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Enumeration of N-rooted maps using quantum field theory

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Abstract

A one-to-one correspondence is proved between the N-rooted ribbon graphs, or maps, with e edges and the $(e - N + 1)$-loop Feynman diagrams of a certain quantum field theory. This result is used to obtain explicit expressions and relations for the generating functions of N-rooted maps and for the numbers of N-rooted maps with a given number of edges using the path integral approach applied to the corresponding quantum field theory.

1. Introduction

Enumeration of rooted maps started with the work by Tutte [14–16] on counting planar maps, followed by [17–19] and [9,2,3] in arbitrary genus, and more recent results, see [1,4,5] and references therein. Beginning with the seminal work of 't Hooft [8] on the applications of matrix integrals to Yang–Mills gauge theories and in particular quantum chromodynamics, maps have

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become a major tool in quantum field theory and in string theory. See for example [7] for a review on matrix models and the enumeration of maps.

In this article, we consider rooted ribbon graphs, that is graphs embedded into a compact oriented surface in such a way that each face is a topological disc and one half-edge is distinguished, that can also be seen as rooted maps. The main aim of this article is to introduce N-rooted ribbon graphs extending the notion of rooted ribbon graphs, where N half-edges at N distinct vertices of the graph are rooted, and to solve the enumeration problem for these graphs. This is a continuation of [1,14–19] where the corresponding enumeration problem for 1-rooted ribbon graphs was solved. Our main idea is to apply methods of quantum field theory to enumeration of graphs. This is possible due to the bijection that we establish between N-rooted maps and Feynman diagrams for $2N$-point functions in a quantum field theory of two interacting scalar fields with a cubic interaction. We will refer to this theory as scalar quantum electrodynamics (scalar QED) to follow the notation of [6], even though this is an abuse of language because our theory does not contain a spin one gauge field.

The coincidence of the number of two-point Feynman diagrams with regard to the perturbative order in scalar QED and the number of rooted maps as a function of the number of edges has already been observed and in [13] an intuitive association between the two objects was proposed. This was verified up to third order by starting from the Feynman diagrams and using the proposed association to generate the corresponding rooted maps. However, the formal proof of a bijection between the two classes of objects to all orders has not yet appeared. Verifying the bijection explicitly to higher order becomes impractical very quickly owing to the rapid increase in the number of Feynman diagrams and rooted maps at higher orders. For instance, the number of Feynman diagrams for the propagator of the complex scalar field with 4 loops or rooted maps with 4 edges is 706, while at 5 loops or edges there are 8162 diagrams or graphs.

In this paper, we prove the equality of the number of two-point Feynman diagrams in scalar QED and the number of rooted maps in two ways. First we notice that the differential equation for the number of rooted maps as function of the number of edges derived in [1] coincides with the differential equation from quantum field theory that governs the number of two-point Feynman diagrams. Our second proof establishes the direct bijective correspondence between Feynman diagrams in question and the rooted maps by using Wick's theorem. The use of Wick's theorem and the technique of ribbon graphs turns out to be the formalization of the correspondence put forward in [13]. It is this second proof that admits a generalization to the general case of $2N$-point Feynman diagrams and leads naturally to the definition of N-rooted maps. We thus prove the correspondence observed in [13] and generalize it to the bijection between $2N$-point Feynman diagrams and N-rooted maps; this is one of the main results of the present article and the statement of Theorem 1.

We would like to point out that the correspondence between rooted ribbon graphs and Feynman diagrams that we establish here is very different from the one obtained via the matrix model approach and widely exploited after the seminal work of 't Hooft [8] since we are not dealing with a non abelian gauge theory and there are therefore no colour indices defining ribbon graphs through the double line notation. In [11] and [12] an approach similar to ours was used for the vacuum diagrams of the QED theory we consider here.

The result of Theorem 1 allows us to use the methods of quantum field theory to enumerate N-rooted graphs. Thus we find the number $m_N(e)$ of N-rooted graphs with given number e of edges by first solving the corresponding enumeration problem for connected $2N$-point Feynman diagrams along the lines of [6]. This appears to be a very powerful approach as it reproduces the result of [1] for enumeration of one-rooted maps ($N = 1$) with little effort, as explained in

Section 5. In addition to bypassing the laborious derivation from [1] of the formula for the number $m_1(e)$ by recursively reconstructing rooted graphs from simpler rooted graphs, our approach yields a closed form expression for generating functions of numbers of more general N-rooted graphs.

More precisely, introducing the generating function for the numbers $m_N(e)$ by

$$M_N(\lambda) = \sum_{e=0}^{\infty} m_N(e)\, \lambda^{2e}\,,$$

we find the second main result of the paper, formulated in Theorem 2, namely the following closed form algebraic expression for these functions with $N \geq 1$:

$$M_N(\lambda) = \sum_{\substack{\alpha_1+2\alpha_2+\ldots+N\alpha_N=N \\ \alpha_1,\ldots,\alpha_N\geq 0}} \frac{N!}{\alpha_1!\alpha_2!\ldots\alpha_N!}\, \frac{(-1)^{\alpha_1+\ldots+\alpha_N-1}\ (\alpha_1+\ldots+\alpha_N-1)!}{\mathcal{Z}_0^{\alpha_1+\ldots+\alpha_N}} \times \prod_{1\leq j\leq N} \left(\frac{\mathcal{Z}_j}{(j!)^2}\right)^{\alpha_j}\,,$$

where

$$\mathcal{Z}_j = \sum_{k=0}^{\infty} \frac{(2k+j)!\,(2k-1)!!}{(2k)!}\, \lambda^{2k}\,, \qquad j \geq 0.$$

Finally, in Theorem 3 it is shown that the generating functions $M_N(\lambda)$ for N-rooted maps are degree N polynomial expressions in $M_1(\lambda)$ with λ-dependent coefficients.

The paper is organized as follows. In Section 2 we collect known definitions relevant to rooted ribbon graphs as well as introduce the definition of N-rooted ribbon graphs. We also define a generating function M_N of the numbers of N-rooted ribbon graphs with a given number of edges and review relevant known results on the generating function in the case $N = 1$. In Section 3 we prove the bijection between the Feynman diagrams of our quantum field theory with N external electron lines and l loops on one side and N-rooted ribbon graphs with $l + N - 1$ edges on the other. In Section 4 we set up the quantum field theory calculation in zero dimension which allows us to count Feynman diagrams. In Section 5 we apply the theory developed in two preceding sections to rederive the known results of [1] on the generating function for one-rooted maps. In Section 6 we derive a closed form expression for the generating function of N-rooted ribbon graphs, or N-rooted maps, using the technique of path integration of the described quantum field theory. Furthermore, in the case $N = 2$ and $N = 3$ we do the calculation leading to a closed form formula for the number of N-rooted maps as a function of the number of edges. This calculation presents an algorithm that can be extended to an arbitrary given N in a straightforward way. Finally, in Section 7 we prove that the generating function $M_N(\lambda)$ of the numbers of N-rooted maps can be expressed as a degree N polynomial in $M_1(\lambda)$. In the Appendix, we show a quantum field theory derivation of a differential equation which plays a central role in the proof of Theorem 3.

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2. N-rooted graphs

A map is a cellular graph, that is a graph embedded into a connected compact orientable surface in such a way that each face is homeomorphic to an open disc, see [10] for more details. The orientation of the underlying surface leads to a cyclic (counterclockwise) ordering on the half-edges incident to each vertex of a map. The notion of a map is equivalent to that of a ribbon graph; we use these two terms interchangeably.

Definition 1. A *ribbon graph*, or a map, is the data $\Gamma = (H, \alpha, \sigma)$ consisting of a set of half-edges $H = \{1, \dots, 2e\}$ with e a positive integer and two permutations $\alpha, \sigma \in S_{2e}$ on the set of half-edges such that

- α is a fixed point free involution,
- the subgroup of S_{2e} generated by α and σ acts transitively on H.

The involution α is a set of transpositions each of which pairs two half-edges that form an edge. Cycles of the permutation σ correspond to vertices of the ribbon graph Γ; each cycle gives the ordering of half-edges at the corresponding vertex. Cycles of the permutation $\sigma^{-1} \circ \alpha$ correspond to faces of Γ. The condition of transitivity of the group $\langle \sigma, \alpha \rangle$ on the set of half-edges ensures the connectedness of the graph Γ.

A ribbon graph defines a connected compact orientable surface. This surface is reconstructed by gluing discs to the faces of the ribbon graph. The genus of the surface is called the genus of the ribbon graph. Let us denote the set of vertices of a map (a ribbon graph) $\Gamma = (H, \alpha, \sigma)$ by V and the set of faces by F. Recall that the set V is in a bijection with the set of cycles of the permutation σ and the set F is in a bijection with the cycles of $\sigma^{-1} \circ \alpha$. Associated to each map is its *Euler characteristic* defined by

$$\chi(\Gamma) = |V| - |E| + |F| .$$

The Euler characteristic of a map is an invariant of the map, it depends only on the genus g of the map and is given by $\chi(\Gamma) = 2 - 2g$.

Definition 2. An isomorphism between two ribbon graphs $\Gamma = (H, \alpha, \sigma)$ and $\Gamma' = (H, \alpha', \sigma')$ is a permutation $\psi \in S_{2e}$, that is $\psi : H \to H$, such that $\alpha' \circ \psi = \psi \circ \alpha$ and $\sigma' \circ \psi = \psi \circ \sigma$.

Two isomorphic ribbon graphs are identified. For a given graph $\Gamma = (H, \alpha, \sigma)$, the automorphisms are permutations on the set of half-edges, $\psi \in S_{2e}$ which commute with σ and α.

In terms of embeddings into a surface, two maps are equivalent if they can be transformed one into another by an orientation preserving homeomorphism of the underlying surface. For example, the two maps 1 and 2 in Fig. 1 are equivalent, both corresponding to the ribbon graph 3 from the same figure.

Automorphisms of ribbon graphs make their enumeration difficult and a way to deal with this is to introduce the idea of a rooted ribbon graph.

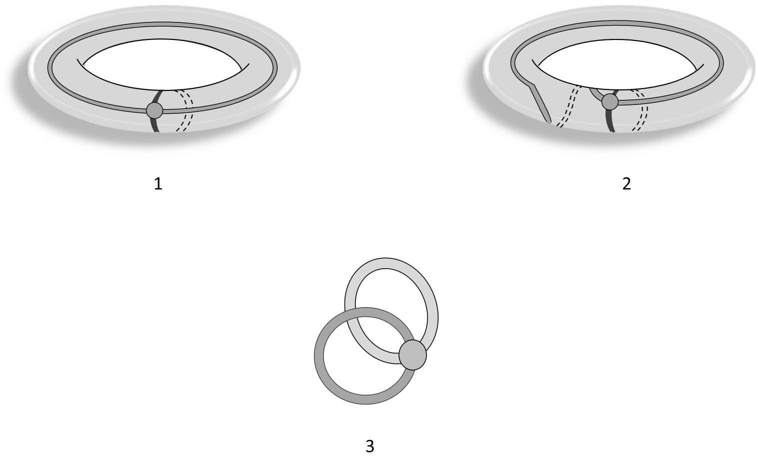

Fig. 1. A genus one map.

Fig. 2. Two-rooted graphs.

Definition 3. A rooted graph is a ribbon graph with a distinguished half-edge, the root of the graph. The vertex to which the root is incident is called the *root vertex*.

Remark 1. If there are no half-edges, the graph consists of one point. This is also considered a rooted graph.

An isomorphism between two rooted ribbon graphs is an isomorphism of the ribbon graphs that maps the root to the root, a *rooted isomorphism* between the two graphs. For a given rooted graph, the group of its rooted automorphisms is trivial, see [17].

We propose the following generalization of the concept of a rooted graph.

Definition 4. An N-rooted graph is the data of a ribbon graph, $\Gamma = (H, \alpha, \sigma)$, with the choice of N distinct ordered elements $\hat{h}_1, \ldots, \hat{h}_N$ of H, called *root half-edges*, or *roots*, belonging to N distinct cycles of σ, that is incident to N distinct ordered vertices, called *root vertices*.

In other words, an N-rooted graph is obtained from a ribbon graph by choosing N distinct vertices, labelling them with numbers from 1 to N, and at each of the chosen vertices placing an arrow on one of the half-edges incident to it. In Fig. 2 examples of two-rooted graphs of genus zero are shown.

An isomorphism between two N-rooted ribbon graphs is an isomorphism of the ribbon graphs that preserves the labelling of the N root vertices and maps roots to roots. We call such an isomorphism an *N-rooted isomorphism* between the two graphs. Similarly, an N-rooted automorphism of an N-rooted graph is an automorphism of the underlying ribbon graph which preserves the set of N root vertices pointwise and maps roots to roots. Clearly, the only N-rooted automorphism of an N-rooted graph is the identity.

2.1. Enumerating rooted ribbon graphs

We are interested in computing the number of N-rooted ribbon graphs and for that it is useful to arrange the graphs by the number of edges and to define the generating function $M_N(\lambda)$ such that

$$M_N(\lambda) = \sum_{e=0}^{\infty} m_N(e)\,\lambda^{2e}, \tag{1}$$

where $m_N(e)$ is the number of N-rooted graphs with e edges regardless of genus. The generating function $M_1(\lambda)$ of the single rooted ribbon graphs has been well studied in the literature, see [1]. From that reference, we have

$$M_1(\lambda) = 1 + 2\lambda^2 + 10\lambda^4 + 74\lambda^6 + 706\lambda^8 + 8\,162\lambda^{10} + 110\,410\lambda^{12} + \dots .$$

Arquès and Béraud [1] also showed that the generating function for one-rooted graphs satisfies the following differential equation[1]:

$$M_1(\lambda) = 1 + \lambda^2 M_1(\lambda)^2 + \lambda^2 M_1(\lambda) + \lambda^3\,\frac{\partial M_1(\lambda)}{\partial \lambda}\,. \tag{2}$$

This equation is similar to the one obtained by Tutte for planar maps [14]. It is a recursive relation where one-rooted ribbon graphs with a given number of edges are constructed recursively from one-rooted ribbon graphs with fewer edges. The above recursive relation leads to an expression for the number of rooted maps with e edges [1]:

$$m_1(e) = \frac{1}{2^{e+1}} \sum_{i=0}^{e} (-1)^i \sum_{\substack{k_1+\cdots+k_{i+1}=e+1\\ k_1,\dots,k_{i+1}>0}} \prod_{j=1}^{i+1} \frac{(2k_j)!}{k_j!}. \tag{3}$$

3. Correspondence between Feynman diagrams and N-rooted ribbon graphs

The quantum field theory we need in the present work is the theory of a neutral scalar field A coupled to a complex charged scalar field ϕ with coupling $\lambda A\phi\phi^*$. By abuse of language and following [6] we will refer to this theory as scalar quantum electrodynamics (QED), to the A field as the photon field and to the ϕ field as the electron field.

In this section we establish the bijective correspondence between the $2N$-point electron correlation functions in scalar QED and N-rooted ribbon graphs. Feynman diagrams are generated by Wick's theorem, while rooted ribbon graphs can be realized as pairs of permutations on the set H of half-edges of the ribbon graph. Our strategy will be to associate to each Wick contraction

[1] The different form of this equation in [1], namely $M_1(z) = 1 + zM_1(z)^2 + zM_1(z) + 2z^2\,\frac{\partial M_1(z)}{\partial z}$, is due to their use of another variable, $z = \lambda^2$, in definition (1) of the generating function.

leading to a connected $2N$-point Feynman diagram with e photon lines a pair of permutations (α, σ) on the set of $2e$ half-edges that establishes the correspondence with a rooted ribbon graph.

Before we prove the exact bijection, let us try to understand intuitively why such a bijection can be anticipated. Both Feynman diagrams and rooted ribbon graphs can be generated as combinatorial objects out of sub-structures (vertices, edges, propagators, etc.) with a set of rules giving the relation between the sub-structures.

Let us, for the sake of simplicity, focus on the case of the two point function and its correspondence to one-rooted ribbon graphs. The generalization to $2N$-point functions is straightforward. In the case of Feynman diagrams we have two kinds of lines, the photon and electron lines, and one kind of vertex where a photon line ends on an electron line. The number of vertices occurring in a Feynman diagram with no external photon lines is already determined and given by twice the number of internal photon lines in the diagram, leaving only two combinatorial objects to consider and a rule for how they combine. In addition, in the case of the 2-point function, there is one special electron line which extends from $-\infty$ to ∞ along the vertical time axis, while all the other electron lines are closed loops. In the case of one-rooted ribbon graphs, we again have two combinatorial objects, vertices and edges, which combine in a given way. In addition, there is again one specific vertex which is special – the root vertex. Thus, combinatorially, both structures are determined by the exact same amount of combinatorial data and we have just the right amount of it to expect a correspondence. Finally, previous results on enumeration of maps and Feynman diagrams suggest, see [13], that the number of 2-point functions arranged by the number of photon lines and the number of one-rooted ribbon graphs arranged by the number of edges are exactly the same. This makes it plausible that one can make such a correspondence concrete by making the right identifications on each side.

To understand which structures relate to which, we note that the simplest object we can consider for a 2-point function is just an external electron propagator with no photon lines. On the other hand, in the case of one-rooted ribbon graphs the simplest object is a single vertex and no edges. Thus, the correspondence should relate electron propagators to vertices in rooted ribbon graphs, implying photon lines get mapped to edges in a rooted ribbon graph. In addition, the correspondence of the external line to the root vertex in the ribbon graph generalizes for the case of $2N$-point functions to the relation between the N external electron lines and the N root vertices in the N-rooted ribbon graph. With this intuitive idea, we can hope to establish the exact bijection between the two sides.

3.1. Bijection between Feynman diagrams and N-rooted ribbon graphs

In this subsection we make precise the correspondence between Feynman diagrams of scalar QED and N-rooted ribbon graphs.

To get a handle on the number of Feynman diagrams in the theory, we use the fact that all Feynman diagrams are generated by Wick's theorem and use this to define a bijection between Feynman diagrams for $2N$-point functions and N-rooted graphs. Here we consider only contractions leading to connected Feynman diagrams.

Theorem 1. *There is a one-to-one correspondence between the set of connected Feynman diagrams of scalar QED with N external electron lines and e internal photon lines on one side and the set of N-rooted maps with e edges on the other.*

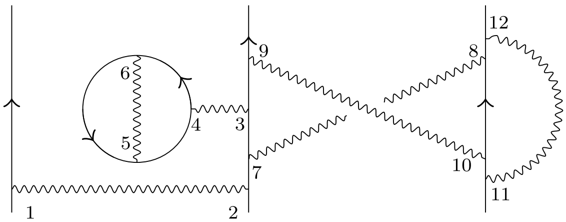

Fig. 3. Feynman diagram corresponding to the permutations $\alpha = (1\,2)(3\,4)(5\,6)(7\,8)(9\,10)(11\,12)$, $\sigma = (\hat{1})(\hat{2}\,7\,3\,9)(\hat{11}\,10\,8\,12)(4\,6\,5)$ on the set of half-edges $H = \{1, 2, \dots, 12\}$. A different choice of labelling of half edges and the resulting permutations define an equivalent ribbon graph.

Example 1. Under the bijection from Theorem 1 the Feynman diagram in Fig. 3 corresponds to the ribbon graph defined by the set of half-edges $H = \{1, 2, \dots, 12\}$ and the permutations $\alpha = (1\,2)(3\,4)(5\,6)(7\,8)(9\,10)(11\,12)$ and $\sigma = (\hat{1})(\hat{2}\,7\,3\,9)(\hat{11}\,10\,8\,12)(4\,6\,5)$ where the first three cycles of the permutation σ are labelled with the numbers from one to three, respectively, and the hat symbol marks the root half-edges.

Proof. In general, a Wick contraction will contain N strings of pairings linking each of the fields ϕ^* appearing in the bra to one and only one of the fields ϕ appearing in the ket. We label the pairs of fields thus linked by the same index:

$$\langle \phi_1^* \dots \phi_k^* \dots \phi_N^* \mid (A_1 \phi_1^* \phi_1)(A_2 \phi_2^* \phi_2) \dots \dots (A_s \phi_s^* \phi_s) \dots (A_{2e} \phi_{2e}^* \phi_{2e}) \mid \phi_1 \dots \phi_k \dots \phi_N \rangle ,$$

where for later convenience we have labelled the N external electron fields in the bra and ket with integers $1, \dots, N$, and labelled the set of vertices in the diagram with integers $1, \dots, 2e$ appearing as subscripts of the corresponding fields.

Given such a Feynman diagram, we construct an N-rooted ribbon graph (H, α, σ) as follows. First, the set H of half-edges is given by the set of vertices in the Wick contraction: define $H = \{1, \dots, 2e\}$. Next we define the following two permutations based on the two kinds of pairings that appear in the corresponding Wick contraction.

For each photon propagator obtained by pairing the fields A_i and A_j between the ith and jth vertices with $i \neq j$, define an involution $\alpha_{ij} = (ij)$. Since there is just one A field in each vertex, the pairing between the A_i and A_j fields determines the involution α_{ij} uniquely. In a Feynman diagram with $2e$ vertices of the kind $(A\phi^*\phi)$, there will be e such pairings and hence e transpositions α_{ij}. These transpositions form a permutation $\alpha \in S_{2e}$ on the set H of half-edges. It is a fixed point free involution by construction.

Now we follow the electron pairings. There are two kinds of electron pairings – pairings involving the electrons in the bra and ket leading to external electron lines in the Feynman diagram and pairings that lead to internal electron loops in the Feynman diagram. Let us consider each separately.

- In the former case, the line starts with the pairing of the field $\langle \dots \phi_k^* \dots \mid$ with a field, say ϕ_p, in the pth vertex. We then follow the sequence formed by the pairing of ϕ_p^* with the next field, say ϕ_q, and so on until the sequence leads us to a pairing of some ϕ_s^* to the electron $\mid \dots \phi_k \dots \rangle$ in the ket, i.e.

$$\langle \phi_1^* \dots \phi_k^* \dots \phi_N^* \mid \dots (A_p \phi_p^* \phi_p) \dots (A_q \phi_q^* \phi_q) \dots \dots (A_s \phi_s^* \phi_s) \dots \mid \phi_1 \dots \phi_k \dots \phi_N \rangle.$$

To the external electron line formed by this sequence we associate the cycle $(p, q, \ldots, s)$ and root the half-edge $p \in H$ corresponding to the first vertex $(A_p \phi_p^* \phi_p)$ linked to ϕ_k^*. The half-edge p becomes thus the kth root and we denote it by $\hat{h}_k$. There is only one ϕ^* and ϕ field in each vertex $(A\phi^*\phi)$ and hence the cycle permutation associated to each sequence of pairings is necessarily unique. Since there are N electrons in the bra or ket, there are N such cycles with N labelled half-edges, $\hat{h}_1, \ldots, \hat{h}_N$.

- In the latter case, the sequence of pairings both starts and ends at the same vertex leading to an internal electron loop, e.g.

$$\langle \phi_1^* \ldots \phi_k^* \ldots \phi_N^* \mid \ldots (A_k \phi_k^* \phi_k) \ldots (A_l \phi_l^* \phi_l) \ldots \ldots (A_n \phi_n^* \phi_n) \ldots \mid \phi_1 \ldots \phi_k \ldots \phi_N \rangle.$$

  To this internal electron loop we associate the cycle $(k, l, \ldots, n)$. The same argument as before implies the cycle associated to the sequence of pairings is also unique.

Since none of the fields must be left unpaired, and since each field appears only once outside of the bra and ket, the set of all such cycles gives a permutation σ on the set of $2e$ vertices, that is on the set H.

Since we only consider connected Feynman diagrams, the obtained permutations α and σ generate a subgroup of S_{2e} that acts transitively on H. This completes the construction of a ribbon graph (H, α, σ) with N roots $\hat{h}_1, \ldots, \hat{h}_N$ corresponding to a given Wick contraction.

Conversely, let (H, α, σ) be an N-rooted ribbon graph with e edges and roots $\hat{h}_1, \ldots, \hat{h}_N$. The set $H = \{1, \ldots, 2e\}$ of half edges determines the set of vertices in a Wick contraction. The number N of roots determines the number of fields ϕ^* and ϕ in the bra and ket. The permutation $\alpha \in S_{2e}$ gives the pairings of the A-fields. The permutation $\sigma \in S_{2e}$ has N special cycles each of which contains a root $\hat{h}_i$ for $i = 1, \ldots, N$. Let the cycle containing the root $\hat{h}_k$ have the form $(\hat{h}_k = p, q, \ldots, s)$. This cycle determines a string of pairings that connects ϕ_k^* in the bra to ϕ_p, followed by pairing ϕ_p^* to ϕ_q, and so on up to the pairing of ϕ_s^* to ϕ_k in the ket. The cycles of σ not containing a root determine in the obvious way pairings of the fields ϕ^* and ϕ in the terms sandwiched between the bra and the ket.

To see that equivalent Wick contractions arise from equivalent rooted ribbon graphs, recall that two Wick contractions are identified if and only if they can be obtained from one another by relabelling of vertices. Denote such a relabelling by $r \in S_{2e}$. By our construction, vertices in a Wick contraction form the set H of half-edges in the corresponding ribbon graph. Therefore a relabelling of vertices in a contraction results in a relabelling of the half-edges, $r : H \to H$, and the new permutations (α', σ') are related to the original permutations (α, σ) as follows:

$$\alpha' = r \circ \alpha \circ r^{-1}, \qquad \sigma' = r \circ \sigma \circ r^{-1}.$$

We want to show that the two ribbon graphs (H, α, σ) and (H, α', σ') are isomorphic. The relabelling permutation r is the bijection $\psi : H \to H$ from Definition 2 and thus the required commutation relations are satisfied.

The roots are mapped to the roots by the relabelling r as the kth root half-edge corresponds to the vertex of the Wick contraction which is paired to ϕ_k^* in the bra-part. Since the relabelling does not affect the fields in the bra and ket, we get that r sends kth root of the graph (H, α, σ) to the kth root of the graph (H, α', σ'). □

Remark 2. Let V denote the set of cycles of the permutation σ. Under the bijection of Theorem 1 between $2N$-point Feynman diagrams in scalar QED and N-rooted graphs, the set of N external electron lines, the set of $(|V| - N)$ internal electron loops, the $2e$ vertices, and the e photon lines in the Feynman diagram correspond bijectively to the N root vertices, the $(|V| - N)$ non-root vertices, the $2e$ half-edges and the e edges of the N-rooted graph, respectively. As is easy to see, the number of loops in the Feynman diagram is $e - N + 1$. Thus the number of Feynman diagrams with l loops is equal to the number of N-rooted graphs with $l + N - 1$ edges.

4. Counting Feynman diagrams using quantum field theory

In this work we are not interested in calculating actual correlation functions, only in counting Feynman diagrams. Then one can simplify greatly the problem by considering our quantum field theory in zero spacetime dimension. What this means in practice is that our fields are now taken to be spacetime independent and the action does not contain an integral over spacetime anymore. From the point of view of the path integral, the fields become now ordinary real or complex variables and the path integrals with respect to the fields reduce to ordinary integrals over $\mathbb{R}^3$. More precisely, the field A and the corresponding source J are now real variables, and $\phi, \phi^* \in \mathbb{C}$ are complex conjugated to each other, which implies the same relationship for their sources $\eta, \eta^* \in \mathbb{C}$. Assuming $\phi = x + iy$ with $x, y \in \mathbb{R}$ we then regard $d\phi d\phi^*$ as $dxdy$. We therefore consider from now on the following integral of a real function

$$Z\left(J, \eta, \eta^*, \lambda\right) = \int_{\mathbb{R}^3} d\phi d\phi^* dA \ \exp\left(- \phi\phi^* - \frac{1}{2}A^2 + \lambda\phi\phi^* A + JA + \eta\phi^* + \eta^*\phi \right). \quad (4)$$

As usual, we expand the exponential of the interaction term into a Taylor series and treat the result as a formal series which can be integrated term by term:

$$\begin{aligned} Z\left(J, \eta, \eta^*, \lambda\right) &= \int_{\mathbb{R}^3} d\phi d\phi^* dA \sum_{k=0} \frac{(\lambda\phi\phi^* A)^k}{k!} \exp\left(- \phi\phi^* - \frac{1}{2}A^2 + JA + \eta\phi^* + \eta^*\phi \right), \\ &= \int_{\mathbb{R}^3} d\phi d\phi^* dA \\ &\quad \times \sum_{k=0} \frac{\lambda^k}{k!} \left(\frac{d^3}{d\eta d\eta^* dJ} \right)^k \exp\left(- \phi\phi^* - \frac{1}{2}A^2 + JA + \eta\phi^* + \eta^*\phi \right), \\ &= \sum_{k=0} \frac{\lambda^k}{k!} \left(\frac{d^3}{d\eta d\eta^* dJ} \right)^k \\ &\quad \times \int_{\mathbb{R}^3} d\phi d\phi^* dA \ \exp\left(- \phi\phi^* - \frac{1}{2}A^2 + JA + \eta\phi^* + \eta^*\phi \right), \\ &= \pi\sqrt{2\pi} \sum_{k=0} \frac{\lambda^k}{k!} \left(\frac{d^3}{d\eta d\eta^* dJ} \right)^k \exp\left(\eta\eta^* + \frac{J^2}{2} \right). \end{aligned} \quad (5)$$

Note that $Z(0)$ is equal to

$$Z(0) = \pi\sqrt{2\pi}\,.$$

We may now obtain generating functions for the Feynman diagrams of relevance to the present work. The generating function for the diagrams with p external photon lines and N external electron lines is given by

$$
\begin{aligned}
Z_{N,p}(\lambda) &:= \langle (\phi\phi^*)^N A^p \rangle \\
&= \frac{1}{Z(0)} \left(\frac{d^2}{d\eta d\eta^*}\right)^N \left(\frac{d}{dJ}\right)^p Z\left(J,\eta,\eta^*,\lambda\right)\Bigg|_{J=\eta=\eta^*=0} \\
&= \left(\frac{d^2}{d\eta d\eta^*}\right)^N \left(\frac{d}{dJ}\right)^p \sum_{k=0} \frac{\lambda^k}{k!}\left(\frac{d^3}{d\eta d\eta^* dJ}\right)^k \exp\left(\eta\eta^* + \frac{J^2}{2}\right)\Bigg|_{J=\eta=\eta^*=0} .
\end{aligned}
$$

This is a generating function in the following sense: in the expansion of $Z_{N,p}(\lambda)$, the coefficient of λ^k gives the number of Feynman diagrams with p external photon lines, N external electron lines, and k vertices. Figure (3) shows one of the Feynman diagram corresponding to $p = 0$, $N = 3$, $k = 12$.

Recall that both connected and disconnected diagrams are counted by these generating functions.

It is possible to evaluate explicitly the derivatives using

$$
\begin{aligned}
&\frac{d^n}{dJ^n} e^{\frac{J^2}{2}}\Bigg|_{J=0} = \begin{cases} (n-1)!! & \text{if n is even} \\ 0 & \text{if n is odd} \end{cases}, \\
&\left(\frac{d}{d\eta}\right)^m \left(\frac{d}{d\eta^*}\right)^n e^{\eta\eta^*}\Bigg|_{\eta=\eta^*=0} = n!\,\delta_{n,m} ,
\end{aligned}
\tag{6}
$$

where the convention $(-1)!! = 1$ is assumed. This leads to

$$
Z_{N,p}(\lambda) = \begin{cases} \sum_{k=0}^{\infty} \frac{(2k+N)!\,(2k+p-1)!!}{(2k)!}\,\lambda^{2k} & \text{if } p \text{ is even,} \\[2ex] \sum_{k=0}^{\infty} \frac{(2k+N+1)!\,(2k+p)!!}{(2k+1)!}\,\lambda^{2k+1} & \text{if } p \text{ is odd.} \end{cases}
$$

As we saw in Section 3, the photon lines in the Feynman diagrams map to edges of the rooted maps. We are therefore interested only in Feynman diagrams with no external photon lines and N external electron lines, in which case the number of vertices is equal to twice the number of photon lines, $k = 2e$, and the formula simplifies to

$$
\begin{aligned}
\mathcal{Z}_N(\lambda) &:= Z_{N,0}(\lambda) \\
&= \frac{1}{Z(0)} \left(\frac{d}{d\eta}\frac{d}{d\eta^*}\right)^N Z\left(J,\eta,\eta^*,\lambda\right)\Bigg|_{J=\eta=\eta^*=0} &(7) \\
&= \sum_{k=0}^{\infty} \frac{(2k+N)!\,(2k-1)!!}{(2k)!}\,\lambda^{2k} . &(8)
\end{aligned}
$$

As noted before, these generating functions produce both connected and disconnected diagrams. Only connected diagrams are relevant, both for physics and for the enumeration of graphs, so we take the natural logarithm of Eq. (4) before taking the derivatives with respect to the sources. We will use the notation $W_{N,p}$ for the generating functions of the connected diagrams with N external electron lines and p external photon lines. They are given by

$$W_{N,p}(\lambda) = \frac{1}{p!N!} \left(\frac{d}{dJ}\right)^p \left(\frac{d}{d\eta}\frac{d}{d\eta^*}\right)^N \ln\left(\frac{Z(J,\eta,\eta^*,\lambda)}{Z(0)}\right)\Bigg|_{J=\eta=\eta^*=0}.$$

As mentioned previously, we focus on the diagrams with no external photon lines, corresponding to $W_{N,0}(\lambda)$. These are the generating functions of the connected Feynman diagrams with N external electron lines and from Theorem 1, they are equal to $M_N(\lambda)$ of Eq. (1). We may therefore write

$$\begin{aligned} M_N(\lambda) &= W_{N,0}(\lambda) \\ &= \frac{1}{N!}\left(\frac{d}{d\eta}\frac{d}{d\eta^*}\right)^N \ln\left(\frac{Z(J,\eta,\eta^*,\lambda)}{Z(0)}\right)\Bigg|_{J=\eta=\eta^*=0}. \end{aligned} \tag{9}$$

In the following we will not indicate explicitly the λ dependence of the $\mathcal{Z}_N$ and M_N to ease the notation.

5. The generating function of one-rooted maps

The first quantity of interest for us is the generating function of the one-rooted maps M_1. Using Eq. (9), this is given by

$$\begin{aligned} M_1 &= \frac{d^2}{d\eta^* d\eta} \ln\left(\frac{Z(J,\eta,\eta^*,\lambda)}{Z(0)}\right)\Bigg|_{J=\eta=\eta^*=0} \\ &= \frac{\mathcal{Z}_1}{\mathcal{Z}_0} \\ &= 1 + 2\lambda^2 + 10\lambda^4 + 74\lambda^6 + 706\lambda^8 + 8\,162\lambda^{10} + 110\,410\lambda^{12} + \ldots, \end{aligned} \tag{10}$$

where we have used Eqs. (8) and (6).

5.1. The number of one-rooted maps with e edges

Even though one can Taylor expand the ratio of two sums $\mathcal{Z}_1$ and $\mathcal{Z}_0$ as above to obtain the power series (10) and then to read off the numbers $m_1(e)$ of one-rooted maps with e of edges from the coefficients of the series, this does not give the closed form expression for these coefficients. In order to recover Eq. (3) of Arquès–Béraud for $m_1(e)$, we need to express our result for M_1 as a single sum over powers of λ instead of a ratio of two sums $\mathcal{Z}_1$ and $\mathcal{Z}_0$. To achieve this, let us first express $\mathcal{Z}_1$ in terms of $\mathcal{Z}_0$ and its derivative with respect to the coupling constant. This is done by noting that

$$\begin{aligned} \mathcal{Z}_1 &= \sum_{n=0} (2n+1)!!\lambda^{2n} \\ &= \left(\lambda\frac{d}{d\lambda} + 1\right)\sum_{n=0}(2n-1)!!\lambda^{2n} \\ &= \lambda\,\mathcal{Z}_0' + \mathcal{Z}_0, \end{aligned}$$

where a prime indicates a derivative with respect to the coupling constant λ. This allows us to write

$$M_1 = \frac{\mathcal{Z}_1}{\mathcal{Z}_0} = \lambda\frac{\mathcal{Z}_0'}{\mathcal{Z}_0} + 1. \tag{11}$$

To recover Eq. (3), it proves convenient to rewrite this in terms of

$$M_0 = \ln \mathcal{Z}_0 \,. \tag{12}$$

Doing so is also useful for the next section where we will obtain equations relating the various M_N directly. Using the previous two equations, we write

$$M_1 = \lambda\, M_0' + 1 \,. \tag{13}$$

The connection with Eq. (3) is made by using the identity

$$\ln\left(1 + \sum_{n=1}^{\infty} A_n \lambda^{2n}\right) = \sum_{i=1}^{\infty} \lambda^{2i} \sum_{k=1}^{i} \frac{(-1)^{k+1}}{k} \sum_{\substack{\mu_1+\ldots+\mu_k=i \\ \mu_i \neq 0}} \prod_{j=1}^{k} A_{\mu_j} \,, \tag{14}$$

giving us

$$M_0 = \left(\sum_{e=1}^{\infty} \lambda^{2e} \sum_{k=1}^{e} \frac{(-1)^{k+1}}{k} \sum_{\substack{\mu_1+\ldots+\mu_k=e \\ \mu_i \neq 0}} \prod_{j=1}^{k} (2\mu_j - 1)!!\right). \tag{15}$$

Using Eqs. (13) and (15), we get

$$\begin{aligned} M_1 &= 1 + \sum_{e=1}^{\infty} (2e)\, \lambda^{2e} \sum_{k=1}^{e} \frac{(-1)^{k+1}}{k} \sum_{\substack{\mu_1+\ldots+\mu_k=e \\ \mu_i \neq 0}} \prod_{j=1}^{k} (2\mu_j - 1)!! \\ &= \sum_{e=0}^{\infty} \lambda^{2e} \sum_{k=0}^{e} (-1)^k \sum_{\substack{\mu_1+\ldots+\mu_{k+1}=e+1 \\ \mu_i \neq 0}} \prod_{j=1}^{k+1} (2\mu_j - 1)!! \,. \end{aligned}$$

From this we can read off the number of rooted maps with e edges since by Eq. (1) we have $M_1 = \sum_{e=0}^{\infty} m_1(e) \lambda^{2e}$. This gives us

$$m_1(e) = \sum_{k=0}^{e} (-1)^k \sum_{\substack{\mu_1+\ldots+\mu_{k+1}=e+1 \\ \mu_i \neq 0}} \prod_{j=1}^{k+1} (2\mu_j - 1)!! \,. \tag{16}$$

We have therefore recovered the known expression for $m_1(e)$ of Arquès and Béraud, Eq. (3), using quantum field theory.

5.2. *Differential equation for $M_1(\lambda)$*

As mentioned in Section 2.1, a differential equation for the generating function of one-rooted maps M_1 is known. In this section we will recover it from quantum field theory.

For the theory we are considering, one can derive a differential equation for M_0 [6]. The details are presented in the appendix and the result is

$$M_0' = 2\lambda + 4\lambda^2 M_0' + \lambda^3 M_0'' + \lambda^3 (M_0')^2 \,, \tag{17}$$

where a prime indicates a derivative with respect to λ.

We can now use this to generate a differential equation for M_1. Isolating M_0' in terms of M_1 in Eq. (13) and plugging into Eq. (17), we get

$$M_1 = 1 + \lambda^2 M_1 + \lambda^3 M_1' + \lambda^2 M_1^2, \tag{18}$$

which is the known equation, Eq. (2).

6. Generating functions of N-rooted maps

Recall that $M_N(\lambda)$ is the generating function of N-rooted maps defined in (1) by

$$M_N(\lambda) = \sum_{e=0}^{\infty} m_N(e)\lambda^{2e},$$

with $m_N(e)$ being the number of N-rooted maps with e edges regardless of genus. Theorem 2 in this section gives a closed form expression for $M_N(\lambda)$ for all values of N.

6.1. A closed form expression for the generating function of N-rooted maps

Theorem 2. *The generating function $M_N(\lambda)$ of N-rooted maps, where $N \geq 1$, is given by*

$$M_N(\lambda) = \sum_{\substack{\alpha_1+2\alpha_2+\ldots+N\alpha_N=N \\ \alpha_1\ldots\alpha_N\geq 0}} \frac{N!}{\alpha_1!\alpha_2!\ldots\alpha_N!} \frac{(-1)^{\alpha_1+\ldots+\alpha_N-1}\ (\alpha_1+\ldots+\alpha_N-1)!}{\mathcal{Z}_0^{\alpha_1+\ldots+\alpha_N}} \times \prod_{1\leq j\leq N} \left(\frac{\mathcal{Z}_j}{(j!)^2}\right)^{\alpha_j}, \tag{19}$$

where

$$\mathcal{Z}_j = \sum_{k=0}^{\infty} \frac{(2k+j)!\,(2k-1)!!}{(2k)!} \lambda^{2k}, \qquad j \geq 0. \tag{20}$$

The number $m_N(e)$ of N-rooted maps with e edges can then be obtained using

$$m_N(e) = \frac{1}{(2e)!} \lim_{\lambda\to 0} \frac{d^{2e}}{d\lambda^{2e}} M_N(\lambda)\,. \tag{21}$$

Proof. We first note that $Z(J, \eta, \eta^*, \lambda)$ as given in Eq. (5) depends on η and η^* only through their product $\eta\eta^*$. Therefore in this proof we will write $Z(J, \eta\eta^*, \lambda)$ instead of $Z(J, \eta, \eta^*, \lambda)$. For a differentiable function depending only on the product of two variables η and η^*, we have

$$\left(\frac{d^2}{d\eta d\eta^*}\right)^N \ln f(\eta\eta^*)\bigg|_{\eta=\eta^*=0} = N!\,\frac{d^N}{dx^N} \ln f(x)\bigg|_{x=0}, \tag{22}$$

which is valid at the condition that $f(0) \neq 0$. We can apply this identity to Eq. (9) with $f(\eta\eta^*) = Z(J, \eta\eta^*, \lambda)/Z(0)$, which does not vanish at $\eta\eta^* = 0$, to obtain

$$M_N(\lambda) = \frac{d^N}{dx^N} \ln\left(\frac{Z(J, x, \lambda)}{Z(0)}\right)\bigg|_{J=x=0},$$

where it is understood that the product $\eta\eta^*$ has been replaced by x.

For $N \geq 1$, we apply Faà di Bruno's formula to the Nth derivative of the logarithm of a function, giving us

$$M_N(\lambda) = \sum_{\substack{\alpha_1+2\alpha_2+\ldots+N\alpha_N=N \\ \alpha_1\ldots\alpha_N\geq 0}} \frac{N!}{\alpha_1!\alpha_2!\ldots\alpha_N!} \frac{(-1)^{\alpha_1+\ldots+\alpha_N-1}\ (\alpha_1+\ldots+\alpha_N-1)!}{\mathcal{Z}_0(\lambda)^{\alpha_1+\ldots+\alpha_N}} \times \prod_{j=1}^{N} \left(\frac{1}{Z(0)} \frac{1}{j!} \frac{d^j}{dx^j} Z(J,x,\lambda) \right)^{\alpha_j} \Bigg|_{J=x=0},$$

where we have used the notation of Eq. (7) to write

$$\frac{Z(J,x,\lambda)}{Z(0)}\Bigg|_{J=x=0} = \mathcal{Z}_0(\lambda)\,.$$

Now we use once more Eq. (22) to rewrite the expression in terms of η and η^* instead of x, giving us

$$M_N(\lambda) = \sum_{\substack{\alpha_1+2\alpha_2+\ldots+N\alpha_N=N \\ \alpha_1\ldots\alpha_N\geq 0}} \frac{N!}{\alpha_1!\alpha_2!\ldots\alpha_N!} \frac{(-1)^{\alpha_1+\ldots+\alpha_N-1}\ (\alpha_1+\ldots+\alpha_N-1)!}{\mathcal{Z}_0(\lambda)^{\alpha_1+\ldots+\alpha_N}} \times \prod_{j=1}^{N} \left(\frac{1}{Z(0)} \frac{1}{(j!)^2} \frac{d^{2j}}{d\eta^j d\eta^{*j}} Z(J,\eta,\eta^*,\lambda) \right)^{\alpha_j} \Bigg|_{J=\eta=\eta^*=0}.$$

Using again the notation of Eq. (7), we obtain Eq. (19) (where the dependence on λ of $\mathcal{Z}_0$ and $\mathcal{Z}_j$ is omitted). As for the expression for $\mathcal{Z}_j$ of Eq. (20), it is already calculated in Eq. (8). □

6.2. An algorithm to derive a formula for $m_N(e)$

Here we discuss an alternative to Eq. (21) from Theorem 2 to compute generating functions for N-rooted graphs. This is a generalization of the approach used in Section 5.1 to obtain $M_1(\lambda)$ and $m_1(e)$. Recall that the first step in Section 5.1 was to express $\mathcal{Z}_1$ in terms of $\mathcal{Z}_0$ and its derivative with respect to λ. It is also simple to write an explicit expression for $\mathcal{Z}_N$ in terms of $\mathcal{Z}_0$ and derivatives of $\mathcal{Z}_0$. From expression (8) for $\mathcal{Z}_N$, we have

$$\begin{aligned}\mathcal{Z}_N &= \sum_{k=0}^{\infty} \frac{(2k+N)!(2k-1)!!}{(2k)!} \lambda^{2k} \\ &= \left(N + \lambda \frac{d}{d\lambda} \right) \mathcal{Z}_{N-1}\,,\end{aligned}$$

so that

$$\begin{aligned}\mathcal{Z}_N &= \left(N + \lambda \frac{d}{d\lambda} \right) \left(N-1 + \lambda \frac{d}{d\lambda} \right) \ldots \left(1 + \lambda \frac{d}{d\lambda} \right) \mathcal{Z}_0 \\ &= \sum_{k=0}^{N} \binom{N}{k} \frac{N!}{k!}\, \lambda^k\, \frac{d^k \mathcal{Z}_0}{d\lambda^k} \qquad (23) \\ &= \frac{d^N}{d\lambda^N} \left(\lambda^N \mathcal{Z}_0 \right). \qquad (24)\end{aligned}$$

Using this, in the remaining part of this section, we compute M_2 and M_3.

Calculating M_2 From Eq. (19) for $M_N(\lambda)$ from Theorem 2, we have

$$M_2 = \frac{1}{2}\frac{\mathcal{Z}_2}{\mathcal{Z}_0} - \left(\frac{\mathcal{Z}_1}{\mathcal{Z}_0}\right)^2 . \tag{25}$$

Using again expression (8) for $\mathcal{Z}_N$, this leads to the following Taylor expansion

$$M_2 = \lambda^2 + 13\,\lambda^4 + 165\,\lambda^6 + 2\,273\,\lambda^8 + 34\,577\,\lambda^{10} + 581\,133\,\lambda^{12} \dots .$$

In order to obtain an explicit expression for $m_2(e)$ we may rewrite formula (25) for M_2 as an expansion in powers of λ symbolically. From (24) we have

$$\mathcal{Z}_2 = \lambda^2 \mathcal{Z}_0'' + 4\lambda \mathcal{Z}_0' + 2\mathcal{Z}_0 \qquad \text{and} \qquad \mathcal{Z}_1 = \lambda \mathcal{Z}_0' + \mathcal{Z}_0 .$$

With this and Eq. (12), Eq. (25) becomes

$$\begin{aligned} M_2 &= \frac{1}{2}\frac{\lambda^2}{\mathcal{Z}_0}\mathcal{Z}_0'' - \frac{\lambda^2}{\mathcal{Z}_0^2}\left(\mathcal{Z}_0'\right)^2 , \\ &= \frac{\lambda^2}{2} M_0'' - \frac{\lambda^2}{2}\left(M_0'\right)^2 . \end{aligned} \tag{26}$$

Using the explicit expression (15) for M_0 and the relation $\lambda\, M_0' = M_1 - 1$ from (13), we obtain

$$M_2 = \sum_{e=1}^{\infty}\left[e(2e-1)\,\lambda^{2e} \sum_{k=1}^{e} \frac{(-1)^{k+1}}{k} \sum_{\substack{\mu_1+\ldots+\mu_k=e \\ \mu_i\neq 0}} \prod_{j=1}^{k} (2\mu_j - 1)!! \right] - \frac{1}{2}\left(M_1 - 1\right)^2 . \tag{27}$$

Rewriting this in terms of $M_1 = \sum_{k=0} m_1(k)\lambda^{2k}$ with $m_1(k)$ given in Eq. (16), we obtain

$$\begin{aligned} M_2 = \lambda^2 + \sum_{e=2}^{\infty}\lambda^{2e}\Bigg[& e(2e-1) \sum_{k=1}^{e} \frac{(-1)^{k+1}}{k} \sum_{\substack{\mu_1+\ldots+\mu_k=e \\ \mu_i\neq 0}} \prod_{j=1}^{k} (2\mu_j - 1)!! \\ & - \frac{1}{2}\sum_{k=1}^{e-1} m_1(k) m_1(e-k) \Bigg]. \end{aligned}$$

This may be written as

$$\begin{aligned} M_2 = \lambda^2 + \sum_{e=2}^{\infty}\lambda^{2e}\Bigg[& \sum_{k=0}^{e} (-1)^k \sum_{\substack{\mu_1+\ldots+\mu_{k+1}=e+1 \\ \mu_i\neq 0}} \mu_{k+1} \prod_{j=1}^{k+1} (2\mu_j - 1)!! \\ & - \frac{1}{2}\sum_{k=1}^{e-1} m_1(k) m_1(e-k) \Bigg]. \end{aligned} \tag{28}$$

Now we can read off the values of $m_2(e)$ from Eq. (28) since by definition (1) we have $M_2 = \sum_{e=0}^{\infty} m_2(e)\,\lambda^{2e}$.

Calculating M_3 As an another example, one finds from Eq. (19) of Theorem 2

$$M_3 = \frac{1}{6}\frac{\mathcal{Z}_3}{\mathcal{Z}_0} - \frac{3}{2}\frac{\mathcal{Z}_1\mathcal{Z}_2}{\mathcal{Z}_0^2} + 2\left(\frac{\mathcal{Z}_1}{\mathcal{Z}_0}\right)^3$$
$$= 6\,\lambda^4 + 172\,\lambda^6 + 3\,834\,\lambda^8 + 81\,720\,\lambda^{10} + 1\,775\,198\,\lambda^{12} + \dots . \tag{29}$$

We may express $\mathcal{Z}_3$ in terms of $\mathcal{Z}_0$ using Eq. (24):

$$\mathcal{Z}_3 = \lambda^3 \mathcal{Z}_0''' + 9\lambda^2 \mathcal{Z}_0'' + 18\lambda \mathcal{Z}_0' + 6\mathcal{Z}_0, \tag{30}$$

giving us

$$M_3 = 2\lambda^3\left(\frac{\mathcal{Z}_0'}{\mathcal{Z}_0}\right)^3 - \frac{3}{2}\lambda^3\frac{\mathcal{Z}_0'\mathcal{Z}_0''}{\mathcal{Z}_0^2} + \frac{\lambda^3}{6}\frac{\mathcal{Z}_0'''}{\mathcal{Z}_0},$$
$$= \frac{2}{3}\lambda^3\left(M_0'\right)^3 - \lambda^3 M_0' M_0'' + \frac{\lambda^3}{6}M_0'''. \tag{31}$$

This could be written as an explicit expansion in λ using again Eq. (15).

7. Relating generating functions of N-rooted maps to M_1

Recall that the generating function $M_1(\lambda)$ of one-rooted maps satisfies differential equation (18). As a generalization of this equation to the case of N-rooted maps, we find that all generating functions $M_N(\lambda)$ can be expressed in terms of $M_1(\lambda)$.

Theorem 3. *The generating function for N-rooted maps $M_N(\lambda)$ can be expressed as a polynomial of degree N in the generating function $M_1(\lambda)$ for one-rooted maps. This polynomial has λ-dependent coefficients and is obtained by substituting the following expression for $\mathcal{Z}_j/\mathcal{Z}_0$ into Eq. (19) of Theorem 2:*

$$\frac{\mathcal{Z}_j}{\mathcal{Z}_0} = \sum_{n=0}^{j}\binom{j}{n}\frac{j!}{n!}\sum_{k=0}^{n}(-1)^{n-k}\, B_{n,2k-1}$$
$$\times\left[\delta_{k,0} + H(k-1)\,\frac{M_1}{\lambda^{2k-2}} - H(k-2)\,(1-M_1\lambda^2)\sum_{m=0}^{k-2}\frac{(2m+1)!!}{\lambda^{2k-2m-2}}\right]. \tag{32}$$

Here H is the Heaviside function with the convention $H(k) = 1$ for $k \geq 0$ and the coefficients B are obtained from the recursion formula

$$B_{n+1,2k-1} = B_{n,2k-3} + (2k+n+1)B_{n,2k-1}, \quad n \geq 1, \quad n \geq k \geq 0 \tag{33}$$

with initial conditions

$$B_{0,-1} = B_{1,-1} = B_{1,1} = 1, \tag{34}$$
$$B_{n,-3} = B_{n,2n+1} = 0. \tag{35}$$

Remark 3. The recursion formula (33) can be solved for each given value of k. For example,

$$\begin{aligned} B_{n,-1} &= n!, \\ B_{n,2n-1} &= 1, \\ B_{n,2n-3} &= \frac{(3n-1)n}{2}. \end{aligned}$$

Proof. We only need to prove relation (32) for $\mathcal{Z}_j/\mathcal{Z}_0$. Throughout this proof we will indicate explicitly the λ dependence of the various quantities. We first define the following quantity for odd i only

$$R_i(\lambda) := \sum_{k=0}^{\infty} (2k+i)!!\, \lambda^{2k}\,, \quad i \geq -1, \quad \text{odd } i. \tag{36}$$

In particular,

$$R_{-1}(\lambda) = \mathcal{Z}_0(\lambda)\,. \tag{37}$$

We may express all the other $R_i(\lambda)$ in terms of $\mathcal{Z}_0$. It is easy to check that

$$R_1(\lambda) = \frac{\mathcal{Z}_0(\lambda) - 1}{\lambda^2}\,, \tag{38}$$

and

$$R_j(\lambda) = \frac{\mathcal{Z}_0(\lambda) - 1}{\lambda^{j+1}} - \sum_{m=0}^{\frac{j-3}{2}} \frac{(2m+1)!!}{\lambda^{j-2m-1}}, \qquad j \geq 3. \tag{39}$$

It follows that

$$\lambda \frac{dR_k(\lambda)}{d\lambda} = R_{k+2}(\lambda) - (k+2)R_k(\lambda), \qquad k \geq -1\,. \tag{40}$$

We now consider $\lambda^n \mathcal{Z}_0^{(n)}(\lambda)$ where the index (n) indicates the number of derivatives with respect to λ. For example

$$\begin{aligned} \lambda \mathcal{Z}_0^{(1)} &= \lambda \frac{d}{d\lambda} R_{-1}(\lambda) \\ &= R_1(\lambda) - R_{-1}(\lambda)\,. \end{aligned} \tag{41}$$

We see that $\lambda^n \mathcal{Z}_0^{(n)}$ will contain $n+1$ terms and will take the form

$$\lambda^n \mathcal{Z}_0^{(n)}(\lambda) = \sum_{k=0}^{n} (-1)^{n-k}\, B_{n,2k-1}\; R_{2k-1}(\lambda)\,, \tag{42}$$

where the sign has been introduced to make the coefficients $B_{n,2k-1}$ positive. We obtain a recursion formula for the coefficients B by taking a derivative of both sides with respect to λ and then multiplying the result by one power of λ. This gives

$$n\lambda^n \mathcal{Z}_0^{(n)}(\lambda) + \lambda^{n+1} \mathcal{Z}_0^{(n+1)}(\lambda) = \sum_{k=0}^{n} (-1)^{n-k}\, B_{n,2k-1}\, \lambda \frac{d}{d\lambda} R_{2k-1}(\lambda)\,. \tag{43}$$

Using Eq. (42) for the terms on the left side and Eq. (40) to evaluate the derivative of $R_{2k-1}(\lambda)$, we obtain the recursion formula (33) with initial conditions (34), (35).

Using Eqs. (37), (38), and (39), we write Eq. (42) as

$$\lambda^n \mathcal{Z}_0^{(n)}(\lambda) = \sum_{k=0}^{n} (-1)^{n-k} \; B_{n,2k-1}(\lambda) \times \left[\delta_{k,0}\, \mathcal{Z}_0(\lambda) + H(k-1)\, \frac{\mathcal{Z}_0(\lambda)-1}{\lambda^{2k}} - H(k-2) \sum_{m=0}^{k-2} \frac{(2m+1)!!}{\lambda^{2k-2m-2}} \right]. \tag{44}$$

From Eq. (11), we have

$$M_1 = \lambda \frac{\mathcal{Z}_0'(\lambda)}{\mathcal{Z}_0(\lambda)} + 1\,. \tag{45}$$

The derivative $\mathcal{Z}_0'(\lambda)$ can be expressed in terms of $\mathcal{Z}_0(\lambda)$ using Eqs. (37), (38), and (41). This leads to

$$\mathcal{Z}_0'(\lambda) = \frac{\mathcal{Z}_0(\lambda) - 1 - \lambda^2 \mathcal{Z}_0(\lambda)}{\lambda^3}\,.$$

Using this result and Eq. (45), we may write

$$\mathcal{Z}_0(\lambda) - 1 = \lambda^2 M_1(\lambda) \mathcal{Z}_0(\lambda) \tag{46}$$

and therefore

$$\frac{1}{\mathcal{Z}_0(\lambda)} = 1 - \lambda^2 M_1(\lambda)\,. \tag{47}$$

Using Eqs. (46), (47), and (44) we obtain

$$\frac{1}{\mathcal{Z}_0(\lambda)} \lambda^n \mathcal{Z}_0^{(n)}(\lambda) = \sum_{k=0}^{n} (-1)^{n-k} \; B_{n,2k-1} \times \left[\delta_{k,0} \; + H(k-1)\, \frac{M_1(\lambda)}{\lambda^{2k-2}} - H(k-2) \left(1 - \lambda^2 M_1(\lambda)\right) \sum_{m=0}^{k-2} \frac{(2m+1)!!}{\lambda^{2k-2m-2}} \right]. \tag{48}$$

Recall that from expression (23) of $\mathcal{Z}_N$ in terms of derivatives of $\mathcal{Z}_0$, we have

$$\frac{\mathcal{Z}_j(\lambda)}{\mathcal{Z}_0(\lambda)} = \sum_{n=0}^{j} \binom{j}{n} \frac{j!}{n!} \frac{1}{\mathcal{Z}_0(\lambda)} \lambda^n \mathcal{Z}_0^{(n)}(\lambda)\,.$$

Now inserting Eq. (48) into this last equation, we obtain the required expression (32) for the ratio $\mathcal{Z}_j/\mathcal{Z}_0$.

It is now a simple matter to show that the result for M_N is a polynomial of degree N in M_1. From Eq. (32) we see that $\mathcal{Z}_j/\mathcal{Z}_0$ is a polynomial of degree one in M_1, for any value of j. According to Eq. (19), $M_N(\lambda)$ thus contains a linear combination of terms of the form

$$M_1^{\alpha_1+\alpha_2+\ldots+\alpha_N}\,,$$

with the condition $\alpha_1 + 2\alpha_2 + \ldots + N\alpha_N = N$. The largest power of M_1 is thus found by maximizing the sum $\alpha_1 + \ldots + \alpha_N$ while satisfying this condition. It is clear that this corresponds to choosing $\alpha_1 = N$ and all the other indices equal to zero, giving us that M_N is a polynomial of degree N. □

Example 2.

Here are the generating functions of N-rooted maps in terms of M_1 for $N = 1, \ldots, 5$.

$$
\begin{aligned}
2\lambda^2 M_2 &= M_1 - 1 - 2\lambda^2 M_1^2 \,, \\
6\lambda^4 M_3 &= M_1 - 1 - 9\lambda^2 M_1^2 + 7\lambda^2 M_1 + 12\lambda^4 M_1^3 \,, \\
24\lambda^6 M_4 &= M_1 - 1 - 15\lambda^2 + 47\lambda^2 M_1 - 34\lambda^2 M_1^2 - 112\lambda^4 M_1^2 + 144\lambda^4 M_1^3 - 144\lambda^6 M_1^4 \,, \\
120\lambda^8 M_5 &= M_1 - 1 - 93\lambda^2 + 216\lambda^2 M_1 + 633\lambda^4 M_1 - 125\lambda^2 M_1^2 \\
&\quad - 1\,875\lambda^4 M_1^2 + 1\,300\lambda^4 M_1^3 + 2\,800\lambda^6 M_1^3 - 3\,600\lambda^6 M_1^4 + 2\,880\lambda^8 M_1^5 \,.
\end{aligned}
$$

Appendix A. The differential equation for M_0

Here we present the derivation of the differential equation for M_0 given in Eq. (17) using quantum field theory. Let us recall that M_0 is defined by (see Eq. (9))

$$M_0(\lambda) = \ln\left(\frac{Z\,(J,\eta,\eta^*,\lambda)}{Z(0)}\right)\Bigg|_{J=\eta=\eta^*=0}\,, \tag{A.1}$$

where $Z(0) = \pi\sqrt{2\pi}$.

Our starting point is Eq. (5) which may be written as

$$
\begin{aligned}
Z(J,\eta,\eta^*,\lambda) &= \pi\sqrt{2\pi}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\left(\hat{A}\hat{\phi}\hat{\phi}^*\right)^k \exp\left(\eta\eta^* + \frac{J^2}{2}\right) \\
&= \pi\sqrt{2\pi}\,\exp\left(\lambda\hat{A}\hat{\phi}\hat{\phi}^*\right)\,\exp\left(\eta\eta^* + \frac{J^2}{2}\right),
\end{aligned}
\tag{A.2}
$$

where we have defined the operators

$$
\begin{aligned}
\hat{A} &:= \frac{d}{dJ}, \\
\hat{\phi} &:= \frac{d}{d\eta}, \\
\hat{\phi}^* &:= \frac{d}{d\eta^*} \,.
\end{aligned}
$$

Our goal is to obtain a differential equation for the partition function Z and its derivatives with respect to the coupling constant λ. Let us then consider the derivative of Z with respect to λ:

$$\frac{dZ(J,\eta,\eta^*,\lambda)}{d\lambda} = \hat{A}\hat{\phi}\hat{\phi}^*\,Z(J,\eta,\eta^*,\lambda)\,. \tag{A.3}$$

The next step is to express the right hand side in terms of Z and its derivatives. For this, we begin by calculating the application of $\hat{\phi}$ on the partition function:

$$\begin{aligned}\hat{\phi}\; Z(J,\eta,\eta^*,\lambda) &= \pi\sqrt{2\pi}\,\exp\left(\lambda\hat{A}\hat{\phi}\hat{\phi}^*\right)\,\hat{\phi}\,\exp\left(\eta\eta^*+\frac{J^2}{2}\right)\\ &= \pi\sqrt{2\pi}\sum_{k=0}^{\infty}\frac{\left(\lambda\hat{A}\hat{\phi}\hat{\phi}^*\right)^k}{k!}\;\eta^*\,\exp\left(\eta\eta^*+\frac{J^2}{2}\right)\\ &= \pi\sqrt{2\pi}\sum_{k=0}^{\infty}\frac{\left(\lambda\hat{A}\hat{\phi}\right)^k}{k!}\;\left(k(\hat{\phi}^*)^{k-1}+\eta^*(\hat{\phi}^*)^k\right)\,\exp\left(\eta\eta^*+\frac{J^2}{2}\right)\\ &= \pi\sqrt{2\pi}(\lambda\hat{A}\hat{\phi}+\eta^*)\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\left(\hat{A}\hat{\phi}\hat{\phi}^*\right)^k\,\exp\left(\eta\eta^*+\frac{J^2}{2}\right)\\ &= \left(\lambda\hat{A}\hat{\phi}+\eta^*\right)Z(J,\eta,\eta^*,\lambda)\,. \end{aligned} \tag{A.4}$$

Following a similar approach, we find

$$\hat{\phi}^*\; Z(J,\eta,\eta^*,\lambda) = \left(\lambda\hat{A}\hat{\phi}^*+\eta\right)Z(J,\eta,\eta^*,\lambda)\,, \tag{A.5}$$

$$\hat{A}\; Z(J,\eta,\eta^*,\lambda) = \left(\lambda\hat{\phi}\hat{\phi}^*+J\right)Z(J,\eta,\eta^*,\lambda)\,. \tag{A.6}$$

Equations (A.4), (A.5) and (A.6) are examples of Dyson–Schwinger equations. Now consider

$$\begin{aligned}\hat{\phi}\hat{\phi}^*\; Z(J,\eta,\eta^*,\lambda) &= \hat{\phi}\left(\lambda\hat{A}\hat{\phi}^*+\eta\right)\,Z(J,\eta,\eta^*,\lambda)\\ &= \left(\lambda\hat{A}\hat{\phi}\hat{\phi}^*+1+\eta\hat{\phi}\right)\,Z(J,\eta,\eta^*,\lambda)\,.\end{aligned} \tag{A.7}$$

At first sight this result may appear suspicious. Indeed, the operators $\hat{\phi}$ and $\hat{\phi}^*$ commute but the right hand side would have contained a term $\eta^*\hat{\phi}^*Z$ if we had calculated $\hat{\phi}^*\hat{\phi}\;Z$ instead. The two expressions would be different if Z was an arbitrary function of η and η^* but they coincide when applied to the partition function of Eq. (A.2). Of course, equations (A.4), (A.5) and (A.6) are also valid only because of the form (A.2) of the partition function.

We now apply $\hat{A}$ to Eq. (A.7):

$$\begin{aligned}\hat{A}\hat{\phi}\hat{\phi}^*Z(J,\eta,\eta^*,\lambda) &= \hat{A}\left(\lambda\hat{A}\hat{\phi}\hat{\phi}^*+1+\eta\hat{\phi}\right)\,Z(J,\eta,\eta^*,\lambda)\\ &= \left(\lambda\hat{A}\hat{\phi}\hat{\phi}^*+1+\eta\hat{\phi}\right)\,\left(\lambda\hat{\phi}\hat{\phi}^*+J\right)Z(J,\eta,\eta^*,\lambda)\\ &= \left(\lambda\hat{A}\hat{\phi}\hat{\phi}^*+1+\eta\hat{\phi}\right)\,\left(\lambda\hat{\phi}\left(\lambda\hat{A}\hat{\phi}^*+\eta\right)+J\right)Z(J,\eta,\eta^*,\lambda)\\ &= \left(\lambda\hat{A}\hat{\phi}\hat{\phi}^*+1+\eta\hat{\phi}\right)\,\left(\lambda^2\hat{A}\hat{\phi}\hat{\phi}^*+\lambda+\lambda\eta\hat{\phi}+J\right)Z(J,\eta,\eta^*,\lambda)\,,\end{aligned} \tag{A.8}$$

where in the third step we have used Eq. (A.5).

The only nontrivial terms to calculate are

$$\begin{aligned}\lambda\hat{A}\hat{\phi}\hat{\phi}^*\left(\lambda\eta\hat{\phi}\right)Z(J,\eta,\eta^*,\lambda) &= \left(\lambda^2\hat{A}\hat{\phi}\hat{\phi}^*+\lambda^2\eta\hat{A}\hat{\phi}^2\hat{\phi}^*\right)Z(J,\eta,\eta^*,\lambda)\,,\\ \lambda\hat{A}\hat{\phi}\hat{\phi}^*\,J\,Z(J,\eta,\eta^*\lambda) &= \left(\lambda\hat{\phi}\hat{\phi}^*+\lambda J\hat{A}\hat{\phi}\hat{\phi}^*\right)Z(J,\eta,\eta^*,\lambda)\\ &= \left(\lambda^2\hat{A}\hat{\phi}\hat{\phi}^*+\lambda+\lambda\eta\hat{\phi}+\lambda J\hat{A}\hat{\phi}\hat{\phi}^*\right)Z(J,\eta,\eta^*,\lambda)\,,\\ \eta\hat{\phi}\left(\lambda\eta\hat{\phi}\right)Z(J,\eta,\eta^*,\lambda) &= \left(\lambda\eta\hat{\phi}+\lambda\eta^2\hat{\phi}\right)Z(J,\eta,\eta^*,\lambda)\,.\end{aligned}$$

Using these relations, Eq. (A.8) may be written as

$$\hat{A}\hat{\phi}\hat{\phi}^* Z(J,\eta,\eta^*,\lambda) = \Bigg(\lambda^3(\hat{A}\hat{\phi}\hat{\phi}^*)^2 + 4\lambda^2\hat{A}\hat{\phi}\hat{\phi}^* + 2\lambda + 2\lambda^2\eta\hat{A}\hat{\phi}^2\hat{\phi}^* + 4\lambda\eta\hat{\phi} + \lambda\eta^2\hat{\phi} + J\eta\hat{\phi} + \lambda J\hat{A}\hat{\phi}\hat{\phi}^* + J\Bigg) Z(J,\eta,\eta^*,\lambda)\,.$$

Setting now all the sources to zero and using Eq. (A.3), one finds

$$\frac{dZ(\lambda)}{d\lambda} = \lambda^3\frac{d^2Z}{d\lambda^2}(\lambda) + 4\lambda^2\frac{dZ(\lambda)}{d\lambda} + 2\lambda Z(\lambda)\,, \tag{A.9}$$

where it is understood that

$$Z(\lambda) := Z(J,\eta,\eta^*,\lambda)\Big|_{J=\eta=\eta^*=0}\,. \tag{A.10}$$

The last step is to rewrite Eq. (A.9) as a differential equation for M_0 by using Eq. (A.1) to replace

$$Z(\lambda) = Z(0)\exp M_0(\lambda)\,. \tag{A.11}$$

Using this into Eq. (A.9), we finally obtain Eq. (17).

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