ON DEFORMATIONS OF \( \mathbb{Q} \)-FANO THREEFOLDS II

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Abstract. We investigate some coboundary map associated to a 3-fold terminal singularity which is important in the study of deformations of singular 3-folds. We prove that this map vanishes only for quotient singularities and a \( A_{1,2}/4 \)-singularity, that is, a terminal singularity analytically isomorphic to a \( \mathbb{Z}_4 \)-quotient of the singularity \( (x^2 + y^2 + z^3 + u^2 = 0) \).

As an application, we prove that a \( \mathbb{Q} \)-Fano 3-fold with terminal singularities can be deformed to one with only quotient singularities and \( A_{1,2}/4 \)-singularities. We also treat the \( \mathbb{Q} \)-smoothability problem on \( \mathbb{Q} \)-Calabi–Yau 3-folds.

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1. INTRODUCTION

We consider algebraic varieties over the complex number field \( \mathbb{C} \).

This paper is a continuation of [12]. We study the \( \mathbb{Q} \)-smoothability of a \( \mathbb{Q} \)-Fano 3-fold \( X \) via certain coboundary maps of local cohomology groups associated to the singularities on \( X \).

1.1. \( \mathbb{Q} \)-smoothing of \( \mathbb{Q} \)-Fano 3-folds. In this paper, a \( \mathbb{Q} \)-Fano 3-fold means a projective 3-fold with only terminal singularities whose anticanonical divisor is ample. A \( \mathbb{Q} \)-Fano 3-fold is an important object in the classification theory of algebraic 3-folds. It is one of the end products of the Minimal Model Program. Toward the classification of \( \mathbb{Q} \)-Fano 3-folds, it is fundamental to study their deformations.

Locally, a 3-fold terminal singularity has a \( \mathbb{Q} \)-smoothing, that is, it can be deformed to a variety with only quotient singularities. In general, local deformations of singularities may not lift to a global deformation of a projective 3-fold as shown for Calabi–Yau 3-folds (cf. [8, Example 5.8]). Nevertheless, Altınok–Brown–Reid

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Tarō Sano ([1] 4.8.3]) conjectured that a $\mathbb{Q}$-Fano 3-fold has a $\mathbb{Q}$-smoothing. (See Example 3.2 for an example of a $\mathbb{Q}$-smoothing.) This conjecture aims to reduce the classification of $\mathbb{Q}$-Fano 3-folds to those with only quotient singularities. For example, there are several papers (cf. [2], [15]) on the classification of certain $\mathbb{Q}$-Fano 3-folds with only quotient singularities.

Previously, deformations of $\mathbb{Q}$-Fano 3-folds are treated in several papers (cf. [9], [6], [16], [12]). In [12, Theorem 1.5], the author proved that a $\mathbb{Q}$-Fano 3-fold with only “ordinary” terminal singularities has a $\mathbb{Q}$-smoothing. (See Definition 2.1 for the ordinariness of the singularity.) In this article, we treat the remaining case, that is, a $\mathbb{Q}$-Fano 3-fold with non-ordinary terminal singularities. We can deform the non-ordinary terminal singularities except one special singularity as follows.

Theorem 1.1. A $\mathbb{Q}$-Fano 3-fold can be deformed to one with only quotient singularities and $A_{1,2}/4$-singularities.

Here, an $A_{1,2}/4$-singularity means a singularity analytically isomorphic to
\[
0 \in (x^2 + y^2 + z^3 + u^2 = 0)/\mathbb{Z}_4 \subset \mathbb{C}^4/\mathbb{Z}_4(1,3,2,1),
\]
where $x, y, z, u$ are coordinates on $\mathbb{C}^4$ and $\mathbb{C}^4/\mathbb{Z}_4(1,3,2,1)$ is the quotient of $\mathbb{C}^4$ by an action of $\mathbb{Z}_4 = \langle \sigma \rangle$ as follows:
\[
\sigma \cdot (x, y, z, u) = (\sqrt{-1}x, -\sqrt{-1}y, -z, \sqrt{-1}u).
\]

Although we do not know how to deal with $A_{1,2}/4$-singularities, we believe that Theorem 1.1 is useful for the classification.

Remark 1.2. The author studied a deformation of a $\mathbb{Q}$-Fano 3-fold with its anticanonical element in [12] and [13]. In [13, Theorem 1.3], it is proved that, if a $\mathbb{Q}$-Fano 3-fold $X$ has a member $D \in |-K_X|$ with only isolated singularities, then $X$ has a $\mathbb{Q}$-smoothing. In the proof, it is necessary to use [12, Theorem 1.9] and Theorem 1.1 in this paper.

The existence of an elephant with mild singularities is discussed in [13, Section 4] by showing several examples of $\mathbb{Q}$-Fano 3-folds.

1.2. Methods of the proof. We use a method which is used in [12, Theorem 3.5]. Let $(U, p)$ be a germ of a 3-fold terminal singularity. The key tool of our method is the coboundary map $\phi_U$ associated to some local cohomology group on a birational modification $\tilde{U} \to U$. (See [2] for the definition of $\phi_U$.) If this map is nonzero, it is useful for finding a smoothing or a $\mathbb{Q}$-smoothing of a projective 3-fold. (cf. [10], [6], [12]) The following purely local statement is the main result of Section 2.

Theorem 1.3. Let $(U, p)$ be a germ of a 3-fold terminal singularity which is not a quotient singularity.

Then $\phi_U = 0$ if and only if $(U, p)$ is an $A_{1,2}/4$-singularity.

The map $\phi_U$ is known to be nonzero when $(U, p)$ is Gorenstein ([10, Theorem 1.1]) or $(U, p)$ is an ordinary singularity ([6], [12]). We calculate the coboundary map for a non-ordinary singularity.

Let us mention about the proof of Theorem 1.3. Since a terminal singularity $(U, p)$ of index $r$ is a $\mathbb{Z}_r$-quotient of a hypersurface singularity $(V, q)$, the set $T_{(U, p)}^{(1)}$ of first order deformations of $(U, p)$ is the $\mathbb{Z}_r$-invariant part of $T_{(V, q)}^{(1)}$. The set $T_{(V, q)}^{(1)}$ can be written as $O_{V, q}/J_{V, q}$ for the Jacobian ideal of $(V, q)$. We calculate the map $\phi_U$ by using this structure and the inequality [4] proved in [10].
By Theorem 1.4 (ii), the map \( \phi_U \) vanishes for a neighborhood \( U \) of an \( A_{1,2}/4 \)-singularity. It seems that we need a new method to treat a \( \mathbb{Q} \)-Fano 3-fold with \( A_{1,2}/4 \)-singularities. (See Remark 3.3)

1.3. \( \mathbb{Q} \)-smoothing of \( \mathbb{Q} \)-Calabi–Yau 3-folds. As another corollary of Theorem 1.3 we obtain a similar result for \( \mathbb{Q} \)-Calabi–Yau 3-folds. Here, a \( \mathbb{Q} \)-Calabi–Yau 3-fold is a normal projective 3-fold with only terminal singularities whose canonical divisor is a torsion class. Let \( r \) be the Gorenstein index of \( X \), that is, the minimal positive integer such that \( \mathcal{O}_X(rK_X) \cong \mathcal{O}_X \). The isomorphism \( \mathcal{O}_X(rK_X) \cong \mathcal{O}_X \) determines the global index one cover \( \pi : Y := \text{Spec} \oplus_{j=0}^{r-1} \mathcal{O}_X(jK_X) \to X \).

As a consequence of Theorem 1.3 and the proof of [6, Main Theorem 1], we obtain the following.

**Theorem 1.4.** Let \( X \) be a \( \mathbb{Q} \)-Calabi–Yau 3-fold. Assume that the global index one cover \( Y \to X \) is \( \mathbb{Q} \)-factorial.

Then a \( \mathbb{Q} \)-Calabi–Yau 3-fold \( X \) can be deformed to one with only quotient singularities and \( A_{1,2}/4 \)-singularities.

**Remark 1.5.** Namikawa studied another invariant for terminal singularities and \( \mathbb{Q} \)-smoothability of \( \mathbb{Q} \)-Calabi–Yau 3-folds in his unpublished note. The invariant is \( \mu(X,x) \) defined in [11 Section 2]. It seems that this invariant also vanishes for a \( A_{1,2}/4 \)-singularity \( (X,x) \). So we do not know the \( \mathbb{Q} \)-smoothability of a \( \mathbb{Q} \)-Calabi–Yau 3-fold with \( A_{1,2}/4 \)-singularities.

2. Calculation of coboundary maps

First, we introduce the coboundary map of local cohomology which is used in [12, 3.2] to find a \( \mathbb{Q} \)-smoothing of a \( \mathbb{Q} \)-Fano 3-fold. (See also [11 Section 1], [6, Section 4].)

Let \( (U,p) \) be a germ of a 3-fold terminal singularity. Let \( \pi_U : (V,q) \to (U,p) \) be the index one cover. By the classification ([7], [11]), we see that \( (V,q) \) is a hypersurface singularity and \( \pi_U \) is étale outside \( p \). Moreover, we have

\[
(V,q) \simeq ((f = 0), 0) \subset (\mathbb{C}^4, 0)
\]

for some \( f \in \mathbb{C}[x,y,z,u] \), where \( x, y, z, u \) are coordinate functions on \( \mathbb{C}^4 \) and \( f \) satisfies \( \sigma \cdot f = \zeta_U f \) for the generator \( \sigma \in G := \text{Gal}(V/U) \cong \mathbb{Z}_r \) and \( \zeta_U = \pm 1 \).

We define the ordinarity of a terminal singularity as follows.

**Definition 2.1.** Let \( (U,p) \) be a germ of a 3-fold terminal singularity. The germ \( (U,p) \) is called **ordinary** (resp. **non-ordinary**) if \( \zeta_U = 1 \) (resp. \( \zeta_U = -1 \)).

**Remark 2.2.** Let \( (U,p) \) be a germ of a non-ordinary terminal singularity. By the classification ([7], [11]), we have

\[
(U,p) \simeq ((x^2 + y^2 + g(z,u) = 0), 0)/\mathbb{Z}_4 \subset (\mathbb{C}^4/\mathbb{Z}_4, 0),
\]

where \( g(z,u) \in \mathbb{Z}_4 \)-semi-invariant polynomial in \( z, u \) and \( \sigma \in \mathbb{Z}_4 \) acts on \( \mathbb{C}^4 \) by \( \sigma \cdot (x,y,z,u) \mapsto (\sqrt{-1}x, -\sqrt{-1}y, -z, \sqrt{-1}u) \).

Let \( (U,p) \) be a germ of a 3-fold terminal singularity and \( V \) its index one cover with the \( \mathbb{Z}_r \)-action as above. Let \( \nu : \tilde{V} \to V \) be a \( \mathbb{Z}_r \)-equivariant resolution such that its exceptional divisor \( F \subset \tilde{V} \) has SNC support and \( \tilde{V} \setminus F \simeq V \setminus \{q\} \). Let \( V' := V \setminus \{q\} \) and

\[
\tau_V : H^1(V, \Omega^1_{V'}(-K_{V'})) \to H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F - \nu^*K_V))
\]
the coboundary map of the local cohomology. Note that the sheaf \( \mathcal{O}_V(-K_V) \) and \( \mathcal{O}_V \) are isomorphic as sheaves, but not isomorphic as \( \mathbb{Z}_r \)-equivariant sheaves. Let \( \tilde{\pi} : \tilde{V} \rightarrow \tilde{U} := V/\mathbb{Z}_r \) be the finite morphism induced by \( \pi \) and \( E \subset \tilde{U} \) the exceptional locus of the birational morphism \( \mu : \tilde{U} \rightarrow U \) induced by \( \nu \). Let \( U' := U \setminus \{p\} \) and \( \mathcal{F}_U^{(0)} \) the \( \mathbb{Z}_r \)-invariant part of \( \tilde{\pi}_*\Omega^2_{\tilde{V}}(\log F)(-F - \nu^*K_V) \). Then we have the coboundary map

\[
(2) \quad \phi_U : H^1(U', \Omega^2_{\tilde{U}}(-K_{U'})) \rightarrow H^2(\tilde{U}, \mathcal{F}_U^{(0)})
\]

which is the \( \mathbb{Z}_r \)-invariant part of \( \tau_V \). We shall study these coboundary maps \( \tau_V \) and \( \phi_U \) in this section.

For an ordinary terminal singularity, we can calculate the map \( \phi_U \) as follows.

**Theorem 2.3.** (cf. [12] Lemma 3.4) Let \((U, p)\) be a germ of a 3-fold ordinary terminal singularity which is not a quotient singularity. Then we have \( \phi_U \neq 0 \).

In the following, we prepare ingredients for calculating \( \phi_U \) for a germ \((U, p)\) of a non-ordinary terminal singularities.

We have \( H^2_F(V, \Omega^2_V(\log F)(-F)) = 0 \) by the proof of [13] Theorem 4. We also have \( H^2(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) = 0 \) by the Guillén–Navarro Aznar–Puerta–Steenbrink vanishing theorem. Thus we have an exact sequence

\[
(3) \quad 0 \rightarrow H^1(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F - \nu^*K_V)) \rightarrow H^1(V', \Omega^2_{\tilde{V}}(-K_V)) \rightarrow H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F - \nu^*K_V)) \rightarrow 0
\]

The following inequality proved in [10] is useful for the calculation of the coboundary maps.

**Proposition 2.4.** We have

\[
\dim \text{Ker} \tau_V \leq \dim \text{Im} \tau_V.
\]

**Proof.** This is proved in Remark after [10] Theorem (1.1). Let us recall the proof for the convenience of the reader.

By the exact sequence \( (29) \), it is enough to show that

\[
h^1(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) \leq h^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)).
\]

We have a surjection

\[
H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) \rightarrow H^2_F(V, \Omega^2_V(\log F))
\]

since we have \( H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F) \otimes \mathcal{O}_F) = \text{Gr}^2_F H^2_{\{\eta\}}(V, \mathbb{C}) = 0 \). By the local duality, we have

\[
H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F))^* \simeq H^1(\tilde{V}, \Omega^1_V(\log F)(-F)).
\]

Moreover we see that the differential homomorphism

\[
d : H^1(\tilde{V}, \Omega^1_V(\log F)(-F)) \rightarrow H^1(\tilde{V}, \Omega^2_V(\log F)(-F))
\]

is surjective by studying the spectral sequence

\[
H^q(\tilde{V}, \Omega^p_V(\log F)(-F)) \Rightarrow \oplus^{p+q}(\tilde{V}, \Omega^{p+q}_V(\log F)(-F)) = 0
\]

as in the proof of [10] Theorem (1.1). Thus we obtain relations

\[
(5) \quad h^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F)) \geq h^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)) = h^1(\tilde{V}, \Omega^1_V(\log F)(-F)) \geq h^1(\tilde{V}, \Omega^2_V(\log F)(-F))
\]
Let $T^1_{(V,q)}$, $T^1_{(U,p)}$ be the sets of first order deformations of the germs $(V,q)$ and $(U,p)$ respectively. Recall that we have an isomorphism $T^1_{(V,q)} \simeq \mathcal{O}_{V,q}/J_{V,q}$ of $\mathcal{O}_{V,q}$-modules for the Jacobian ideal $J_{V,q} \subset \mathcal{O}_{V,q}$. Hence we have a surjective $\mathcal{O}_{V,q}$-module homomorphism $\varepsilon : \mathcal{O}_{V,q} \to T^1_{(V,q)}$ which sends $h \in \mathcal{O}_{V,q}$ to the corresponding deformation $\varepsilon h \in T^1_{(V,q)}$. Also we have a commutative diagram

$$
\begin{array}{ccc}
T^1_{(U,p)} & \rightarrow & H^1(U', \Omega^2_{U'}(-K_{U'})) \\
\downarrow & & \downarrow \\
T^1_{(V,q)} & \rightarrow & H^1(V', \Omega^2_{V'}(-K_{V'}))
\end{array}
$$

where the horizontal isomorphisms are restrictions by open immersions and the upper terms inject into the lower terms as the $\mathbb{Z}_r$-invariant parts. Note that we have the horizontal isomorphisms since $\{p\} \mapsto U$ and $\{q\} \mapsto V$ have codimensions 3, and the spaces $U$ and $V$ are Cohen-Macaulay. Thus we identify $T^1_{(V,q)}$, $T^1_{(U,p)}$ and $H^1(V', \Omega^2_{V'}(-K_{V'}))$, $H^1(U', \Omega^2_{U'}(-K_{U'}))$ respectively via these isomorphisms.

We use the following notion of right equivalence ([4, Definition 2.9]).

**Definition 2.5.** Let $\mathbb{C}\{x_1, \ldots, x_n\}$ be the convergent power series ring of $n$ variables. Let $f, g \in \mathbb{C}\{x_1, \ldots, x_n\}$.

We say that $f$ is right equivalent to $g$ if there exists an automorphism $\varphi$ of $\mathbb{C}\{x_1, \ldots, x_n\}$ such that $\varphi(f) = g$. We write this as $f \overset{\sim}{\simeq} g$.

By using these ingredients, we calculate the coboundary map for a non-ordinary singularity. The following theorem and Theorem 2.3 imply Theorem 1.3.

**Theorem 2.6.** Let $(U,p)$ be a germ of a non-ordinary 3-fold terminal singularity which is not a quotient singularity.

(i) Assume that the index one cover $(V,q) \not\simeq ((x^2 + y^2 + z^3 + u^2 = 0), 0)$. Then we have $\phi_U \not\simeq 0$.

(ii) Assume that $(V,q) \simeq ((x^2 + y^2 + z^3 + u^2 = 0), 0)$. Then $\phi_U = 0$.

**Proof.** (i) Suppose that $\phi_U = 0$. We show the claim by contradiction. We can write $g(z,u) = \sum a_{i,j} z^i u^j \in \mathbb{C}[z,u]$ for some $a_{i,j} \in \mathbb{C}$ for $i,j \geq 0$. Since the generator $\sigma \in \mathbb{Z}_4$ acts on $g$ by $\sigma \cdot g = -g$ and on $z^i u^j$ by $\sigma \cdot z^i u^j = \sqrt{-1}^{2i+j} z^i u^j$, we see that $a_{i,j} \neq 0$ only if

$$
2i + j \equiv 2 \mod 4.
$$

(6)

Let $J_g := (\frac{\partial g}{\partial z}, \frac{\partial g}{\partial u}) \subset \mathbb{C}[z,u]$ be the Jacobian ideal of the polynomial $g$. Note that we have $T^1_{(V,q)} \simeq \mathbb{C}[z,u]/(g, J_g)$ since $\varepsilon_x = \varepsilon_y = 0 \in T^1_{(V,q)}$.

**Case 1** Assume that $a_{0,2} \neq 0$. We can write

$$
g(z,u) = u^2(1 + h_1(z,u)) + h_2(z)
$$

for some polynomials $h_1(z,u) \in (z,u) \subset \mathbb{C}[z,u]$ and $h_2(z) \in (z) \subset \mathbb{C}[z]$. Thus $g(z,u) \in \mathcal{O}_{\mathbb{C},0}$ is right equivalent to $u^2 + h_2(z)$. We see that $h_2(z) \in \mathcal{O}_{\mathbb{C},0}$ is right equivalent to $z^{2i_0+1}$ for some positive integer $i_0$ since $(g = 0)$ has an isolated singularity and by the condition (6). Thus we have

$$
(V,q) \simeq ((x^2 + y^2 + z^{2i_0+1} + u^2 = 0), 0),
$$

and this implies [4].
If \( i_0 = 1 \), it contradicts the assumption \((V, q) \not\cong ((x^2 + y^2 + z^3 + u^2 = 0), 0)\). Hence we have \( i_0 \geq 2 \). By calculating the partial derivatives of \( x^2 + y^2 + z^2 + u^2 \), we see that \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \in T_{(V, q)}^1 \) are linearly independent and
\[
\dim T_{(V, q)}^1 \geq 3.
\]

On the other hand, we see that \( \tau_V(\varepsilon_z) = 0 \) since we assumed \( \phi_U = 0 \) and \( \varepsilon_z \in T_{(U, p)}^1 \). By this and the fact that \( \tau_V \) is an \( \mathcal{O}_{V, q} \)-module homomorphism, we obtain a surjection \( \mathbb{C}[z, u]/(z, u) \to \text{Im } \tau_V \) since \( \varepsilon_u = 0 \). By this surjection and \( \mathbb{C}[z, u]/(z, u) \cong \mathbb{C} \), we obtain \( \dim \text{Im } \tau_V \leq 1 \). By this and the inequality (10), we obtain an inequality
\[
\dim T_{(V, q)}^1 = \dim \text{Im } \tau_V + \dim \text{Ker } \tau_V \leq 1 + 1 = 2
\]
and it is a contradiction.

(Case 2) Assume that \( a_{0, 2} = 0 \). Then we see that \( a_{i, j} \neq 0 \) only if \( 2i + j \geq 6 \) by (10). Note that a monomial \( z^i u^j \) with \( 2i + j \geq 6 \) is some multiple of either \( z^3, z^2 u^2, z u^4 \) or \( u^6 \). By computing partial derivatives of these monomials, we see that \( (g, J_g) \subset (z^2, z u^2, u^4) \). Thus we see that \( \varepsilon_1, \varepsilon_z, \varepsilon_{zu}, \varepsilon_u, \varepsilon_{u^2}, \varepsilon_{u^3} \in T_{(V, q)}^1 \) are linearly independent and we obtain
\[
\dim T_{(V, q)}^1 \geq 6.
\]

On the other hand, by the assumption \( \phi_U = 0 \), we have \( \tau_V(\varepsilon_z) = 0 \), \( \tau_V(\varepsilon_{u^2}) = 0 \) since \( \varepsilon_z, \varepsilon_{u^2} \in T_{(U, p)}^1 \). Thus we have a relation \( (z, u^2) \subset \text{Ker } \tau_V \circ \varepsilon \subset \mathcal{O}_{V, q} \) and obtain a surjection \( \mathbb{C}[z, u]/(z, u^2) \to \text{Im } \tau_V \). This implies an inequality \( \dim \text{Im } \tau_V \leq \dim \mathbb{C}[z, u]/(z, u^2) = 2 \). By this inequality and the inequality (10), we have an inequality
\[
\dim T_{(V, q)}^1 = \dim \text{Ker } \tau_V + \dim \text{Im } \tau_V \leq 2 + 2 = 4.
\]
This contradicts (10).

Hence we obtain \( \phi_U \neq 0 \) and finish the proof of (i).

(ii) For non-negative integers \( i, j \), we set
\[
b^{i,j} := \dim H^i(\tilde{V}, \Omega^j(\log F)(-F)),
\]
\[
l^{i,j} := \dim H^j(F, \Omega^i(\log F) \otimes \mathcal{O}_F).
\]
Let \( s_k(V, q) \) for \( k = 0, 1, 2, 3 \) be the Hodge number of the Milnor fiber of \((V, q)\) as in [14] Section 4. By [14] Theorem 6, we have \( s_0 = 0, s_1 = b^{1,1}, s_2 = b^{1,1} + l^{1,1} \) and \( s_3 = l^{0,2} \). We see that \( l^{0,2} = 0 \) by [14] Lemma 2. Since the sum \( \sum_{k=0}^3 s_k(V, q) \) is the Milnor number of \((V, q)\), we obtain \( 2b^{1,1} + l^{1,1} = 2 \). Since \( b^{1,1} \neq 0 \) by [10] Theorem 2.2, we obtain
\[
b^{1,1} = 1, \quad l^{1,1} = 0.
\]

There exists an exact sequence
\[
H^0(F, \Omega^1(\log F) \otimes \mathcal{O}_F) \to H^1(\tilde{V}, \Omega^1(\log F)(-F)) \to H^1(\tilde{V}, \Omega^1(\log F))
\]
\[
\quad \to H^1(F, \Omega^1(\log F) \otimes \mathcal{O}_F).
\]
Since \( l^{1,0} = 0 \) by [14] Lemma 1, the both outer terms are zero and the homomorphism in the middle is an isomorphism. By this and (8), we have
\[
\mathbb{C} \cong H^1(\tilde{V}, \Omega^1(\log F)) \cong H^2_F(\tilde{V}, \Omega^2(\log F)(-F))^*.
\]
Suppose that $\tau_V(\varepsilon_z) \neq 0$. Then $\varepsilon_z \not\in \ker \tau_V$. This implies that $\ker \tau_V = 0$ since $T_1^{1}(V,q) \simeq \mathbb{C}[z]/(z)$ as $\mathbb{C}[z]$-modules. Thus $\mathbb{C}^2 \simeq \text{Im} \tau_V \simeq H^2_F(\tilde{V}, \Omega^2_F(-\nu^*K_V))$. This contradicts (11).

Thus we obtain $\tau_V(\varepsilon_z) = 0$. Since $T_1^{1}(U,p) \simeq \mathbb{C}$ is generated by $\varepsilon_z$, we see that $\phi_U = 0$. Thus we finish the proof of (ii). \hfill \Box

Now we prepare another coboundary map to study $\mathbb{Q}$-smoothability of a $\mathbb{Q}$-Calabi–Yau 3-fold.

Let $(U, p)$ be a germ of a 3-fold terminal singularity and $V, \tilde{V}, F, \tilde{U}$ as before. We have the coboundary map

$$\bar{\tau}_V : H^1(V', \Omega^2_{V'}(-K_{V'})) \to H^2_F(\tilde{V}, \Omega^2_F(-\nu^*K_V))$$

and this fits in the commutative diagram

\begin{equation}
\begin{aligned}
H^1(V', \Omega^2_{V'}(-K_{V'})) & \xrightarrow{\bar{\tau}_V} H^2_F(\tilde{V}, \Omega^2_F(-\nu^*K_V)) \\
& \downarrow \tau_V \quad \uparrow \tau_{V'} \\
H^2_F(\tilde{V}, \Omega^2_F(-\nu^*K_V)) & \to H^2_F(\tilde{V}, \Omega^2_F(-F - \nu^*K_V)),
\end{aligned}
\end{equation}

where the injectivity of $\tau_{V'}$ is proved in the proof of [10, Theorem 1.1].

Let $\mathcal{F}^{(0)}_U := (\tilde{\sigma}_* \Omega^2_{\tilde{V}}(-\nu^*K_V))^{Z_\tau}$ be the $\mathbb{Z}_\tau$-invariant part. Let

$$\tilde{\phi}_U : H^1(U', \Omega^2_{U'}(-K_{U'})) \to H^2_F(\tilde{U}, \mathcal{F}^{(0)}_U)$$

be the coboundary map. It is the $\mathbb{Z}_\tau$-invariant part of $\bar{\tau}_V$. As the $\mathbb{Z}_\tau$-invariant part of the diagram (11), we obtain the following diagram:

\begin{equation}
\begin{aligned}
H^1(U', \Omega^2_{U'}(-K_{U'})) & \xrightarrow{\tilde{\phi}_U} H^2_F(\tilde{U}, \mathcal{F}^{(0)}_U) \\
& \downarrow \phi_U \quad \uparrow \phi_{U'} \\
H^2_F(\tilde{U}, \mathcal{F}^{(0)}_U).
\end{aligned}
\end{equation}

By these arguments, we obtain the following result as a corollary of Theorem 2.3 and Theorem 2.6

**Corollary 2.7.** Let $(U, p)$ be a germ of a 3-fold terminal singularity which is not a quotient singularity.

Then $\tilde{\phi}_U = 0$ if and only if the germ $(U, p)$ is an $A_{1,2}/4$-singularity.
be the restriction homomorphism by the open immersion $V' \hookrightarrow \tilde{V}$. We use this notation since there is a commutative diagram

\[ H^1(\tilde{V}, \Omega^2_{\tilde{V}}(-K_{\tilde{V}})) \xrightarrow{\nu^*} H^1(V', \Omega^2_{V'}(-K_{V'})) \cong T_V^1 \xrightarrow{\sim} H^1(V, \Omega^2_V(-K_V)) \]

where the lower horizontal homomorphism is the blow-down homomorphism of deformations (17). We can prove the relation

\[ \text{Im} \nu^* \subset \text{Ker} \tau_V = \text{Ker} \tilde{\tau}_V \]

by the same argument as in [12, Claim 3.7].

### 3. Application to $\mathbb{Q}$-smoothing problems

In [12, Theorem 3.2], we proved the following.

**Theorem 3.1.** Let $X$ be a $\mathbb{Q}$-Fano 3-fold.

Then there exists a deformation $\mathcal{X} \to \Delta^1$ of $X$ over a unit disc $\Delta^1$ such that the general fiber $\mathcal{X}_t$ for $t \in \Delta^1 \setminus \{0\}$ satisfies the following: For each singular point $p \in \mathcal{X}_t$ and its Stein neighborhood $U_p$, the coboundary map $\phi_{U_p}$ vanishes.

As an application of this result and Theorem 2.6, we obtain a proof of Theorem 1.1 as follows.

**Proof of Theorem 1.1.** By Theorem 3.1, we can deform a $\mathbb{Q}$-Fano 3-fold $X$ to one with only singularities $p_1, \ldots, p_l$ such that $\phi_{U_i} = 0$, where $U_i$ is a Stein neighborhood of $p_i$ for $i = 1, \ldots, l$. By Theorem 3.3, such a terminal singularity is either a quotient singularity or an $A_{1,2}/4$-singularity. Thus we finish the proof. \qed

**Example 3.2.** There exists an example of a $\mathbb{Q}$-Fano 3-fold with an $A_{1,2}/4$-singularity. This example has a $\mathbb{Q}$-smoothing.

Let $X := X_{10} \subset \mathbb{P}(1, 1, 2, 3, 4)$ be a weighted hypersurface of degree 10 defined by the polynomial

\[ f_{X_{10}} := w^2(x_1^2 + x_2^2) + w(y^3 + z^2) + x_1^{10} + x_2^{10} + y^5 + z^3 x_1, \]

where $x_1, x_2, y, z, w$ are coordinates of weights 1, 1, 2, 3, 4, respectively. By perturbing the coefficients of the polynomial, we obtain that

$\text{Sing} X = \{[0 : 0 : 0 : 1 : 0], [0 : 0 : 0 : 0 : 1]\}$,

$p_z := [0 : 0 : 0 : 1 : 0]$ is a $1/3(1, 1, 2)$-singularity and $p_w := [0 : 0 : 0 : 0 : 1]$ is an $A_{1,2}/4$-singularity. Let

\[ \mathcal{X} := (f_{X_{10}} + t \cdot yw^2 = 0) \subset \mathbb{P}(1, 1, 2, 3, 4) \times \mathbb{A}^1 \to \mathbb{A}^1 \]

be a deformation of $X$, where $t$ is a coordinate of $\mathbb{A}^1$. Then we see that $\mathcal{X}$ is a $\mathbb{Q}$-smoothing of $X$. The general fiber $\mathcal{X}_t$ has two $1/2(1, 1, 1)$-singularities, a $1/3(1, 1, 2)$-singularity and a $1/4(1, 3, 1)$-singularity.
Remark 3.3. We give a comment on a \( \mathbb{Q} \)-Fano 3-fold with \( A_{1,2}/4 \)-singularities.

Let \( X \) be a \( \mathbb{Q} \)-Fano 3-fold. The local-to-global spectral sequence of Ext groups induces an exact sequence

\[ \text{Ext}^1(\Omega_X^1, \mathcal{O}_X) \rightarrow H^0(X, \text{Ext}^1(\Omega_X^1, \mathcal{O}_X)) \rightarrow H^2(X, \Theta_X), \]

where \( \text{Ext}^1 \) is a sheaf of Ext groups. Recall that \( \text{Ext}^1(\Omega_X^1, \mathcal{O}_X) \) and \( H^0(X, \text{Ext}^1(\Omega_X^1, \mathcal{O}_X)) \) are the sets of first order deformations of \( X \) and the singularities on \( X \), respectively. Thus, if we have \( H^2(X, \Theta_X) = 0 \), we see that \( X \) is \( \mathbb{Q} \)-smoothable.

However, this approach does not work in general. Namikawa constructed an example of a Fano 3-fold \( X \) with \( A_{1,2} \)-singularities such that \( H^2(X, \Theta_X) \neq 0 \) ([9, Example 5]). Here an \( A \)-example of a Fano 3-fold. The local-to-global spectral sequence of Ext groups follows.

Thus we do not know \( \mathbb{Q} \)-smoothability of a \( \mathbb{Q} \)-Fano 3-fold with \( A_{1,2}/4 \)-singularities.

As another application of Theorem 2.6 we obtain a proof of Theorem 1.4 as follows.

Proof of Theorem 1.4. The proof is a modification of the proof of [9, Main Theorem 1]. We sketch the proof for the convenience of the reader.

First we prepare notations to define the diagram (13).

Let \( p_1, \ldots, p_l \in X \) be the non-quotient singularities and \( U_1, \ldots, U_l \) their Stein neighborhoods. Let \( \nu: \tilde{Y} \rightarrow Y \) be a \( \mathbb{Z}_r \)-equivariant resolution such that its exceptional divisor \( F \) is a SNC divisor and \( \tilde{\nu}^{-1}(\{p_1, \ldots, p_l\}) \). Let \( \tilde{\pi}: \tilde{Y} \rightarrow \tilde{X} := \tilde{Y}/\mathbb{Z}_r \) be the quotient morphism and \( \mu: \tilde{X} \rightarrow X \) the induced birational morphism with the exceptional divisor \( E \).

Let \( V_i := \pi^{-1}(U_i) \), \( \tilde{V}_i :=\tilde{\nu}^{-1}(V_i) \), \( F_i := F \cap \tilde{V}_i \) and \( \nu_i := \nu|_{\tilde{V}_i} : \tilde{V}_i \rightarrow V_i \) be the restrictions. Let \( \tilde{U}_i := \mu^{-1}(U_i) \), \( E_i := E \cap \tilde{U}_i \) and \( \tilde{\pi}_i := \tilde{\pi}|_{\tilde{V}_i} : \tilde{V}_i \rightarrow \tilde{U}_i \) the induced finite morphism. Let \( \mathcal{F}^{(0)} := (\tilde{\pi}_* \Omega_Y^2(-\nu^* K_Y))^\mathbb{Z}_r \) be the \( \mathbb{Z}_r \)-invariant part and \( \mathcal{F}_i^{(0)} := \mathcal{F}^{(0)}|_{\tilde{U}_i} \) its restriction.

Then we have the diagram

\[
\begin{align*}
H^1(X', \Omega_{X'}^2(-K_{X'})) \oplus_{\varphi_i} &\rightarrow \oplus_{i=1}^l H^2_E(\tilde{X}, \mathcal{F}^{(0)}) \oplus B_i \rightarrow H^2(\tilde{X}, \mathcal{F}^{(0)}) \\
\oplus_{i=1}^l H^1(U_i', \Omega_{U_i'}^2(-K_{U_i'})) \otimes \varphi_i &\rightarrow \oplus_{i=1}^l H^2_E(\tilde{U}_i, \mathcal{F}_i^{(0)}),
\end{align*}
\]

where \( X' := X \setminus \{p_1, \ldots, p_l\} \) and \( U_i' := U_i \cap X' \).

Let \( V_i' := \pi^{-1}(U_i') \). Note that \( B_i \circ \varphi_i^{-1} \circ \tilde{\phi}_i \) is the \( \mathbb{Z}_r \)-invariant part of the composition

\[
\begin{align*}
H^1(V_i', \Omega_{U_i'}^2(-K_{U_i'})) &\rightarrow H^2_E(\tilde{V}_i, \Omega_{\tilde{V}_i}^2(-\nu_i^* K_{V_i})) \rightarrow H^2(\tilde{Y}, \Omega_{\tilde{V}_i}^2(-\nu^* K_Y)) \rightarrow H^2(\tilde{Y}, \Omega_{\tilde{V}_i}^2(-\nu^* K_Y)).
\end{align*}
\]

We see that this is zero by [10, Proposition 1.2] since we assumed that \( Y \) is \( \mathbb{Q} \)-factorial. Thus we also see that \( B_i \circ \varphi_i^{-1} \circ \tilde{\phi}_i = 0 \).
There exists an element $\eta_i \in H^1(U_i', \Omega^2_{U_i'}(-K_{U_i'}))$ such that $\tilde{\phi}_i(\eta_i) \neq 0$ by Theorem 1.3. Since $B_i \circ \varphi_i^{-1} \circ \tilde{\phi}_i(\eta_i) = 0$, there exists $\eta \in H^1(X', \Omega^2_{X'}(-K_{X'}))$ such that $\psi(\eta) = \varphi_i^{-1}(\tilde{\phi}_i(\eta_i))$. By the relation (12) and $p_{U_i}(\eta) - \eta_i \in \text{Ker} \tilde{\phi}_i$, we see that $p_{U_i}(\eta) \notin \text{Im}(\nu_i)_*$, where we use the inclusion $H^1(U_i', \Omega^2_{U_i'}(-K_{U_i'})) \subset H^1(V_i', \Omega^2_{V_i'}(-K_{V_i'}))$. By arguing as in the proof of [12, Theorem 3.5], we can deform singularity $p_i \in U_i$ as long as $\phi_i \neq 0$. By Corollary 2.7, we obtain a required deformation since the deformations of a $\mathbb{Q}$-Calabi–Yau 3-fold are unobstructed ([8, Theorem A]).

\[ \square \]

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