + h_1 O(r^{-l_1}) + \ldots + h_n O(r^{-l_n})$ is defined recursively.

Lemma 15. *There are positive constants $\bar{\epsilon}$, ϵ_{e^2}, and C such that if $|a| \le \bar{\epsilon}M$, $0 < \epsilon_{e^2} \le \bar{\epsilon_{e^2}}$ and $f : \mathcal{C}^+ \to [0, \infty)$ is a solution of the Vlasov equation, then*

$$C\Omega^2 \Pi_{\mathbf{A}} \ge M\frac{\Delta^2}{(r^2+a^2)^2} v_r^2 |f|_2 + r^5 \tilde{\mathcal{R}}'\tilde{\mathcal{R}}'\mathcal{L}f. \tag{3.8}$$

and

$$\begin{aligned}\tilde{\mathcal{R}}' = & -2r^{-4}(r - 3M)\mathcal{L}_{\epsilon_{e^2}} \\ & + aMO(r^{-4})el_z \\ & + a^2\left(O(r^{-5})q + O(r^{-5})l_z^2\right) \\ & + M^2\epsilon_{e^2}\left(a^2 O(r^{-5})e^2 + O(r^{-5})q + O(r^{-5})l_z^2\right).\end{aligned} \tag{3.9}$$

Proof. Direct calculation of $\tilde{\mathcal{R}}'$ with our choices of z and w gives

$$\begin{aligned}\tilde{\mathcal{R}}' = &-M^2\epsilon_{e^2}(2(r-3M)r^{-4}+a^2O(r^{-5}))e^2\\&+aMO(r^{-4})el_z\\&-(2(r-3M)r^{-4}+a^2O(r^{-5})+M^2\epsilon_{e^2}O(r^{-5})q\\&-(2(r-3M)r^{-4}+a^2O(r^{-5})+M^2\epsilon_{e^2}O(r^{-5})l_z^2.\end{aligned}$$

Grouping the terms in orders of ϵ_{e^2} and a gives equation (3.9). From this,

$$\begin{aligned}-\tilde{\tilde{\mathcal{R}}}'' = &M^3\epsilon_{e^2}(r^{-2}+a^2O(r^{-3})+M^2\epsilon_{e^2}O(r^{-3})e^2\\&+aMO(r^{-2})el_z\\&+M(r^{-2}+a^2O(r^{-3})+M^2\epsilon_{e^2}O(r^{-3}))q\\&+M(r^{-2}+a^2O(r^{-3})+M^2\epsilon_{e^2}O(r^{-3}))l_z^2.\end{aligned}$$

In the coefficients of $M^2\epsilon_{e^2}e^2$, q, and l_z^2, the Mr^{-2} term dominates the remaining terms for sufficiently small $|a|$ and ϵ_{e^2}. From the Cauchy-Schwarz inequality, the el_z term is dominated by the e^2 and l_z^2 terms for sufficiently small $|a|$. Thus, fixing ϵ_{e^2} sufficiently small and choosing a constant accordingly,

$$-\tilde{\tilde{\mathcal{R}}}'' \geq CM(r^2+a^2)^{-1}(M^2e^2+q+l_z^2).$$

Thus, for $|a|$ sufficiently small and ϵ_{e^2} as above,

$$\Omega^2\Pi_{\mathbf{A}} \geq CM\frac{\Delta^2}{(r^2+a^2)^2}v_r^2|f|_2+\frac{1}{4r}\frac{(r^2+a^2)^4}{3r^2-a^2}\tilde{\mathcal{R}}'\tilde{\mathcal{R}}'\mathcal{L}f.$$

Thus, there is a new constant C, such that

$$C\Omega^2\Pi_{\mathbf{A}} \geq M\frac{\Delta^2}{(r^2+a^2)^2}v_r^2|f|_2+r^5\tilde{\mathcal{R}}'\tilde{\mathcal{R}}'\mathcal{L}f.$$

□

Lemma 16 (Controlling the boundary terms). *With ϵ_{e^2} as in Lemma 15, there is a constant C such that for any $f:\mathcal{C}^+\to\mathbb{R}$ and $t\in\mathbb{R}$,*

$$|E_{\mathbf{A}}[f](\Sigma_t)| \leq C|E_{\mathbf{T}_\chi}[f](\Sigma_t)|.$$

Proof. This follows from lemma 3.11 of [2]. Assume both energies are defined. By direct computation,

$$\begin{aligned}E_{\mathbf{A}} &= -\int_{\Sigma_t}\left(\mathrm{T}_{ab\underline{a}\underline{b}}\mathbf{A}^{a\underline{a}\underline{b}}\right)_a T_\perp^a\frac{\Pi}{\Delta}\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi,\\|E_{\mathbf{A}}| &\leq C\int_{\Sigma_t}\left(|T_\perp^a v_a||\mathbf{A}^{rab}||S_{\underline{a}}S_{\underline{b}}||v_r|\frac{\Pi}{\Delta}\right)\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi\\&\leq C\int_{\Sigma_t}\left(\frac{\Pi}{\Delta}|T_\perp^a v_a|_2^2+\frac{\Pi}{\Delta}\left(\sum_{\underline{a},\underline{b}}|\mathbf{A}^{r\underline{a}\underline{b}}|^2\right)|v_r|_2^2\right)\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi.\end{aligned}$$

Since Π/Δ, $(r^2+a^2)^2/\Delta$, and r^4/Δ are all uniformly equivalent and since $\sum_{\underline{a},\underline{b}}\mathbf{A}^{r\underline{a}\underline{b}}$ is bounded by a multiple of Δr^{-2}, it follows from estimate (3.3) that $|E_{\mathbf{A}}| \leq CE_{\mathbf{T}_\chi}$.

Since this bound followed from the Cauchy-Schwarz inequality, if $E_{\mathbf{T}_\chi}$ is finite, then the absolute value of the integrand in $E_{\mathbf{A}}$ is integrable, and $E_{\mathbf{A}}$. If $E_{\mathbf{T}_\chi}$ is infinite, then the desired estimate holds trivially. Thus, the initial assumption that both energies are finite is redundant. □

3.4. Closing the argument.

Proof of Theorems 1 and 2. Let $t_1, t_2 \in \mathbb{R}$. Initially, assume that f restricted to $\pi^{-1}(\Sigma_{t_1})$ has compact support in $\pi^{-1}(\Sigma_{t_1})$. By standard results for the Vlasov equation, this means f restricted to $\pi^{-1}(\Sigma_t)$ has compact support in each $\pi^{-1}(\Sigma_t)$. From integrating the result of Lemma 15, one finds

$$\begin{aligned} &E_{\mathbf{A}}[f](t_2) - E_{\mathbf{A}}[f](t_1) \\ &\geq \int_{t_1}^{t_2} \int_{\Sigma_t} \int_{\mathcal{C}^+} \left(M \frac{\Delta^2}{(r^2+a^2)^2} v_r^2 |f|_2 + r^5 \tilde{\mathcal{R}}' \tilde{\mathcal{R}}' \mathcal{L} f \right) \mathrm{d}\mu_{\mathcal{C}_x^+} \mathrm{d}\mu_g . \end{aligned}$$

Applying Lemma 16, one finds

$$\begin{aligned} &E_{\mathbf{T}_\chi}[f](t_2) + E_{\mathbf{T}_\chi}[f](t_1) \\ &\geq C \int_{t_1}^{t_2} \int_{\Sigma_t} \int_{\mathcal{C}^+} \left(M \frac{\Delta^2}{(r^2+a^2)^2} v_r^2 |f|_2 + r^5 \tilde{\mathcal{R}}' \tilde{\mathcal{R}}' \mathcal{L} f \right) \mathrm{d}\mu_{\mathcal{C}_x^+} \mathrm{d}\mu_g . \end{aligned} \tag{3.10}$$

From multiplying equation (3.6) for Π_{T_χ} by $(M^2 v_t^2 + q + l_z^2)^2$, one obtains the bulk term for $\Pi_{\mathbf{T}_\chi}$. Integrating this over $\bigcup_{t \in [t_1, t_2]} \Sigma_t$, and observing that $|\partial_r \chi|$ is compactly supported and that $\omega_{\mathcal{H}}$ vanishes linearly in a, one finds

$$\begin{aligned} &E_{\mathbf{T}_\chi}[f](t_2) - E_{\mathbf{T}_\chi}[f](t_1) \\ &\leq \int_{t_1}^{t_2} \int_{\Sigma_t} \int_{\mathcal{C}_x^+} (M^2 v_t^2 + q + l_z^2)^2 \Delta |\partial_r \chi| |v_r| |v_\phi| f \mathrm{d}\mu_{\mathcal{C}_x^+} \Sigma^{-1} \mathrm{d}\mu_g \\ &\leq \frac{|a|}{M} C \int_{t_1}^{t_2} \int_{\Sigma_t} \int_{\mathcal{C}^+} M \frac{\Delta^2}{(r^2+a^2)^2} v_r^2 |f|_2 + r^5 \tilde{\mathcal{R}}' \tilde{\mathcal{R}}' \mathcal{L} f \mathrm{d}\mu_{\mathcal{C}_x^+} \Sigma^{-1} \mathrm{d}\mu_g . \end{aligned} \tag{3.11}$$

Combining equations (3.10) and (3.11) and taking $|a|/M$ sufficiently small, one finds that there is a constant C such that

$$E_{\mathbf{T}_\chi}[f](\Sigma_{t_2}) \leq C E_{\mathbf{T}_\chi}[f](\Sigma_{t_1}).$$

Taking $t_2 = t$ and $t_1 = 0$ proves Theorem 1 for solutions with compactly supported data. From this, Estimate (3.10), and taking the limits $t_2 \to \infty$ with $t_1 = 0$ and $t_1 \to -\infty$ with $t_2 = 0$, one finds equation (1.6). Observing that $\tilde{\mathcal{R}}'$ grows like $O(r^{-3})(M^2 e^2 q + l_z^2)$ for large r and has a simple root near $r = 3M$ allows us to replace $r^5 \tilde{\mathcal{R}}' \tilde{\mathcal{R}}' \mathcal{L} f$ by $r^{-1} \mathbf{1}_{r \not\approx 3M} (M v_t^2 + v_\theta^2 + v_\phi^2) |f|_2$. This proves estimate (1.5) and completes the proof of Theorem 2 for solutions with compactly supported data. Since the bounds do not depend on the support of the initial data, by density Theorems 1 and 2 hold for all functions for which $E_{\mathbf{T}_\chi}$ is finite. The theorems follow trivially when this energy is infinite. □

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