THE A–THEORETIC FARRELL–JONES CONJECTURE FOR VIRTUALLY SOLVABLE GROUPS

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Abstract. We prove the A–theoretic Farrell–Jones Conjecture for virtually solvable groups. As a corollary, we obtain that the conjecture holds for S–arithmetic groups and lattices in almost connected Lie groups.

1. Introduction

For every group G there is a functor $\mathbb{A}\colon \mathrm{Or}G \to \mathrm{Spectra}$ from the orbit category of G to the category of spectra sending G/H to (a spectrum weakly equivalent to) the non-connective A-theory spectrum $\mathbb{A}(BH)$. For any such functor $\mathbb{F}\colon \mathrm{Or}G \to \mathrm{Spectra}$, a G–homology theory $H_{\mathbb{F}}$ can be constructed via

$$H_{\mathbb{F}}(X) := \mathrm{Map}_G(\_, X_+) \wedge_{\mathrm{Or}G} \mathbb{F},$$

see Davis and Lück [DL98]. We will denote its homotopy groups by $H_n^G(X;\mathbb{F}) := \pi_n H_{\mathbb{F}}(X)$. The assembly map for the family of virtually cyclic subgroups (in A–theory) is the map

$$H_n^G(E_{\mathcal{VC}yc}G;\mathbb{A}) \to H_n^G(\mathrm{pt};\mathbb{A}) \cong A_n(BG)$$

induced by the map $E_{\mathcal{VC}yc}G \to \mathrm{pt}$. Here, $E_{\mathcal{VC}yc}G$ denotes the classifying space for the family of virtually cyclic subgroups, see Lück [Lüc05]. The assembly map can more generally be defined with coefficients, cf. [UW, Conjecture 7.1]. In this note, we consider the *A–theoretic Farrell–Jones Conjecture with coefficients and finite wreath products*, which predicts for a discrete group G that the assembly map with coefficients is an isomorphism for every wreath product $G \wr F$ of G with a finite group F.

Our main result is the following:

Theorem 1.1. *Let G be a virtually solvable group. Then G satisfies the Farrell–Jones Conjecture for A–theory with coefficients and finite wreath products.*

Using this, we can adapt previous work by Rüping [Rüp16] and Kammeyer, Lück and Rüping [KLR16] to A–theory:

Corollary 1.2. *The A–theoretic Farrell–Jones Conjecture with coefficients and finite wreath products holds for subgroups of $\mathrm{GL}_n(\mathbb{Q})$ or $\mathrm{GL}_n(F(t))$, where F is a finite field.*

In particular, the conjecture holds for S–arithmetic groups.

Proof. The proof works as the one of [Rüp16, Theorem 8.13]: Since the conjecture is inherited under directed colimits [ELP$^+$, Theorem 1.1(ii)], it suffices to consider linear groups over localizations at finitely many primes. Then [Rüp16, Proposition 2.2] together with [ELP$^+$, Corollary 6.6] shows that such a group satisfies the conjecture relative to a certain family of subgroups, all whose members in turn

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satisfy the conjecture relative to the class of virtually solvable groups [Rüp16, Theorem 8.12]. The corollary follows from Theorem 1.1 together with the Transitivity Principle [UW, Proposition 11.2]. □

Corollary 1.3. *The A–theoretic Farrell–Jones Conjecture with coefficients and finite wreath products holds for arbitrary lattices in almost connected Lie groups.*

More generally, it holds for lattices Γ in second countable, locally compact Hausdorff groups G whose group of path components $\pi_0(G)$ is discrete and satisfies the A–theoretic Farrell–Jones Conjecture with coefficients and finite wreath products.

Proof. In [KLR16], it is shown that a class of groups satisfying the list of properties from [KLR16, Theorem 2] also contains the groups considered in the corollary.

The statement of [KLR16, Theorem 2] holds for the class of groups satisfying the A–theoretic Farrell–Jones Conjecture with coefficients and finite wreath products by [ELP+, Theorem 1.1], Theorem 1.1 and Corollary 1.2. □

As explained in [ELP+, Section 3], the analogous statements of Theorem 1.1, Corollary 1.2 and Corollary 1.3 for (topological, PL or smooth) Whitehead spectra and pseudoisotopy spectra also hold true.

Remark 1.4. We have been informed that Thomas Farrell and Xiaolei Wu have independently obtained a proof of Theorem 1.1.

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2. Dress–Farrell–Hsiang–Jones groups

The proof of the A–theoretic Farrell–Jones Conjecture for solvable groups relies on a concoction of the Farrell–Hsiang method [UW] and transfer reducibility [ELP+] which mimics the combination of the methods from [BL12b] and [BL12a, Weg12] in [Weg15].

Definition 2.1. Let F be a finite group. We call F a *Dress group* if there exists a normal series $P \trianglelefteq H \trianglelefteq F$ such that P is a p–group for some prime p, H/P is cyclic and F/H is a q–group for some prime q.

We refer to [Weg15, Definition 2.7] and [Weg15, Definition 2.12] for the definitions of "homotopy coherent G–action" and "controlled domination".

Definition 2.2. Let G be a discrete group and let $S \subseteq G$ be a finite and symmetric generating set of G which contains the trivial element. Let $\mathcal{F}$ be a family of subgroups of G.

Then G is a *Dress–Farrell–Hsiang–Jones group with respect to $\mathcal{F}$*, or *DFHJ-group (with respect to $\mathcal{F}$)* for short, if there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there is a homomorphism $\pi\colon G \to F$ to a finite group with the property that for every Dress subgroup $D \leq F$ there exist

(1) a compact, contractible metric space X_D such that for every $\varepsilon > 0$ there is an ε–controlled domination of X_D by an at most N–dimensional finite simplicial complex;
(2) a homotopy coherent G–action Γ_D on X_D;
(3) a $\pi^{-1}(D)$–simplicial complex Σ_D of dimension at most N whose isotropy is contained in $\mathcal{F}$;
(4) a $\pi^{-1}(D)$–equivariant map $f_D\colon G \times X_D \to \Sigma_D$ such that

- for all $g \in G$, $x \in X_D$ and $s \in S^n$

$$d^{l^1}\big(f_D(g,x), f_D(gs^{-1}, \Gamma_D(s,x))\big) \leq \frac{1}{n},$$

- for all $g \in G$, $x \in X_D$ and $s_0, \ldots, s_n \in S^n$

$$\operatorname{diam}\big\{f_D(g, \Gamma_D(s_n, t_n, \ldots, s_0, x)) \,\big|\, t_1, \ldots, t_n \in [0,1]\big\} \leq \frac{2}{n}.$$

Remark 2.3.

(1) If G is homotopy transfer reducible with respect to $\mathcal{F}$ [ELP$^+$, Definition 6.2], then it is a DFHJ-group with respect to $\mathcal{F}$: Choose the finite quotient to be trivial for all n.

(2) If G is a Dress–Farrell–Hsiang group with respect to $\mathcal{F}$ [UW, Definition 7.3], then it is a DFHJ-group with respect to $\mathcal{F}$: Choose the transfer space X_D to be a point for all n and D.

Remark 2.4. Condition (4) in Definition 2.2 looks a bit different than [Weg15, Definition 4.1]. The difference lies mostly in notation. As we argue in the proof of Proposition 3.3 below, the condition in [Weg15, Definition 4.1] implies ours. Conversely, the proof showing the existence of the functor F in diagram (2.1) (cf. [ELP$^+$, Lemma 6.11]) shows that condition (4) also yields the condition in [Weg15, Definition 4.1], up to some constants.

Theorem 2.5. *Suppose that G is a DFHJ-group with respect to a family $\mathcal{F}$ of subgroups of G.*

Then the A–theoretic isomorphism conjecture with coefficients relative $\mathcal{F}$ holds for G.

The remainder of this section is dedicated to a proof of Theorem 2.5 and is modelled on [Weg15, Section 4.2]. Just like the proofs in [UW, ELP$^+$], we show that the fiber of the assembly map is weakly contractible. This uses the fact that this fiber can be modelled by the K–theory of certain categories of controlled retractive spaces, whose definition we recall next (cf. also [UW, Sections 2 and 3]).

A *coarse structure* is a triple $\mathfrak{Z} = (Z, \mathfrak{C}, \mathfrak{S})$ such that Z is a Hausdorff G–space, $\mathfrak{C}$ is a collection of reflexive, symmetric and G–invariant relations on Z which is closed under taking finite unions and compositions, and $\mathfrak{S}$ is a collection of G–invariant subsets of Z which is closed under taking finite unions. See [UW, Definition 3.23] for the notion of a *morphism of coarse structures*.

Fix a coarse structure $\mathfrak{Z}$.

A *labeled G–CW–complex relative W*, see [UW, Definition 2.3], is a pair (Y, κ), where Y is a free G–CW–complex relative W together with a G–equivariant function $\kappa\colon \diamond Y \to Z$. Here, $\diamond Y$ denotes the (discrete) set of relative cells of Y.

A *$\mathfrak{Z}$–controlled map* $f\colon (Y_1, \kappa_1) \to (Y_2, \kappa_2)$ is a G–equivariant, cellular map $f\colon Y_1 \to Y_2$ relative W such that for all $k \in \mathbb{N}$ there is some $C \in \mathfrak{C}$ for which

$$(\kappa_2, \kappa_1)(\{(e_2, e_1) \mid e_1 \in \diamond_k Y_1, e_2 \in \diamond Y_2, \langle f(e_1)\rangle \cap e_2 \neq \emptyset\}) \subseteq C$$

holds.

A *$\mathfrak{Z}$–controlled G–CW–complex relative W* is a labeled G–CW–complex (Y, κ) relative W, such that the identity is a $\mathfrak{Z}$–controlled map and for all $k \in \mathbb{N}$ there is some $S \in \mathfrak{S}$ such that

$$\kappa(\diamond_k Y) \subseteq S.$$

A *$\mathfrak{Z}$–controlled retractive space relative W* is a $\mathfrak{Z}$–controlled G–CW–complex (Y, κ) relative W together with a G–equivariant retraction $r\colon Y \to W$, ie. a left inverse to the structural inclusion $W \hookrightarrow Y$. The $\mathfrak{Z}$–controlled retractive spaces relative W form a category $\mathcal{R}^G(W,\mathfrak{Z})$ in which *morphisms* are $\mathfrak{Z}$–controlled maps which additionally respect the chosen retractions.

The category of controlled G–CW–complexes (relative W) and controlled maps admits a notion of *controlled homotopies*, see [UW, Definition 2.5] via the objects $(Y \rightthreetimes [0,1], \kappa \circ pr_Y)$, where $Y \rightthreetimes [0,1]$ denotes the reduced product which identifies $W \times [0,1] \subseteq Y \times [0,1]$ to a single copy of W and $pr_Y : \diamond Y \rightthreetimes [0,1] \to \diamond Y$ is the canonical projection. In particular, we obtain a notion of *controlled homotopy equivalence* (or *h–equivalence*).

A $\mathfrak{Z}$–controlled retractive space (Y,κ) is called *finite* if it is finite-dimensional, the image of $Y \backslash W$ under the retraction meets the orbits of only finitely many path components of W and for each $z \in Z$ there is some open neighborhood U of z such that $\kappa^{-1}(U)$ is finite, see [UW, Definition 3.3].

A $\mathfrak{Z}$–controlled retractive space (Y,κ) is called *finitely dominated*, if there are a finite $\mathfrak{Z}$–controlled, retractive space D, a morphism $p\colon D \to Y$ and a $\mathfrak{Z}$–controlled map $i\colon Y \to D$ such that $p \circ i$ is controlled homotopic to id_Y.

The finite, respectively finitely dominated, $\mathfrak{Z}$–controlled retractive spaces form full subcategories $\mathcal{R}^G_f(W,\mathfrak{Z}) \subset \mathcal{R}^G_{fd}(W,\mathfrak{Z}) \subset \mathcal{R}^G(W,\mathfrak{Z})$. All three of these categories support a Waldhausen category structure in which inclusions of G–invariant subcomplexes up to isomorphism are the cofibrations and controlled homotopy equivalences are the weak equivalences, see [UW, Corollary 3.22].

Let X be a G–CW–complex and let M be a metric space with free, isometric G–action. Define $\mathfrak{C}_{bdd}(M)$ to be the collection of all subsets $C \subset M \times M$ which are of the form

$$C = \{(m,m') \in M \times M \mid d(m,m') \le \alpha\}$$

for some $\alpha \ge 0$. Define further $\mathfrak{C}_{Gcc}(X)$ to be the collection of all $C \subset (X \times [1,\infty[) \times (X \times [1,\infty[)$ which satisfy the following:

1. For every $x \in X$ and every G_x–invariant open neighborhood U of (x,∞) in $X \times [1,\infty]$, there exists a G_x–invariant open neighborhood $V \subset U$ of (x,∞) such that $(((X \times [1,\infty[) \setminus U) \times V) \cap C = \emptyset$.
2. Let $p_{[1,\infty[}\colon X \times [1,\infty[\to [1,\infty[$ be the projection map. Equip $[1,\infty[$ with the Euclidean metric. Then there exists some $B \in \mathfrak{C}_{bdd}([1,\infty[)$ such that $C \subset p^{-1}_{[1,\infty[}(B)$.
3. C is symmetric, G–invariant and contains the diagonal.

Next define $\mathfrak{C}(M,X)$: Let $p_M\colon M \times X \times [1,\infty[\to M$ and $p_{X\times[1,\infty[}\colon M \times X \times [1,\infty[\to X \times [1,\infty[$ denote the projection maps. Then $\mathfrak{C}(M,X)$ is the collection of all subsets $C \subset (M \times X \times [1,\infty[)^2$ which are of the form

$$C = p_M^{-1}(B) \cap p^{-1}_{X\times[1,\infty[}(C')$$

for some $B \in \mathfrak{C}_{bdd}(M)$ and $C' \in \mathfrak{C}_{Gcc}(X)$.

Finally, define $\mathfrak{S}(M,X)$ to be the collection of all subsets $S \subset M \times X \times [1,\infty[$ which are of the form $S = K \times [1,\infty[$ for some G–compact subset $K \subset M \times X$.

All these data combine to a coarse structure

$$\mathbb{J}(M,X) := (M \times X \times [1,\infty[, \mathfrak{C}(M,X), \mathfrak{S}(M,X))$$

which serves to define the "obstruction category" $\mathcal{R}^G_f(W,\mathbb{J}(G,E_{\mathcal{F}}(G))),h)$, cf. [UW, Example 2.2 and Definition 6.1]. The spectrum $\mathbb{F}(G,W,E_{\mathcal{F}}(G))$ alluded to above is the non-connective K–theory spectrum of $\mathcal{R}^G_f(W,\mathbb{J}(G,E_{\mathcal{F}}(G)))$ with respect to the h–equivalences, cf. [UW, Section 5]. By [UW, Corollary 6.11], a group G satisfies the Farrell–Jones Conjecture with coefficients in A–theory with respect to $\mathcal{F}$ if and only if $\mathbb{F}(G,W,E_{\mathcal{F}}(G))$ is weakly contractible for every free G–CW–complex W.

Suppose now that G is a DFHJ-group. By definition, there exists some $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there is a homomorphism $\pi_n \colon G \to F_n$ to a finite group with the property that for every Dress subgroup $D \leq F_n$ there exist

(1) a compact, contractible metric space $X_{n,D}$ such that for every $\varepsilon > 0$ there is an ε–controlled domination of $X_{n,D}$ by an at most N–dimensional finite simplicial complex;
(2) a homotopy coherent G–action $\Gamma_{n,D}$ on $X_{n,D}$;
(3) a $\pi_n^{-1}(D)$–simplicial complex $\Sigma_{n,D}$ of dimension at most N whose isotropy is contained in $\mathcal{F}$;
(4) a $\pi_n^{-1}(D)$–equivariant map $f_{n,D} \colon G \times X_{n,D} \to \Sigma_{n,D}$ such that
   - for all $g \in G$, $x \in X_D$ and $s \in S^n$
$$d^{l^1}\left(f_{n,D}(g,x), f_{n,D}(gs^{-1}, \Gamma_D(s,x))\right) \leq \frac{1}{n},$$
   - for all $g \in G$, $x \in X_D$ and $s_0, \ldots, s_n \in S^n$
$$\operatorname{diam}\left\{f_{n,D}(g, \Gamma_{n,D}(s_n, t_n, \ldots, s_0, x)) \,\middle|\, t_1, \ldots, t_n \in [0,1]\right\} \leq \frac{2}{n}.$$

Assume we have chosen all of this. Then the proof is organized around the following diagram, in which we abbreviate $E := E_{\mathcal{F}}(G)$ (further explanations follow below):
(2.1)

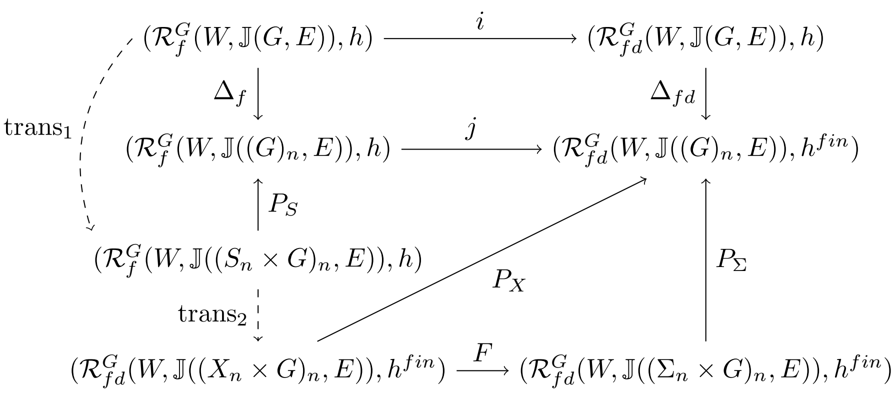

Diagram (2.1) involves some additional notation which we explain first.

Suppose that $(M_n)_n$ is a sequence of metric spaces with a free, isometric G–action. Let X be a G–CW–complex. Following [UW, Section 7], define the coarse structure
$$\mathbb{J}((M_n)_n, X) := \Big(\coprod_n M_n \times X \times [1,\infty[, \mathfrak{C}((M_n)_n, X), \mathfrak{S}((M_n)_n, X)\Big)$$
as follows: Members of $\mathfrak{C}((M_n)_n, X)$ are of the form $C = \coprod_n C_n$ with $C_n \in \mathfrak{C}(M_n, X)$, and we additionally require that C satisfies the *uniform metric control conditon*: There is some $\alpha > 0$, independent of n, such that for all $((m,x,t),(m',x',t')) \in C$ we have $d(m,m') < \alpha$. Members of $\mathfrak{S}((M_n)_n, X)$ are sets of the form $T = \coprod_n T_n$ with $T_n \in \mathfrak{S}(M_n, X)$. The resulting category $\mathcal{R}^G(W, \mathbb{J}((M_n)_n, X))$ is canonically a subcategory of the product category $\prod_n \mathcal{R}^G(W, \mathbb{J}(M_n, X))$.

Some instances of the category $\mathcal{R}^G(W, \mathbb{J}((M_n)_n, X))$ we consider in diagram (2.1) come equipped with another notion of weak equivalence: Let $(Y_n)_n$ be an object of $\mathcal{R}^G(W, \mathbb{J}((M_n)_n, E))$. For $\nu \in \mathbb{N}$, we denote by $(-)_{n>\nu}$ the endofunctor which sends $(Y_n)_n$ to the sequence $(\widetilde{Y}_n)_n$ with $\widetilde{Y}_n = *$ for $n \leq \nu$ and $\widetilde{Y}_n = Y_n$ for $n > \nu$. A morphism $(f_n)_n \colon (Y_n)_n \to (Y'_n)_n$ is an h^{fin}*–equivalence* if there is some $\nu \in \mathbb{N}$, such that $(f_n)_{n>\nu} \colon (Y_n)_{n>\nu} \to (Y'_n)_{n>\nu}$ is an h–equivalence.

Next, we define the families of metric spaces that we plug into the coarse structure $\mathbb{J}(-, E)$. As a shorthand, we denote the preimage $\pi_n^{-1}(D)$ of any Dress group $D \leq F_n$ by $\overline{D}$.

(1) The family $(G)_n$ is the constant family in which we equip each component with the word metric on G with respect to S.

(2) Let $\mathcal{D}r_n$ denote the family of Dress subgroups of F_n. Then define the G–space $S_n := \coprod_{D \in \mathcal{D}r_n} G/\overline{D}$. We equip $S_n \times G$ with the diagonal G–action and the quasi-metric d_{S_n} given by

$$d_{S_n}((g_1\overline{D}, g_2), (h_1\overline{D'}, h_2)) := \begin{cases} d_G(g_2, h_2) & \overline{D} = \overline{D'}, g_1\overline{D} = h_1\overline{D}, \\ \infty & \text{otherwise.} \end{cases}$$

(3) The space X_n is defined to be $\coprod_{D \in \mathcal{D}r_n} X_{n,D} \times G/\overline{D}$. Define for each $D \in \mathcal{D}r_n$ the constant $\Lambda_{n,D}$ as in [ELP+, Section 6]. We equip $X_n \times G$ with the G–action $\gamma \cdot (x, g_1\overline{D}, g_2) := (x, \gamma g_1\overline{D}, \gamma g_2)$ and the metric d_{X_n} given by

$$d_{X_n}((x, g_1\overline{D}, g_2), (y, h_1\overline{D'}, h_2))$$
$$:= \begin{cases} d_G(g_2, h_2) + d_{\Gamma_{n,D}, S^n, n, \Lambda_{n,D}}((x, g_2), (y, h_2)) & \overline{D} = \overline{D'}, g_1\overline{D} = g_2\overline{D} \\ \infty & \text{otherwise,} \end{cases}$$

where we use the metric $d_{\Gamma_{n,D}, S^n, n, \Lambda_{n,D}}$ defined in [Weg15, Definition 2.9].

(4) Finally, Σ_n is defined to be the G–simplicial complex $\coprod_{D \in \mathcal{D}r_n} G \times_{\overline{D}} \Sigma_{n,D}$, equipped with the metric $n \cdot d^{\ell^1}$, where d^{ℓ^1} denotes the ℓ^1–metric of a simplicial complex.

When crossing one of the above metric spaces with the group G, we regard the resulting space as a metric space by equipping it with the sum of the given metric and the word metric on G. This defines all categories appearing in diagram (2.1).

Let us now define the functors connecting these categories. The functors i and j are the exact inclusions functors from finite to finitely dominated objects. The functors Δ_f and Δ_{fd} are the diagonal functors sending a given object Y to the constant sequence $(Y)_n$. Note that $j \circ \Delta_f = \Delta_{fd} \circ i$. The functors P_S, P_X and P_Σ are induced the projection maps from $S_n \times G$, $X_n \times G$ and $\Sigma_n \times G$ to G. The functor F is induced by the sequence of maps $(f_n \colon X_n \times G \to \Sigma_n \times G)_n$, which we define by

$$f_n(x, g_1\overline{D}, g_2) := (g_1, f_{n,D}(g_1^{-1}g_2, x)).$$

The formula uses secretly the identification $G/\overline{D} \times G \cong G \times_{\overline{D}} G$. Using the contracting properties Definition 2.2 (4), one checks that the functor F is well-defined, the proof being completely analogous to [ELP+, Lemma 6.11]. Moreover, $P_X = P_\Sigma \circ F$.

We make the following claims:

Proposition 2.6.

(1) *After applying K–theory, the dashed arrow* trans_1 *exists such that* $K_m(\Delta_f) = K_m(P_S) \circ \mathrm{trans}_1$.

(2) *After applying K–theory, the dashed arrow* trans_2 *exists such that* $K_m(j \circ P_S) = K_m(P_X) \circ \mathrm{trans}_2$.

(3) *The K–theory of $(\mathcal{R}^G_{fd}(W, \mathbb{J}((\Sigma_n \times G)_n, E)), h^{fin})$ is trivial.*

(4) *$K_m(\Delta_{fd} \circ i)$ is injective for all m.*

Theorem 2.5 follows from Proposition 2.6 by an easy diagram chase.

Proof of Proposition 2.6. Claim (1) is an immediate consequence of [UW, Proposition 9.2]. Claim (3) is established in [UW, Section 10]. Claim (4) is [ELP+, Lemma 6.12]. So all that is left to show is claim (2).

The map trans_2 arises as a slight modification of the transfer constructed in [ELP+, Section 7], whose notation we will also use in the following discussion.

Let $\mathcal{R}^G_f(W, \mathbb{J}((S_n \times G)_n, E))_{\alpha,d}$ denote the subcategory of $\mathcal{R}^G_f(W, \mathbb{J}((S_n \times G)_n, E))$ containing only those objects $(Y_n, \kappa_n)_n$ such that Y_n has dimension at most d and is α–controlled over $S_n \times G$, together with morphisms $(\varphi_n : (Y_n, \kappa_n) \to (Y'_n, \kappa'_n))_n$ which are *cellwise* 0*–controlled* in the following sense: Each φ_n is a regular map (ie. it maps open cells onto open cells), and for each cell $c \in \diamond Y_n$, we have $\kappa'_n(\varphi_n(c)) = \kappa_n(c)$. Note that such morphisms automatically satisfy the uniform metric control condition.

Arguing as in [ELP+, Section 7.1], we observe that it suffices to construct compatible transfers on each $\mathcal{R}^G_f(W, \mathbb{J}((S_n \times G)_n, E))_{\alpha,d}$ individually.

Let $(Y_n, \kappa_n)_n$ be an object in $\mathcal{R}^G_f(W, \mathbb{J}((S_n \times G)_n, E))_{\alpha,d}$. By the definition of the metric d_{S_n}, the complex Y_n decomposes G–equivariantly as $Y_n = \coprod_{D \in \mathcal{D}r_n} Y_{n,D}$, with $Y_{n,D}$ living over the metric component $G/\pi_n^{-1}(D) \times G$. Let $\kappa_{n,D}$ denote the restriction of κ_n to the set of cells of $Y_{n,D}$. Then define

$$\mathrm{trans}^{\alpha,d}_n(Y_n) := \coprod_{D \in \mathcal{D}r_n} \mathrm{trans}^{\alpha,d}_{X_{n,D}}(Y_{n,D}),$$

cf. [ELP+, Definition 7.9]. The control map $\mathrm{trans}^{\alpha,d}_n(\kappa_n)$ of $\mathrm{trans}^{\alpha,d}_n(Y_n)$ is defined as in *loc. cit.* (formula directly before Lemma 7.10), replacing G by $S_n \times G$. Then the obvious analog of [ELP+, Lemma 7.10] holds, so that

$$\mathrm{trans}^{\alpha,d}((Y_n, \kappa_n)_n) := (\mathrm{trans}^{\alpha,d}_n(Y_n), \mathrm{trans}^{\alpha,d}_n(\kappa_n))_n$$

is indeed an object in $\mathcal{R}^G_{fd}(W, \mathbb{J}(X_n \times G)_n, E))$. By the obvious analog of [ELP+, Lemma 7.11], $\mathrm{trans}^{\alpha,d}$ defines a functor

$$\mathrm{trans}^{\alpha,d} \colon \mathcal{R}^G_f(W, \mathbb{J}((S_n \times G)_n, E))_{\alpha,d} \to \mathcal{R}^G_{fd}(W, \mathbb{J}(X_n \times G)_n, E)).$$

Since we leave the $S_n \times G \times E \times [1, \infty[$–component of each κ_n unchanged, the rest of [ELP+, Section 7] carries over to show the existence of the map trans_2, and thus claim (2). □

Remark 2.7. In fact, the discussion we have given so far only establishes the vanishing of $K_m(\mathcal{R}^G_f(W, \mathbb{J}(E)), h)$ for $m > 0$. In order to show vanishing in all degrees, we need to consider appropriate deloopings constructed by introducing another metric coordinate $\mathbb{R}^k$. Since this coordinate remains unchanged throughout, the previous discussion applies verbatim. Cf. also [UW, Section 9] and the discussion in Section 6 of [ELP+].

3. Proof of the main theorem

As in [Weg15, Section 3], the first step in proving Theorem 1.1 lies in reducing the general theorem to some special cases. For any non-zero algebraic number w, set $G_w := \mathbb{Z}[w, w^{-1}] \rtimes_{\cdot w} \mathbb{Z}$.

Lemma 3.1. *If G_w satisfies the A–theoretic Farrell–Jones Conjecture with coefficients and finite wreath products for every non-zero algebraic number w, then so does every virtually solvable group.*

Proof. We claim that the arguments in [Weg15, Section 3] carry over to A–theory. Indeed, the argument relies only on the following statements about the Farrell–Jones Conjecture with coefficients and finite wreath products:

(1) The class of groups satisfying the conjecture has the following closure properties [ELP+, Theorem 1.1(ii)]:
- If a group satisfies the conjecture, so does every subgroup.

- If two groups satisfy the conjecture, so do their direct and free products.
- If $\{G_i\}_{i \in I}$ is a directed system of groups satisfying the conjecture, so does the colimit.
- If $p\colon G \twoheadrightarrow Q$ is an epimorphism, and Q as well as every preimage $p^{-1}(C)$ of virtually cyclic subgroups of Q satisfy the conjecture, so does G.

(2) The following groups satisfy the conjecture:
- Semidirect products $A \rtimes \mathbb{Z}$ with A torsion abelian: This case follows from the case of hyperbolic groups [ELP+, Theorem 1.1(i)], cf. [FL03, Lemma 4.1].
- The wreath product $\mathbb{Z} \wr \mathbb{Z}$: This is, for example, a directed colimit of CAT(0)–groups, and hence satisfies the conjecture by [ELP+, Theorem 1.1(i)]. Alternatively, one can argue as in [FL03, Lemma 4.3].
- Virtually abelian groups [UW, Corollary 11.11], [ELP+, Theorem 1.1(i)].

For details, we refer to [Weg15, Section 3]. □

If w is a root of unity, G_w is a virtually abelian group (cf. [Weg15, Lemma 5.32]) and satisfies the A–theoretic Farrell Jones Conjecture with coefficients and finite wreath products by [UW, Corollary 11.11]. So we may assume that w is not a root of unity in the sequel.

We recall some notation from [Weg15, Section 5]. In what follows, we fix a non-zero algebraic number w which is not a root of unity. Let $\mathcal{O}$ be the ring of integers in $\mathbb{Q}(w)$. Define the ring $\mathcal{O}_w$ to be

$$\mathcal{O}_w := \{x \in \mathbb{Q}(w) \mid v_{\mathfrak{p}}(x) \geq 0 \text{ for all prime ideals } \mathfrak{p} \subset \mathcal{O} \text{ with } v_{\mathfrak{p}}(w) = 0.\},$$

so that $\mathcal{O} \subseteq \mathcal{O}_w$ and w, $w^{-1} \in \mathcal{O}_w$.

For $s \in \mathbb{N}$ we define $t_w(s) \geq 0$ to be the number determined by

$$t_w(s)\mathbb{Z} = \{z \in \mathbb{Z} \mid w^z \equiv 1 \mod s\mathcal{O}_w\}.$$

Lemma 3.2. *Let q_1, q_2 be prime numbers satisfying $q_1 \neq q_2$ and $v_{\mathfrak{p}}(w) = 0$ for all prime factors $\mathfrak{p}$ of q_1 or q_2 in $\mathcal{O}$. Let m_1, m_2 be natural numbers.*

Consider the finite group $F := (\mathcal{O}_w/q_1^{m_1}q_2^{m_2}\mathcal{O}_w) \rtimes \mathbb{Z}/t_w(q_1^{m_1}q_2^{m_2})\mathbb{Z}$.

For every Dress group $D \leq F$, there exists $i \in \{1,2\}$ such that the image of D under the canonical projection $\eta_i\colon F \twoheadrightarrow \mathcal{O}_w/q_i^{m_i}\mathcal{O}_w \rtimes \mathbb{Z}/t_w(q_i^{m_i})\mathbb{Z}$ is hyperelementary.

Proof. Let D be a Dress subgroup of F. Then D fits into a normal series $P \trianglelefteq H \trianglelefteq D$ such that P is a p–group, D/H is a p'–group, P is normal in D and $|H/P|$ is coprime to both p and p' [Win15, Lemma 5.1].

The prime p cannot be q_1 and q_2 at the same time; without loss of generality, assume that $p \neq q_1$. Set $t := t_w(q_1^{m_1}q_2^{m_2})$ and $t_1 := t_w(q_1^{m_1})$. Consider the normal subgroup $N := q_1^{m_1}\mathcal{O}_w/q_1^{m_1}q_2^{m_2}\mathcal{O}_w \rtimes t_1\mathbb{Z}/t\mathbb{Z}$ and let η_1 denote the projection map

$$\eta_1\colon F \twoheadrightarrow F/N \cong \mathcal{O}_w/q_1^{m_1}\mathcal{O}_w \rtimes \mathbb{Z}/t_1\mathbb{Z}.$$

Then $\eta_1(P) \cap \mathcal{O}_w/q_1^{m_1}\mathcal{O}_w = \{0\}$ since the latter is a q_1–group and $p \neq q_1$. Hence, $\eta_1(P)$ is mapped isomorphically to a subgroup of $\mathbb{Z}/t_1\mathbb{Z}$ by the projection map $\mathcal{O}_w/q_1^{m_1}\mathcal{O}_w \rtimes \mathbb{Z}/t_1\mathbb{Z} \twoheadrightarrow \mathbb{Z}/t_1\mathbb{Z}$. So $\eta_1(P)$ is cyclic. Since p is coprime to $|H/P|$ and H/P is cyclic, the image $\eta_1(H)$ is also cyclic. It follows that $\eta_1(D)$ is hyperelementary. □

Proposition 3.3. *Let $w \neq 0$ be an algebraic number which is no root of unity. Then $G_w = \mathbb{Z}[w, w^{-1}] \rtimes \mathbb{Z}$ is a DFHJ–group with respect to the family of virtually abelian subgroups.*

Proof. Let N be the natural number determined by [Weg15, Proposition 5.26]. Let $S \subseteq G_w$ be a finite, symmetric generating set containing the trivial element.

In the proof of [Weg15, Proposition 5.33], it is shown that for every $n \in \mathbb{N}$ and for every sufficiently large prime number q (depending on n) there is a natural number $m \in N$ such that for every hyperelementary subgroup

$$H \leq F_n := \mathcal{O}_w/q^m\mathcal{O}_w \rtimes \mathbb{Z}/t_w(q^m)\mathbb{Z}$$

there exist

(1) a compact, contractible metric space $X_{n,H}$ such that for every $\varepsilon > 0$ there is an ε–controlled domination of $X_{n,H}$ by an at most N–dimensional finite simplicial complex;[1]
(2) a homotopy coherent G_w–action $\Psi_{n,H}$ on $X_{n,H}$;
(3) a positive real number $\Lambda_{n,H}$;
(4) a $\alpha_n^{-1}(H)$–simplicial complex $E_{n,H}$ of dimension at most N whose isotropy groups are virtually cyclic or abelian;
(5) a $\alpha_n^{-1}(H)$–equivariant map $f_{n,H}\colon G_w \times X_{n,H} \to E_{n,H}$ such that
$$n \cdot d^{l^1}\big(f_{n,H}(g,x), f_{n,H}(h,y)\big) \leq d_{\Psi_{n,H},S^n,n,\Lambda_{n,H}}\big((g,x),(h,y)\big)$$
for all $(g,x),(h,y) \in G_w \times X_{n,H}$ with $h^{-1}g \in S^n$.

Here, $\alpha_n\colon G_w \to F_n$ denotes the composition of the inclusion $G_w \hookrightarrow \mathcal{O}_w \rtimes \mathbb{Z}$ with the quotient map $\mathcal{O}_w \rtimes \mathbb{Z} \twoheadrightarrow F_n$. The metric $d_{\Psi_{n,H},S^n,n,\Lambda_{n,H}}$ on $G_w \times X_{n,H}$ is defined in [Weg15, Definition 2.9]. It has the property

$$d_{\Psi_{n,H},S^n,n,\Lambda_{n,H}}\big((g,x),(g(s_n\cdots s_0)^{-1}, \Psi_{n,H}(s_n,t_n,\ldots,s_0,x))\big) \leq 1$$

for all $g \in G_w$, $x \in X_{n,H}$ and $s_0,\ldots,s_n \in S^n$. Hence,

$$d^{l^1}\big(f_{n,H}(g,x), f_{n,H}(gs^{-1}, \Psi_{n,H}(s,x))\big) \leq \frac{1}{n}$$

for all $g \in G_w$, $x \in X_{n,H}$ and $s \in S^n$, and

$$\operatorname{diam}\big\{f_{n,H}(g, \Psi_{n,H}(s_n,t_n,\ldots,s_0,x)) \,\big|\, t_1,\ldots,t_n \in [0,1]\big\} \leq \frac{2}{n}$$

for all $g \in G_w$, $x \in X_{n,H}$ and $s_0,\ldots,s_n \in S^n$.

Now let us come to the actual proof. For a given $n \in \mathbb{N}$ we choose two distinct (large) prime numbers q_1, q_2 with appropriate natural numbers $m_1, m_2 \in N$ (as described above). Consider the finite group

$$F := \mathcal{O}_w/q_1^{m_1}q_2^{m_2}\mathcal{O}_w \rtimes \mathbb{Z}/t_w(q_1^{m_1}q_2^{m_2})\mathbb{Z}.$$

Let $D \leq F$ be a Dress subgroup. By Lemma 3.2, there exists $i \in \{1,2\}$ such that $\eta_i(D)$ is hyperelementary. We have a finite group $F_n := \mathcal{O}_w/q_i^{m_1}\mathcal{O}_w \rtimes \mathbb{Z}/t_w(q_i^{m_i})\mathbb{Z} = \operatorname{im}(\eta_i)$ with a hyperelementary subgroup $H := \eta_i(D) \leq F_n$. As mentioned at the beginning of the proof, we obtain a homotopy coherent G_w–action $\Gamma_{n,H}$ on a metric space $X_{n,H}$, an $\alpha_n^{-1}(H)$–simplicial complex $E_{n,H}$ and an $\alpha_n^{-1}(H)$–equivariant map $f_{n,H}$ with the properties described above. We define $\pi\colon G_w \to F$ as the composition of the inclusion $G_w \hookrightarrow \mathcal{O}_w \rtimes \mathbb{Z}$ with the quotient map $\mathcal{O}_w \rtimes \mathbb{Z} \twoheadrightarrow F$. Then $\pi^{-1}(D)$ is a subgroup of $\alpha_n^{-1}(H)$. We finally set $X_D := X_{n,H}$, $\Gamma_D := \Psi_{n,H}$, $\Sigma_D := E_{n,H}$, $f_D := f_{n,H}$. □

Since virtually abelian groups satisfy the A–theoretic Farrell–Jones Conjecture with coefficients and finite wreath products, Theorem 1.1 follows from Lemma 3.1, Proposition 3.3 and Theorem 2.5 together with the Transititvity Principle [UW, Proposition 11.2] in view of the following:

[1] In the proof of [Weg15, Proposition 5.33] the space $X_{n,H}$ is denoted by X_w^R.

Lemma 3.4. *Suppose that G is a DFHJ-group with respect to the family of all subgroups which satisfy the A–theoretic Farrell–Jones Conjecture with coefficients and finite wreath products. Let F be a finite group.*

Then $G \wr F$ is a DFHJ-group with respect to the family of all subgroups which satisfy the A–theoretic Farrell–Jones Conjecture with coefficients and finite wreath products.

Proof. The proof is analogous to that of [Weg15, Lemma 4.3], replacing "hyperelementary" by "Dress" and using the fact that the collection of Dress groups is also closed under taking subgroups and quotients. □

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