Fractional parts of Dedekind sums

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Abstract

Using a recent improvement by Bettin and Chandee to a bound of Duke, Friedlander and Iwaniec (1997) on double exponential sums with Kloosterman fractions, we establish a uniformity of distribution result for the fractional parts of Dedekind sums $s(m,n)$ with m and n running over rather general sets. Our result extends earlier work of Myerson (1988) and Vardi (1987). Using different techniques, we also study the least denominator of the collection of Dedekind sums $\{s(m,n) : m \in (\mathbb{Z}/n\mathbb{Z})^*\}$ on average for $n \in [1,N]$.

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1 Introduction

For any integers $n \geqslant m \geqslant 1$ the *Dedekind sum* $s(m,n)$ is defined by

$$s(m,n) = \sum_{k \bmod n} \left(\!\!\left(\frac{km}{n}\right)\!\!\right)\left(\!\!\left(\frac{m}{n}\right)\!\!\right),$$

where $((t))$ denotes the distance from the real number t to the closest integer. Vardi [23, Theorem 1.2] observes that the bound

$$\sum_{n \leqslant x} \sum_{\substack{1 \leqslant m \leqslant n \\ \gcd(m,n)=1}} \mathbf{e}(12k\, s(m,n)) \ll x^{3/2+o(1)} \qquad (x \to \infty),$$

where

$$\mathbf{e}(t) = \exp(2\pi i t) \qquad (t \in \mathbb{R}),$$

is an easy consequence of the Weil bound on Kloosterman sums. This implies that for any fixed integer $k \neq 0$ the collection of fractional parts $\{12k\, s(m,n)\}$ with $n \in \mathbb{N}$ and $m \in (\mathbb{Z}/n\mathbb{Z})^*$ is uniformly distributed over the interval $[0,1)$; see [23, Theorem 1.3]. Vardi further shows (cf. [23, Theorem 1.6]) that the fractional parts $\{\rho\, s(m,n)\}$ are uniformly distributed over $[0,1)$ for every fixed real number $\rho \neq 0$.

Myerson [20, Theorem 3] extends the latter result by showing that for any fixed $\rho \neq 0$ the set of pairs $\big(m/n, \{\rho\, s(m,n)\}\big)$ with $n \in \mathbb{N}$ and $m \in (\mathbb{Z}/n\mathbb{Z})^*$ is uniformly distributed over the square $[0,1) \times [0,1)$. This can be naturally interpreted as a statement about the number of fractional parts $\{\rho\, s(m,n)\}$ with $1 \leqslant n \leqslant N$, $1 \leqslant m \leqslant L_n$ and $\gcd(m,n) = 1$ that fall into any given connected interval in $[0,1)$, where the numbers L_n are arbitrary integers for which the sequence $(L_n/n)_{n\in\mathbb{N}}$ has a positive limit.

In the present paper we give another extension of [23, Theorem 1.6]. More precisely, suppose we are given a real number $\rho \neq 0$, positive integers $M \leqslant N$, and two sequences of integers $\mathcal{K} = (K_n)$ and $\mathcal{L} = (L_n)$ for which

$$M \leqslant K_n < K_n + L_n \leqslant 2M \qquad (n \sim N), \tag{1}$$

where the notation $n \sim N$ is used here and elsewhere as an abbreviation for $N < n \leqslant 2N$. Furthermore, suppose that we are given two sets

$$\mathfrak{M} \subseteq (M, 2M] \qquad \text{and} \qquad \mathfrak{N} \subseteq (N, 2N]. \tag{2}$$

For a given choice of the data $\mathfrak{D} = (\rho, M, N, \mathcal{K}, \mathcal{L}, \mathfrak{M}, \mathfrak{N})$ as above, we use $\mathcal{A}_{\mathfrak{D}}(\lambda)$ to denote the number of pairs $(m, n) \in \mathfrak{M} \times \mathfrak{N}$ such that

$$K_n < m \leqslant K_n + L_n, \qquad \gcd(m, n) = 1, \tag{3}$$

and for which

$$\{\rho\, s(m, n)\} \in [0, \lambda].$$

We also denote by $\mathcal{N}_{\mathfrak{D}} = \mathcal{A}_{\mathfrak{D}}(1)$ the number of pairs $(m, n) \in \mathfrak{M} \times \mathfrak{N}$ satisfying only (3) (in particular, note that $\mathcal{N}_{\mathfrak{D}} = \mathcal{A}_{\mathfrak{D}}(1)$ regardless of the value of ρ). It is natural to expect that if the sets $\mathfrak{M}$ and $\mathfrak{N}$ are reasonably dense in the intervals $(M, 2M]$ and $(N, 2N]$, respectively, and do not have any local obstructions to the condition $\gcd(m, n) = 1$ (such as containing only even numbers), then one might expect that

$$\mathcal{N}_{\mathfrak{D}} = N^{o(1)} \sum_{n \in \mathfrak{N}} \# \left([K_n, K_n + L_n] \cap \mathfrak{M} \right). \tag{4}$$

For instance, using a version of the prime number theorem for short intervals (see [17, Section 10.5] and also [3]) the bound (4) holds if $\mathfrak{M}$ is the *set of prime numbers* and most of the interval lengths L_n are reasonably long relative to M. A similar result can also be obtained when $\mathfrak{M}$ is the *set of Q-smooth numbers* (in different ranges of the smoothness level Q depending on the sizes of the interval lengths L_n); necessary results can be found in the surveys [14, 16].

If $\mathcal{N}_{\mathfrak{D}}$ is sufficiently large (in particular if (4) holds) then it is reasonable to expect that $\mathcal{A}_{\mathfrak{D}}(\lambda) \approx \lambda\, \mathcal{N}_{\mathfrak{D}}$ in many situations. To make a quantitative statement, we denote by $\Delta_{\mathfrak{D}}$ the largest deviation of $\mathcal{A}_{\mathfrak{D}}(\lambda)$ from its expected value as λ varies over the interval $[0, 1]$; that is, we set

$$\Delta_{\mathfrak{D}} = \sup_{\lambda \in [0,1]} \left| \mathcal{A}_{\mathfrak{D}}(\lambda) - \lambda\, \mathcal{N}_{\mathfrak{D}} \right|.$$

Here, we demonstrate how recent results of Bettin and Chandee [4], which improve earlier estimates of Duke, Friedlander and Iwaniec [8], can be used to estimate $\Delta_{\mathfrak{D}}$ for a wide variety of the data $\mathfrak{D} = (\rho, M, N, \mathcal{K}, \mathcal{L}, \mathfrak{M}, \mathfrak{N})$. To illustrate the ideas, we focus on the simplest case in which $\rho = 12$; however, our approach works in much greater generality. The following result is proved below in §4.

Theorem 1. *Let $\mathfrak{D} = (12, M, N, \mathcal{K}, \mathcal{L}, \mathfrak{M}, \mathfrak{N})$ be a given tuple of data with positive integers $M \leqslant N$, two sequences of integers $\mathcal{K} = (K_n)$ and $\mathcal{L} = (L_n)$ satisfying (1), and two sets $\mathfrak{M}$ and $\mathfrak{N}$ satisfying (2). Then the bound*

$$\Delta_{\mathfrak{D}} \ll |\mathfrak{M} \times \mathfrak{N}|^{1/2} M^{3/10} N^{13/20+o(1)} + \mathfrak{N}_{\mathfrak{D}} M^{1/2} N^{-1/2}$$

holds as $M \to \infty$.

Using the bound $\mathfrak{N}_{\mathfrak{D}} \leqslant |\mathfrak{M} \times \mathfrak{N}|$ we simplify Theorem 1 as follows.

Corollary 2. *Under the conditions of Theorem 1 the bound*

$$\Delta_{\mathfrak{D}} \ll |\mathfrak{M} \times \mathfrak{N}|^{1/2} M^{3/10} N^{13/20+o(1)} + |\mathfrak{M} \times \mathfrak{N}| M^{1/2} N^{-1/2}$$

holds as $M \to \infty$.

Moreover, using $|\mathfrak{M} \times \mathfrak{N}| \leqslant MN$ we can simplify Theorem 1 further.

Corollary 3. *Under the conditions of Theorem 1 the bound*

$$\Delta_{\mathfrak{D}} \ll M^{4/5} N^{23/20} + M^{3/2} N^{1/2}$$

holds as $M \to \infty$.

In the case that $\mathfrak{N}_{\mathfrak{D}} = (MN)^{1+o(1)}$ one sees that Corollary 3 improves the trivial bound $\Delta_{\mathfrak{D}} \leqslant \mathfrak{N}_{\mathfrak{D}}$ provided that the inequalities $N^{3/4+\varepsilon} \leqslant M \leqslant N^{1-\varepsilon}$ hold with some fixed $\varepsilon > 0$ as $M \to \infty$.

In this paper we also study the distribution of the least denominator $q(n)$ of the Dedekind sums to modulus n. More precisely, expressing each Dedekind sum $s(m, n)$ as a reduced fraction $a(m, n)/q(m, n)$, let

$$q(n) = \min\{q(m, n) \ : \ m \in (\mathbb{Z}/n\mathbb{Z})^*\}.$$

In §5 we prove the following result.

Theorem 4. *We have*

$$\sum_{n=1}^{N} q(n) = (C + o(1)) \frac{N^2}{(\log N)^{1/2}} \qquad (N \to \infty),$$

where

$$C = \frac{3\sqrt{2}}{8\pi} \prod_{p \equiv 1 \bmod 4} (1 - p^{-2})^{-1} \prod_{p \equiv 3 \bmod 4} (1 - p^{-2})^{-1/2}.$$

For other recent results about the distribution and other properties of Dedekind sums, we refer the interested reader to [1, 2, 9, 10, 11, 12, 13, 18, 21].

2 Preliminaries

Throughout the paper, the implied constants in the symbols "O" and "$\ll$" are absolute. We recall that the expressions $A \ll B$ and $A = O(B)$ are each equivalent to the statement that $|A| \leqslant cB$ for some constant c.

We use $\|\cdot\|_1$ and $\|\cdot\|_2$ to denote the L^1 and L^2 norms, respectively, for finite sequences of complex numbers.

Given coprime integers $m, n \geqslant 1$ we use m_n^* to denote the unique integer defined by the conditions

$$mm_n^* \equiv 1 \bmod n \qquad \text{and} \qquad 1 \leqslant m_n^* \leqslant n.$$

As mentioned in §1, our investigation of the distribution of the fractional parts $\{\rho\, s(m,n)\}$ focuses on the special case $\rho = 12$; accordingly, following Girstmair [10] we denote

$$S(m,n) = 12\, s(m,n).$$

The next well known result is due to Hickerson [15].

Lemma 5. *For any coprime integers $m, n \geqslant 1$ we have*

$$S(m,n) - \frac{m + m_n^*}{n} \in \mathbb{Z}.$$

We also need the following estimate which is the special case $A = 1$ of the more general bound of Bettin and Chandee [4] on their sum $\mathcal{C}_1(M, N, A; \beta, \nu)$, which is obtained in the proof of their main result; see also [8, Theorem 6].

Lemma 6. *For any positive integer b and any complex numbers β_n, the sum*

$$\mathcal{C}(M, N; \beta, b) = \sum_{m \sim M} \Bigg| \sum_{\substack{n \sim N \\ \gcd(n,m)=1}} \beta_n \, \mathbf{e}\left(b \, \frac{m_n^*}{n}\right) \Bigg|^2$$

is bounded by

$$\mathcal{C}(M, N; \beta, b) \leqslant \|\beta\|_2^2 \left(\frac{b}{MN} + 1\right)^{1/2} (MN^{3/4} + N^{7/4} + M^{6/5}N^{7/10} + M^{3/5}N^{13/10})(MN)^{o(1)}$$

as $MN \to \infty$.

In the case $M \leqslant N$, which is relevant to our situation, Lemma 6 simplifies as follows.

Corollary 7. *For any positive integer b and any complex numbers β_n, for $M \leqslant N^{1+o(1)}$ the sum, in the notation of Lemma 6 we have*

$$\mathcal{C}(M,N;\beta,b) \leqslant \|\beta\|_2^2 \left(\frac{b}{MN}+1\right)^{1/2} (N^{7/4}+M^{3/5}N^{13/10})N^{o(1)}$$

as $N \to \infty$.

Given a sequence $\mathcal{G} = (\gamma_j)_{j=1}^J$ of real numbers in the interval $[0,1)$ we use $\Delta_{\mathcal{G}}$ to denote its *discrepancy*; this quantity is defined by

$$\Delta_{\mathcal{G}} = \sup_{\lambda\in[0,1]} \left| J_{\mathcal{G}}(\lambda) - \lambda J \right|,$$

where $J_{\mathcal{G}}(\lambda)$ denotes the cardinality of $\mathcal{G} \cap [0,\lambda]$ (in particular, $J_{\mathcal{G}}(1) = J$).

The celebrated *Erdős–Turán inequality* (see [7, 19]) provides a means for deriving distributional properties of a sequence from nontrivial bounds on corresponding exponential sums.

Lemma 8. *For any integer $H \geqslant 1$ the discrepancy $\Delta_{\mathcal{G}}$ of the real sequence $\mathcal{G} = (\gamma_j)_{j=1}^J$ satisfies the bound*

$$\Delta_{\mathcal{G}} \ll \frac{J}{H} + \sum_{h=1}^{H} \frac{1}{h} \left| \sum_{j=1}^{J} \mathbf{e}(h\gamma_j) \right|.$$

Let $v_p(\cdot)$ be the standard p-adic valuation; that is, for any integer $n \geqslant 1$, $v_p(n)$ is the largest integer e for which $p^e \mid n$. To study the distribution of the least denominator $q(n)$ of the Dedekind sums to modulus n, we make use of the following explicit formula of Girstmair [9, Corollary 1].

Lemma 9. *For any positive integer n we have*

$$q(n) = \begin{cases} q_0(n) & \text{if } n \text{ is odd,} \\ 2^{v_2(n)-1} q_0(n) & \text{if } n \text{ is even,} \end{cases}$$

where

$$q_0(n) = \prod_{\substack{p \mid n \\ p \equiv 3 \bmod 4}} p^{v_p(n)}.$$

To prove Theorem 4 we apply Lemma 9 in conjunction with the following classical theorem of Wirsing [24].

Lemma 10. *Assume that a real-valued multiplicative function $f(n)$ satisfies the following conditions:*

(i) $f(n) \geqslant 0$ *for all* $n \in \mathbb{N}$*;*

(ii) *for some constants* $a, b > 0$ *with* $b < 2$ *the inequality* $f(p^\alpha) \leqslant ab^\alpha$ *holds for all primes* p *and integers* $\alpha \geqslant 2$*;*

(iii) *there exists a constant* $\nu > 0$ *such that*

$$\sum_{p \leqslant N} f(p) = (\nu + o(1)) \frac{N}{\log N} \qquad (N \to \infty).$$

Then

$$\sum_{n \leqslant N} f(n) = \left(\frac{1}{e^{\gamma\nu}\Gamma(\nu)} + o(1) \right) \frac{N}{\log N} \prod_{p \leqslant N} \sum_{\alpha=0}^{\infty} \frac{f(p^\alpha)}{p^\alpha} \qquad (N \to \infty),$$

where γ *is the Euler-Mascheroni constant, and* $\Gamma(\cdot)$ *is the gamma function of Euler.*

3 Double sums with Kloosterman fractions

In what follows, we use the notation

$$\mathbf{e}_k(t) = \exp(2\pi i t/k) \qquad (k \in \mathbb{N},\ t \in \mathbb{R}).$$

In the next lemma, we establish a variant of [8, Theorem 2]; however, we use Corollary 7 to get a quantitatively stronger result.

Lemma 11. *Given arbitrary integers* a *and* $b \neq 0$*, positive integers* $M \leqslant N$*, two sequences of integers* $\mathcal{K} = (K_n)$ *and* $\mathcal{L} = (L_n)$ *satisfying* (1)*, and two sets* $\mathfrak{M}$ *and* $\mathfrak{N}$ *satisfying* (2)*, the sum*

$$\mathcal{S} = \sum_{\substack{(m,n) \in \mathfrak{M} \times \mathfrak{N} \\ K_n < m \leqslant K_n + L_n \\ \gcd(m,n)=1}} \mathbf{e}_n(am + bm_n^*)$$

satisfies the uniform bound

$$\mathcal{S} \ll |\mathfrak{M}|^{1/2}|\mathfrak{N}|^{1/2}\left(\frac{b}{MN}+1\right)^{1/4}(N^{7/8}+M^{3/10}N^{13/20})N^{o(1)}+|a|JMN^{-1}$$

provided that $|a|M \leqslant N$*, where* J *is the number of pairs* $(m,n) \in \mathfrak{M} \times \mathfrak{N}$ *that satisfy* (3)*.*

Proof. Since $|a|M \leqslant N$ by hypothesis, the estimate $\mathbf{e}_n(am) = 1+O(|a|M/N)$ holds uniformly for all terms in the sum under consideration, hence

$$\mathcal{S} = \mathcal{S}_0 + O\left(|a|JMN^{-1}\right), \tag{5}$$

where

$$\mathcal{S}_0 = \sum_{\substack{(m,n)\in\mathfrak{M}\times\mathfrak{N} \\ K_n<m\leqslant K_n+L_n \\ \gcd(m,n)=1}} \mathbf{e}_n(bm_n^*).$$

We can assume $b \geqslant 1$. Since $L_n \leqslant M$ for every $n \sim N$ by (1), taking $M_0 = (M-1)/2$ we have the following relation for all $m \sim M$:

$$M^{-1}\sum_{-M_0<c\leqslant M_0}\ \sum_{K_n<k\leqslant K_n+L_n} \mathbf{e}_M(c(k-m)) = \begin{cases} 1 & \text{if } K_n < m \leqslant K_n+L_n, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\mathcal{S}_0 = M^{-1}\sum_{-M_0<c\leqslant M_0} \mathcal{S}_0(c), \tag{6}$$

where

$$\mathcal{S}_0(c) = \sum_{\substack{(m,n)\in\mathfrak{M}\times\mathfrak{N} \\ \gcd(m,n)=1}} \alpha_m^{(c)}\beta_n^{(c)}\,\mathbf{e}_n(bm_n^*)$$

with

$$\alpha_m^{(c)} = \mathbf{e}_M(-cm) \qquad (m \in \mathfrak{M})$$

and

$$\beta_n^{(c)} = \sum_{K_n<k\leqslant K_n+L_n} \mathbf{e}_M(ck) \qquad (n \in \mathfrak{N}).$$

We also put

$$\beta_n^{(c)} = 0 \qquad (n \sim N,\ n \notin \mathfrak{N}).$$

Using Cauchy's inequality we see that

$$\left|\mathcal{S}(c)\right|^2 \leqslant |\mathfrak{M}| \sum_{m\sim M} \Bigg| \sum_{\substack{n\sim N\\ \gcd(m,n)=1}} \beta_n^{(c)}\, \mathbf{e}_n(bm_n^*)\Bigg|^2 = |\mathfrak{M}|\, \mathcal{C}(M,N;\beta^{(c)},b),$$

where $\mathcal{C}(M,N;\beta^{(c)},b)$ is defined as in Lemma 6. Applying the bound of Corollary 7 it follows that

$$\mathcal{S}_0(c) \ll |\mathfrak{M}|^{1/2}\|\beta^{(c)}\|_2 \left(\frac{b}{MN}+1\right)^{1/4} (N^{7/8}+M^{3/10}N^{13/20})N^{o(1)},$$

hence by (6) we have

$$\mathcal{S}_0 \ll |\mathfrak{M}|^{1/2} \left(\frac{b}{MN}+1\right)^{1/4} M^{-1}(N^{7/8}+M^{3/10}N^{13/20})N^{o(1)} \sum_{-M_0<c\leqslant M_0} \|\beta^{(c)}\|_2.$$

We now recall the well-known bound

$$\beta_n^{(c)} \ll \min\left\{L_n, \frac{M}{|c|}\right\}, \tag{7}$$

which holds for any integer c, with $0<|c|\leqslant M_0$; see [17, Bound (8.6)]. Since $L_n \leqslant M$, from (7) we immediately derive that

$$\|\beta^{(c)}\|_2 \ll \frac{M|\mathfrak{N}|^{1/2}}{|c|+1} \qquad (-M_0<c\leqslant M_0),$$

and thus

$$\mathcal{S}_0 \ll |\mathfrak{M}|^{1/2}|\mathfrak{N}|^{1/2} \left(\frac{b}{MN}+1\right)^{1/4} (N^{7/8}+M^{3/10}N^{13/20})N^{o(1)}.$$

Using this relation in (5) we complete the proof. □

4 Proof of Theorem 1

Let $\mathcal{J}$ be the set of pairs (m,n) in $\mathfrak{M}\times\mathfrak{N}$ that satisfy (3). Put $J=|\mathcal{J}|=\mathcal{N}_{\mathfrak{D}}$. Applying Lemma 8 to the sequence $\mathcal{G}$ of fractional parts $\{S(m,n)\}$ with

$(m,n) \in \mathcal{J}$, we see that the bound

$$\begin{aligned} \Delta_{\mathfrak{D}} &\ll \frac{J}{H} + \sum_{h=1}^{H} \frac{1}{h} \Big| \sum_{(m,n)\in\mathcal{J}} \mathbf{e}(h\,S(m,n)) \Big| \\ &= \frac{J}{H} + \sum_{h=1}^{H} \frac{1}{h} \Big| \sum_{(m,n)\in\mathcal{J}} \mathbf{e}_n(hm + hm_n^*) \Big| \end{aligned} \tag{8}$$

holds uniformly for any integer $H \in [1,N]$, where we have used Lemma 5 in the second step. Next, we apply Lemma 11 with $a = b = h$ to bound each inner sum on the right side of (8), and we sum over h; using the notation $K = |\mathfrak{M} \times \mathfrak{N}| = |\mathfrak{M}||\mathfrak{N}|$ we see that

$$\Delta_{\mathfrak{D}} \ll JH^{-1} + K^{1/2}(N^{7/8} + M^{3/10}N^{13/20})N^{o(1)} \sum_{h=1}^{H} \frac{1}{h} \left(\frac{b}{MN} + 1 \right)^{1/4} + HJMN^{-1}.$$

Clearly, we can assume that $H \leqslant MN$ as otherwise the last term $HJMN^{-1}$ exceeds the trivial bound $\Delta_{\mathfrak{D}} \leqslant J$. Under this condition the above bound simplifies as

$$\Delta_{\mathfrak{D}} \ll JH^{-1} + K^{1/2}(N^{7/8} + M^{3/10}N^{13/20})N^{o(1)} + HJMN^{-1}.$$

We now choose $H = \lfloor (N/M)^{1/2} \rfloor$ and note that since $N \geqslant M$ we have $H \asymp (N/M)^{1/2}$. Hence, with this choice we derive that

$$\Delta_{\mathfrak{D}} \ll JM^{1/2}N^{-1/2} + K^{1/2}(N^{7/8} + M^{3/10}N^{13/20})N^{o(1)}. \tag{9}$$

Since $K \leqslant MN$ we see that the bound (9) can be nontrivial only in the case that $N^{7/8} \leqslant K^{1/2} \leqslant (MN)^{1/2}$ and thus only for $M \geqslant N^{3/4}$. However, in this case we have $N^{7/8} \leqslant M^{3/10}N^{13/20}$, and (9) simplifies to

$$\Delta_{\mathfrak{D}} \ll JM^{1/2}N^{-1/2} + K^{1/2}M^{3/10}N^{13/20+o(1)}.$$

This concludes the proof.

5 Proof of Theorem 4

Let $f(n) = q(n)/n$ for all $n \in \mathbb{N}$. Note that $f(1) = 1$, and

$$f(n) = \delta_n \prod_{\substack{p \mid n \\ p \equiv 1 \bmod 4}} p^{-v_p(n)} \qquad (n > 1),$$

where

$$\delta_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 1/2 & \text{if } n \text{ is even.} \end{cases}$$

It is easy to see that $f(n)$ is a multiplicative function satisfying the conditions of Lemma 10 with $\nu = 1/2$. Since $\Gamma(1/2) = \sqrt{\pi}$ we have

$$\begin{aligned} \sum_{n \leqslant N} \frac{q(n)}{n} = \sum_{n \leqslant N} f(n) &= \left(\frac{1}{e^{\gamma/2}\sqrt{\pi}} + o(1) \right) \frac{N}{\log N} \prod_{p \leqslant N} \sum_{\alpha=0}^{\infty} \frac{f(p^\alpha)}{p^\alpha} \\ &= \left(\frac{1}{e^{\gamma/2}\sqrt{\pi}} + o(1) \right) \frac{N}{\log N}\, Q_2(N)\, Q_{4,1}(N)\, Q_{4,3}(N), \end{aligned}$$

where

$$\begin{aligned} Q_2(N) &= \frac{1}{4} + \sum_{\alpha=1}^{\infty} \frac{1}{2^{\alpha+1}} = \frac{3}{4}, \\ Q_{4,1}(N) &= \prod_{\substack{p \leqslant N \\ p \equiv 1 \bmod 4}} \sum_{\alpha=0}^{\infty} \frac{1}{p^{2\alpha}} = \prod_{\substack{p \leqslant N \\ p \equiv 1 \bmod 4}} \frac{1}{1 - p^{-2}}, \\ Q_{4,3}(N) &= \prod_{\substack{p \leqslant N \\ p \equiv 3 \bmod 4}} \sum_{\alpha=0}^{\infty} \frac{1}{p^{\alpha}} = \prod_{\substack{p \leqslant N \\ p \equiv 3 \bmod 4}} \frac{1}{1 - p^{-1}}. \end{aligned}$$

Clearly, the product for $Q_{4,1}(N)$ converges to

$$\varpi_{4,1} = \prod_{p \equiv 1 \bmod 4} \frac{1}{1 - p^{-2}}.$$

Furthermore, by a result of Uchiyama [22] we have

$$Q_{4,3}(N) = (\varpi_{4,3} + o(1))(\log N)^{1/2},$$

where

$$\varpi_{4,3} = \left(\frac{2e^{\gamma}}{\pi} \prod_{p \equiv 3 \bmod 4} \frac{1}{1 - p^{-2}} \right)^{1/2}.$$

Collecting the above results we deduce that

$$\sum_{n \leqslant N} \frac{q(n)}{n} = \left(\frac{3\,\varpi_{4,1}\varpi_{4,3}}{4\,e^{\gamma/2}\sqrt{\pi}} + o(1) \right) \frac{N}{(\log N)^{1/2}}.$$

The result now follows by partial summation.

6 Comments

We note that in the case that m runs through a sufficiently long interval of consecutive integers, that is, for pairs (m, n) with m running through *all* integers satisfying (3), the standard application of the Weil bound (see, for example, [17, Chapter 11]) leads to a stronger bound than that of Theorem 1. More specifically, this approach works when $L_n \geqslant n^{1/2+\varepsilon}$ for most values of n under consideration (with an arbitrary fixed $\varepsilon > 0$).

On the other hand, using recent bounds of very short Kloosterman sums due to Bourgain and Garaev [5, 6] one can obtain similar results in some cases in which $\mathfrak{M}$ is a fairly short interval of consecutive integers. For instance, this approach works when $K_n = 0$ and $L_n \geqslant n^{\varepsilon}$ for most values of n. This approach, however, yields only logarithmic savings over the trivial bound.

Of course, neither of the approaches just mentioned can be applied in the general setting considered in this paper, in which the sets $\mathfrak{M}$ and $\mathfrak{N}$ are essentially arbitrary.

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