Abstract

F. Labourie [arXiv:1212.5015] characterized the Hitchin components for $\text{PSL}(n, \mathbb{R})$ for any $n > 1$ by using the swapping algebra, where the swapping algebra should be understood as a ring equipped with a Poisson bracket. We introduce the rank $n$ swapping algebra, which is the quotient of the swapping algebra by the $(n+1) \times (n+1)$ determinant relations. The main results are the well-definedness of the rank $n$ swapping algebra and the “cross-ratio” in its fraction algebra. As a consequence, we use the subfraction algebra of the rank $n$ swapping algebra generated by these “cross-ratios” to characterize the $\text{PSL}(n, \mathbb{R})$ Hitchin component for a fixed $n > 1$. We also show the relation between the rank 2 swapping algebra and the cluster $\text{APGL}(2, \mathbb{R}), D_k$-space.

1. Introduction

1.1 Background

Let $S$ be a connected oriented closed surface of genus $g > 1$. When $G$ is a reductive Lie group, the character variety is

$$R(S, G) := \{ \text{homomorphisms } \rho : \pi_1(S) \to G \} // G,$$

where the group $G$ acts on homomorphisms above by conjugation, and the quotient is taken in the sense of geometric invariant theory [MFK94]. When $G = \text{PSL}(2, \mathbb{R})$, the character variety $R(S, \text{PSL}(2, \mathbb{R}))$ has $4g - 3$ connected components [G88]. Two of these components correspond to all discrete faithful homomorphisms from $\pi_1(S)$ to $\text{PSL}(2, \mathbb{R})$. By the uniformization theorem, any one of the two components is diffeomorphic to the Teichmüller space of complex structures on $S$ up to isotopy. For $n \geq 2$, we define $n$-Fuchsian representation to be a representation $\rho$, which can be written as $\rho = i \circ \rho_0$, where $\rho_0$ is a discrete faithful representation of $\pi_1(S)$ with values in $\text{PSL}(2, \mathbb{R})$ and $i$ is the irreducible representation of $\text{PSL}(2, \mathbb{R})$ in $\text{PSL}(n, \mathbb{R})$. In [H92], N. Hitchin found one of the connected components of the character variety $R(S, \text{PSL}(n, \mathbb{R}))$, which contains the $n$-Fuchsian representations, called Hitchin component and denoted by $H_n(S)$. By N. Hitchin [H92], the GIT quotient of the Hitchin component $H_n(S)$ coincides with its usual topological quotient. Furthermore, the Hitchin component $H_n(S)$ is diffeomorphic to a ball $\mathbb{R}^{(2g-2)(n^2-1)}$.

A decade later, F. Labourie [L06] and O. Guichard [Gu08] showed that every $\rho$ in the Hitchin component $H_n(S)$ is one to one associated to a $\rho$-equivariant $(\xi_\rho(\gamma x) = \rho(\gamma) \xi_\rho(x))$ hyperconvex Frenet curve $\xi_\rho$ from the boundary at infinity of $\pi_1(S)$—$\partial_\infty \pi_1(S)$ to $\mathbb{R}P^{n-1}$, where hyperconvex...
means that for any pairwise distinct points \((x_1, ..., x_p)\) with \(p \leq n\), the sum \(\xi(x_1) + ... + \xi(x_p)\) is direct. Let \(\xi^*_\rho\) be its associated \(\rho\)-equivariant osculating hyperplane curve from \(\partial_\infty \pi_1(S)\) to \(\mathbb{R} P^{n-1}\). Let \(\tilde{\xi}_\rho, (\xi^*_\rho)\) be the lifts of \(\xi, (\xi^*_\rho)\) with values in \(\mathbb{R}^n, (\mathbb{R}^n)^*\). F. Labourie defined the weak cross ratio \(\mathbb{B}_\rho\) of four different points \(x, y, z, t\) in \(\partial_\infty \pi_1(S)\):

\[
\mathbb{B}_\rho(x, y, z, t) = \frac{\langle \tilde{\xi}(x) \tilde{\xi}^*(z) \rangle}{\langle \tilde{\xi}(x) \tilde{\xi}^*(t) \rangle} \cdot \frac{\langle \tilde{\xi}(y) \tilde{\xi}^*(t) \rangle}{\langle \tilde{\xi}(y) \tilde{\xi}^*(z) \rangle}.
\]

Such cross ratios are the only cross ratios, called the rank \(n\) cross ratios, that satisfy some symmetry properties, normalisation properties, multiplicative cocycle identities, \(\pi_1(S)\)-invariant properties and \(\mathbb{R}^n\)-linear algebraic properties \([L07]\). Therefore, the space of the rank \(n\) cross ratios identifies with the Hitchin component \(H_n(S)\).

Later on, F. Labourie \([L12]\) defined the swapping algebra to characterize the union of the Hitchin components \(\bigcup_{n=2}^\infty H_n(S)\). The swapping algebra is defined on the ordered pair of points of a subset \(\mathcal{P} \subseteq S^1\). More precisely, we represent an ordered pair \((x, y)\) of \(\mathcal{P}\) by the expression \(xy\), and we consider the ring \(\mathcal{Z}(\mathcal{P}) := \mathbb{K}[\{xy\}_{x, y \in \mathcal{P}}]/\langle xy | \forall x \in \mathcal{P} \rangle\) over a field \(\mathbb{K}\) of characteristic zero. Then we equip \(\mathcal{Z}(\mathcal{P})\) with a Poisson bracket \(\{\cdot, \cdot\}\), called the swapping bracket, by extending the formula on generators for any \(rx, sy \in \mathcal{P}\):

\[
\{rx, sy\} = J(rx, sy) \cdot ry \cdot sx,
\]

\(J\) is defined by Leibniz’s rule. We will define the linking number \(J(rx, sy)\) in Section 2. Therefore, the swapping algebra of \(\mathcal{P}\) is \((\mathcal{Z}(\mathcal{P}), \{\cdot, \cdot\})\). Let \(x, y, z, t\) belong to \(\mathcal{P}\) so that \(x \neq t\) and \(y \neq z\). The cross fraction determined by \((x, y, z, t)\) is the element:

\[
[x, y, z, t] := \frac{xz}{xt} \cdot \frac{yt}{yz}.
\]

Let \(\mathcal{B}(\mathcal{P})\) be the sub fraction ring of \(\mathcal{Z}(\mathcal{P})\) generated by all the cross fractions. Then, the swapping multifraction algebra of \(\mathcal{P}\) is \((\mathcal{B}(\mathcal{P}), \{\cdot, \cdot\})\). Let \(\mathcal{R}\) be the subset of \(\partial_\infty \pi_1(S)\) given by the end points of periodic geodesics. F. Labourie consider a natural homomorphism \(I\) from \(\mathcal{B}(\mathcal{R})\) to \(C^\infty(H_n(S))\) by extending the following formula on generators to \(\mathcal{B}(\mathcal{R})\):

\[
I([x, y, z, t]) = \mathbb{B}_\rho(x, y, z, t).
\]

\textbf{Theorem 1.1} \([F. \text{ Labourie} \ [L12]]\) Let \(S\) be a connected oriented closed surface of genus \(g > 1\). Let \(\{\cdot, \cdot\}\) be the swapping bracket. For \(n \geq 2\), let \(\{\cdot, \cdot\}_S\) be the Atiyah-Bott-Goldman Poisson bracket \([AB83]/[G84]\) of the Hitchin component \(H_n(S)\). If \(\Gamma_1, ..., \Gamma_k, ...\) is a vanishing sequence of finite index subgroups of \(\pi_1(S)\). Let \(S_k = \mathbb{H}^2/\Gamma_k\), vanishing means that any two primitive representatives of \(\pi_1(S)\) in the sequence \(S_1, ..., S_k, ...\) intersect simply at zero or one point at last. For any \(b_0, b_1 \in \mathcal{B}(\mathcal{R})\), we have

\[
\lim_{k \to \infty} \{I(b_0), I(b_1)\}_{S_k} = I \circ \{b_0, b_1\}.
\]

The above theorem is true for any integer \(n > 1\), therefore the swapping multifraction algebra \((\mathcal{B}(\mathcal{R}), \{\cdot, \cdot\})\) asymptotically characterizes the union of Hitchin components \(\bigcup_{n=2}^\infty H_n(S)\).

F. Labourie also showed that, for the space \(\mathcal{L}_n\) of the Drinfeld-Sokolov reduction \([DS85]/[Se91]\) on the space of \(\text{PSL}(n, \mathbb{R})\)-Hitchin opers with trivial holonomy, the natural homomorphism \(i\) from the swapping multifraction algebra \(\mathcal{B}(S^1)\) to the function space \(C^\infty(\mathcal{L}_n)\) is Poisson with respect to the swapping bracket and the Poisson bracket corresponding to second Gelfand-Dickey symplectic structure.
Both the homomorphism \( I \) and the homomorphism \( i \) have large kernels arising from linear algebra of \( \mathbb{R}^n \). Is the swapping algebra \( (\mathcal{Z}(P), \{\cdot, \cdot\}) \) still well-defined after divided by these corresponding linear algebraic relations? Is the associated sub fraction algebra generated by all the cross fractions well-defined? These two questions are the main focus of this paper.

### 1.2 Rank \( n \) swapping algebra and the main results

For \( n \geq 2 \), let \( R_n(P) \) be the ideal of \( \mathcal{Z}(P) \) generated by

\[
D \in \mathcal{Z}(P) \mid D = \det \begin{pmatrix} x_1y_1 & \ldots & x_1y_{n+1} \\ \ldots & \ldots & \ldots \\ x_{n+1}y_1 & \ldots & x_{n+1}y_{n+1} \end{pmatrix}, \forall x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \in P.
\]

Let \( \mathcal{Z}_n(P) \) be the quotient ring \( \mathcal{Z}(P)/R_n(P) \). The following two theorems are the main results of this paper, which will be proven in Section 3 and 4. By induction on corresponding positions of the points on the circle, we prove the following theorem.

**Theorem 1.2** For \( n \geq 2 \), \( R_n(P) \) is a Poisson ideal with respect to the swapping bracket, thus \( \mathcal{Z}_n(P) \) inherits a Poisson bracket from the swapping bracket.

It then follows the Theorem 1.2 that the rank \( n \) swapping algebra of \( P \) is the compatible pair \( (\mathcal{Z}_n(P), \{\cdot, \cdot\}) \). For the well-definedness of the cross fractions of the ring \( \mathcal{Z}_n(P) \), by using very classical geometric invariant theory [CP76] [W39] and Lie group cohomology [CE48], we prove the following theorem.

**Theorem 1.3** For \( n \geq 2 \), the quotient ring \( \mathcal{Z}_n(P) \) is an integral domain.

Let \( \mathcal{B}_n(P) \) be the sub fraction ring of \( \mathcal{Z}_n(P) \) generated by all the cross fractions. Then, the rank \( n \) swapping multifraction algebra of \( P \) is the pair \( (\mathcal{B}_n(P), \{\cdot, \cdot\}) \). Thus, the homomorphism \( I \) naturally factors through \( \mathcal{B}_n(P) \) because of the rank \( n \) cross ratio conditions [L07], which provides a homomorphism

\[
I_n : \mathcal{B}_n(R) \to C^\infty(H_n(S)).
\]

Then we can replace \( I \) by \( I_n \) in Theorem 1.1. Therefore, for a fixed \( n \geq 2 \), the rank \( n \) swapping multifraction algebra \( (\mathcal{B}_n(P), \{\cdot, \cdot\}) \) is the Poisson algebra which characterizes \( H_n(S) \). But the homomorphism \( I_n \) is not injective, since the image of the cross fractions are \( \pi_1(S) \) invariant. Still, the non-injectivity and the asymptotic behavior of \( I_n \) are two obstructions to characterize \( (H_n(S), \omega_{ABG}) \) exactly. We suggest that these two obstructions are worth of being investigated.

For the homomorphism \( i \), we do not have these two obstructions. Similarly, we also have a homomorphism

\[
i_n : \mathcal{B}_n(S^1) \to C^\infty(\mathcal{L}_n)
\]

induced from the homomorphism \( i \). The homomorphism \( i_n \) is Poisson by Theorem 10.7.2 in [L12] and Theorem 1.2, and injective by Theorem 4.6. As a consequence, the rank \( n \) swapping multifraction algebra \( (\mathcal{B}_n(S^1), \{\cdot, \cdot\}) \) should be regarded as the dual of \( \mathcal{W}_n \) algebra.

### 1.3 Rank 2 swapping algebra and the cluster \( \mathcal{X}_{\text{PSL}(2,\mathbb{R}),D_k}\)-space

Let \( S \) be a connected oriented surface with non-empty boundary and a finite set \( P \) of special points on boundary, considered modulo isotopy. The rank \( n \) swapping algebra also relates to the Fock-Goncharov’s cluster-\( \mathcal{X}_{\text{PSL}(n,\mathbb{R}),S} \)-space. V. Fock and A. Goncharov [FG06] introduced the positive structure in sense of [L94] [L98] and the cluster algebraic structure for the moduli space.
$\mathcal{X}_{\text{PGL}(n,\mathbb{R}),S}$ of framed local systems of the surface $S$. The positive part of the moduli space $\mathcal{X}_{\text{PGL}(n,\mathbb{R}),S}$ is related to the Hitchin component $H_n(S)$. (For the surface $S$ with boundary or punctures, we can still define $H_n(S)$, but the monodromy around a boundary component is conjugated to an upper or lower triangular totally positive matrix.) Moreover, they introduced a special coordinate system for the cluster $\mathcal{X}_{\text{PGL}(n,\mathbb{R}),S}$-space in [FG06] Section 9, which generalizes Thurston’s shearing coordinates for Teichmüller space [T86]. (The Fock-Goncharov coordinates are also used in case of the closed surface $S$ of genus $g > 1$ [FD14].) This coordinate system is local, because it depends on the ideal triangulation $T$. Moreover, the coordinate system for $T$ gives us a split torus $T_T$ of $\mathcal{X}_{\text{PGL}(n,\mathbb{R}),S}$. The space $\mathcal{X}_{\text{PGL}(n,\mathbb{R}),S}$ is a variety glued by all these $T_T$, and the transition function from $T_T$ to another $T'_T$ is defined by a positive rational transformation corresponding to a composition of flips, where each flip is a composition of mutations in its cluster algebraic structure. The positive structure of $\mathcal{X}_{\text{PGL}(n,\mathbb{R}),S}$ arises from the positivity of the rational transformations.

Let $D_k$ be a disc with $k$ special points on the boundary. In the last section, we will prove the following theorem.

**Theorem 1.4** Given an ideal triangulation $T$ of $D_k$, there is an injective and Poisson homomorphims from the fraction algebra generated by the Fock-Goncharov coordinates for the cluster $\mathcal{X}_{\text{PGL}(2,\mathbb{R}),D_k}$-space to the rank 2 swapping multifraction algebra $(B_n(\mathcal{P}),\{\cdot,\cdot\})$, with respect to the natural Fock-Goncharov Poisson bracket and the swapping bracket.

Then we will show that the cluster dynamic of the cluster $\mathcal{X}_{\text{PGL}(2,\mathbb{R}),D_k}$-space can also be interpreted by the rank 2 swapping algebra. As a consequence, the natural Fock-Goncharov Poisson bracket does not depend on the triangulations. The above theorem is generalized for $\mathcal{X}_{\text{PGL}(n,\mathbb{R}),D_k}$-space in the following papers. For $n = 3$, in Chapter 3 of [Su14], we showed a complicated homomorphism, where $k$ flags of $\mathbb{R}P^2$ correspond to the set $\mathcal{P}$ with $k$ elements. For a general $n$, the homomorphism is discussed in [Su15], where the set $\mathcal{P}$ has $(n-1)\cdot k$ elements, each flag of $\mathbb{R}P^{n-1}$ corresponding to $n-1$ points near each other on the boundary $S^1$.

Therefore, the rank $n$ swapping algebra provides the links among the Hitchin component $H_n(S)$, $W_n$ algebra and the cluster $\mathcal{X}_{\text{PGL}(2,\mathbb{R}),D_k}$-space.

### 1.4 Further discussions

In the upcoming paper [Su1511], we will define a quantized version of the rank $n$ swapping algebra. The quantization of $\mathcal{X}_{D_k,\text{PSL}(n,\mathbb{R})}$ by Fock-Goncharov [FG06] [FG09] is embedded into our quantization of the rank $n$ swapping algebra. We will glue the rank $n$ swapping algebras to characterize the cluster $\mathcal{X}_{S,\text{PSL}(n,\mathbb{R})}$-space for the surface $S$ in general. We expect to build a TQFT and some geometric invariants from the rank $n$ swapping algebra.

In [Su1412], we relate the rank $n$ swapping algebra to the discrete integrable system of the configuration space of $N$-twisted polygon in $\mathbb{R}P^{n-1}$ [FV93][SOT10][KS13]. When $n = 2$, there is a bi-hamiltonian structure for the configuration space of $N$-twisted polygon in $\mathbb{R}P^{n-1}$. This was conjectured in [SOT10] for $n = 3$. We expect that there exists a bi-hamiltonian structure for $n$ in general.

### 2. Swapping algebra revisited

In this section, we will recall some basic definitions about the swapping algebra introduced by F. Labourie in Section 2 of [L12]. The new part of this section is that we take care of the
compatibilities of the rings related to $\mathcal{Z}(\mathcal{P})$ and the swapping bracket, particularly the sub 
fract ring $\mathcal{B}(\mathcal{P})$ generated by “cross ratios”.

2.1 Linking number

Definition 2.1 [LINKING NUMBER] Let $(r, x, s, y)$ be a quadruple of four points in $S^1$. The linking number between $rx$ and $sy$ is

$$J(rx, sy) = \frac{1}{2} \cdot (\sigma(r - x) \cdot \sigma(r - y) \cdot \sigma(y - x) - \sigma(r - x) \cdot \sigma(r - s) \cdot \sigma(s - x)),$$

such that for any $a \in \mathbb{R}$, we define $\sigma(a)$ as follows. Remove any point $o$ different from $r, x, s, y$ in $S^1$ in order to get an interval $[0, 1]$. Then the points $r, x, s, y \in S^1$ correspond to the real numbers in $[0, 1]$, $\sigma(a) = -1; 0; 1$ whenever $a < 0; a = 0; a > 0$ respectively.

In fact, the value of $J(rx, sy)$ belongs to $\{0, \pm 1, \pm \frac{1}{2}\}$, depends on the corresponding positions of $r, x, s, y$ and does not depend on the choice of the point $o$. In Figure 1, we describe five possible values of $J(rx, sy)$.

2.2 Swapping algebra

Let $\mathcal{P}$ be a cyclic subset of $S^1$, we represent an ordered pair $(r, x)$ of $\mathcal{P}$ by the expression $rx$. Then we consider the associative commutative ring

$$\mathcal{Z}(\mathcal{P}) := \mathbb{K}\{xy\}_{x,y \in \mathcal{P}}/\{xx\}_{\forall x \in \mathcal{P}}$$

over a field $\mathbb{K}$ of characteristic 0, where $\{xy\}_{x,y \in \mathcal{P}}$ are variables. Then we equip $\mathcal{Z}(\mathcal{P})$ with a swapping bracket.

Definition 2.2 [SWAPPING BRACKET [L12]] The swapping bracket over $\mathcal{Z}(\mathcal{P})$ is defined by extending the following formula for any $rx, sy$ in $\mathcal{P}$ to $\mathcal{Z}(\mathcal{P})$ by using Leibniz’s rule:

$$\{rx, sy\} = J(rx, sy) \cdot ry \cdot sx.$$

By direct computations, F. Labourie proved the following theorem.

Theorem 2.3 [F. Labourie [L12]] The swapping bracket is Poisson.
Definition 2.4 [SWAPPING ALGEBRA] The swapping algebra of $\mathcal{P}$ is the ring $\mathcal{Z}(\mathcal{P})$ equipped with the swapping bracket, denoted by $(\mathcal{Z}(\mathcal{P}), \{, \})$.

2.3 Swapping multifraction algebra

In this subsection, we consider the rings related to $\mathcal{Z}(\mathcal{P})$ and their compatibilities with the swapping bracket.

Definition 2.5 [CLOSED UNDER SWAPPING BRACKET] For a ring $R$, if $\forall a, b \in R$, we have $\{a, b\} \in R$, then we say that $R$ is closed under swapping bracket.

Since $\mathcal{Z}(\mathcal{P})$ is an integral domain, let $\mathcal{Q}(\mathcal{P})$ be the total fraction ring of $\mathcal{Z}(\mathcal{P})$. By Leibniz’s rule, we have $\{a, \frac{1}{b}\} = -\frac{\{a, b\}}{b^2}$, thus the swapping bracket is well defined on $\mathcal{Q}(\mathcal{P})$. Therefore we have the following definition.

Definition 2.6 [SWAPPING FRACTION ALGEBRA OF $\mathcal{P}$] The swapping fraction algebra of $\mathcal{P}$ is the ring $\mathcal{Q}(\mathcal{P})$ equipped with the induced swapping bracket, denoted by $(\mathcal{Q}(\mathcal{P}), \{, \})$.

Definition 2.7 [CROSS FRACTION] Let $x, y, z, t$ belong to $\mathcal{P}$ so that $x \neq t$ and $y \neq z$. The cross fraction determined by $(x, y, z, t)$ is the element of $\mathcal{Q}(\mathcal{P})$:

$$[x, y, z, t] := \frac{xz}{xt} \cdot \frac{yt}{yz}.$$  \hspace{1cm} (6)

Remark 2.8 Notice that the cross fractions verify the following cross-ratio conditions [L07]:

Symmetry: $[a, b, c, d] = [b, a, c, d]$,

Normalisation: $[a, b, c, d] = 0$ if and only if $a = c$ or $b = d$,

Normalisation: $[a, b, c, d] = 1$ if and only if $a = b$ or $c = d$,

Cocycle identity: $[a, b, c, d] \cdot [a, b, d, e] = [a, b, c, e]$,

Cocycle identity: $[a, b, d, e] \cdot [b, c, d, e] = [a, c, e, f]$.

Let $\mathcal{CR}(\mathcal{P}) = \{[x, y, z, t] \in \mathcal{Q}(\mathcal{P}) \mid \forall x, y, z, t \in \mathcal{P}, x \neq t, y \neq z\}$ be the set of all the cross-fractions in $\mathcal{Q}(\mathcal{P})$. Let $\mathcal{B}(\mathcal{P})$ be the subring of $\mathcal{Q}(\mathcal{P})$ generated by $\mathcal{CR}(\mathcal{P})$.

Proposition 2.9 The ring $\mathcal{B}(\mathcal{P})$ is closed under swapping bracket.

Proof. By Leibniz’s rule, $\forall c_1, \ldots, c_n, d_1, \ldots, d_m \in \mathcal{Z}(\mathcal{P})$

$$\{c_1 \cdots c_n, d_1 \cdots d_m\} = \sum_{i=1}^{n} \sum_{j=1}^{m} \{c_i, d_j\},$$  \hspace{1cm} (7)

we only need to show that for any two elements $[x, y, z, t]$ and $[u, v, w, s]$ in $\mathcal{CR}(\mathcal{P})$, where $x \neq t$, $y \neq z$, $u \neq s$, $v \neq w$ in $\mathcal{P}$, then $\{\frac{x}{x}, \frac{y}{y}, \frac{w}{w}, \frac{s}{s}\} \in \mathcal{B}(\mathcal{P})$. Let $e_1 = xz$, $e_2 = \frac{1}{xz}$, $e_3 = yt$, $e_4 = \frac{1}{yt}$, $h_1 = uw$, $h_2 = \frac{1}{uw}$, $h_3 = vs$, $h_4 = \frac{1}{vs}$. By the definition of the swapping bracket, we have

$$\frac{\{e_1, h_1\}}{e_1 \cdot h_1} = J(xz, uw) \cdot \frac{xw}{xz} \cdot \frac{uz}{uw} \in \mathcal{B}(\mathcal{P}).$$

Then by the Leibniz’s rule, we deduce that for any $e, h \in \mathcal{Z}(\mathcal{P})$, we have

$$\frac{\{e, \frac{1}{h}\}}{e / h} = \frac{\{e, h\}}{e \cdot h} \cdot \frac{1/e}{h / e} = \frac{\{e, h\}}{e \cdot h} \cdot \frac{1/eh}{e \cdot h} = \frac{\{e, h\}}{e \cdot h}.$$
So for any $i, j = 1, 2, 3, 4$, we have $\{e_i, h_j\} \in B(\mathcal{P})$. Since $e_1e_2e_3e_4$ and $h_1h_2h_3h_4$ are also in $B(\mathcal{P})$, so

$$
\{e_1e_2e_3e_4, h_1h_2h_3h_4\} = \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\{e_i, h_j\}}{e_i \cdot h_j} \cdot (e_1e_2e_3e_4h_1h_2h_3h_4) \in B(\mathcal{P}).
$$

Finally, we conclude that $B(\mathcal{P})$ is closed under swapping bracket. \hfill \Box

**Definition 2.10** [SWAPPING MULTIFRACTION ALGEBRA OF $\mathcal{P}$] The swapping multifraction algebra of $\mathcal{P}$ is the ring $B(\mathcal{P})$ equipped with the swapping bracket, denoted by $(B(\mathcal{P}), \{\cdot, \cdot\})$.

### 3. Rank $n$ swapping algebra

Swapping algebra $(Z(\mathcal{P}), \{\cdot, \cdot\})$ corresponds to $\bigcup_{n=2}^{\infty} H_n(S)$. In this section, we define the rank $n$ swapping algebra $Z_n(\mathcal{P})$, in order to restrict the correspondence for a fixed $n$. In theorem 3.4, we will prove that the ring $Z_n(\mathcal{P})$ is compatible with the swapping bracket.

#### 3.1 The rank $n$ swapping ring $Z_n(\mathcal{P})$

**Notation 3.1** Let

$$
\Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) = \det \begin{pmatrix} x_1 y_1 & \ldots & x_1 y_{n+1} \\ \ldots & \ldots & \ldots \\ x_{n+1} y_1 & \ldots & x_{n+1} y_{n+1} \end{pmatrix}.
$$

Inspired by linear algebra for $\mathbb{R}^n$, and the space of the rank $n$ cross-ratios identified with the Hitchin component $H_n(S)$ [L07], we define the rank $n$ swapping ring as follows.

**Definition 3.2** [The rank $n$ swapping ring $Z_n(\mathcal{P})$] For $n \geq 2$, let $R_n(\mathcal{P})$ be the ideal of $Z(\mathcal{P})$ generated by $\{D \in Z(\mathcal{P}) \mid D = \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})), \forall x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \in \mathcal{P} \}.$

The rank $n$ swapping ring $Z_n(\mathcal{P})$ is the quotient ring $Z(\mathcal{P})/R_n(\mathcal{P})$.

**Remark 3.3** Decomposing the determinant $D$ in the first row, we have by induction that

$$R_2(\mathcal{P}) \supseteq R_3(\mathcal{P}) \supseteq \ldots \supseteq R_n(\mathcal{P}). \quad (8)$$

#### 3.2 Swapping bracket over $Z_n(\mathcal{P})$

We will prove by induction the fundamental theorem of the rank $n$ swapping algebra.

**Theorem 3.4** [First main result] For $n \geq 2$, the ideal $R_n(\mathcal{P})$ is a Poisson ideal with respect to the swapping bracket. Thus the ring $Z_n(\mathcal{P})$ inherits a Poisson bracket from the swapping bracket.

**Proof.** The above theorem is equivalent to say that for any $h \in R_n(\mathcal{P})$ and any $f \in Z(\mathcal{P})$, we have $\{f, h\} \in R_n(\mathcal{P})$ where $n \geq 2$. By the Leibniz’s rule of the swapping bracket, it suffices to prove the case where $f = ab \in Z(\mathcal{P})$, $h = \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}))$. The points $x_1, \ldots, x_{n+1}$ ($y_1, \ldots, y_{n+1}$ resp.) should be different from each other in $\mathcal{P}$, otherwise $h = 0$. Therefore the theorem follows from Lemma 3.5. \hfill \Box

**Lemma 3.5** Let $n \geq 2$. Let $x_1, \ldots, x_{n+1}$ ($y_1, \ldots, y_{n+1}$ resp.) in $\mathcal{P}$ be different from each other and ordered anticlockwise, $a, b$ belong to $\mathcal{P}$ and $x_1, \ldots, x_l, y_1, \ldots, y_k$ are on the right side of $ab$ (include
coinciding with \( a \) or \( b \) as illustrated in Figure 2. Let \( u \) (resp.) be strictly on the left (right resp.) side of \( \overrightarrow{ab} \). Let

\[
\Delta^{R}(a, b) = \sum_{d=1}^{l} J(ab, xd) \cdot x_{d} \cdot \Delta((x_1, \ldots, x_{d-1}, a, x_{d+1}, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) \\
+ \sum_{d=1}^{k} J(ab, uy_{d}) \cdot a_{d} \cdot \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{d-1}, b, y_{d+1}, \ldots, y_{n+1}))
\]

We obtain that

\[
\{ab, \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}))\} = \Delta^{R}(a, b).
\]

Proof. The main idea of the proof is to consider the change of \( \{ab, \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}))\} \) when \( ab \) moves topologically in the circle with special points \( a, b, x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \).

We will prove that

\[
\{ab, \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}))\} = \Delta^{R}(a, b)
\]

by induction on the number of elements of \( \{x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}\} \) on the right side of \( \overrightarrow{ab} \) (includes coinciding with \( a \) or \( b \)), which is \( m = l + k \). Let \( S_{n+1} \) be the permutation group of \( \{1, \ldots, n+1\} \), the signature of \( \sigma \in S_{n+1} \) denoted by \( \text{sgn}(\sigma) \), is defined as 1 if \( \sigma \) is even and \(-1\) if \( \sigma \) is odd. Then we have

\[
\Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) = \sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma) \prod_{i=1}^{n+1} x_i y_{\sigma(i)}.
\]

By the Leibniz’s rule, we obtain that

\[
\{ab, \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}))\} = \sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma) \prod_{i=1}^{n+1} x_i y_{\sigma(i)} \sum_{j=1}^{n+1} \frac{\{ab, x_{j} y_{\sigma(j)}\}}{x_{j} y_{\sigma(j)}}
\]

By the Leibniz’s rule, we obtain that

\[
\{ab, \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}))\} = \sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma) \prod_{i=1}^{n+1} x_i y_{\sigma(i)} \left( \sum_{j=1}^{n+1} \frac{J(ab, x_{j} y_{\sigma(j)}) \cdot a_{\sigma(j)} \cdot x_{j} b}{x_{j} y_{\sigma(j)}} \right).
\]
When $m = 0$ as illustrated in Figure 3, since $J(ab, x_j y_{\sigma(j)}) = 0$, we have $\{ab, x_j y_{\sigma(j)}\} = 0$ for any $j = 1, \ldots, n + 1$ and any $\sigma \in S_{n+1}$. By Equation 12, we have

$$\{ab, \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}))\} = 0 = \Delta^R(a, b)$$

in this case.

Suppose

$$\{ab, \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}))\} = \Delta^R(a, b)$$

for $m = q \geq 0$.

When $m = q + 1$, suppose that $x_l$ is the first point of $\{x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}\}$ on the right side of $\overrightarrow{ab}$ (include coinciding with $a$ or $b$) with respect to the clockwise orientation.

(i) If $x_l$ coincides with $a$ as illustrated in Figure 4, then $m = 1$. So we have $J(ab, x_ly_{\sigma(l)}) = \frac{1}{2}$ and $J(ab, x_j y_{\sigma(j)}) = 0$ for $j \neq l$. By Equation 12, we have

$$\{ab, \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}))\} = \frac{1}{2} \cdot ab \cdot \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) = \Delta^R(a, b).$$
(ii) If $x_l$ does not coincide with $a$, we move $b$ clockwise to the point $b'$, such that $b' \neq x_l$ and the intersection between $\{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\}$ and the arc $bb'$ is $x_l$ as illustrated in Figure 5.

Then $\{ab', \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}$ corresponds to the case $m = q$. Thus we have

$$\{ab', \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}$$

$$= \sum_{d=1}^{l-1} J(ab', x_d u) \cdot x_d b \cdot \Delta((x_1, ..., x_{d-1}, a, x_{d+1}, ..., x_{n+1}), (y_1, ..., y_{n+1}))$$

$$+ \sum_{d=1}^{k} J(ab', y_d d) \cdot y_d b \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{d-1}, b, y_{d+1}, ..., y_{n+1})).$$

On the other hand, by Equation 12,

$$\{ab', \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}$$

$$= \sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma) \prod_{i=1}^{n+1} x_i y_{\sigma(i)} \left( \sum_{j=1}^{n+1} \frac{J(ab', x_j y_{\sigma(j)}) \cdot a y_{\sigma(j)} \cdot x_j b'}{x_j y_{\sigma(j)}} \right).$$

(14)

is a polynomial of $ab', x_1 b', ..., x_{n+1} b'$, denoted by $P(ab', x_1 b', ..., x_{n+1} b')$. Then

$$P(ab, x_1 b, ..., x_{n+1} b) = \sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma) \prod_{i=1}^{n+1} x_i y_{\sigma(i)} \left( \sum_{j=1}^{n+1} \frac{J(ab', x_j y_{\sigma(j)}) \cdot a y_{\sigma(j)} \cdot x_j b'}{x_j y_{\sigma(j)}} \right).$$

By the cocycle identity [L12]: $J(ab, xy) - J(ab', xy) = J(b'b, xy)$, we have

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} - P(ab, x_1 b, ..., x_{n+1} b)$$

$$= \sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma) \prod_{i=1}^{n+1} x_i y_{\sigma(i)} \left( \sum_{j=1}^{n+1} \frac{(J(ab, x_j y_{\sigma(j)}) - J(ab', x_j y_{\sigma(j)})) \cdot a y_{\sigma(j)} \cdot x_j b}{x_j y_{\sigma(j)}} \right)$$

(15)
Since $J(b'b, x_jy_{\sigma(j)}) = 0$ when $j \neq l$, $J(b'b, x_l y_{\sigma(l)}) = J(ab, x_l u)$. So the above sum equals to

$$\sum_{\sigma \in S_{n+1}} sgn(\sigma) \prod_{i=1}^{n+1} x_i y_{\sigma(i)} \cdot \frac{J(b'b, x_l y_{\sigma(l)}) \cdot a y_{\sigma(l)} \cdot x_l b}{x_l y_{\sigma(l)}}$$

(16)

Since $J(ab', x_d u) = J(ab, x_d u)$ for $d = 1, ..., l-1$, $J(ab', uy_d) = J(ab, uy_d)$ for $d = 1, ..., k$, by Equations 13 15 16, we have

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \Delta^R(a, b)$$

in this case.

When $y_k$ is the first point of $\{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\}$ on the right side of $\overrightarrow{ab}$ (include coinciding with $a$ or $b$) with respect to clockwise orientation, the result follows the similar argument. By induction, we have

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \Delta^R(a, b)$$

in general.

Finally, we conclude that

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \Delta^R(a, b).$$

\[ \square \]

**Remark 3.6** We can also consider $\Delta^L(a, b)$ similar to $\Delta^R(a, b)$ with respect to the left side of $\overrightarrow{ab}$. The equation $\Delta^R(a, b) = \Delta^L(a, b)$ follows

$$\sum_{i=1}^{n+1} a y_i \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{i-1}, b, y_{i+1}, ..., y_{n+1}))$$

(17)

$$\sum_{i=1}^{n+1} x_i b \cdot \Delta((x_1, ..., x_{i-1}, a, x_{i+1}, ..., x_{n+1}), (y_1, ..., y_{n+1})).$$
Example 3.7  As shown in Figure 6, we have
\[ \{xz, \Delta((x, z, y), (z, x, t))\} = -xt \cdot \Delta((x, z, y), (z, x, z)) = 0. \]  \hspace{1cm} (18)
Therefore, the swapping bracket over \( \mathbb{Z}_n(\mathcal{P}) \) is well defined for \( n \geq 2 \).

Definition 3.8 [Rank \( n \) swapping algebra of \( \mathcal{P} \)] For \( n \geq 2 \), the rank \( n \) swapping algebra of \( \mathcal{P} \) is the rank \( n \) swapping ring \( \mathbb{Z}_n(\mathcal{P}) \) equipped with the swapping bracket, denoted by \( (\mathbb{Z}_n(\mathcal{P}), \{\cdot, \cdot\}) \).

4. The ring \( \mathbb{Z}_n(\mathcal{P}) \) is an integral domain

In this section, we will show that the cross fractions are well-defined in the fraction ring of \( \mathbb{Z}_n(\mathcal{P}) \), by proving that the ring \( \mathbb{Z}_n(\mathcal{P}) \) is an integral domain. The strategy of the proof is the following. First, we introduce a geometric model studied by H. Weyl [W39] and C. D. Concini and C. Procesi [CP76] to characterize the ring \( \mathbb{Z}_n(\mathcal{P}) \) as a \( \text{GL}(n, \mathbb{K}) \)-module. Then, we transfer the integrality of the ring \( \mathbb{Z}_n(\mathcal{P}) \) to another ring \( \mathbb{K}^{\text{GL}(n, \mathbb{K})} \) by an injective ring homomorphism. The homomorphism is recovered by a long exact sequence of Lie group cohomology with values in \( \text{GL}(n, \mathbb{K}) \)-modules by Proposition 4.9. In the end, we prove that the ring \( \mathbb{K}_{n,p} \) is integral, which will complete the proof of the theorem.

4.1 A geometric model for \( \mathbb{Z}_n(\mathcal{P}) \)

Let us introduce a geometric model to characterize \( \mathbb{Z}_n(\mathcal{P}) \). Let \( M_{n,p} = (\mathbb{K}^n \times \mathbb{K}^{n^*})^p \) be the space of \( p \) vectors in \( \mathbb{K}^n \) and \( p \) co-vectors in \( \mathbb{K}^{n^*} \).

Notation 4.1 Let \( a_i = (a_{i,1}, ..., a_{i,n})^T, b_i = \sum_{l=1}^{n} b_{i,l} \sigma_l \) where \( a_{i,l}, b_{i,l} \in \mathbb{K}, \sigma_l \in \mathbb{K}^{n^*} \) and \( \sigma_l(a_i) = a_{i,l} \). We define the product between a vector \( a_i \) in \( \mathbb{K}^n \) and a co-vector \( b_j \) in \( \mathbb{K}^{n^*} \) by
\[ \langle a_i | b_j \rangle := b_j(a_i) = \sum_{k=1}^{n} a_{i,k} \cdot b_{j,k}. \]  \hspace{1cm} (19)

The group \( \text{GL}(n, \mathbb{K}) \) acts naturally on the vectors and the covectors by
\[ g \circ a_i := g \cdot (a_{i,1}, ..., a_{i,n})^T, \]
\[ g \circ b_j := (b_{j,1}, ..., b_{j,n}) \cdot (g^{-1}) \cdot (\sigma_1, ..., \sigma_n)^T \]
where \( T \) is the transpose of the matrix. When we consider the action on their products, we write \( b_j = (b_{j,1}, ..., b_{j,n})^T \) in column as \( a_i \), then
\[ g \circ b_j := (g^{-1})^T \cdot (b_{j,1}, ..., b_{j,n})^T. \]
For any \( g \in \text{GL}(n, \mathbb{K}), a, b \in \mathbb{K}[M_{n,p}] \),
\[ g \circ (a \cdot b) := (g \circ a) \cdot (g \circ b). \]
It induces a \( \text{GL}(n, \mathbb{K}) \) action on \( \mathbb{K}[M_{n,p}] \) satisfying:
- For any \( g \in \text{GL}(n, \mathbb{K}), a, b \in \mathbb{K}[M_{n,p}] \), we have
  \[ g \circ (a + b) = g \circ a + g \circ b, \]
- For any \( g_1, g_2 \in \text{GL}(n, \mathbb{K}), a \in \mathbb{K}[M_{n,p}] \), we have
  \[ g_1 \circ (g_2 \circ a) = (g_1 \cdot g_2) \circ a. \]
Then the polynomial ring $K[M_{n,p}]$ is a $GL(n, K)$-module.

Let $B_{nK}$ be the subring of $K[M_{n,p}]$ generated by $\{\langle a_i | b_j \rangle\}_{i=1,j=1}^p$. We denote the $GL(n, K)$ invariant ring of $K[M_{n,p}]$ by $K[M_{n,p}]^{GL(n,K)}$. Since $\langle a_i | b_j \rangle \in K[M_{n,p}]$ is invariant under $GL(n, K)$ action, we have $B_{nK} \subseteq K[M_{n,p}]^{GL(n,K)}$. Moreover, C. D. Concini and C. Procesi proved that

**Theorem 4.2** [C. D. Concini and C. Procesi [CP76] 1] $B_{nK} = K[M_{n,p}]^{GL(n,K)}$.

Since $K[M_{n,p}]$ is an integral domain, they obtained the following corollary.

**Corollary 4.3** [C. D. Concini and C. Procesi [CP76]] The subring $B_{nK}$ is an integral domain.

H. Weyl describe $B_{nK}$ as a quotient ring.

**Theorem 4.4** [H. Weyl [W39]] All the relations in $B_{nK}$ are generated by $R = \{ f \in B_{nK} \mid f = \det \left( \begin{array}{ccc} \langle a_{i_1} | b_{j_1} \rangle & \ldots & \langle a_{i_1} | b_{j_n+1} \rangle \\ \vdots & \ddots & \vdots \\ \langle a_{i_n+1} | b_{j_1} \rangle & \ldots & \langle a_{i_n+1} | b_{j_n+1} \rangle \end{array} \right), \forall i_k, j_l = 1, \ldots, p \}$.

**Remark 4.5** In other words, let $W$ be the polynomial ring $K[\{x_{i,j}\}_{i,j=1}^p]$, $r = \{ f \in W \mid f = \det \left( \begin{array}{ccc} x_{i_1,j_1} & \ldots & x_{i_1,j_{n+1}} \\ \vdots & \ddots & \vdots \\ x_{i_{n+1},j_1} & \ldots & x_{i_{n+1},j_{n+1}} \end{array} \right), \forall i_k, j_l = 1, \ldots, p \}$, let $T$ be the ideal of $W$ generated by $r$, then we have $B_{nK} \cong W/T$.

Let us recall that $Z_n(P) = Z(P)/R_n(P)$ is the rank $n$ swapping ring where $P = \{x_1, \ldots, x_p\}$. When we identify $a_i$ with $x_i$ on the left and $b_i$ with $x_i$ on the right of the pairs of points in $Z_n(P)$, we obtain the main result of this subsection below.

**Theorem 4.6** Let $Z_n(P)$ be the rank $n$ swapping ring. Let $S_{nK}$ be the ideal of $B_{nK}$ generated by $\{\langle a_i | b_i \rangle\}_{i=1}^p$, then $B_{nK}/S_{nK} \cong Z_n(P)$.

### 4.2 Proof of the second main result

**Theorem 4.7** [Second main result] For $n > 1$, $Z_n(P)$ is an integral domain.

Firstly, let us first consider the following $GL(n, K)$-modules:

(i) Let $L$ be the ideal of $K[M_{n,p}]$ generated by $\{\langle a_i | b_i \rangle\}_{i=1}^p$,

(ii) let $K_{n,p}$ be the quotient ring $K[M_{n,p}]/L$,

(iii) let $S_{nK}$ be the ideal of $B_{nK}$ generated by $\{\langle a_i | b_i \rangle\}_{i=1}^p$.

Thus there is an exact sequence of $GL(n, K)$-modules (the right arrows are not only module homomorphisms, but also ring homomorphisms):

$$0 \rightarrow L \rightarrow K[M_{n,p}] \rightarrow K_{n,p} \rightarrow 0. \quad (20)$$

By Lie group cohomology [CE48], the exact sequence above induces the long exact sequence:

$$0 \rightarrow L^{GL(n,K)} \rightarrow K[M_{n,p}]^{GL(n,K)} \rightarrow K_{n,p}^{GL(n,K)} \rightarrow H^1(GL(n,K), L) \rightarrow \ldots. \quad (21)$$

1Thanks for the reference provided by J. B. Bost.
Lemma 4.8 Let $S$ be the finite subset $\{(a_i | b_i)\}_{i=1}^p$. Let $\mathbb{K}$ be a field of characteristic 0. Then

$$(\mathbb{K}[M_{n,p}] \cdot S)^{GL(n, \mathbb{K})} = \mathbb{K}[M_{n,p}]^{GL(n, \mathbb{K})} \cdot S.$$ 

Proof. The proof follows from Weyl’s unitary trick. Let

$$U(n) = \{g \in GL(n, \mathbb{K}) \mid g \cdot \bar{g}^T = I\}.$$ 

We want to prove that

$$(\mathbb{K}[M_p] \cdot S)^{U(n)} = \mathbb{K}[M_p]^{U(n)} \cdot S.$$ 

Notice first that one inclusion is obvious

$$(\mathbb{K}[M_p] \cdot S)^{U(n)} \supseteq \mathbb{K}[M_p]^{U(n)} \cdot S.$$ 

We next prove the other inclusion:

$$(\mathbb{K}[M_p] \cdot S)^{U(n)} \subseteq \mathbb{K}[M_p]^{U(n)} \cdot S.$$ 

For this, let $dg$ be a Haar measure on $U(n)$. Let $x$ belongs to $(\mathbb{K}[M_p] \cdot S)^{U(n)}$. We represent $x$ by

$$\sum_{l=1}^k (g \circ t_l) \cdot s_l,$$

where $g \circ s_t = s_t$. Thus we have

$$x = g \circ x = \sum_{l=1}^k (g \circ t_l) \cdot (g \circ s_l) = \sum_{l=1}^k (g \circ t_l) \cdot s_l.$$ 

By integrating over $U(n)$:

$$g \circ x = \int_{U(n)} \sum_{l=1}^k (g \circ t_l) \cdot s_l \, dg = \sum_{l=1}^k \left( \int_{U(n)} g \circ t_l \, dg \right) \cdot s_l, \quad (22)$$

where

$$b_l = \int_{U(n)} g \circ t_l \, dg \in \mathbb{K}[M_p].$$

For any $g_1$ in $U(n)$, we have

$$g_1 \circ b_l = \int_{U(n)} g_1 \circ (g \circ t_l) \, dg = \int_{U(n)} ((g_1 \circ g) \circ t_l) \, dg$$

$$= \int_{U(n)} ((g_1 \circ g) \circ t_l) \, d(g_1 \circ g) = b_l. \quad (23)$$

Thus $b_l$ belongs to $\mathbb{K}[M_p]^{U(n)}$, $x$ belongs to $\mathbb{K}[M_p]^{U(n)} \cdot S$. Hence

$$(\mathbb{K}[M_p] \cdot S)^{U(n)} \subseteq \mathbb{K}[M_p]^{U(n)} \cdot S.$$ 

Therefore, we obtain

$$(\mathbb{K}[M_p] \cdot S)^{U(n)} = \mathbb{K}[M_p]^{U(n)} \cdot S.$$ 

By extending the ground field $\mathbb{K}$ of $U(n)$, the property is extended to $GL(n, \mathbb{K})$. Therefore, we conclude that

$$(\mathbb{K}[M_{n,p}] \cdot S)^{GL(n, \mathbb{K})} = \mathbb{K}[M_{n,p}]^{GL(n, \mathbb{K})} \cdot S.$$ 

This complete the proof of the lemma.

Then, the integrality of $Z_n(P)$ is transferred to another ring $K_{n,p}^{GL(n, \mathbb{K})}$ by the following proposition.
Proposition 4.9 There is a ring homomorphism $\theta : B_{nK}/S_{nK} \rightarrow K^{GL(n,K)}_{n,p}$ induced from the long exact sequence:

$$0 \rightarrow L^{GL(n,K)} \rightarrow \mathbb{K}[M_{n,p}]^{GL(n,K)} \rightarrow K^{GL(n,K)}_{n,p} \rightarrow H^1(\text{GL}(n, \mathbb{K}), L) \rightarrow \ldots,$$

which is injective.

Proof. By Theorem 4.2, we have $\mathbb{K}[M_{n,p}]^{GL(n,K)} = B_{nK}$. By Lemma 4.8, we have

$L^{GL(n,K)} = (\mathbb{K}[M_{n,p}] \cdot (\{(a_i|b_i)\}_{i=1}^p))^{GL(n,K)} = \mathbb{K}[M_{n,p}]^{GL(n,K)} \cdot (\{(a_i|b_i)\}_{i=1}^p) = B_{nK} \cdot (\{(a_i|b_i)\}_{i=1}^p) = S_{nK}$.

Hence Long exact sequence 21 becomes into:

$$0 \rightarrow S_{nK} \rightarrow B_{nK} \rightarrow K^{GL(n,K)}_{n,p} \rightarrow H^1(\text{GL}(n, \mathbb{K}), L) \rightarrow \ldots \quad (25)$$

Therefore, there is an injective module homomorphism $\theta$ from $B_{nK}/S_{nK}$ to $K^{GL(n,K)}_{n,p}$. By definition of $\theta$ with respect to Exact sequence 20, for any $a, b \in B_{nK}/S_{nK}$, we have

$$\theta(a \cdot b) = \theta(a) \cdot \theta(b).$$

Thus the module homomorphism $\theta$ is also a ring homomorphism. As such, we conclude that the ring homomorphism $\theta$ is injective. \qed

Therefore, we only need to prove that $K_{n,p}$ is integral. This is the content of the following proposition.

Proposition 4.10 For $n > 1$, $K_{n,p}$ is an integral domain.

Proof. We prove the proposition by induction on the number of the vectors or covectors $p$. Let us start with $p = 1$. When $p = 1$, $K_{n,1} = \mathbb{K}[M_{n,1}]/(\sum_{k=1}^n a_{1,k} \cdot b_{1,k})$. Let us define the degree of a monomial in $\mathbb{K}[M_{n,1}]$ to be the sum of the degrees in all the variables. Let the degree of a polynomial $f$ in $\mathbb{K}[M_{n,1}]$ be the maximal degree of the monomials in $f$, denoted by $\deg(f)$. Suppose that $\sum_{k=1}^n a_{1,k} \cdot b_{1,k}$ is a reducible polynomial in $\mathbb{K}[M_{n,1}]$, we have

$$\sum_{k=1}^n a_{1,k} \cdot b_{1,k} = g \cdot h,$$

where $g, h \in \mathbb{K}[M_{n,1}]$, $\deg(g) > 0$ and $\deg(h) > 0$. Since $\mathbb{K}[M_{n,1}]$ is an integral domain, $2 = \deg(gh) = \deg(g) + \deg(h)$, so we have $\deg(g) = \deg(h) = 1$. Suppose that

$$g = \lambda_0 + \lambda_1 \cdot c_1 + \ldots + \lambda_r \cdot c_r,$$

$$h = \mu_0 + \mu_1 \cdot d_1 + \ldots + \mu_s \cdot d_s,$$

where $\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s$ are non zero elements in $\mathbb{K}$, $c_1, \ldots, c_r$ ($d_1, \ldots, d_s$ resp.) are different elements in $\{a_{1,k}, b_{1,k}\}_{k=1}^n$. Since there is no square in $g \cdot h$, we have

$$\{c_1, \ldots, c_r\} \cap \{d_1, \ldots, d_s\} = \emptyset.$$

Because there are $n$ monomials in $gh$, we obtain

$$r \cdot s = n.$$

Moreover, there are $2n$ variables in $g \cdot h$, we have

$$r + s = 2n,$$

thus

$$r \cdot s \geq 2n - 1.$$
Since \( n > 1 \), we obtain that
\[
r \cdot s \geq 2n - 1 > n = r \cdot s,
\]
which is a contradiction. We therefore conclude that \( \sum_{k=1}^{n} a_{1,k} b_{1,k} \) is an irreducible polynomial in \( \mathbb{K}[M_{n,p}] \). Since \( \mathbb{K}[M_{n,1}] \) is an integral domain, we obtain that \( K_{n,1} \) is an integral domain. Suppose that the proposition is true for \( p = m + 1 \). When \( p = m + 1 \),
\[
K_{n,m+1} = \mathbb{K}\left[\{a_{i,k}, b_{i,k}\}_{i,k=1}^{m+1,n}\right] / \left(\sum_{k=1}^{n} a_{i,k} b_{i,k} \right)_{i=1}^{m+1}
\]
(26)
we have \( K_{n,m} \) is an integral domain by induction, thus \( K_{n,m}\left[\{a_{m+1,k}, b_{m+1,k}\}_{k=1}^{n}\right] \) is an integral domain. By the above argument, the polynomial \( \sum_{k=1}^{n} a_{m+1,k} b_{m+1,k} \) is an irreducible polynomial over \( \mathbb{K}\left[\{a_{m+1,k}, b_{m+1,k}\}_{k=1}^{n}\right] \). Moreover, \( a_{m+1,k}, b_{m+1,k} \) (\( k = 1, \ldots, n \)) are not variables that appear in \( K_{n,m} \), so \( \sum_{k=1}^{n} a_{m+1,k} b_{m+1,k} \) is an irreducible polynomial over \( K_{n,m}\left[\{a_{m+1,k}, b_{m+1,k}\}_{k=1}^{n}\right] \). Hence \( K_{n,m+1} \) is an integral domain.

We therefore conclude that \( K_{n,p} \) is an integral domain for any \( p \geq 1 \) and \( n > 1 \). \( \square \)

**Proof of Theorem 4.7.** By Proposition 4.10, the ring \( K_{n,p} \) is an integral domain, we deduce that the invariant ring \( K_{n,p}^{GL(n, \mathbb{K})} \) is an integral domain. By Proposition 4.9, there is an injective ring homomorphism \( \theta \) from \( B_{n,K}/S_{n,K} \) to \( K_{n,p}^{GL(n, \mathbb{K})} \), so \( B_{n,K}/S_{n,K} \) is an integral domain. Moreover, by Theorem 4.6 \( Z_{n}(\mathcal{P}) \cong B_{n,K}/S_{n,K} \), we conclude that for \( n > 1 \), the rank \( n \) swapping ring \( Z_{n}(\mathcal{P}) \) is an integral domain.

**Remark 4.11** The ring \( Z_{1}(\mathcal{P}) \) is not an integral domain, since
\[
D = xy \cdot yz = \det \begin{pmatrix} xy & xz \\ yz & yz \end{pmatrix}
\]
is zero in \( Z_{1}(\mathcal{P}) \), but we have \( xy \) and \( yz \) are not zero in \( Z_{1}(\mathcal{P}) \) whenever \( x \neq y, y \neq z \).

### 4.3 Rank \( n \) Swapping Multifraction Algebra of \( \mathcal{P} \)

After Theorem 4.7, we define rank \( n \) swapping multifraction algebra of \( \mathcal{P} \) without any obstruction.

**Definition 4.12** [Rank \( n \) Swapping Fraction Algebra of \( \mathcal{P} \)] Let \( Q_{n}(\mathcal{P}) \) be the total fraction ring of \( Z_{n}(\mathcal{P}) \). The rank \( n \) swapping fraction algebra of \( \mathcal{P} \) is the total fraction ring \( Q_{n}(\mathcal{P}) \) equipped with the swapping bracket, denoted by \((Q_{n}(\mathcal{P}), \{\cdot, \cdot\})\).

Let \( CR_{n}(\mathcal{P}) = \{[x, y, z, t] = \frac{xy}{zt} : [y, t] \in Q_{n}(\mathcal{P}) \} \) be the set of all the cross fractions in \( Q_{n}(\mathcal{P}) \). Let \( B_{n}(\mathcal{P}) \) be the sub fraction ring of \( Q_{n}(\mathcal{P}) \) generated by \( CR_{n}(\mathcal{P}) \).

Similar to Proposition 2.9, we have the following.

**Proposition 4.13** The sub fraction ring \( B_{n}(\mathcal{P}) \) is closed under swapping bracket.

**Definition 4.14** [Rank \( n \) Swapping Multifraction Algebra of \( \mathcal{P} \)] Let \( n \geq 2 \), the rank \( n \) swapping multifraction algebra of \( \mathcal{P} \) is the sub fraction ring \( B_{n}(\mathcal{P}) \) equipped with the closed swapping bracket, denoted by \((B_{n}(\mathcal{P}), \{\cdot, \cdot\})\).
Then the ring homomorphism \( I \) from \( \mathcal{B}(\mathcal{R}) \) to \( C^\infty(H_n(S)) \) for any \( n > 1 \), induces the homomorphism \( I_n \) from \( \mathcal{B}_n(\mathcal{R}) \) to \( C^\infty(H_n(S)) \) by extending the following formula on generators to \( \mathcal{B}_n(\mathcal{R}) \):

\[
I_n([x, y, z, t]) = B_0(x, y, z, t),
\]

for any \([x, y, z, t] \in CR_n(\mathcal{R})\). By the rank \( n \) cross ratio condition, the homomorphism \( I_n \) is well-defined. Then we rephrase Theorem 1.1 as follows.

**Theorem 4.15** [F. Labourie [L12]] *With the same conditions as in Theorem 1.1, for any \( b_0, b_1 \in \mathcal{B}_n(\mathcal{R}) \), we have*

\[
\lim_{n \to \infty} \{I_n(b_0), I_n(b_1)\}_{S_n} = I_n \circ \{b_0, b_1\}.
\]

Hence the rank \( n \) swapping multifraction algebra \((\mathcal{B}_n(\mathcal{R}), \{\cdot, \cdot\})\) characterizes the Hitchin component \( H_n(S) \) for a fixed \( n > 1 \).

5. Cluster \( \mathcal{X}_{PGL(2,\mathbb{R}),D_k}\)-space

Even though the rank \( n \) swapping multifraction algebra \((\mathcal{B}_n(\mathcal{P}), \{\cdot, \cdot\})\) characterizes the \( PSL(n, \mathbb{R}) \) Hitchin component asymptotically, we still have the rank \( n \) swapping multifraction algebra \((\mathcal{B}_n(\mathcal{P}), \{\cdot, \cdot\})\) characterizes the related object—cluster \( \mathcal{X}_{PGL(n,\mathbb{R}),D_k}\)-space without this asymptotic behavior where \( D_k \) is a disc with \( k \) special points on the boundary. We will show a simple case when \( n = 2 \). We show that the cluster dynamic of \( \mathcal{X}_{PGL(2,\mathbb{R}),D_k}\) can be demonstrated in the rank 2 swapping algebra. As a byproduct, we reprove that the Fock-Goncharov Poisson bracket for \( \mathcal{X}_{PGL(2,\mathbb{R}),D_k}\) is independent of the ideal triangulation.

5.1 Cluster \( \mathcal{X}_{PGL(2,\mathbb{R}),D_k}\)-space and rank 2 swapping algebra

Let \( S \) be an oriented surface with non-empty boundary and a finite set \( P \) of special points on boundary, considered modulo isotopy. In [FG06], Fock and Goncharov introduced the moduli space \( \mathcal{X}_{G,S}(\mathcal{A}_{G,S} \text{ resp.}) \) which is a pair \((\nabla, f)\), where \( \nabla \) is a flat connection on the principal \( G \) bundle on the surface \( S \), \( f \) is a flat section of \( \partial S \backslash P \) with values in the flag variety \( G/B \) (decorated flag variety \( G/U \) resp.). They found that the pair of two moduli spaces \((\mathcal{X}_{G,S}, \mathcal{A}_{G,S})\) is equipped with a cluster ensemble structure. Particularly, the moduli space \( \mathcal{X}_{G,S} \) is called the cluster \( \mathcal{X}_{G,S}\)-space. Moreover, each one of the moduli spaces \( \mathcal{X}_{G,S}, \mathcal{A}_{G,S} \) is equipped with a positive structure. When the set \( P \) is empty, the hole on the surface \( S \) should be regarded as the puncture, the positive part of \( \mathcal{X}_{PGL(2,\mathbb{R}),S} \) is related to the Teichmüller space of \( S \), and the positive part of \( \mathcal{A}_{SL(2,\mathbb{R}),S} \) is related to Penner’s decorated Teichmüller space [P87]. The fact that Penner’s decorated Teichmüller space is related to a cluster algebra was independently observed by M. Gekhtman, M. Shapiro, and A. Vainshtein [GSV05].

When \( D_k \) is a disc with \( k \) special points on the boundary, the generic cluster \( \mathcal{X}_{PGL(2,\mathbb{R}),D_k}\)-space corresponds to the generic configuration space \( Conf_{2,k} \) of \( k \) flags in \( \mathbb{R}^n \) up to projective transformations. Given a generic configuration of \( k \) flags \((m, y, z, t), n, x, \ldots \) in \( \mathbb{R}^n \), let \( P_k \) be the associated convex \( k \)-gon with \( k \) vertices \( m, y, z, t, n, x, \ldots \) as illustrated in Figure 7. The ideal triangulation of \( D_k \) corresponds to the triangulation of \( P_k \). Given a triangulation \( \mathcal{T} \) of the \( k \)-gon \( P_k \), for any pair of triangles \((\Delta_{xyz}, \Delta_{xzt})\) of \( \mathcal{T} \) where \( x, y, z, t \) are anticlockwise ordered, the Fock-Goncharov coordinate [FG06] corresponding to the inner edge \( xz \) is

\[
X_{xz} = -\frac{\Omega(\hat{y}^1 \land \hat{z}^1)}{\Omega(\hat{y}^1 \land \hat{z}^1)} \cdot \frac{\Omega(\hat{t}^1 \land \hat{x}^1)}{\Omega(\hat{y}^1 \land \hat{z}^1)}.
\]
By definition \( X_{xz} = X_{zx} \), so there are \( k-3 \) different coordinates.

**Definition 5.1** [FOCK-GONCHAROV ALGEBRA] Let \( A(T) \) be the fraction ring generated by \( k-3 \) Fock-Goncharov coordinates for the triangulation \( T \), the natural Fock-Goncharov Poisson bracket \( \{\cdot,\cdot\}_2 \) is defined on the fraction ring \( A(T) \) by extending the following map on the generators:

\[
\{X_{ab},X_{cd}\}_2 = \varepsilon_{ab,cd} \cdot X_{ab} \cdot X_{cd}
\]

for any inner edges \( ab, cd \), where the value of \( \varepsilon_{ab,cd} \) only depend on the anticlockwise orientation of \( P_k \) as illustrated in Figure 7. More precisely, \( \varepsilon_{ab,cd} = 1(\varepsilon_{ab,cd} = -1 \text{ resp.}) \) when \( a = c \) and \( \Delta_{abd} \) is a triangle of \( T \) such that \( a, b, d \) are ordered anticlockwise(clockwise resp.) in \( P_k \); otherwise \( \varepsilon_{ab,cd} = 0 \).

The Fock-Goncharov algebra of \( T \) is a pair \( (A(T),\{\cdot,\cdot\}_2) \).

**Definition 5.2** Let \( P \) be the vertices of the convex \( k \)-gon \( P_k \). We define an injective ring homomorphism \( \theta_T \) from \( A(T) \) to \( B_2(P) \), by extending the following map on the generators:

\[
\theta_T(X_{xz}) := -\frac{yz}{tz} \cdot \frac{tx}{yx}
\]

for any inner edge of \( T \).

**Theorem 5.3** The injective ring homomorphism \( \theta_T \) is Poisson with respect to the Poisson bracket \( \{\cdot,\cdot\}_2 \) and the swapping bracket \( \{\cdot,\cdot\} \).

**Proof.** By direct calculations, for any inner edge \( ab \) of the triangulation \( T \), we have

\[
\left\{ab, \frac{yz}{tz} \cdot \frac{tx}{yx}\right\} = \begin{cases} 
1 & \text{if } ab = zx; \\
-1 & \text{if } ab = xz; \\
0 & \text{otherwise.}
\end{cases}
\]

(29)

The theorem follows from the above equation and the Leibniz’s rule. \( \square \)

### 5.2 Cluster dynamic in rank 2 swapping algebra

The cluster \( \mathcal{X} \)-space is introduced by Fock and Goncharov [FG06] by using the same set-up as the cluster algebra [FZ02]. We consider the case for the cluster \( \mathcal{X}_{\text{PGL}(2,\mathbb{R}),D_4} \)-space.
Definition 5.4 [cluster $X_{PGL(2,R),D_k}$-space [FG04] [FG06]] Let $I_T$ be the set of $k - 3$ inner edges of the triangulation $T$ of $D_k$. The function $\varepsilon$ from $I_T \times I_T$ to $\mathbb{Z}$ is defined as in Definition 5.1, a seed is $I_T = (I_T, \varepsilon)$.

A mutation at the edge $e \in I_T$ changes the seed $I_T$ to a new one $I_T' = (I_T', \varepsilon')$, where the edge $e$ of the triangulation $T$ is changed into the edge $e'$ of $T$ by a flip illustrated in Figure 8. We identify $I_T$ with $I_T'$ by identifying $e$ with $e'$, where

$$\varepsilon'_{i,j} = \begin{cases} 
-\varepsilon_{i,j} & \text{if } e \in \{i, j\}; \\
\varepsilon_{i,j} + \varepsilon_{i,e} \max \{0, \varepsilon_{i,e} \varepsilon_{e,j} \} & \text{if } e \notin \{i, j\}.
\end{cases}$$

A cluster transformation is a composition of mutations and automorphisms of seeds.

We assign to the seed $I_T$ ($I_{T'}$, resp.) the split tori $\mathbb{T}_T$ ($\mathbb{T}_{T'}$, resp.) associated to the Fock-Goncharov coordinates $\{X_i\}_{i \in I_T}$ ($\{X'_i\}_{i \in I_{T'}}$, resp.). Then the transition function from $\mathbb{T}_T$ to $\mathbb{T}_{T'}$ is

$$\mu_e(X'_i) = g_e(X_i) = \begin{cases} 
X_i(1 + X_e)^{-\varepsilon_{i,e}} & \text{if } e \neq i, \varepsilon_{i,e} \leq 0; \\
X_i(1 + X_e^{-1})^{-\varepsilon_{i,e}} & \text{if } e \neq i, \varepsilon_{i,e} > 0; \\
x_i^{e-1} & \text{if } i = e.
\end{cases}$$

Any two triangulations are related by a composition of flips, therefore any two split tori are also related by a composition of the rational functions as above.

The cluster $X_{PGL(2,R),D_k}$-space is obtained by gluing all the possible algebraic tori $\mathbb{T}_T$ according to the transition functions described as above.

We show the cluster dynamic of $X_{PGL(2,R),D_k}$ in the rank 2 swapping algebra as follows.

Lemma 5.5 The triangulation $T'$ is the flip of $T$ at the edge $e$. Then

$$\theta_T \circ g_e(X_i) = \theta_{T'}(X'_i).$$

Proof. Let us consider the case where $e = xz$ and the triangulations $T, T'$ is illustrated in Figure 8. For $i = xz$, we have

$$\theta_{T'}(X'_{xz}) = -\frac{zt}{xt} \cdot \frac{xy}{zy}.$$
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\[ \theta_T \circ g_e(X_{xz}) = \theta_T \left( \frac{1}{X_{xz}} \right) = -\frac{tz}{yz} \cdot \frac{yx}{tx}. \]

By the rank 2 swapping algebra relations:

\[
\begin{align*}
yz \cdot tz \cdot ty + tz \cdot zy \cdot yt &= 0, \\
tx \cdot xy \cdot yt + yx \cdot xt \cdot ty &= 0,
\end{align*}
\]

we obtain

\[ \theta_T \circ g_e(X_{xz}) = \theta_T'(X_{xz}). \]

For \( i = xy \), we have

\[ \theta_T'(X_{xy}) = -\frac{my}{ty} \cdot \frac{tx}{mx}, \]

\[ \theta_T \circ g_e(X_{xy}) = \theta_T \left( X_{xy} \cdot (1 + X_{xz}^{-1}) \right) = -\frac{my}{ty} \cdot \frac{zx}{mx} \left( 1 - \frac{tz}{yz} \cdot \frac{yx}{tx} \right). \]

By the rank 2 swapping algebra relation:

\[
\begin{align*}
yz \cdot ty \cdot zx + yx \cdot tz \cdot zy - yz \cdot zy \cdot tx &= 0,
\end{align*}
\]

we obtain

\[ \theta_T \circ g_e(X_{xy}) = \theta_T'(X_{xy}). \]

We have same results for the other inner edges and the other cases different from the one illustrated in Figure 8, by the similar arguments. We therefore conclude that

\[ \theta_T \circ g_e(X_i) = \theta_T'(X_i). \]

\[ \Box \]

**Proposition 5.6** The homomorphism \( \mu_e \) preserves the Poisson bracket, so the Poisson bracket \( \{\cdot,\cdot\}_2 \) does not depend on the triangulation \( \mathcal{T} \).

**Proof.** We need to prove that

\[ \{\mu_e(X'_i),\mu_e(X'_j)\}_2 = \mu_e \left( \{X'_i,X'_j\}_2 \right), \]

which is equivalent to

\[ \{g_e(X_i),g_e(X_j)\}_2 = \epsilon'_{i,j} \cdot g_e(X_i) \cdot g_e(X_j). \]

Since \( \theta_T' \) is injective, we only need to prove that

\[ \theta_T \circ \{g_e(X_i),g_e(X_j)\}_2 = \theta_T \left( \epsilon'_{i,j} \cdot g_e(X_i) \cdot g_e(X_j) \right). \]

By Theorem 5.3 and Lemma 5.5, we have

\[
\begin{align*}
\theta_T \circ \{g_e(X_i),g_e(X_j)\}_2 &= \{\theta_T \circ g_e(X_i),\theta_T \circ g_e(X_j)\} \\
&= \{\theta_T'(X'_i),\theta_T'(X'_j)\}_2 \\
&= \epsilon'_{i,j} \cdot \theta_T'(X_i) \cdot \theta_T'(X_j) = \theta_T \left( \epsilon'_{i,j} \cdot g_e(X_i) \cdot g_e(X_j) \right). 
\end{align*}
\]

We therefore conclude that the homomorphism \( \mu_e \) preserves the Poisson bracket. \( \Box \)

**Remark 5.7** For \( n \) in general, the generalized injective ring homomorphism \( \theta_{T_n} \) is shown in [Su15], where the set \( \mathcal{P} \) has \( (n-1) \cdot k \) elements, each flag of \( \mathbb{RP}^{n-1} \) corresponds to \( n-1 \) points near each other on the boundary \( S^1 \). We expect to glue the rank \( n \) swapping algebras for the purpose of characterizing \( X_{\text{PGL}(n,\mathbb{R})} \) for the surface case.

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