CONGRUENCE LINK COMPLEMENTS—A 3-DIMENSIONAL RADEMACHER CONJECTURE

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Abstract. In this article we discuss a 3-dimensional version of a conjecture of Rademacher concerning genus 0 congruence subgroups $\mathrm{PSL}(2,\mathbb{Z})$. We survey known results, as well as including some new results that make partial progress on the conjecture.

Dedicated to Joachim Schwermer on the occasion of his 66th birthday

1. Introduction

Let k be a number field with ring of integers R_k. A subgroup $\Gamma < \mathrm{PSL}(2,\mathrm{R_k})$ is called a *congruence subgroup* if there exists an ideal $I \subset R_k$ so that Γ contains the *principal congruence group*:

$$\Gamma(I) = \ker\{\mathrm{PSL}(2,\mathrm{R_k}) \to \mathrm{PSL}(2,\mathrm{R_k/I})\},$$

where $\mathrm{PSL}(2,\mathrm{R_k/I}) = \mathrm{SL}(2,\mathrm{R_k/I})/\{\pm\mathrm{Id}\}$. The largest ideal I for which $\Gamma(I) < \Gamma$ is called the *level* of Γ. For convenience, if $n \in \mathbb{Z} \subset R_k$, we will denote the principal R_k-ideal $< n >$ simply by n.

For a variety of reasons (geometric, number theoretic and topological), congruence subgroups are perhaps the most studied class of arithmetic subgroups. For example, in the case when $R_k = \mathbb{Z}$, and $\Gamma < \mathrm{PSL}(2,\mathbb{Z})$, the genus, number of cone points and number of cusps of $\mathbb{H}^2/\Gamma$ has been well-studied. This was perhaps best articulated in a conjecture of Rademacher which posited that there are only finitely many congruence subgroups $\Gamma < \mathrm{PSL}(2,\mathbb{Z})$ of genus 0 (i.e. when $\mathbb{H}^2/\Gamma$ has genus 0). The proof of this was completed by Denin in a sequence of papers [16], [17] and [18]. Different proofs of this (actually of a slightly stronger version of this result) were also given by Thompson [40] and Zograf [44]. Indeed, in these two papers it is proved that there are only finitely congruence subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ of any fixed genus.

The complete enumeration of congruence subgroups of genus 0 was completed in [15], where 132 groups are listed. The list of torsion-free congruence subgroups of genus 0 was completed in [36] (there are 33 and the levels are all of the form $2^a3^b5^c7$ with $a \le 5$, $b \le 3$, and $c \le 2$ with 2^5 being the largest level). As is easy to see (and we describe this in §4 below) of those 33, only 4 are principal congruence subgroups (of levels 2, 3, 4 and 5).

Turning to dimension 3, let d be a square-free positive integer, let O_d denote the ring of integers in $\mathbb{Q}(\sqrt{-d})$, and let Q_d denote the Bianchi orbifold $\mathbb{H}^3/\mathrm{PSL}(2,\mathrm{O_d})$. A non-compact finite volume hyperbolic 3-manifold X is called *arithmetic* if X and Q_d are commensurable, that is to say they share a common finite sheeted cover (see [30, Chapter 8] for more on this). If N is a closed orientable 3-manifold and $L \subset N$ a link, then L is called *arithmetic* if $N \setminus L$ is an arithmetic hyperbolic 3-manifold.

In his list of problems in his Bulletin of the AMS article [42], Thurston states as Question 19:

Find topological and geometric properties of quotient spaces of arithmetic subgroups of $\mathrm{PSL}(2,\mathbb{C})$. *These manifolds often seem to have special beauty.*

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For example, many of the key examples in the development of the theory of geometric structures on 3-manifolds (e.g. the figure-eight knot complement, the Whitehead link complement, the complement of the Borromean rings and the Magic manifold) are arithmetic.

The "beauty" referred to by Thurston is captured particularly well by congruence manifolds (which includes all of the above examples); i.e. manifolds $M = \mathbb{H}^3/\Gamma$ where Γ is congruence. Similarly, a manifold $\mathbb{H}^3/\Gamma$ is called principal congruence if $\Gamma = \Gamma(I)$ for some ideal I. As above we will also refer to a link $L \subset N$ as congruence (resp. principal congruence) if the manifold $N \setminus L$ is so.

In this paper we will survey what is known about the following question, which as we discuss below, can be viewed as the most natural generalization of the Rademacher conjecture to dimension 3.

Question 1.1. Are there only finitely many congruence link complements in S^3?

As noted above the work of Zograf shows that there are only finitely many congruence surfaces of any fixed genus. Thus a far reaching generalization of this to dimension 3 is the following question:

Question 1.2. Let N be a fixed closed orientable 3-manifold. Are there only finitely many congruence link complements in N?

We finish the introduction with a brief discussion of the remainder of the paper. In §3 we describe early progress on Question 1.1 in the context of the "Cuspidal Cohomology Problem". This was given particular impetus in work of Schwermer (with Fritz Grunewald) in [24]. In §4–6 we survey recent work of the authors ([7], [8] and [9]), which together with the work of Goerner [21] and [22] (which completed the enumeration in the cases of $d = 1, 3$) answers a question of Thurston about the complete list of principal congruence link complements asked in an email to the authors in 2009:

"Although there are infinitely many arithmetic link complements, there are only finitely many that come from principal congruence subgroups. Some of the examples known seem to be among the most general (given their volume) for producing lots of exceptional manifolds by Dehn filling, so I'm curious about the complete list."

To that end we discuss the proof of the following theorem (the final details of which will appear in [9]).

Theorem 1.3. *There are 48 principal congruence link complements in S^3. The values of d and the levels I are listed in Table 1 in §5.*

Note that since links with at least 2 components are not generally determined by their complements (see [23]), one cannot just say "finitely many principal congruence links".

Also in §4 we describe what is known about Question 1.2 for principal congruence link complements as well as Question 1.1 for principal congruence link complements in S^3 arising from other maximal orders. In §7 we discuss some levels where Question 1.1 can be answered positively, and in §8 we finish with some comments and speculations.

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2. Preliminaries

Let h_d denote the class number of $\mathbb{Q}(\sqrt{-d})$. Then as is well-known the quotient orbifold $Q_d = \mathbb{H}^3/\mathrm{PSL}(2, \mathrm{O_d})$ has h_d cusps. Apart from the cases of $d = 1, 3$, the cusp ends of the Q_d all have the form $T^2 \times [0, \infty)$ where T^2 is the 2-torus. When $d = 1$, the cusp end is $[S, \infty)$ where S is the Euclidean 2-orbifold which is a 2-sphere with 4 cone points of cone angle π, and when $d = 3$, the cusp end is $[B, \infty)$ where B is the Euclidean 2-orbifold which is a 2-sphere with 3 cone points of cone angle $2\pi/3$.

Let $\Gamma \leq \mathrm{PSL}(2, \mathrm{O_d})$ be a finite index subgroup. Then

- A *cusp*, $[c]$, of Γ is a Γ-orbit of points in $\mathbb{P}^1(\mathbb{Q}(\sqrt{-d}))$
- A *peripheral subgroup* of Γ for $[c]$ is a maximal parabolic subgroup, $P_x < \Gamma$, fixing $x \in [c]$. Note that if $y \in [c]$, then P_x and P_y are conjugate; hence a peripheral subgroup for $[c]$ is determined up to conjugacy.
- A *set of peripheral subgroups for* Γ is the choice of one peripheral subgroup for each cusp of Γ.

We will use the term *cusp* to mean $[c]$, a choice of point x in $[c]$, as well as the end of $\mathbb{H}^3/\Gamma$ corresponding to $[c]$. Which one is meant should be clear from the context.

Recall that if $p \in \mathbb{Z}$ then p is called *inert* if the O_d-ideal $< p >$ remains prime, and p is said to *split* if the O_d-ideal $< p >= \mathcal{P}_1\mathcal{P}_2$ (and also say that $\mathcal{P}_i$ is a split prime for $i = 1, 2$). If $I \subset O_d$ is an ideal then $N(I) = |O_d/I|$ denotes the norm of the ideal I.

3. The Cuspidal Cohomology Problem

Let Γ be a non-cocompact Kleinian (resp. Fuchsian) group acting on $\mathbb{H}^3$ (resp. $\mathbb{H}^2$) with finite co-volume, and set $X = \mathbb{H}^n/\Gamma$ with $n = 2, 3$. Let $\mathcal{U}(\Gamma)$ denote the subgroup of Γ generated by parabolic elements of Γ. Note that $\mathcal{U}(\Gamma)$ is visibly a normal subgroup of Γ, and we may define:

$$V(X) = V(\Gamma) = (\Gamma/\mathcal{U}(\Gamma))^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Then the subspace of $H_1(X, \mathbb{Q})$ which defines the (degree 1) cuspidal cohomology of X (or Γ) can be identified with $V(\Gamma)$.

Setting $r(\Gamma) = \dim_{\mathbb{Q}}(V(\Gamma))$, we see that in the case when Γ is a torsion-free Fuchsian group $r(\Gamma) = 0$ if and only if the underlying space of X is a punctured S^2, which in turn is equivalent to Γ being generated by parabolic elements.

When Γ is Kleinian and $X \cong S^3 \setminus L$, then $r(\Gamma) = 0$ and Γ is generated by parabolic elements. Of course, in dimension 3, other closed manifolds provide examples of link complements $\mathbb{H}^3/\Gamma \cong \Sigma \setminus L$ satisfying $r(\Gamma) = 0$, e.g. when Σ is an integral homology 3-sphere. It is also not the case that being generated by parabolic elements forces the link complement to be contained in S^3 (see [22]).

In the setting of the Bianchi groups, $\mathrm{PSL}(2, \mathrm{O_d})$, *The Cuspidal Cohomology Problem* posed in the 1980's asked which Bianchi groups have $r(\mathrm{PSL}(2, \mathrm{O_d})) = 0$. Building on work of many people (see for example [24], [25], [35]), the solution of the Cuspidal Cohomology Problem was provided by Vogtmann [43] who determined the list of all values of d (see Theorem 3.1 below) with $r(Q_d) = 0$. In particular, this provided the list of those d for which Q_d can have a cover homeomorphic to an arithmetic link complement in S^3. Moreover in [5] it was shown that for every such d there does exist an arithmetic link complement. We summarize this discussion in the following result:

Theorem 3.1. *Q_d is covered by an arithmetic link complement in S^3 if and only if*

$$d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}.$$

4. Finiteness of principal congruence link complements

In this section we discuss the proof of finiteness of principal congruence link complements; the case of dimension 2 is well-known, and a proof in dimension 3 is given in [7] (see also [8]).

4.1. **The case of** $\mathrm{PSL}(2,\mathbb{Z})$**.**

Proposition 4.1. *The only principal congruence subgroups of* $\mathrm{PSL}(2,\mathbb{Z})$ *of genus* 0 *have level* $n = 2, 3, 4, 5$.

Proof. An easy argument shows that $\Gamma(n)$ is torsion-free for all $n \geq 2$. The proof is completed using the following straightforward observations:

(1) $\Gamma(n)$ has genus zero if and only if $\Gamma(n)$ is generated by parabolic elements.

(2) Let $T(n)$ denote the cyclic subgroup of $\Gamma(n)$ fixing the cusp at ∞; i.e. $T(n)$ is the cyclic group generated by $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Hence the normal closure $N(n)$ of $T(n)$ in $\mathrm{PSL}(2,\mathbb{Z})$ is a subgroup of $\Gamma(n)$. It is easy to see that $\Gamma(n)$ is generated by parabolic elements if and only $N(n) = \Gamma(n)$.

(3) Now note that $\mathrm{PSL}(2,\mathbb{Z})/\mathrm{N(n)}$ is isomorphic to the $(2, 3, n)$ triangle group, and this is finite if and only if n is as in the statement of Proposition 4.1. □

4.2. **The Bianchi groups.**

Although we will also use the idea of proof of Proposition 4.1, the case of dimension 3 requires additional technology, and handled somewhat differently; since, amongst other reasons, there are cases where the class number is greater than 1. Following [7] and [8], we will appeal to systole bounds. Recall that if M is a finite volume orientable hyperbolic 3-manifold, the *systole of* M is the length of the shortest closed geodesic in M, and will be denoted by sys(M). The following is proved in [8, Lemma 4.1] (using [1] and the improvement in [33]). The solution to the Cuspidal Cohomology Problem yields only finitely many d to consider, and so the finiteness of the number of principal congruence link complements easily follows from Lemma 4.2. We include the proof since we will refer to it later.

Lemma 4.2. *Suppose that* $I \subset O_d$ *is an ideal such that* $\mathbb{H}^3/\Gamma(I)$ *is homeomorphic to a link complement in* S^3*. Then* $N(I) < 39$.

Proof. If $\gamma \in \Gamma(I)$ is a hyperbolic element, its complex length is $\ell(\gamma) = \ell_0(\gamma) + i\theta(\gamma)$, where $\ell_0(\gamma)$ is the translation length of γ and $\theta(\gamma)$ is the angle incurred in translating along the axis of γ by distance $\ell_0(\gamma)$. Now, as is well-known $\cosh(\ell(\gamma)/2) = \pm\mathrm{tr}\,(\gamma)/2$, and so we get the following inequality for $\ell_0(\gamma)$: $|\mathrm{tr}\,(\gamma)|/2 \leq \cosh(\ell_0(\gamma)/2)$. With the systole bound given in [33] of $7.171646\ldots$, the argument of [1] used in [7] can be reworked and gives:

$$|\mathrm{tr}\,(\gamma)/2| \leq \cosh(7.1717/2) \leq 18.1 \text{ and so } |\mathrm{tr}\,(\gamma)| < 37.$$

From [7, Lemma 2.5] (see also [8, Lemma 4.1]) we have $\mathrm{tr}\,(\gamma) \pm 2 \in I^2$, and so the bound on $|\mathrm{tr}\,(\gamma)|$ given above implies that $N(I) < 39$. □

Remark 4.3. The proof of Lemma 4.2 actually shows more: if $\Gamma(I)$ is a link group, then there exists $x \in I$ such that $|x|^2 < 39$. This is relevant in the case when $h_d > 1$, since there are ideals I of norm less than 39 for which no such element exists and hence $\Gamma(I)$ is not a link group.

Finally in this section we discuss the case of principal congruence link complements in arbitrary closed orientable 3-manifolds. The first remark is that in the case when N is a closed orientable 3-manifold that does not support a metric of negative curvature, then the systole bound from [1] (or the improvement in [33]) and the argument of Lemma 4.2 also proves finiteness for principal congruence link complements $N \setminus L$.

To discuss the hyperbolic case, let $X = N \setminus L$; then the dimension of $V(X)$ can be shown to be bounded above by $\dim(H_2(N;\mathbb{Q}))$, whilst on the other hand, it is known that (see [24]) as $d \to \infty$ the dimension of $V(Q_d)$ goes to infinity.

Thus we deduce the following corollary of this discussion.

Corollary 4.4. *Suppose that N is a fixed closed orientable 3-manifold. If $N \setminus L = \mathbb{H}^3/\Gamma(I)$, then the there are at most finitely many d such that $I \subset O_d$.*

Thus as in the case of S^3, it remains to control the levels in these finitely many d. This was done by Lakeland and Leininger [28] using a more refined analysis of systole bounds.

Theorem 4.5. *Let N be a closed orientable 3-manifold. Then there are only finitely many principal congruence link complements in N.*

4.3. **Other maximal orders.** The Bianchi groups can be thought of as arising from the maximal order $M(2,O_d) \subset M(2,\mathbb{Q}(\sqrt{-d}))$. In the case when h_d is even, there are maximal orders $\mathcal{O} \subset M(2,\mathbb{Q}(\sqrt{-d}))$ that are not $\mathrm{GL}(2,\mathbb{Q}(\sqrt{-\mathrm{d}}))$-conjugate to $M(2,O_d)$ and whose group of elements of norm one thereby define arithmetic Kleinian groups $\Gamma^1_{\mathcal{O}}$ commensurable with $\mathrm{PSL}(2,\mathrm{O_d})$, with the same co-volume as $\mathrm{PSL}(2,\mathrm{O_d})$, but not conjugate to $\mathrm{PSL}(2,\mathrm{O_d})$ (see below and [30] for further details). A version of the cuspidal cohomology problem was solved for these groups, and the following additional values of d (beyond those listed in Theorem 3.1) provide groups $\Gamma^1_{\mathcal{O}}$ with $r(\Gamma^1_{\mathcal{O}}) = 0$: namely $d \in \{10, 14, 35, 55, 95, 119\}$. In addition, for $d \in \{5, 6, 15, 39\}$ (which do arise in Theorem 3.1) groups $\Gamma^1_{\mathcal{O}}$ with $\mathcal{O} \neq M(2,O_d)$ were also shown to have $r(\Gamma^1_{\mathcal{O}}) = 0$ (see [13]).

Turning to link groups, it is known that link groups do appear as subgroups of finite index in some of these additional groups $\Gamma^1_{\mathcal{O}}$; namely in the cases of $d \in \{5, 6, 10, 15, 35, 39, 55\}$ ([6],[37],[38]). One can also make sense of the notion of congruence subgroups of the groups $\Gamma^1_{\mathcal{O}}$, however nothing is known about the existence of (principal) congruence link complements or more generally Question 1.1 in this setting. However, we can establish finiteness of principal congruence link complements in this broader sense. Before stating the result carefully we recall the construction of these other types of maximal orders. To that end, let $J \subset O_d$ be an ideal and define $\mathcal{O}_d(J)$ by:

$$\mathcal{O}_d(J) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in O_d, c \in J, b \in J^{-1}\},$$

where $J^{-1} = \{x \in \mathbb{Q}(\sqrt{-d}) : xy \in O_d, \forall y \text{ in } J\}$ is the inverse ideal.

It follows from [30, Chapter 6.7] that the *type number of maximal orders* of $M(2,\mathbb{Q}(\sqrt{-d})$ (which is the number of distinct $\mathrm{GL}(2,\mathbb{Q}(\sqrt{-\mathrm{d}}))$-conjugacy classes of maximal orders) is one when h_d is odd and and when h_d is even it is equal to $t_d = |H_d/H_d^{(2)}|$ where H_d is the class group and $H_d^{(2)}$ the group generated by squares of elements in H_d. In particular, in the case when h_d is even, we may find a finite collection of ideals $J_1, \ldots, J_{t_d}$ so that every maximal order of $M(2,\mathbb{Q}(\sqrt{-d}))$ is $\mathrm{GL}(2,\mathbb{Q}(\sqrt{-\mathrm{d}}))$-conjugate to one of $\mathcal{O}_d(J_i)$. By convention we take $J_1 = O_d$ and identify the maximal order $\mathcal{O}_d(J_1) = M_2(O_d)$.

Following the notation above, the maximal orders $\mathcal{O}_d(J_i)$ give rise to arithmetic Kleinian groups $\Gamma^1_{\mathcal{O}_d(J_i)}$ commensurable with $\mathrm{PSL}(2,\mathrm{O_d})$. Note that from the previous discussion and [30, Theorem 9.2.2] every arithmetic link group is conjugate into some group $\Gamma^1_{\mathcal{O}_d(J_i)}$

For an ideal $I \subset O_d$ with $I \neq J_i$ for $i = 1, \ldots, t_d$, we can construct the principal congruence subgroup $\Gamma_i(I)$ by reducing modulo I (we refer the reader to [30, Chapter 6.6] for more details about this). Call $\Gamma < \Gamma^1_{\mathcal{O}_d(J_i)}$ congruence if $\Gamma > \Gamma_i(I)$ for some $i = 1, \ldots, t_d$ and $I \subset O_d$.

Theorem 4.6. *There are only finitely many principal congruence link complements arising as $\mathbb{H}^3/\Gamma_i(I)$.*

Proof. The proof follows the idea in the proof of Lemma 4.2. From above, there is a finite list of possible values of d, and moreover, we need only consider those d for which h_d is even—which reduces to $d \in \{5, 6, 10, 14, 15, 35, 39, 55, 95, 119\}$. Since the type number is finite, it follows, as before, that it remains to bound the norm of the ideal I. Following the proof of [7, Lemma 2.5] if $\gamma \in \Gamma_i(I)$ is a hyperbolic element then it has the form

$$\gamma = \begin{pmatrix} \pm 1 + a & b \\ c & \pm 1 + d \end{pmatrix}$$

where $a, d \in I$, $c \in I.J$ and $b \in I.J^{-1}$. Following the argument in the proof of [7, Lemma 2.5] we deduce that $\pm(a+d) = -ad+bc$, but this time we have $bc \in IJ \cdot IJ^{-1} \subset I^2$. Regardless, we can still deduce that $\operatorname{tr}\gamma \equiv \pm 2 \bmod \mathrm{I}^2$. Given this we can complete the proof as in Lemma 4.2 to deduce that $N(I) < 39$. □

5. Techniques to determine the list

Table 1 below gives the complete list of 48 pairs (d, I) describing all principal congruence subgroups $\Gamma(I) < \mathrm{PSL}(2, \mathrm{O_d})$ such that $\mathbb{H}^3/\Gamma(I)$ is a link complement in S^3. Note that if $\Gamma(I)$ determines a principal congruence link group, so does $\Gamma(\overline{I})$.

(1) **$d = 1$:** $I \in \{2, < 2 \pm i >, < (1 \pm i)^3 >, 3, < 3 \pm i >, < 3 \pm 2i >, < 4 \pm i >\}$.
(2) **$d = 2$:** $I \in \{2, < 1 \pm \sqrt{-2} >, < 1 \pm 2\sqrt{-2} >, < 2 \pm \sqrt{-2} >, < 3 \pm \sqrt{-2} >, < 1 \pm 3\sqrt{-2} >\}$.
(3) **$d = 3$:** $I \in \{2, 3, < (5 \pm \sqrt{-3})/2 >, < 3 \pm \sqrt{-3} >, < (7 \pm \sqrt{-3})/2 >, < 4 \pm \sqrt{-3} >, < (9 \pm \sqrt{-3})/2 >\}$.
(4) **$d = 5$:** $I =< 3, (1 \pm \sqrt{-5}) >$.
(5) **$d = 7$:** $I \in \{< (1 \pm \sqrt{-7})/2 >, 2, < (3 \pm \sqrt{-7})/2 >, < \sqrt{-7} >, < 1 \pm \sqrt{-7} >, < (-5 \pm \sqrt{-7})/2 >, < 2 \pm \sqrt{-7} >, < (1 \pm 3\sqrt{-7})/2 >\}$.
(6) **$d = 11$:** $I \in \{< (1 \pm \sqrt{-11})/2 >, < (3 \pm \sqrt{-11})/2 >, < (5 \pm \sqrt{-11})/2 >\}$.
(7) **$d = 15$:** $I \in \{< 2, (1 \pm \sqrt{-15})/2 >, < 3, (3 \pm \sqrt{-15})/2 >, < 4, (1 \pm \sqrt{-15})/2 >, < 5, (5 \pm \sqrt{-15})/2 >, < 6, (-3 \pm \sqrt{-15})/2 >\}$.
(8) **$d = 19$:** $I =< (1 \pm \sqrt{-19})/2 >$.
(9) **$d = 23$:** $I \in \{< 2, (1 \pm \sqrt{-23})/2 >, < 3, (1 \pm \sqrt{-23})/2 >, < 4, (-3 \pm \sqrt{-23})/2 >\}$.
(10) **$d = 31$:** $I \in \{< 2, (1 \pm \sqrt{-31})/2 >, < 4, (1 \pm \sqrt{-31})/2 >, < 5, (3 \pm \sqrt{-31})/2 >\}$.
(11) **$d = 47$:** $I \in \{< 2, (1 \pm \sqrt{-47})/2 >, < 3, (1 \pm \sqrt{-47})/2 >, < 4, (1 \pm \sqrt{-47})/2 >\}$.
(12) **$d = 71$:** $I =< 2, (1 \pm \sqrt{-71})/2 >$.

Table 1

Comparing with Theorem 3.1, the reader will note that the cases of $d = 6$ and $d = 39$ do not occur in Table 1. Indeed, there are no principal congruence link complements for these values of d, although as we show in [8], there are congruence link complements.

To determine the list of those d and levels I, there are two main issues. First, proving that those on the list are link complements in S^3, and secondly, eliminating those that are not on this list. We refer the reader to [7], [8] and [9] for further details. Sample computations are given in §6.

We also note that these principal congruence link complements in S^3 also give rise to many more examples of principal congruence link complements in other closed non-hyperbolic 3-manifolds. For example, the case of $(7, (1 \pm \sqrt{-7})/2)$ gives a principal congruence link complement known as the Magic manifold. This manifold has many "exceptional" (i.e. non-hyperbolic) Dehn surgeries, which thereby gives closed 3-manifolds N containing a link L with $N \setminus L \cong \mathbb{H}^3/\Gamma(< (1 \pm \sqrt{-7})/2 >)$; for example infinitely many small Seifert fibered spaces, and infinitely many torus bundles over the circle admitting SOL geometry (see [31]).

5.1. **Establishing $\mathbb{H}^3/\Gamma(I)$ is a link complement.** To establish the principal congruence link groups in Table 1, we typically invoke the following strategy, which closely mimics that used in the case of $\mathrm{PSL}(2, \mathbb{Z})$. Note that only in a small number of cases do we have explicit links, indeed some of these explicit links in S^3 were in the literature (e.g. in [2], [4], [26] and [41]) and were shown to give principal congruence link complements directly (see [7] and [8] for further details).

(1) If $\mathbb{H}^3/\Gamma(I) \cong S^3 \setminus L$, then $\Gamma(I)$ is generated by parabolic elements.
(2) $\mathbb{H}^3/\Gamma(I) \cong S^3 \setminus L$ if and only if $\Gamma(I)$ can be trivialized by setting one parabolic from each cusp of $\Gamma(I)$ equal to 1.

Some further commentary: If $\mathbb{H}^3/\Gamma(I) \cong S^3 \setminus L$, then for each component L_i of L, there is a meridian curve x_i so that Dehn filling $S^3 \setminus L$ along the totality of these curves gives S^3. Thus, trivializing the corresponding parabolic elements $[x_i]$ in $\Gamma(I)$ gives the trivial group. Conversely, given Perelman's resolution of the Geometrization Conjecture, if $\Gamma(I)$ can be trivialized by setting one parabolic from each cusp of $\Gamma(I)$ equal to 1, then $\mathbb{H}^3/\Gamma(I)$ is homeomorphic to a link complement in S^3.

To check whether $\Gamma(I)$ is generated by parabolic elements we can proceed as follows.

Let $\Gamma(I) < \mathrm{PSL}(2, \mathrm{O_d})$, and let P_i be the peripheral subgroup of $\mathrm{PSL}(2, \mathrm{O_d})$ fixing the cusp $c_i \in \mathbb{Q}(\sqrt{-d}) \cup \{\infty\}$ for $i = 1, \ldots, h_d$. Set $P_i(I) = \mathrm{P}_i \cap \Gamma(I)$ be the peripheral subgroup of $\Gamma(I)$ fixing c_i. Let $N_d(I)$ denote the normal closure in $\mathrm{PSL}(2, \mathrm{O_d})$ of $\{P_1(I), \ldots, P_{h_d}(I)\}$. Note that $N_d(I) < \Gamma(I)$ since $\Gamma(I)$ is a normal subgroup of $\mathrm{PSL}(2, \mathrm{O_d})$. It is clear that $\Gamma(I)$ is generated by parabolic elements if and only if $N_d(I) = \Gamma(I)$. Now we can try to use Magma [14] to test whether $\Gamma(I) = N_d(I)$. However, this does not always succeed in allowing us to decide on way or the other, and additional methods are required (we refer the reader to [7] and [8] for further details).

To execute the second part of the strategy described above, we need to find parabolic elements in $\Gamma(I)$, one for each cusp, so that trivializing these elements trivializes the group. From above, we obtain a partial set $S = \{P_1(I), \ldots, P_{h_d}(I)\}$ of peripheral subgroups for $\Gamma(I)$. To obtain a full set of peripheral subgroups for $\Gamma(I)$ we need to add certain conjugates of the $P_i(I)$ to the partial set S.

Next, given this full set of peripheral subgroups for $\Gamma(I)$, we choose one parabolic from each of these peripheral subgroups and use Magma to check that trivializing these elements trivializes $\Gamma(I)$. This choice of parabolic elements involves trial and error. It is worth emphasizing that finding these peripheral subgroups and expressing them in terms of generators for a presentation of $\mathrm{PSL}(2, \mathrm{O_d})$ was a highly non-trivial exercise.

5.2. **Eliminating $\mathbb{H}^3/\Gamma(I)$ as a link complement.** We begin by discussing the case when $I =< n >$ and $n \in \mathbb{Z}$. To that end we recall the following result from [3] that places severe restrictions on the list of possible d's.

Theorem 5.1. *If $h_d > 1$, and $\Gamma(n) < \mathrm{PSL}(2, \mathrm{O_d})$, then $\mathbb{H}^3/\Gamma(n)$ is not homeomorphic to a link complement in S^3.*

This result, together with the discussion in [7, Section 4.1] allows one to deduce:

Corollary 5.2. *Suppose that $\Gamma(n) < \mathrm{PSL}(2, \mathrm{O_d})$ and $\mathbb{H}^3/\Gamma(n)$ is homeomorphic to a link complement in S^3. Then $d \in \{1, 2, 3, 7, 11, 19\}$ and $n \in \{2, 3, 4, 5\}$.*

Since we are reduced to the case when $h_d = 1$ some pairs (d, n) can be eliminated quite quickly using Magma. If P denotes the peripheral subgroup of $\Gamma(n)$ fixing ∞, we show the normal closure $< P > \neq \Gamma(n)$. Table 2 below shows the cases for which this works. In this table, N is a normal subgroup of $\mathrm{PSL}(2, \mathrm{O_d})$ that contains the group $< P >$. Note the orders in the final two columns are different, so we can conclude that $< P > \neq \Gamma(n)$ in each case.

d	n	**Order of** $\mathrm{PSL(2, O_d)/N}$	**Order of** $\mathrm{PSL(2, O_d/I)}$
2	3	2304	288
3	4	3840	1920
7	3	1080	360
11	2	120	60
11	4	7680	1920

Table 2

To handle the remaining integral levels a combination of methods are used, we summarize these in the following combination of results from [7].

Proposition 5.3. *Assume that* $d \in \{2, 7, 11, 19\}$*, then* $\Gamma(p) < \mathrm{PSL(2, O_d)}$ *is not a link group in the following two cases:*

- p *is an inert prime in* O_d*.*
- $p \geq 5$ *splits in* O_d*.*

Using this we can eliminate the following cases.

Corollary 5.4. *For* $(d, n) \in \{(2, 5), (7, 3), (7, 5), (11, 2), (11, 5), (19, 2), (19, 3), (19, 5)\}$*, the groups* $\Gamma(n)$ *are not link groups.*

Finally, using cuspidal cohomology calculations (see [7, Section 4]), one can show.

Proposition 5.5. *For those pairs* (d, n) *listed below, the principal congruence subgroups* $\Gamma(n)$ *satisfy* $r(\Gamma(n)) \neq 0$.

$$\{(1, 4), (1, 5), (2, 4), (3, 5), (7, 4), (11, 3), (19, 4)\}.$$

Other levels: In [8], a strengthening of Theorem 5.1 proved useful, namely:

Theorem 5.6. *If* $h_d > 1$*, then* $\Gamma(n) < \mathrm{PSL(2, O_d)}$ *satisfies* $r(\Gamma(n)) \neq 0$ *in the following cases (using the notation introduced earlier to indicate d and the level):*

$$(23, 3), (23, 5), (31, 2), (47, 2), (47, 3), (71, 2), (71, 3).$$

Putting this together, in [8] it is shown that:

Corollary 5.7. *Let* $d \in \{23, 31, 47, 71\}$*,* $I \subset O_d$ *an ideal and* $p = 2, 3, 5$*. Suppose that* (d, p) *is as in Theorem 5.6 and* I *is divisble by* $< p >$*. Then* $\Gamma(I)$ *has non-trivial cuspidal cohomology. In particular* $\mathbb{H}^3/\Gamma(I)$ *is not homeomorphic to a link complement in* S^3*.*

Most of the eliminations relied on Magma computations once again and we refer the reader to §6 for some sample calculations. However, one particular case $(2, < 1 + 3\sqrt{-2} >)$ proved particularly stubborn, and we required additional help in the form of programs that can compute automatic structures on groups to eliminate this case (see [9] for details).

The strategy more generally is this (which is an extension of some of the ideas used for dealing with some of the integral levels). Using presentations for the the Bianchi groups, as well as matrix representatives for the peripheral subgroups P_i of $\mathrm{PSL(2, O_d)}$ (as above), we identify the peripheral subgroups $P_i(I)$ for $i = 1, \ldots, h_d$, and consider the quotient group $B_d(I) = \mathrm{PSL(2, O_d)/N_d(I)}$.

If $\Gamma(I)$ is a link group then $B_d(I)$ is a finite group with order equal to $|\mathrm{PSL(2, O_d/I)}|$. Hence if $B_d(I)$ is infinite or has order greater than $|\mathrm{PSL(2, O_d/I)}|$, then $\Gamma(I)$ cannot be a link group. We can input $B_d(I)$ in the Magma routines as:

$$B_d(I) =< \mathrm{PSL(2, O_d)} | \mathrm{P_1(I)} = \ldots = \mathrm{P_{h_d}(I)} = 1 >$$

that is by adding the peripheral subgroups $P_i(I)$ to the relations of $\mathrm{PSL(2, O_d)}$. We distinguish two cases:

Case 1: *$B_d(I)$ is a finite group but has order larger than* $|\mathrm{PSL}(\mathrm{O_d}/\mathrm{I})|$.

Case 2: *$B_d(I)$ has a finite index subgroup with "large" abelianization, i.e. of very large order compared to the size of* $\mathrm{PSL}(2,\mathrm{O_d}/\mathrm{I})$ *or infinite.*

In particular, in either case we can deduce that $\Gamma(I)$ cannot be a link group. Also note that if $B_d(I)$ is infinite or of order larger than $|\mathrm{PSL}(2,\mathrm{O_d}/\mathrm{J})|$ for an ideal $J \subset I$, then so is $B_d(J)$, and hence $\Gamma(J)$ is also not a link group.

6. Sample calculations

6.1. **Establishing principal congruence link complements.** We include two examples which are representative of the methods used, one from [7] and one from [8].

Case of $d = 1, I = 3$: We will sketch some of the ideas from [7] to show that the principal congruence subgroup $\Gamma(3) < \mathrm{PSL}(2,\mathrm{O_1})$ is a twenty component link group. To that end, from [39], $\mathrm{PSL}(2,\mathrm{O_1})$ has the following presentation:

$$\mathrm{PSL}(2,\mathrm{O_1}) =< \mathrm{a}, \ell, \mathrm{t}, \mathrm{u} \mid \ell^2 = (\mathrm{t}\ell)^2 = (\mathrm{u}\ell)^2 = (\mathrm{a}\ell)^2 = \mathrm{a}^2 = (\mathrm{ta})^3 = (\mathrm{ua}\ell)^3 = 1, [\mathrm{t},\mathrm{u}] = 1 >,$$

where $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $u = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$ and $\ell = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ (with the obvious abuse of notation between SL and PSL). The peripheral subgroup $P < \Gamma(3)$ fixing ∞ in this case is $< t^3, u^3 >$. A Magma routine from [7] is included below and shows $< P >= \Gamma(3)$.

We include some additional preamble before the Magma computation. It can be checked that $\Gamma(3)$ is a normal subgroup of $\mathrm{PSL}(2,\mathrm{O_1})$ of index 360 and the peripheral subgroup $P_\infty < \mathrm{PSL}(2,\mathrm{O_1})$ fixing ∞ maps to a group of order 18. Hence $\mathbb{H}^3/\Gamma(3)$ has 20 cusps.

In this case (as with others in [7] and [8]) it is helpful to work with an intermediate subgroup $\Gamma(3) < \Gamma < \mathrm{PSL}(2,\mathrm{O_1})$, where Γ is defined to be the group $< \Gamma(3), \delta >= \Gamma(3).\delta$ where $\delta = atu^{-1}$. As shown by Magma, $[\Gamma : \Gamma(3)] = 5$, and so we may deduce that the cover $\mathbb{H}^3/\Gamma(3) \to \mathbb{H}^3/\Gamma$ is a regular 5-fold cyclic cover with $\mathbb{H}^3/\Gamma$ having four cusps, and each cusp of $\mathbb{H}^3/\Gamma(3)$ projecting one-to-one to a cusp of $\mathbb{H}^3/\Gamma$.

As was alluded to above, to determine appropriate parabolic elements is a somewhat tedious but straightforward computation. Briefly, in the case at hand, the four parabolic fixed points ∞, ± 1 and $1-i$ (the set of which we denote by S) can be shown all to be mutually inequivalent under the action of Γ. In addition, the following parabolic elements in Γ fix the points in S:

$$S' = \{t^3u^3, tat^3u^{-3}at^{-1}, t^{-1}au^3at, u^{-1}tau^3at^{-1}u\}.$$

These can be easily shown to be primitive parabolic elements in Γ.

Magma now shows that the normal closure of S' in Γ is $\Gamma(3)$. Since the parabolic elements listed above represent inequivalent cusps of $\mathbb{H}^3/\Gamma$, if we now perform Dehn filling on $\mathbb{H}^3/\Gamma$ along the curves corresponding to these parabolic elements, the normal closure computation shows that we obtain a group of order 5. Since these are primitive parabolic elements, this group is the fundamental group of a closed 3-manifold, namely some lens space L (by Geometrization). Hence we deduce that $\mathbb{H}^3/\Gamma$ is a 4 component link in L with fundamental group of order 5. From above we can compatibly fill the cusps of $\mathbb{H}^3/\Gamma(3) \to \mathbb{H}^3/\Gamma$ resulting in a 5-fold cover $N \to L$, and so $N \cong S^3$ as required.

Magma routine for $\Gamma(3)$

```
G<a,l,t,u>:=Group<a,l,t,u|l^2,a^2,(t*l)^2,(u*l)^2,(a*l)^2,(t*a)^3,(u*a*l)^3,
(t,u)>;
h:=sub<G|t^3,u^3>;
```

```
n:=NormalClosure(G,h);
print Index(G,n);
\\360
print AbelianQuotientInvariants(n);
\\[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
d:=sub<G|n,a*t*u^-1>;
print Index(G,d);
\\72
print AbelianQuotientInvariants(d);
\\[ 5, 0, 0, 0, 0 ]
d1:=sub<d|t^3*u^3,t*a*t^3*u^-3*a*t^-1,t^-1*a*u^3*a*t,u^-1*t*a*u^3*a*t^-1*u>;
d2:=NormalClosure(d,d1);
print Index(d,d2);
\\5
d2 eq n;
\\true
```

Case of $d = 15, I =< 2, \omega_{15} >$: Set $\omega_{15} = \frac{1+\sqrt{-15}}{2}$, and $I =< 2, \omega_{15} >$, an ideal of norm 2. From [39], a presentation for $\mathrm{PSL}(2, \mathrm{O}_{15})$ is given by:

$$\mathrm{PSL}(2,\mathrm{O}_{15}) =< \mathrm{a,t,u,c} \mid \mathrm{a}^2 = (\mathrm{ta})^3 = \mathrm{ucuatu}^{-1}\mathrm{c}^{-1}\mathrm{u}^{-1}\mathrm{a}^{-1}\mathrm{t}^{-1} = 1, [\mathrm{t,u}] = [\mathrm{a,c}] = 1 >,$$

where $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $u = \begin{pmatrix} 1 & \omega_{15} \\ 0 & 1 \end{pmatrix}$ and $c = \begin{pmatrix} 4 & 1-2\omega_{15} \\ 2\omega_{15}-1 & 4 \end{pmatrix}$. Since $h_{15} = 2$, the quotient orbifold Q_{15} has two cusps, and equivalence classes can be taken to be ∞ and $\frac{1-\sqrt{-5}}{2}$ with cusp stabilizers $P_1 =< t, u >$, and $P_2 =< tb, tu^{-1}ct^{-1} >$. Since $N(I) = 2$, $[\mathrm{PSL}(2,\mathrm{O}_{15}) : \Gamma(\mathrm{I})] = 6$ and it is easy to see that $\mathbb{H}^3/\Gamma(I)$ has 6 cusps. Now it can be shown that (in the notation above) $P_1(I) =< t^2, u >$ and $P_2(I) =< uca, (c^{-1}au^{-1}c^{-1}u^{-1}ta)^2 >$. Since $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = ta$, conjugating $P_1(I)$ and $P_2(I)$ by the elements $\{Id, ta, (ta)^2\}$ gives a set of 6 peripheral subgroups for $\Gamma(I)$. Now, we choose one element from each of these 6 peripheral subgroups:

$$\{t^2, (ta)u(ta)^{-1}, (ta)^2u(ta)^{-2}, uca, (ta)uca(ta)^{-1}(ta)^2(c^{-1}au^{-1}c^{-1}u^{-1}ta)^2(ta)^{-2}\}$$

In the Magma routine, Q denotes the quotient of $\Gamma(I)$ by the normal closure of these 6 parabolic elements, and Magma calculates that $Q =< 1 >$ which shows that $\Gamma(I)$ is trivialized by setting these 6 elements equal to 1. Thus $\Gamma(< 2, \omega_{15} >)$ is indeed a 6 component link group.

```
G<a,c,t,u>:=Group<a,c,t,u|a^2,(t*a)^3,u*c*u*a*t*u^-1*c^-1*u^-1*a*t^-1, (t,u),(a,c)>;
H:=sub<G|t^2,u,(c^-1*a*u^-1*c^-1*u^-1*t*a)^2,u*c*a>;
N:=NormalClosure(G,H);
print Index(G,N);
6
\\
Q:=quo<N|t^2,(t*a)*u*(t*a)^-1,(t*a)^2*u*(t*a)^-2,
u*c*a,(t*a)*u*c*a*(t*a)^-1,(t*a)^2*(c^-1*a*u^-1*c^-1*u^-1*t*a)^2*(t*a)^-2>;
print Order(Q);
1
\\
```

6.2. **Eliminating principal congruence link complements.** We now give some sample calculations which illustrates the discussion in §5.1.

From [7], Magma rules out the following values of d and levels. Here P denotes the subgroup fixing ∞ in $\Gamma(I)$ for the ideal I in question.

(i) $|\mathrm{PSL}(2, \mathrm{O}_1/ < 5+\mathrm{i} >)| = 6552$, and $< P >$ is contained in a normal subgroup of index 46800.

(ii) $|\mathrm{PSL}(2, \mathrm{O}_2/ < 4+\sqrt{-2} >)| = 1944$, and $< P >$ is contained in a normal subgroup of index 2654208.

(iii) $|\mathrm{PSL}(2, \mathrm{O}_2/ < 2+3\sqrt{-2} >)| = 3960$, and $< P >$ is contained in a normal subgroup of index 36432.

(iv) $|\mathrm{PSL}(2, \mathrm{O}_7/ < 3+\sqrt{-7} >)| = 1152$, and $< P >$ is contained in a normal subgroup of index 4608.

We next consider the example from [8] with $d = 6$ (so the class number is 2) and $I =< 11, 4+\sqrt{-6} >$. The norm of I is 11 and so $\mathrm{PSL}(2, \mathrm{O}_6/\mathrm{I})$ has order 660. The Magma routine included below shows that the quotient group of $\mathrm{PSL}(2, \mathrm{O}_6)$ obtained by quotienting by the normal closure of the parabolic subgroups $P_1(I)$ and $P_2(I)$ (as described above) provides a group denoted $B < a, t, u, b, c >$ in the Magma routine that has order which is too large since it has a finite index subgroup with an abelian quotient group of very large order. The relevant information needed is a presentation for $\mathrm{PSL}(2, \mathrm{O}_6)$ (from [39]) given below, and the generators for the groups $P_1(I) =< t^{11}, t^4u >$ and $P_2(I) =< (tb)^{11}, (tb)^4(cu)^{-1} >$.

$$\mathrm{PSL}(2, \mathrm{O}_6) =< \mathrm{a, t, u, b, c} \mid \mathrm{a}^2 = \mathrm{b}^2 = (\mathrm{ta})^3 = (\mathrm{atb})^3 = (\mathrm{atubu}^{-1})^3 =$$
$$t^{-1}ctubu^{-1}c^{-1}b^{-1} = 1, [t, u] = [a, c] = 1 >,$$

where $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $u = \begin{pmatrix} 1 & \sqrt{-6} \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} -1\sqrt{-6} & 2-\sqrt{-6} \\ 2 & 1+\sqrt{-6} \end{pmatrix}$, and $c = \begin{pmatrix} 5 & -2\sqrt{-6} \\ 2\sqrt{-6} & 5 \end{pmatrix}$.

Magma routine for $I =< 11, 4+\omega_6 >$**:**

```
B<a,t,u,b,c>:=Group<a,t,u,b,c|a^2,b^2,(t*a)^3,(a*t*b)^3, (a*t*u*b*u^-1)^3,
t^-1*c*t*u*b*u^-1*c^-1*b^-1, (t,u),(a,c),
t^11,t^4*u,(t*b)^11,(t*b)^4*(c*u)^-1>;
L:=LowIndexNormalSubgroups(B,660);
print #L;
2
\\
print Index(B,L[2]‘Group);
660
\\
print AbelianQuotientInvariants(L[2]‘Group);
[ 4, 120, 120, 120, 120, 120, 120, 120, 120, 120, 120 ]
\\
```

7. Congruence link complements

We now discuss what is known in the direction of Question 1.1. In the subsections below we prove the following theorem for which it will be convenient to recall the following.

Definition 7.1. Let $I \subset O_d$ be an ideal and let $\Gamma_0(I) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL(2, O_d)|c \equiv 0 \bmod I}\}$. and let $\Gamma_1(I) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL(2, O_d)|c \equiv 0 \bmod I, a, d \equiv 1 \bmod I}\}$

Theorem 7.2. *There are only finitely many link groups* $\Gamma < \mathrm{PSL(2, O_d)}$ *such that* $\Gamma(I) < \Gamma$ *in the following cases:*

(1) $\Gamma = \Gamma_1(I)$.
(2) $I = \mathcal{P}$ *is a prime ideal.*
(3) $I = \mathcal{P}_1^{n_1} \dots \mathcal{P}_k^{n_k} \subset O_d$, *where* $h_d = 1$ *and the ideals* $\mathcal{P}_i$ *are split primes of norm* p_i, *such that* $p_i \neq p_j$ *for* $i \neq j$.

Before commencing with the proof, we note that whenever $N(I) \geq 5$ the group $\Gamma_1(I)$ is torsion-free. Moreover, in [8] examples of link groups arising as $\Gamma_1(I)$ are given when $d = 6, 39$.

7.1. **Proof of Part 1.** We begin with some definitions (recall the proof of Lemma 4.2).

Definition 7.3. A group Γ has a small (resp. large) systole if its systole is at most (resp. greater than) $7.171646\dots$

The proof of Lemma 4.2 shows that if Γ is a link group it has a small systole. Part (1) of Theorem 7.2 will follow immediately from our next lemma (since there are only finitely many d we need only bound the norm of the ideal I).

Lemma 7.4. *Assume that* Γ *is generated by parabolic elements and has a small systole, then* $\Gamma = \Gamma_1(I)$ *for only finitely many ideals* $I \subset O_d$.

Proof. Suppose that $A \in \Gamma$ is a hyperbolic element whose translation length achieves the systole bound. Since $\Gamma = \Gamma_1(I)$ then $\mathrm{tr}\,(A) \pm 2 \in I$. But then this, coupled with the argument in the proof of Lemma 4.2, now shows that $N(I) \leq 39^2$. □

7.2. **Proof of Part 2.** Since there are only finitely many d, there are only finitely many ramified primes, and so we can assume that either $\mathcal{P}$ is a split prime or $\mathcal{P} =< p >$ and p is inert. We can also assume that $\mathcal{P}$ does not divide 2. In the case of a split prime, $\mathrm{PSL(2, O_d/\mathcal{P}) \cong PSL(2, \mathbb{F}_p)}$ and in the inert prime case $\mathrm{PSL(2, O_d/\mathcal{P}) \cong PSL(2, \mathbb{F}_{p^2})}$, where $\mathbb{F}_p$ (resp. $\mathbb{F}_{p^2}$) is the field of p elements (resp. p^2 elements). Since Γ is generated by parabolic elements, its image in $\mathrm{PSL(2, O_d/\mathcal{P})}$ is generated by parabolic elements, and it follows from the classification of subgroups of $\mathrm{PSL(2, \mathbb{F}_p)}$ and $\mathrm{PSL(2, \mathbb{F}_{p^2})}$ that this image group is conjugate into the image of $\Gamma_1(\mathcal{P})$ or it generates an isomorphic copy of $\mathrm{PSL(2, \mathbb{F}_p)} < \mathrm{PSL(2, \mathbb{F}_{p^2})}$ (see [19]), which only occurs in the case when p is inert. Now in the former case, we deduce that $\Gamma_1(\mathcal{P})$ has a small systole, and so we are done by Part (1). In the latter case we can argue as follows.

The orders of $\mathrm{PSL(2, \mathbb{F}_p)}$ and $\mathrm{PSL(2, \mathbb{F}_{p^2})}$ are $p(p^2-1)/2$ and $p^2(p^4-1)/2$ respectively. Hence the index $\mathrm{[PSL(2, O_d) : \Gamma] = p(p^2 + 1)}$. Now Γ is a link group and so in particular is torsion-free. Note that since -1 is not a square modulo 3, it follows that 3 does not divide p^2+1 for any prime p. Since $\mathrm{PSL(2, O_d)}$ has elements of order 3, then unless $p = 3$ we deduce that 3 does not divide the index $\mathrm{[PSL(2, O_d) : \Gamma]}$ which is impossible if Γ is torsion-free. But there are only finitely many possible primes $\mathcal{P}$ that can divide 3 and this proves finiteness. □

7.3. **Proof of Part 3.**

Definition 7.5. An ideal I will be referred to as large if $N(I) \geq 39^2$. Otherwise it is called small.

A variation of the proof Lemma 7.4 proves Lemma 7.6 (below) on noting that if I is an ideal and $A \in \Gamma_0(I)$ is a parabolic element, then $A \in \Gamma_1(I)$. The reason for this is that the image of A in $\mathrm{PSL}(2, \mathrm{O_d/I})$ has the form $\pm \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, with $ad = 1$ and $\mathrm{tr}\,(A) = a + d \equiv \pm 2 \bmod \mathrm{I}$. This forces $A \in \Gamma_1(I)$.

Lemma 7.6. *Suppose that $\mathcal{P} \subset O_d$ is a split prime, and assume that Γ is generated by parabolic elements and has a small systole. Then Γ cannot be a subgroup of $\Gamma_0(\mathcal{P}^n)$ if $\mathcal{P}^n$ is large.*

The proof of Theorem 7.2(3) will be completed using the following propositions whose proofs are included below. Henceforth, we assume that Γ has a small systole and is generated by parabolic elements; as noted earlier, both of these properties hold for link groups.

Proposition 7.7. *If $\Gamma(\mathcal{P}_1^{n_1} \ldots \mathcal{P}_k^{n_k}) < \Gamma$, then all the primes $\mathcal{P}_i$ can be assumed to be small.*

Given this we can assume that the ideals $\mathcal{P}_i$ are small.

Proposition 7.8. *Suppose that $\Gamma(\mathcal{P}_1^{n_1} \ldots \mathcal{P}_k^{n_k}) < \Gamma$, and that $m_1 \geq 2$ is the smallest integer for which $\mathcal{P}_1^{m_1}$ is large. Then n_1 can be chosen to be less than $8m_1$.*

In fact the role of $\mathcal{P}_1$ is not important here, and the same argument proves.

Corollary 7.9. *Suppose that $\Gamma(\mathcal{P}_1^{n_1} \ldots \mathcal{P}_k^{n_k}) < \Gamma$, and that $m_i \geq 2$ is the smallest integer for which $\mathcal{P}_i^{m_i}$ is large. Then each n_i can be chosen to be less than $8m_i$.*

The proof of Theorem 7.2 is now complete, since there are only finitely many values of d, only finitely many split prime ideals of norm less than 39^2 and, for each of these ideals, the exponent is bounded using Corollary 7.9. □

Proof of Proposition 7.7: The proof of Proposition 7.7 will follow immediately from the claim below. For given this claim, if there is a prime that is not small, then after possibly relabelling this prime by $\mathcal{P}_1$, we can lower the level of the principal congruence subgroup contained in Γ. We can now repeat this process, but since Γ is assumed congruence, this must stop at some $\Gamma(I)$ where I is an ideal that can only be a product of powers of small primes. □

Claim: *Suppose that $\Gamma(\mathcal{P}_1^{n_1} \ldots \mathcal{P}_k^{n_k}) < \Gamma$ and that $\mathcal{P}_1$ is large. Then $\Gamma(\mathcal{P}_2^{n_2} \ldots \mathcal{P}_k^{n_k}) < \Gamma$.*

The proof of the claim will require some additional lemmas.

Lemma 7.10. *i) Any matrix $M \in \mathrm{PSL}(2, \mathrm{O_d})$ is equivalent to a matrix $M' \in \mathrm{PSL}(2, \mathbb{Z})$ modulo $\Gamma(\mathcal{P}_1^{n_1} \ldots \mathcal{P}_k^{n_k})$. Furthermore, if $M \in \Gamma(\mathcal{P}_1^{m_1} \ldots \mathcal{P}_k^{m_k})$, where $0 \leq m_i \leq n_i$ then $M' \in \Gamma(p_1^{m_1} \ldots p_k^{m_k}) < \mathrm{PSL}(2, \mathbb{Z})$.*

ii) If M is a parabolic matrix, then M' can be chosen to be parabolic as well.

Proof. Since $O_d/\mathcal{P}_1^{m_1} \ldots \mathcal{P}_k^{m_k} \cong \mathbb{Z}/(p_1^{n_1} \ldots p_k^{n_k})\mathbb{Z}$, we have that $\mathrm{PSL}(2, \mathrm{O_d}/\mathcal{P}_1^{\mathrm{m_1}} \ldots \mathcal{P}_\mathrm{k}^{\mathrm{m_k}}) \cong \mathrm{PSL}(2, \mathbb{Z}/(\mathrm{p}_1^{\mathrm{n_1}} \ldots \mathrm{p}_\mathrm{k}^{\mathrm{n_k}})\mathbb{Z})$ and (i) follows easily.

Since $h_d = 1$, all parabolic elements of $\mathrm{PSL}(2, \mathrm{O_d})$ are conjugate into the peripheral subgroup fixing ∞. Hence $M = A \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} A^{-1}$ for some $b \in O_d$. This in turn is equivalent modulo $\Gamma(\mathcal{P}_1^{n_1} \ldots \mathcal{P}_k^{n_k})$ to $A' \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} (A')^{-1}$ where $A' \in \mathrm{PSL}(2, \mathbb{Z})$ and $n \in \mathbb{Z}$. This proves (ii). □

Lemma 7.11. *Let* $\Gamma =< \Gamma(I), M_1, M_2 >< \mathrm{PSL}(2, \mathrm{O_d})$ *where* M_1, M_2 *are parabolic matrices such that* $M_1 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \bmod \mathcal{P}_1^{\mathrm{n_1}}$ *and* $M_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathcal{P}_2^{\mathrm{n_2}} \dots \mathcal{P}_{\mathrm{k}}^{\mathrm{n_k}}$*, while* $M_2 \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \bmod \mathcal{P}_1^{\mathrm{n_1}}$ *and* $M_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathcal{P}_2^{\mathrm{n_2}} \dots \mathcal{P}_{\mathrm{k}}^{\mathrm{n_k}}$*. Then* $\Gamma(\mathcal{P}_2^{n_2} \dots \mathcal{P}_k^{n_k}) < \Gamma$.

Proof. Consider the map

$$\mathrm{PSL}(2, \mathrm{O_d}) \to \mathrm{P}(\mathrm{SL}(2, \mathrm{O_d/I})) \cong \mathrm{P}\big(\mathrm{SL}(2, \mathrm{O_d}/\mathcal{P}_1^{\mathrm{n_1}}) \times \mathrm{SL}(2, \mathrm{O_d}/\mathcal{P}_2^{\mathrm{n_2}} \dots \mathcal{P}_{\mathrm{k}}^{\mathrm{n_k}})\big)$$

Now note that $\Gamma(\mathcal{P}_2^{n_2} \dots \mathcal{P}_k^{n_k})$ maps to $P\big(\mathrm{SL}(2, \mathrm{O_d}/\mathcal{P}_1^{\mathrm{n_1}}) \times \mathrm{Id}\big)$ and that the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ generate $\mathrm{SL}(2, \mathrm{O_d}/\mathcal{P}_1^{\mathrm{n_1}})$. □

Before commencing with the proof of the claim it will be convenient to introduce the following definition.

Definition 7.12. A parabolic M has $\mathcal{P}$-level n if $n \geq 0$ is the largest integer for which $M \in \Gamma(\mathcal{P}^n)$.

Proof of Claim: Since Γ is generated by parabolics, it can be expressed as

$$\Gamma =< \Gamma(I), M_1, \dots, M_l >$$

where the M_i are parabolic matrices. We can assume that the M_i are in $\mathrm{PSL}(2, \mathbb{Z})$ by Lemma 7.10. Renumbering if necessary, let M_1 be of smallest $\mathcal{P}_1$-level among the M_i. Replacing Γ by a $\mathrm{PSL}(2, \mathbb{Z})$-conjugate, we can further assume that M_1 fixes ∞, and hence $M_1 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for some $b \in O_d$. Now $(b, p_1) = 1$ else all the M_i would have $\mathcal{P}_1$-level ≥ 1 and hence $\Gamma < \Gamma_0(\mathcal{P}_1)$, which contradicts Lemma 7.6.

Note that Γ must contain a second parabolic matrix, $M_2 = \begin{pmatrix} 1+a & b \\ c & 1-a \end{pmatrix}$, not in $\Gamma_0(\mathcal{P}_1)$; hence $(c, p_1) = 1$. Replacing M_1, M_2 by suitable powers, we can further assume that $M_1 \equiv M_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathrm{p}_2^{\mathrm{n_2}} \dots \mathrm{p}_{\mathrm{k}}^{\mathrm{n_k}}$ and that $M_1 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $M_2 \equiv \begin{pmatrix} 1+a & -a^2 \\ 1 & 1-a \end{pmatrix} \bmod \mathrm{p}_1^{\mathrm{n_1}}$.

Since the product ${M_1}^{-a} M_2 {M_1}^{a} \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \bmod \mathrm{p}_1^{\mathrm{n_1}}$, we have that a subgroup of Γ satisfies the hypotheses of Lemma 7.11; thus $\Gamma(\mathcal{P}_2^{n_2} \dots \mathcal{P}_k^{n_k}) < \Gamma$. □

Remark 7.13. The above proof shows that if $\Gamma(\mathcal{P}_1^{n_1}) < \Gamma$, then $\Gamma = \mathrm{PSL}(2, \mathrm{O_d})$, which is not possible for Γ a (torsion-free) link group.

Proof of Proposition 7.8: The following lemmas will be used in the proof of Proposition 7.8.

Lemma 7.14. *Let* $M_1 = \begin{pmatrix} 1 & p^r \\ 0 & 1 \end{pmatrix}$, $M_2 = \begin{pmatrix} 1 & 0 \\ p^r & 1 \end{pmatrix}$, *and* $M_3 = \begin{pmatrix} 1+p^r & -p^r \\ p^r & 1-p^r \end{pmatrix}$*. Then*

i) $\Gamma(\mathcal{P}^r) =< \Gamma(\mathcal{P}^{2r}), M_1, M_2, M_3 >$.

ii) $\Gamma(\mathcal{P}^r) =< \Gamma(\mathcal{P}^s), M_1, M_2, M_3 >$ *for* $s > r > 1$.

Proof. By Lemma 7.10 any matrix of $\Gamma(\mathcal{P}^r)$ is equivalent modulo $\Gamma(\mathcal{P}^{2r})$ to a matrix in $\mathrm{PSL}(2,\mathbb{Z})$ of the form $\begin{pmatrix} 1+ap^r & bp^r \\ cp^r & 1+dp^r \end{pmatrix}$ which is congruent modulo p^{2r} to the product $M_1^{b+a} M_2^{c-a} M_3^a$. This proves i). Part ii) is an easy consequence of i). □

Lemma 7.15. *Let* $A = \begin{pmatrix} 1 & p^\alpha \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1+p^\beta & -p^{2\beta-\gamma} \\ p^\gamma & 1-p^\beta \end{pmatrix}$, *such that* $0 \le \alpha < r$, $4r \le \gamma \le 5r$, *and* $4r \le \beta$. *Then* $\Gamma =< \Gamma(\mathcal{P}^{8r}), A, B >$ *is of* $\mathcal{P}$*-level* $\le 7r$.

Proof. In what follows we will work modulo $\mathcal{P}^{8r}$. Since A, B are in $\mathrm{PSL}(2,\mathbb{Z})$, this is the same as modulo p^{8r}.

By Lemma 7.14, it suffices to show that Γ contains the matrices $C = \begin{pmatrix} 1 & p^{7r} \\ 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ p^{7r} & 1 \end{pmatrix}$, and $E = \begin{pmatrix} 1+p^{7r} & -p^{7r} \\ p^{7r} & 1-p^{7r} \end{pmatrix}$. Note that $F = CD^{-1}E \equiv \begin{pmatrix} 1+p^{7r} & 0 \\ 0 & 1-p^{7r} \end{pmatrix}$.

Now $C \in \Gamma$ since C is a power of A. Also $F \in \Gamma$ since $ABA^{-1}B^{-1} \equiv \begin{pmatrix} 1+p^{\alpha+\gamma} & -2p^{\alpha+\beta} - p^{2\alpha+\gamma} \\ 0 & 1-p^{\alpha+\gamma} \end{pmatrix}$ and multiplying this matrix on the left by a power of A yields $G = \begin{pmatrix} 1+p^{\alpha+\gamma} & 0 \\ 0 & 1-p^{\alpha+\gamma} \end{pmatrix}$, with $4r \le \alpha+\gamma < 6r$. Thus it remains to show that $D \in \Gamma$.

Consider the matrix B. If $6r \le \beta \le 8r$ then $B^{p^{8r-\beta}} \equiv \begin{pmatrix} 1 & 0 \\ p^{\gamma+8r-\beta} & 1 \end{pmatrix}$, where $\gamma + 8r - \beta \le 7r$; hence $D \in \Gamma$. Now suppose that $4r \le \beta < 6r$. Since $4r \le \alpha+\gamma < 6r$, replacing B and G by powers we can assume that the diagonals of B and G are the same, and that $\beta < 6r$ while $\gamma < 7r$. Thus $G^{-1}B \equiv \begin{pmatrix} 1 & -p^{2\beta-\gamma} + p^{3\beta-\gamma} \\ p^\gamma & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ p^\gamma & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & p^\gamma \\ 0 & 1 \end{pmatrix} G^{-1}B \in \Gamma$, which implies $D \in \Gamma$.
□

Lemma 7.16. *Let* $m \ge 2$ *be the smallest integer for which* $\mathcal{P}$ *is large. If* $\Gamma(\mathcal{P}^{8m}) < \Gamma$, *then* $\Gamma(\mathcal{P}^{7m}) < \Gamma$ *as well.*

Proof. Since Γ is generated by parabolic elements, it can be written as $\Gamma =< \Gamma(\mathcal{P}^{8m}), M_1, \ldots, M_l >$, with the M_i parabolic. By Lemma 7.15, it suffices to show that Γ contains the matrices $A = \begin{pmatrix} 1 & p^\alpha \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1+p^\beta & -p^{2\beta-\gamma} \\ p^\gamma & 1-p^\beta \end{pmatrix}$, such that $0 \le \alpha < m$, $4m \le \gamma \le 5m$, and $4m \le \beta$.

As in the proof of the Claim, we can assume that the M_i are in $\mathrm{PSL}(2,\mathbb{Z})$ and that $M_1 = \begin{pmatrix} 1 & bp^\alpha \\ 0 & 1 \end{pmatrix}$ is of smallest $\mathcal{P}$-level among the M_i. Thus $0 \le \alpha < m$. Since $(b,p) = 1$, we can assume (taking a power of M_1 and calculating mod p^{8m})) that Γ contains $A = \begin{pmatrix} 1 & p^\alpha \\ 0 & 1 \end{pmatrix}$.

Now Γ must contain a second parabolic element, $M_2 = \begin{pmatrix} 1+ap^\beta & bp^{2\beta-\gamma} \\ cp^\gamma & 1-ap^\beta \end{pmatrix}$ with $(a,p) = (b,p) = (c,p) = 1$ that is not in $\Gamma_0(\mathcal{P}^m)$. Thus $0 \leq \alpha \leq \gamma < m$. Suppose first that M_2 fixes the cusp at 0, so that $M_2 = \begin{pmatrix} 1 & 0 \\ cp^\gamma & 1 \end{pmatrix}$. Thus Γ contains the matrix $D = \begin{pmatrix} 1 & 0 \\ p^\delta & 1 \end{pmatrix}$ with $4m \leq \delta \leq 5m$. Since $D(ADA^{-1}D^{-1}) \equiv B = \begin{pmatrix} 1+p^{\alpha+\delta} & -p^{2\alpha+\delta} \\ p^\delta & 1-p^{\alpha+\delta} \end{pmatrix}$, it follows that $\Gamma(\mathcal{P}^{7m}) < \Gamma$ by Lemma 7.15.

Finally, consider the case $M_2 = \begin{pmatrix} 1+ap^\beta & bp^{2\beta-\gamma} \\ cp^\gamma & 1-ap^\beta \end{pmatrix}$ fixing a cusp other than 0 or ∞. Replacing M_2 by a power, we can assume that $M_2 = \begin{pmatrix} 1+p^\beta & p^{2\beta-\gamma} \\ p^\gamma & 1-p^\beta \end{pmatrix}$ such that $4m \leq \gamma \leq 5m$ and $4m \leq \beta$. Thus, again by Lemma 7.15, $\Gamma(\mathcal{P}^{7m}) < \Gamma$ as required. □

Lemma 7.17. *Let $m \geq 2$ be the smallest integer for which $\mathcal{P}$ is large. If $\Gamma(\mathcal{P}^n) < \Gamma$ and $n \geq 8m$, then $\Gamma(\mathcal{P}^{n-m}) < \Gamma$ as well.*

Proof. Let $s = (n-8m)/2$ (resp. $(n-8m+1)/2$) if $n - 8m$ is even (resp. odd). As in the proof of Lemma 7.16, one shows that Γ contains the matrices $A = \begin{pmatrix} 1 & p^\alpha \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1+p^\beta & -p^{2\beta-\gamma} \\ p^\gamma & 1-p^\beta \end{pmatrix}$, such that $0 \leq \alpha < m$, $4m+s \leq \gamma \leq 5m+s$, and $4m+s \leq \beta$. A slight modification of the proof of Lemma 7.15 gives $\Gamma(\mathcal{P}^{n-m}) < \Gamma$. □

We can now complete the proof of Proposition 7.8. With the notation of Proposition 7.8, if $\Gamma(\mathcal{P}^n) < \Gamma$ and $n \geq 8m$ then Lemma 7.17 implies $\Gamma(\mathcal{P}^{n-m}) < \Gamma$. Repeating this argument until the exponent of $\mathcal{P}$ is less than $8m$ proves the proposition in this case.

Now suppose that $\Gamma(\mathcal{P}_1^{n_1} \ldots \mathcal{P}_k^{n_k}) < \Gamma$, for $k > 1$. Following the proof of Lemma 7.16, Γ contains the matrices $M_1 = \begin{pmatrix} 1 & bp_1^\alpha \\ 0 & 1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 1+ap_1^\beta & dp_1^{2\beta-\gamma} \\ cp_1^\gamma & 1-ap_1^\beta \end{pmatrix}$. Furthermore, by taking powers, we can assume that $M_1 \equiv M_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ mod $p_2^{n_2} \ldots p_k^{n_k}$. The proof of Lemma 7.16 then shows that Γ contains matrices A, B such that $A \equiv \begin{pmatrix} 1 & p^\alpha \\ 0 & 1 \end{pmatrix}$ mod $p_1^{n_1}$ and $A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ mod $p_2^{n_2} \ldots p_k^{n_k}$ while $B \equiv \begin{pmatrix} 1+p^\beta & -p^{2\beta-\gamma} \\ p^\gamma & 1-p^\beta \end{pmatrix}$ mod $p_1^{n_1}$ and $B \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ mod $p_2^{n_2} \ldots p_k^{n_k}$. Also, $0 \leq \alpha < m$, $4m+s \leq \gamma \leq 5m+s$, and $4m+s \leq \beta$. Thus it follows that $\Gamma(\mathcal{P}_1^{n_1-m_1}\mathcal{P}_2^{n_2} \ldots \mathcal{P}_k^{n_k}) < \Gamma$. Repeating the argument until the exponent of $\mathcal{P}_1$ is less than $8m_1$ completes the proof of Proposition 7.8. □

8. Final comments and speculations

We finish with some discussion about possible approaches to Question 1.1.

8.1. **Spectral gap.** An important property of congruence manifolds is that they admit a spectral gap; i.e. there exists a number $C > 0$ (conjectured to be 1 in dimension 3) so that if $M = \mathbb{H}^3/\Gamma$ (or $\mathbb{H}^2/\Gamma$) is any congruence manifold, then the first non-zero eigenvalue of the Laplacian on M, denoted $\lambda_1(M)$, satisfies $\lambda_1(M) > C$.

The argument of [44] to prove the finiteness result in dimension 2 for congruence surfaces of genus 0 discussed in §1 is to play off the spectral gap for congruence manifolds in dimension 2, together with a result proved in [44] that says that for a sequence of genus 0 manifolds with increasing numbers of punctures we must have $\lambda_1 \to 0$.

Thus a natural question is whether there exists a "Zograf type result" in dimension 3. The answer to this in general is no since Lackenby and Souto (unpublished) have shown that there exists a sequence of hyperbolic link complements in S^3 (say M_n) with $\mathrm{Vol(M_n)} \to \infty$ and a constant $C_1 > 0$ such that $\lambda_1(M_n) > C_1$.

On the other hand there are classes of links known for which sequences as above do not arise (see [20] and [27]). In particular, the result below follows from [27].

Theorem 8.1 (Lackenby)**.** *There are only finitely many alternating links in S^3 whose complements are congruence link complements.*

8.2. **Torsion.** If $L \subset S^3$ is a link of n components, then $H_1(S^3 \setminus L) \cong \mathbb{Z}^n$, so that an "easy" way to exclude a congruence subgroup $\Gamma < \mathrm{PSL(2, O_d)}$ from being a link group is to prove the existence of torsion in $H_1(\Gamma, \mathbb{Z})$. Indeed, this was used in [7] to rule out certain levels as allowable for a principal congruence link complement (using computer computations of M. H. Sengun). A recent emerging theme, both in low-dimensional topology and in automorphic forms, is that "*sequences of congruence subgroups should develop torsion in H_1*". More precisely, in the light of the results, numerics and conjectures in [11], [12], [29], and [34], a reasonable conjecture might be the following (see also [10, Conjecture 6.1]):

Conjecture 8.2. Let $\{\Gamma_n\}$ be a sequence of congruence subgroups of $\mathrm{PSL(2, O_d)}$ with $\mathrm{Vol(\mathbb{H}^3/\Gamma_n)} \to \infty$. Then:

$$\frac{\log|\mathrm{Tor(H_1(\Gamma_n, \mathbb{Z}))}|}{[\mathrm{PSL(2, O_d) : \Gamma_n}]} \to \frac{1}{6\pi}\mathrm{Vol(Q_d)} \text{ as } \mathrm{n} \to \infty.$$

A positive answer to Conjecture 8.2 would of course establish the finiteness stated in Questions 1.1 and 1.2.

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