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The category $\mathcal{MF}$ in the semistable case

The categories $\mathcal{MF}$ over discrete valuation rings were introduced by J. M. Fontaine as crystalline objects one might hope to associate with Galois representations. The definition was later extended to smooth base-schemes. Here we give a further extension to semistable schemes. As an application we show that certain Shimura varieties have semistable models.

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§ 1. Introduction

J. M. Fontaine pioneered a theory relating p-adic étale and crystalline cohomologies. For discrete valuation rings he defined a certain category $\mathcal{MF}$ and a fully faithful functor from it into Galois representations. The objects of $\mathcal{MF}$ are modules with filtration and Frobenius-action. Originally defined ([6]) for unramified p-adic valuation rings, the theory was extended to smooth extensions R of them in [4]. Here our goal is a further extension to semistable schemes. This involves additional complications because the base-ring does not admit any Frobenius-lift. Instead we have to pass to a divided power thickening R_{crys}. This makes the commutative algebra more difficult than in [4]. However for objects annihilated by p we can find a canonical lift to a regular ring R_{inf} whose relation to R_{crys} is similar to that between Fontaine's rings A_{inf} and A_{crys}. The ring R_{inf} is Noetherian and regular and thus more accessible to techniques from commutative algebra. Finally we should mention that working over R_{crys} has some similarity with the considerations in [1]. As a drawback our method does not generalise well to derived categories. As a consequence we cannot show that under log-smooth (maybe with some extra conditions) proper maps of semistable schemes, étale direct images are crystalline.

More precisely, assume that V_0 is a discrete complete valuation ring with field of fractions K of characteristic zero, uniformiser a prime p, and perfect residue field k of characteristic p. The base-ring R is p-adically complete, and formally étale over $V_0[x_0,\dots,x_d]/(\eta)$ where

$$\eta = x_0\cdot x_1\cdots x_d - p.$$

For most considerations we may assume that it is local. We can find a ring R_{inf} (local, p-adically complete) which is formally smooth over $V[\check{x}_0,\dots,\check{x}_d]$ with

$$R = R_{\mathrm{inf}}/(\check{x}_0\cdots\check{x}_d - p),$$

and denote by R_{crys} the p-adically complete divided power-hull of the ideal generated by

$$\eta = \check{x}_0 \cdots \check{x}_d - p \in R_{\mathrm{inf}}.$$

It is obtained by adjoining divided powers $\eta^n/n!$ and completing p-adically, and admits a Hodge-filtration by ideals $F^r(R_{\mathrm{crys}})$ which are generated by the divided powers of order $\geqslant r$.

R_{inf} admits various Frobenius lifts. For example we may send all the $\check{x}_i$ to their p-th powers. It induces an endomorphism ϕ on R_{crys}. Note that $\phi(\eta)$ is divisible by p and the quotient

$$\frac{\phi(\eta)}{p} = -1 + \frac{(\check{x}_0, \ldots, \check{x}_d)^p}{p}$$

is a unit. Thus if $F^i(R_{\mathrm{crys}})$ denotes the DP-filtration, then the restriction of ϕ to F^i is divisible by p^i for $i < p$.

Furthermore we denote by Ω_R the logarithmic (finite) module of differentials, which is generated over R by the dx_i/x_i with the single relation

$$\sum_{i=0}^{d} \frac{dx_i}{x_i} = 0.$$

Also Ω_{inf} denotes the corresponding module for R_{inf}. It is the free R_{inf}-module with basis $d\log(\check{x}_i)$, $1 \leqslant i \leqslant d$. We denote by ∂_i the dual basis of logarithmic derivations of R_{inf} or R_{crys} ($\partial_i = \check{x}_i\,\partial/\partial\check{x}_i$). They satisfy the identity (for $0 \leqslant i \leqslant d$ and $\phi(x_i) = x_i^p$)

$$\partial_i(\phi(r)) = p\phi(\partial_i(r)).$$

A logarithmic connection ∇ on an R_{inf}-module M is given by endomorphisms ∇_i satisfying the Leibniz rule. It is integrable if the ∇_i all commute.

§ 2. The category $\mathcal{MF}$ modulo p (without ∇)

We first treat objects annihilated by p, as many questions in the general case are reduced by devissage to this. We denote by $\overline{R}$ the ring

$$\overline{R} = R_{\mathrm{inf}}/(p, \eta^p) = R_{\mathrm{crys}}/(p, F^p).$$

Furthermore we fix an integer e, $0 \leqslant e \leqslant p-2$. Consider triples

$$\bigl(\overline{L}, \overline{M}, \Phi\bigr),$$

where $\overline{M} \subseteq \overline{L}$ are finitely generated $\overline{R}$-modules, $\eta^e\overline{L} \subseteq \overline{M}$, $\overline{L}$ is flat over $\overline{R}$, and

$$\Phi\colon \overline{M} \otimes_{\overline{R},\phi} \overline{R} \cong \overline{L}$$

is an isomorphism between $\overline{L}$ and the Frobenius-transform of $\overline{M}$. Such triples are usually abbreviated to $\underline{E}$.

We lift $\underline{E}$ to a triple $\underline{E}_\infty$ of $R_{\mathrm{inf}}/(p)$-modules $(L_\infty, M_\infty, \Phi_\infty)$.

Define inductively projective systems L_n, M_n by

$$M_{-1} = \overline{M}/(\eta), \quad M_0 = \overline{M}, \quad L_0 = \overline{L},$$
$$L_{n+1} = M_n \otimes_{R_{\mathrm{inf}},\phi} R_{\mathrm{inf}}/(p), \quad M_{n+1} = M_n \times^{L_n} L_{n+1}.$$

Then by induction

$$L_{n+1}/M_{n+1} = L_n/M_n, \quad L_{n+1}/(\eta^p) \cong L_n/(\eta^p), \quad M_{n+1}/(\eta) = M_n/(\eta).$$

The first equality is obvious, the second follows from the third (for n instead of $n+1$) by applying Frobenius, and for the third we need that the kernel of $M_{n+1} \to M_n$ (equal to the kernel of $L_{n+1} \to L_n$) lies in ηM_{n+1}, and this holds because it is contained in $\eta^p L_{n+1}$ and M_n contains $\eta^e L_n$. As η lies in the maximal ideal $\mathfrak{m}$ of $R_{\mathrm{inf}}/(p)$, this implies that all transition maps are surjective, and that the projective limits L_∞ and M_∞ are finitely generated over $R_{\mathrm{inf}}/(p)$. Also

$$M_\infty \otimes_\phi R_{\mathrm{inf}}/(p) \cong L_\infty$$

and M_∞ contains $\eta^e L_\infty$.

We want to show that L_∞ and M_∞ are projective $R_{\mathrm{inf}}/(p)$-modules. For this it suffices to show that $\mathrm{Ext}^i_{R_{\mathrm{inf}}/(p)}(M_\infty, R_{\mathrm{inf}}/(p))$ vanishes for $i > 0$, and similarly for L_∞. However the Ext-group for L_∞ is obtained from that for M_∞ by Frobenius pushout, that is, tensoring with $R_\infty/(p)$ considered as a module over itself via Frobenius (more precisely the dual module of $R_\infty/(p)$ occurs, but both are isomorphic).

Assume $\mathfrak{p} \neq (0)$ is a minimal prime in the support of $\mathrm{Ext}^i(M_\infty, R_\infty/(p))$. As M_∞ contains $\eta^e L_\infty$, the multiplication by η^e on $\mathrm{Ext}^i(L_\infty, R_\infty/(p))_\mathfrak{p}$ factors over $\mathrm{Ext}^i(M_\infty, R_\infty/(p))_\mathfrak{p}$. In $R_\infty/(p)$ the element η is the product of the $\check{x}_j$, and those contained in $\mathfrak{p}$ form part of a regular system of parameters of $\mathfrak{p}$. It follows that for any $R_\infty/(p)_\mathfrak{p}$-module N of finite length the image of multiplication by η^e on $N \otimes_\phi R_\infty/(p)$ is at least equal to the length of N multiplied by $(p-e)^{\mathrm{height}(\mathfrak{p})}$.

It suffices to check this for the residue-field at $\mathfrak{p}$. As the argument will be used several times we give some details. Namely, assume $R_{\mathrm{inf},\mathfrak{p}}$ has regular parameters $z_1, \ldots, z_s$. Then the Frobenius pushout of the residue-field in $\mathfrak{p}$ is

$$R_{\mathrm{inf},\mathfrak{p}}/(z_1^p, \ldots, z_s^p).$$

The image of multiplication by $(z_1 \cdots z_s)^e$ on this module has length $(p-e)^s$ (multiply monomials with all exponents $< p-e$ by $(z_1 \cdots z_s)^e$).

Applied to N equal to the localisation of the Ext-group we get the desired result (namely, $l(N) \geqslant (p-e)^s l(N)$, thus $N = (0)$). As a corollary we obtain that the original $\overline{L}$ is a free $R_{\mathrm{inf}}/(p, \eta^p)$-module, and $\overline{M}/(\eta)$ is projective over $R_{\mathrm{inf}}/(p, \eta)$. As the quotients L_n/M_n are all isomorphic it follows that $\overline{L}/\overline{M}$ admits a projective resolution of length one over R_{inf}. Equivalently its depth is equal to

$$d = \dim(R_{\mathrm{inf}}/(p, \eta^p)).$$

By abuse of notation we call elements of $\overline{M}$ a "basis" if they induce a basis of $\overline{M}/(\eta)$.

Remark. Instead of the Ext-group we could use local cohomology $H^i_{\mathfrak{m}}$, that is, its dual under Matlis duality.

The same arguments apply to maps. If

$$\overline{L} \to \overline{L'}, \quad \overline{M} \to \overline{M'}$$

are maps compatible with all structures $(\overline{L}, \overline{M}, \Phi)$, these maps extend to L_∞ and M_∞, and by the Ext-arguments the cokernels are free over $R_{\mathrm{inf}}/(p)$, and thus also the kernels. So the same holds for the original maps

$$\overline{L} \to \overline{L'}, \quad \overline{M}/(\eta) \to \overline{M'}/(\eta).$$

Thus our category of triples $(\overline{L}, \overline{M}, \Phi)$ is an abelian category and the functors $\overline{L}$, $\overline{M}$, $\overline{L}/\overline{M}$ are exact.

Before we introduce connections we first change the base-ring from $R_{\mathrm{inf}}/(p)$ to $R_{\mathrm{crys}}/(p)$. For this we replace L_∞ by its tensor product with $R_{\mathrm{crys}}/(p)$ and M_∞ by the R_{crys}-submodule generated by M_∞ and $F^p(R_{\mathrm{crys}})L_\infty$. In other words we consider $L_{\mathrm{crys}} = \overline{L} \otimes R_{\mathrm{crys}}/(p)$ and its submodule $M_{\mathrm{crys}} \subseteq L_{\mathrm{crys}}$ with

$$L_{\mathrm{crys}}/M_{\mathrm{crys}} = \overline{L}/\overline{M}.$$

In all this we use the inclusion

$$R_{\mathrm{inf}}/(p, \eta^p) \subset R_{\mathrm{crys}}/(p).$$

The "crys-objects" have the property that L_{crys} is a projective $R_{\mathrm{crys}}/(p)$-module, and

$$L_{\mathrm{crys}} = M_{\mathrm{crys}}/(\eta) \otimes_\phi R_{\mathrm{crys}}/(p).$$

We recover the original objects over $\overline{R}$ by

$$\overline{L} = L_{\mathrm{crys}}/\big(F^p(R_{\mathrm{crys}})L_{\mathrm{crys}}\big), \quad \overline{M} = M_{\mathrm{crys}}/\big(F^p(R_{\mathrm{crys}})L_{\mathrm{crys}}\big),$$

and they also form an abelian category such that L_{crys}, M_{crys}, $L_{\mathrm{crys}}/M_{\mathrm{crys}}$ are exact functors. In addition, L_{crys} is free over $R_{\mathrm{crys}}/(p)$, and the quotient $L_{\mathrm{crys}}/M_{\mathrm{crys}}$ has depth d. Naturally such objects are denoted by $\underline{E}_{\mathrm{crys}}$. Thus our original data are equivalent to giving objects over R_{crys} satisfying the conditions above. In short we obtain various equivalent categories over $\overline{R}$, $R_{\mathrm{inf}}/(p)$, or $R_{\mathrm{crys}}/(p)$. They will be the objects annihilated by p in a category $\mathcal{MF}(R)$ except that we still need to introduce connections.

§ 3. Connections, descent data

Now we can define connections. On $\overline{L}$ we assume given a logarithmic integrable connection, that is, commuting operators ∇_i satisfying the Leibniz-rule. Concerning $\overline{M}$ we require that this submodule is stable under the operators $\eta\nabla_i$, and that the Frobenius Φ is parallel in the sense that (for $m \in \overline{M}$)

$$\Phi(\eta\nabla_i(m)) = \frac{\phi(\eta)}{p} \cdot \nabla_i(\Phi(m)).$$

Note that $\phi(\eta)/p \equiv -1$ modulo $(p, F^p(R_{\text{crys}}))$. Also the inclusion $\overline{M} \subseteq \overline{L}$ should commute with $\eta\nabla_i$. These induce unique connections on L_∞ and (after multiplication by η) on M_∞, and thus also connections on L_{crys} and M_{crys}. As usual this is independent of the logarithmic Frobenius-lift ϕ, and allows us to define a functor $\mathbb{D}$ from $\mathcal{MF}$ to Galois representations.

Denote by S the integral closure of R in the maximal étale extension of $R[1/p]$. The Frobenius is surjective on $S/(p)$ and the projective limit

$$\mathcal{R} = \lim.\operatorname{proj.}\big(S/(p)\big)$$

(transition maps Frobenius) is a perfect ring. It consists of sequences $(x_n \mid n \geqslant 0)$ with

$$x_n = x_{n+1}^p,$$

where the x_n are either elements of $S/(p)$ or of the p-adic completion $\widehat{S}$. Prominent elements are $\underline{1}$ (with $x_0 = 1$, x_n a primitive p^n-th root of unity) and $\underline{p}$ (with x_n a p^n-th root of p). The ring

$$A_{\inf}(R) = W(\mathcal{R})$$

(Witt-vectors) admits a surjective homomorphism

$$\theta\colon A_{\inf}(R) \to \widehat{S}$$

with kernel generated by one element, for example by $[\underline{p}] - p$. The surjection

$$R_{\inf} \to R \subset \widehat{S}$$

lifts to $R_{\inf} \to A_{\inf}(R)$, for example by mapping the variables $\check{x}_i$ to elements $[\overline{x}_i]$ made up from p-power roots of the x_i. This also maps η to the generator of $\ker(\theta)$ exhibited before. In the following we only consider lifts $R_\infty \to A_\infty$ which differ from the above by multiplying $\overline{x}_i$ by units (logarithmic lifts).

As usual define $A_{\text{crys}}(R)$ as the p-adically completed divided power-hull of

$$\ker(\theta) \subset A_{\inf}(R).$$

Lift

$$R_{\text{crys}} \to R \subset \widehat{S}$$

somehow to a homomorphism into $A_{\text{crys}}(R)$. For example use the lift above which also commutes with Frobenius. Then

Definition 1. For an object

$$\underline{E}_{\text{crys}} \in \mathcal{MF}$$

annihilated by p define

$$\mathbb{D}(\underline{E}_{\text{crys}}) = \operatorname{Hom}\big(\underline{E}_{\text{crys}}, A_{\text{crys}}(R)/(p)\big).$$

Here homomorphisms are R_{crys}-linear maps $L_{\mathrm{crys}} \to A_{\mathrm{crys}}(R)/(p)$ (or $A_{\mathrm{crys}}(R)$-linear maps from $L_{\mathrm{crys}} \otimes_{R_{\mathrm{crys}}} A_{\mathrm{crys}}(R)$ into $A_{\mathrm{crys}}(R)/(p)$) which map M_{crys} into the e-th stage of the divided power filtration $F^e(A_{\mathrm{crys}}(R)/(p))$, respecting Frobenius, which means that for $m \in M_{\mathrm{crys}}$ the image of the element $\Phi(m) \in L_{\mathrm{crys}}$ is equal to the divided power "ϕ/p^e" applied to the image of m. As usual the connection ∇ makes this independent of choices (map $R_{\mathrm{crys}} \to A_{\mathrm{crys}}(R)$ and Frobenius-lift on R_{crys}). By transport of structure the Galois group of S/R, equal to the étale fundamental group of $\mathrm{Spec}(R[1/p])$, acts on this.

The description of these homomorphisms can be simplified. Firstly it suffices to consider Frobenius-linear maps into $A_{\mathrm{crys}}(R)/(p)$ modulo F^p, that is, into

$$A_{\mathrm{crys}}/((p), F^p) \cong \mathcal{R}/(\eta^p),$$

which lift uniquely by Frobenius invariance. Furthermore

$$M_{\mathrm{crys}}/F^p(R_{\mathrm{crys}})L_{\mathrm{crys}}$$

is generated by h (the rank of L_{crys}) elements m_μ (a "basis", images of generators of M_∞) and the $\Phi(m_\mu)$ form a basis for L_{inf}. Also, multiplication by η^e is given in these bases by a matrix $b_{\mu,\nu}$ dividing η^e, with coefficients in $R_{\mathrm{crys}}/(p)$ such that modulo $F^p(R_{\mathrm{crys}})L_{\mathrm{crys}}$,

$$\eta^e(\Phi(m_\mu)) = \sum_\nu b_{\mu,\nu} m_\nu.$$

Now if a map sends the m_μ to elements $\eta^e x_\mu \in \eta^e\mathcal{R}/(\eta^p)$, each $\Phi(m_\mu)$ goes to $(-1)^e x_\mu^p$. As

$$\eta^e(\Phi(m_\mu)) \in M_{\mathrm{inf}}$$

maps to the image of $\Phi(m_\mu)$ multiplied by η^e, it follows that the x_μ satisfy the equation

$$x_\mu^p = (-1)^e \sum_\nu b_{\mu,\nu} x_\nu.$$

Conversely, such solutions define maps. The x_μ determine where to send the m_μ, which determine the images of $\Phi(m_\mu)$ (by divided Frobenius), and the corresponding maps on M_{inf} and L_{inf} are compatible with multiplication by η^e. We can multiply the equations for the x_μ by the matrix $a_{\lambda,\mu}$ describing the inclusion $\overline{M} \subseteq \overline{L}$ to obtain

$$(-1)^e \sum_\mu a_{\lambda,\mu} x_\mu^p = \eta^e x_\lambda,$$

that is, our map is compatible with the inclusion $M_{\mathrm{crys}} \subset L_{\mathrm{crys}}$.

Now solutions in $\mathcal{R}/(\eta^p) = \mathcal{R}/(\underline{p}^p)$ of the equations above correspond via p-th roots to solutions (where we replace the $b_{\mu,\nu}$ by their p-th roots) in

$$\mathcal{R}/(\underline{p}) = S/(p),$$

and these lift uniquely to S or $\widehat{S}$ (if we first lift somehow the $a_{\mu,\nu}$ and $b_{\mu,\nu}$). But over S the equations define a finite flat algebra of rank p^h which becomes étale if we invert p. Thus we have precisely p^h solutions. It follows that the functor $\mathbb{D}$ is exact and faithful. In fact we have as in [4, Theorem 2.6]:

THEOREM 2. *The functor* $\mathbb{D}$ *is* (*on objects annihilated by* p) *exact and fully faithful. The essential image is stable under subobjects and quotients.*

PROOF. We need some general discussion. We go back to the original modules

$$\overline{L} = L_{\mathrm{crys}}/F^p(R_{\mathrm{crys}}/(p))L_{\mathrm{crys}}, \quad \overline{M} = M_{\mathrm{crys}}/F^p(R_{\mathrm{crys}}/(p))L_{\mathrm{crys}}.$$

The connections $\eta\nabla_i$ are nilpotent on $\overline{M}/(\eta)$ $((\eta\nabla_i)^{e+1} = 0)$ and thus ∇_i^{e+1} vanishes on the image

$$\Phi\big(\overline{M}/(\eta)\big) \subset \overline{L}.$$

Thus $\overline{M}/(\eta)$ defines a descent of the $R_{\mathrm{crys}}/(p, F^p)$-module $\overline{L}$ with respect to the flat Frobenius

$$\phi\colon R_{\mathrm{crys}}/(p, F^1) \to R_{\mathrm{crys}}/(p, F^p).$$

Now we show that the functor $\mathbb{D}$ is fully faithful. It turns out that the method of [4] also works here. Firstly we consider the case when R is a discrete valuation ring, with residue-field k, that is, we localise at one of the minimal primes containing some x_i. By reordering we may assume that $i = 0$, and the other x_i become units in R. At the beginning we adjoin all p-power roots of the remaining x_j, making the residue-field k perfect, and finally make it algebraically closed by passing to an étale cover. Thus now

$$R = V_0, \quad R_\infty = V_0[[\eta]], \quad \overline{R} = k[\eta]/(\eta^p).$$

Also a lift $R_{\mathrm{inf}} \to A_{\mathrm{crys}}(R)/(p, F^p)$ may be chosen invariant under the Galois group $\mathrm{Gal}(\overline{R}/R[p^{1/p}])$.

As in [4], we first determine the simple objects in $\mathcal{MF}$. In [4] this is done citing [6], but it is known that this can be replaced by simpler arguments because we restrict the range of the filtration. The simple objects will be indexed by rational numbers β, $0 \leqslant \beta < 1$, with denominator prime to p. If h is minimal with $(p^h - 1)\beta$ integral, write

$$(p^h - 1)\beta = j_0 + j_1 p + \cdots + j_{h-1}p^{h-1}$$

with integers $0 \leqslant j_i < p$ (this is the "decimal expansion" with the number 10 replaced by p). We furthermore assume that all digits $j_i \leqslant e \leqslant p - 2$. Finally we define new numbers β_i as the fractional part of $p^i\beta$. The digits in the p-expansion of β_i are a cyclic permutation of the j_i. Then the corresponding simple object $\underline{E}(\beta)$ has as $\underline{L}$ the free $\overline{R}$-module with basis $e(\beta_i)$, $0 \leqslant i < h$. The submodule $\overline{M}$ is generated by the elements $\eta^{e-j_{h-i}}e(\beta_i)$, whose Φ-image is $(-1)^{j_{h-i}}e(\beta_{i+1})$. The connection ∇_0 annihilates the $e(\beta_i)$. The image of the functor $\mathbb{D}(\underline{E}(\beta))$ consists of maps which map $e(\beta_i)$ to $\mathbb{F}_{q^h}$-multiples of $\underline{p}^{\beta_i}$ (all coefficients conjugate). It is the tame Galois representation indexed by β.

Now assume that $\underline{E} = (\underline{L}, \underline{M}, \Phi, \nabla_0)$ is simple, say of rank h. Then $\mathbb{D}(\underline{E})$ is a Galois-module of order p^h, and can be identified with solutions to certain equations

$$x_\mu^p = (-1)^e \sum_\nu b_{\mu,\nu} x_\nu.$$

The equations hold in $S/(p)$, the coefficients $b_{\mu,\nu}$ lie in $R[p^{1/p}]$, and the solutions lift uniquely to S. $\mathbb{D}(\underline{E})$ contains a non-trivial element λ on which the wild inertia operates trivially. The Galois-action involves the connection but if we restrict to the absolute Galois group of $R[p^{1/p}]$ this modification disappears. Thus the solutions x_μ of the equations above are invariant under the wild inertia over $R[p^{1/p}]$ and so are their lifts. But then the lifts must be linear combinations of fractional powers $p^{\alpha/p}$, where the denominator of α is prime to p. Thus also our original map λ has image contained in the subspace of $S/(p)$ spanned by these elements. If we apply Frobenius to such elements they are fixed under wild inertia, and the tame inertia acts by characters with multiplicities one. Thus $\lambda\big(\Phi(\,\overline{M}\,)\big)$ lies in the space spanned by the p^α. As it is Galois-invariant it is the direct sum of certain kp^α, and $\lambda(\,\overline{L}\,)$ has k-basis $p^{\alpha+i/p}$, with i such that the exponent is less than 1. Then the quotient

$$\frac{\lambda(\,\overline{L}\,)\cap(p^{e/p})}{\lambda(\,\overline{L}\,)\cap(p^{(e+1)/p})}$$

has as k-basis elements $p^{i/p+\alpha}$, where the exponent lies between e/p and $(e+1)/p$. As the divided Frobenius ϕ/p^e maps this surjectively onto $\lambda\big(\Phi(\,\overline{M}\,)\big)$, the number of basis-elements must be at least that for $\lambda\big(\Phi(\,\overline{M}\,)\big)$, that is, all $\alpha < 1-1/p$, and Frobenius replaces the element indexed by α by that by the fractional part of $p\alpha$. This implies that all p-digits in the α's are $\leqslant p-2$, and the set of α's is the union of the exponents occurring in a finite collection of $\underline{E}(\beta)$'s. We then define a map from $(\underline{L},\underline{M},\Phi)$ to the sum of the $\underline{E}(\beta)$ by mapping $\Phi(\underline{M})$ to the basis-elements $e(\beta_i)$ as to the linear combinations of the p^{β_i}. It remains to show that at least one of the maps respects connections ∇_0. It is then an isomorphism (by simplicity).

The Galois-operation of wild inertia on elements of $\lambda\big(\Phi(\,\overline{M}\,)\big)$ is trivial, but also determined by the connection. If an element σ of wild inertia maps $p^{1/p}$ to $\zeta p^{1/p}$, with ζ a p-th root of unity, the action of σ on the image of an element $z=\Phi(m)\in\Phi(\,\overline{M}\,)\subset\overline{L}$ is given by the λ-image of

$$\sum_{n=0}^{p-2}\frac{\nabla_0(\nabla_0-1)\cdots(\nabla_0-n+1)(z)(\zeta-1)^n}{n!}.$$

Here we use the fact that $(\eta\nabla_0)^{p-1}(\,\overline{L}\,)\subseteq\eta\overline{M}$, so ∇_0^{p-1} vanishes on $\Phi(\,\overline{M}\,)$. If we replace σ by σ^l we may replace ∇_0 by $l\nabla_0$, so this is a polynomial in l of degree $\leqslant p-1$. If this is constant for $0\leqslant l<p$, the polynomial itself is constant, and λ sends $\nabla_0(z)$ into elements annihilated by $\zeta-1$, or annihilated by $p^{1/(p-1)}$, or elements divisible by $p^{1-1/(p-1)}$.

The λ-image of $\nabla_0(z)$ lies in $\lambda\big(\Phi(\,\overline{M}\,)\big)$, thus it is a linear combination of p^β's, and the only β with p^β annihilated by $p^{1-1/(p-1)}$ is $\beta=(p-2)/(p-1)$. Thus the only map to some $\underline{E}(\beta)$ which might not be ∇_0-linear might be that for this β. The map to $\underline{E}(\beta)$ has rank one and the image of $\overline{L}$ coincides with that of $\overline{M}$. Thus for any element $m\in\overline{M}$ the image of $\nabla_0(m)$ lies in $\overline{M}$ ("the $\overline{M}$ for $\underline{E}(\beta)$"), thus $\Phi(\eta\nabla_0(m))$ maps to 0, and so does $\nabla_0(\Phi(m))$.

We have already noticed that $\mathbb{D}(\underline{E}(\beta))$ is the irreducible Galois representation indexed by β. It follows easily that for two such $\underline{E}(\beta)$, $\underline{E}(\beta')$ the functor $\mathbb{D}$ induces an isomorphism on Hom's. We claim that it induces an injection on Ext^1's. We must check that if for an extension

$$0 \to \underline{E}(\beta) \to \underline{E} \to \underline{E}(\beta') \to 0$$

the extension $\mathbb{D}(\underline{E})$ splits, then so does the original extension. For this note that $\mathbb{D}(\underline{E})$ is a tame Galois representation, so it consists of elements $\lambda\colon \underline{L} \to S/(p)$ which map $\Phi(\underline{M})$ into the tame subspace. As before we can lift to solutions of certain equations in S, and conclude as before. The assertion follows easily.

It follows that $\mathbb{D}$ is fully faithful and its image is closed under forming subobjects and quotients. For the latter, use the fact that $\mathbb{D}$ respects simple objects. So far these assertions have been shown if the residue-field k is algebraically closed. For perfect residue-field the same result follows by Galois-descent. For general k we can make it perfect by adjoining p-power roots of the x_i, $i > 0$. If we have a Galois-map

$$\mathbb{D}(\underline{E}) \to \mathbb{D}(\underline{E}'),$$

it comes over the extension from a map

$$\underline{E}' \to \underline{E}.$$

This map is defined over the original R, except for the commutation with the ∇_i (use Φ-invariance). It then defines a Galois-map over the extension obtained by adjoining the p-th roots of the x_i, and the rest follows as in [4]. For the convenience of the reader we sketch the argument.

We have a map

$$f\colon (\overline{L}_1, \overline{M}_1, \Phi_1) \to (\overline{L}_2, \overline{M}_2, \Phi_2)$$

in $\mathcal{MF}$ after adjoining sufficiently many p-power roots of units, which induces a Galois-linear map (even before we adjoin p-power roots) after we apply $\mathbb{D}$. By Frobenius-invariance it respects connections after we adjoin only p-th roots (no higher p-powers). If we adjoin the p-th root of u where u is part of a p-basis of the residue-field of R we have a derivation ∂_i of R with $\partial_i(u) = 1$, inducing ∇_i on $\overline{L}_1$ and $\overline{L}_2$. The commutator $g = [\nabla_i, f]$ is $\overline{R}$-linear, ηg respects the $\overline{M}_i$, and g is the Φ-transform of ηg on the $\overline{M}$'s. Thus the image $g(\overline{L}_1)$ is a direct summand in $\overline{L}_2$. Furthermore, by a previous argument, the composite of g with any $\lambda \in \mathbb{D}(\underline{E}_2)$ is annihilated by $\zeta - 1$, ζ a primitive p-th root of unity. Thus $\lambda(\eta g(\overline{L}_1))$ is annihilated by $p^{1/p(p-1)}$, and lies in the kernel of "ϕ/p^e". Hence because of Φ-invariance, λ annihilates $g(\overline{L}_1)$. To derive that this image vanishes we may make the residue-field of R algebraically closed, then assume that $\underline{E}_2$ is simple, and finally check that for simple objects $\underline{E}(\beta)$ no non-trivial direct summand of $\overline{L}$ can be annihilated by all λ's.

The same works for subobjects of $\mathbb{D}(\underline{E})$. Over the extension they give quotients of $\underline{E}$ which are defined over the original R, and the Galois-action also makes them ∇_i-stable. In short the theory from [4] carries over to discrete valuation rings.

Over general R's we still follow the strategy of [4, proof of Theorem 2.6]. Assume $\mathbb{L}$ is a representation of $\mathrm{Gal}(\overline{R}/R)$ on an $\mathbb{F}_p$-vector space $\mathbb{L}$ of order p^h. Homomorphisms

$$\mathbb{L} \to \mathbb{D}(\underline{E})$$

correspond to maps

$$\lambda\colon \underline{E} \to \mathrm{Hom}\big(\mathbb{L}, \mathcal{R}/(\underline{p}^p)\big)$$

which map $\underline{M}$ to $(\underline{p}^e)$ and are Galois-invariant in the sense that the image lies in the invariants under the subgroup fixing p-th roots of the x_i, and the remaining Galois-action is determined by the ∇_i. If m_μ denotes a "basis" of $\overline{M}$ there exist matrices $a_{\mu,\nu}$, $b_{\mu,\nu}$ with product η^e such that

$$m_\mu = \sum a_{\mu,\nu}\Phi(m_\nu), \quad \eta^e\Phi(m_\mu) = \sum_\nu b_{\mu,\nu} m_\nu.$$

If their λ-image is $\underline{p}^e z_\mu$, the $z_\mu \in \mathrm{Hom}\big(\mathbb{L}, \mathcal{R}/(\underline{p}^p)\big)$ satisfy equations

$$\phi(z_\mu) = (-1)^e \sum b_{\mu,\nu} z_\nu.$$

The z_μ lift uniquely to solutions in $\mathrm{Hom}(\mathbb{L}, \mathcal{R})$ of the equations above (where we use some Frobenius linear lift $R_\infty/(p) \to \mathcal{R}$). They define an R_{inf}-linear map from L_∞ to $\mathrm{Hom}(\mathbb{L}, \mathcal{R})$ which sends M_∞ into $(\underline{p}^e)\,\mathrm{Hom}(\mathbb{L}, \mathcal{R})$. The images of L_∞ and M_∞ in $\mathrm{Hom}(\mathbb{L}, \mathcal{R})$ then satisfy the conditions in § 2 (the image of the first is ϕ_e applied to the image of the second), so they are both free R_{inf}-modules. It follows that except for the connection ∇ they define a quotient $(\overline{L'}, \overline{M'})$ in the category $\mathcal{MF}$ over which λ factors.

To get a connection we need that the ∇_i respect the kernel. They induce linear maps from the kernel to the cokernel which respect $\overline{M}$'s if multiplied by η. Also they are the Φ-transforms of these maps on $\overline{M}/(\eta)$'s. Finally by Galois-equivariance their image is annihilated by $p^{1/(p-1)}$. This implies (as before) that they vanish. Also L'_∞ injects into $\mathrm{Hom}(\mathbb{L}, \mathcal{R})$ and the map $\mathbb{L} \to \mathbb{D}(\underline{E}')$ is Galois-linear. Next we claim that the rank of $\overline{L'}$ is at most h.

It suffices to show that the rank (over $R_{\mathrm{inf}}/(p)$) of

$$L_\infty \to \mathrm{Hom}(\mathbb{L}, \mathcal{R})$$

is at most h. For this we may localise at minimal primes containing p, thus assume that R is a discrete valuation ring. (If we localise at $\mathfrak{p}$, the new $\mathcal{R}$ contains the $(\underline{p})$-adic completion of the $\mathfrak{p}$-adic localisation of the old $\mathcal{R}$.) Then the image of

$$\mathbb{L} \to \mathbb{D}(\underline{E}')$$

corresponds to a quotient of $\underline{E}$ which has rank $\leqslant h$.

So, as in [4, proof of Theorem 2.6, section g)], we can define an adjoint $\mathbb{E}(\mathbb{L})$ of $\mathbb{D}$ as the inductive limit of $\underline{E}$'s with maps

$$\mathbb{L} \to \mathbb{D}(\underline{E}).$$

It suffices to consider the filtering cofinal system of $\underline{E}$'s such that $\mathbb{L}$ surjects onto $\mathbb{D}(\underline{E})$, and to chose an $\underline{E}$ of maximal rank. Then

$$\mathbb{E}(\mathbb{D}(\underline{E})) = \underline{E},$$

so $\mathbb{D}$ is fully faithful, and its essential image is closed under subobjects and quotients.

§ 4. General objects

We want to extend the theory to more general Frobenius-crystals not necessarily annihilated by p. We have not found a clean general description, so we give a pragmatic one. Namely, consider objects (L, M, Φ), where L, M are R_{crys}-modules,

$$F^e(R_{\mathrm{crys}})L \subseteq M \subseteq L,$$

and Φ a morphism

$$\Phi\colon M \otimes_\phi R_{\mathrm{crys}} \cong L.$$

We further assume that L has a logarithmic connection ∇, $\eta\nabla$ respects M, and the usual identities hold. Finally our object should be a repeated extension (L and M are repeated extensions) of objects $\underline{E}$ annihilated by p as before. That is, L and M admit decreasing filtrations M^μ, L^μ such that the subquotients are annihilated by p and define an object in $\mathcal{MF}$. By devissage, Φ is then an isomorphism. This defines an abelian category, with L and M exact functors.

We show that maps $(L, M) \to (L', M')$ are strict. For this we may assume that L' is annihilated by p. If (L, M) admits a filtration by subobjects (L_n, M_n) with subquotients annihilated by p, denote by L'_n, M'_n the images in L', M'. Then

$$(L'_n/L'_{n-1}, M'_n/M'_{n-1}) \subseteq (L'/M'_{n-1}, M'/M'_{n-1})$$

is a strict subobject (an image of objects annihilated by p), and this implies the same for

$$(L'_n, M'_n) \subseteq (L', M')$$

by induction. For example it applies to objects which are flat over $\mathbb{Z}_p/(p^n)$ and whose reduction modulo p lies in $\mathcal{MF}$.

Define

$$\mathbb{D}(\underline{E}) = \mathrm{Hom}\big(\underline{E} \otimes A_{\mathrm{crys}}, A_{\mathrm{crys}}[1/p]/A_{\mathrm{crys}}\big).$$

Then the functor $\mathbb{D}$ becomes exact and faithful once we show that $\mathbb{D}(\underline{E})$ has the expected order. For this we have to count the number of possible extensions to $\underline{E}$ of a given map

$$\underline{E}' \to A_{\mathrm{crys}}[1/p]/A_{\mathrm{crys}}$$

where

$$\underline{E}' \subset \underline{E}$$

is the maximal subobject annihilated by a smaller p-power than is necessary for $\underline{E}$. If m_μ denotes a basis of M/M', their images correspond to elements in $\mathcal{R}/(\underline{p}^p)$ satisfying certain equations

$$x_\mu^p = \sum_\nu b_{\mu,\nu} x_\nu + y_\mu,$$

which as usual can be reduced to equations in $S/(p)$ or $\widehat{S}$ and have the right number of solutions. To show that $\mathbb{D}$ is fully faithful and that the essential image is closed under subobjects and quotients, use the fact that we have the same simple objects as before, and that $\mathbb{D}$ is still injective on Ext^1's. This follows because if $\mathbb{D}$ of an extension of simple objects splits, then the total space is annihilated by p, thus we can apply the previous theory for objects annihilated by p.

Thus we obtain:

THEOREM 3. *With the above definitions the functor $\underline{D}$ (on general objects) is still exact and fully faithful, and its essential image is closed under subobjects and quotients.*

REMARK. Over smooth base-schemes, objects in $\mathcal{MF}$ with $e = 1$ $(p \geqslant 3)$ correspond to finite flat group-schemes of p-power order. This is no longer the case here, because the construction would involve a "logarithmic descent". For example, the Galois representation for the Tate-curve is given by an object of $\mathcal{MF}$, but does not result from a finite flat group-scheme. Concretely $e = 1$, $\underline{E}$ has basis m_0, m_1 with

$$\phi^0(m_0) = m_0, \quad \phi^1(m_1) = m_1$$

(so M has basis ηm_0, m_1), and ∇ annihilates m_0 while $\nabla(m_1) = m_0\, dx/x$. The Galois representation is an extension of $\mathbb{F}_p$ by μ_p parametrised by p and is not defined by a finite flat group scheme. It would be an extension of $\mathbb{Z}/(p)$ by μ_p, which is necessarily defined by the p-th root of a unit.

§ 5. Examples

Some objects in $\mathcal{MF}$ over a semistable base R are obtained by base change from the old $\mathcal{MF}$ ([4]) over smooth base $\widetilde{R}$ (together with a Frobenius-lift ϕ). Namely, an "old" object in $\mathcal{MF}$ is given by a filtered p-torsion $\widetilde{R}$-module M isomorphic to $\mathrm{gr}_F(M)$, with filtration degrees between 0 and $e \leqslant p-2$, and divided Frobenius-maps

$$\phi_i \colon F^i \otimes_\phi \widetilde{R} \to M$$

inducing

$$\widetilde{M} \otimes_\phi \widetilde{R} \cong M.$$

In addition the Frobenius morphisms should be parallel for a suitable connection ∇. Given such an object over $\widetilde{R}$ and a homomorphism $\widetilde{R} \to R$ (which we lift to $\widetilde{R} \to R_{\mathrm{crys}}$) the pushforward of M is filtered and admits a connection and divided Frobenius morphisms (whose definition involves the connection). We then obtain an object in our present category $\mathcal{MF}$ by considering $M \otimes_{\widetilde{R}} R_{\mathrm{crys}}$ with its subobject $\sum_i F^{e-i}(R_{\mathrm{crys}})F^i$. This construction is compatible with associated Galois representations. It applies for example to crystals associated to abelian schemes, or to finite flat group-schemes of p-power order (which can be étale locally embedded in abelian schemes).

The Galois representation $\mathbb{Z}_p(1)$ is associated to the objects (over the base $\mathbb{Z}_p$) with $e = 1$,

$$L = M = \mathbb{Z}_p,$$

$\Phi(1) = 1$, $\nabla(1) = 0$ (it maps to A_{crys} by sending 1 to the well-known element $t = \log([\underline{1}])$). If x is a monomial in the x_i (which we lift to a monomial $\check{x}$ in the $\check{x}_i$), then p-power roots of x define an extension of $\mathbb{Z}_p$ by $\mathbb{Z}_p(1)$ which is associated to an object of $\mathcal{MF}$ with $e = 1$, L is the free module with basis-elements f, g, and M is spanned by f, ηg with Φ-images given by

$$\Phi(f) = f, \quad \Phi(g) = \left(\frac{\phi(\eta)}{p}\right)g,$$
$$\nabla(f) = 0, \quad \nabla(g) = f \otimes d\log(\check{x}).$$

This example looks very much like the objects in [4] except that the connection is logarithmic.

In the next example we still have $e = 1$, so $p \geqslant 3$. Suppose $G \to \mathrm{Spec}(R)$ is a semiabelian scheme over R_{inf} which is an abelian variety over $\mathrm{Spec}(R[1/p])$. If we assume that R is complete it can be obtained by the Mumford construction;, see [3, Ch. 3, Corollary 7.2]. That is, there exists a semiabelian scheme $\widetilde{G}/R$ (the Raynaud extension) which is globally an extension

$$T \to \widetilde{G} \to A,$$

with T a torus with character group X and A an abelian scheme. We assume for simplicity that T is split, that is, X is constant. There exists a period map

$$\iota\colon Y \to \widetilde{G}(R[1/p])$$

(Y a subgroup of X) such that G is a rigid quotient

$$G = \widetilde{G}/\iota(Y).$$

ι induces a map $Y \to A(R_{\mathrm{inf}})$. More precisely we can write ι as a product

$$\iota = \iota_0\iota_1,$$

where ι_0 maps Y to $\widetilde{G}(R)$, and ι_1 to $T(R[1/p])$, such that the elements of Y map to monomials in the x_i.

The module $T_p(G)$ has a three-step filtration with submodule $X^t \otimes \mathbb{Z}_p(1)$, quotient $Y \otimes \mathbb{Z}_p$, and $T_p(A)$ in the middle. It is defined via p-power roots of elements $\iota(y)$. It is the amalgamated sum of the Tate-module defined by ι_0 and the extension of $Y \otimes \mathbb{Z}_p$ by $X^t \otimes \mathbb{Z}_p(1)$ defined by ι_1. That is, we take the product of both total spaces, in it the pre-image of the diagonal in $Y \otimes \mathbb{Z}_p$, and divide by the diagonal (one factor with a minus sign) $X^t \otimes \mathbb{Z}_p(1)$. We need to explain what we mean by "Tate-module defined by ι_0".

For example a p^n-division point (with coefficients in a suitable ring) in "$\widetilde{G}/\iota_0(Y)$" is defined by a pair $\widetilde{g} \in \widetilde{G}(S)$, $y \in Y$ such that $p^n\widetilde{g} = \iota_0(y)$, modulo the action by elements $z \in Y$ sending $\widetilde{g}$ to $\widetilde{g}\iota_0(z)$ and y to $y + p^n z$. The Tate-module is the projective limit of the p^n-division points.

We show that both summands come from elements of $\mathcal{MF}$: for the extension defined by ι_1 this is obvious from the previous example. Thus consider the extension given by ι_0.

The universal vector extension $E(\widetilde{G})$ of $\widetilde{G}$ is induced from the universal vector extension $E(A)$ of A, and its pullback via ι_0 is trivial. There exists a universal vector extension $E(Y, \widetilde{G})$ of $\widetilde{G}$ with such a trivialisation (that is, a homomorphism $\widetilde{\iota}_0$ with domain Y), namely, the direct sum of $E(\widetilde{G})$ and $Y \otimes \mathbb{G}_a$, where ι_0 lifts in some way into $E(\widetilde{G})$ and by the tautological map of Y into $Y \otimes \mathbb{G}_a$. Its vector part is the direct sum of $Y \otimes \mathbb{G}_a$ and the vector part $\mathrm{Lie}(A^t)^t$ of the universal vector extension of A. It is of crystalline nature, that is, it lifts uniquely modulo any DP-nilpotent ideal I, for example modulo $p \geqslant 3$, or the kernel of $A_{\mathrm{crys}}(R) \to \widehat{S}$. Its Lie-algebra with its Hodge-filtration (of degrees -1, 0) represents the "crystalline homology" of "$\widetilde{G}/\iota_0(Y)$". It has a filtration with subobject $\mathrm{Lie}(T)$, quotient $Y \otimes R_{\mathrm{inf}}$, and the crystalline homology of A in the middle. Its dual admits a Frobenius-action and defines an object of $\mathcal{MF}$ with associated Galois representation the Tate-module of "$\widetilde{G}/\iota_0(Y)$". This follows in the usual way as any element of the Tate-module defines a sequence $\widetilde{g}_n \in \widetilde{G}(S)$ with $\widetilde{g}_0 = 0$,

$$\widetilde{g}_n - p\widetilde{g}_{n+1} = \iota_0(y_n), \qquad y_n \in Y,$$

modulo action of z_n ($z_0 = 0$) sending g_n to

$$g_n + \iota_0(z_n)$$

and y_n to

$$y_n + z_n - pz_{n+1}.$$

These have coefficients in the p-adic completion of S but lift to elements $\widetilde{g}_n$ in $E(Y, \widetilde{G})(A_{\mathrm{crys}}(R))$. Modifying them by the kernel of

$$E(Y, \widetilde{G})(A_{\mathrm{crys}}(R)) \to \widetilde{G}(\widehat{S})$$

(a p-adically complete module) we may assume that they still satisfy

$$\widetilde{g}_n - p\widetilde{g}_{n+1} = \widetilde{\iota}_0(y_n)$$

(but $\widetilde{g}_0$ need not vanish). In fact $\widetilde{g}_0$ lies in the kernel above and its logarithm lies in $F^0\big(\mathrm{Lie}(E(Y, \widetilde{G})(A_{\mathrm{crys}}(R))\big)$. This defines the required period map.

Another example concerns the description by Deligne–Rapoport ([2]) of a local model for $Y_0(p)$. Denote by V_0 the maximal unramified extension of $\mathbb{Z}_p$. A versal deformation $E(x)$ of a supersingular elliptic curve over the residue-field k of V_0 can be described by its associated Frobenius-crystal (see [5]). Over $V_0[[x]]$ (with Frobenius-lift $\phi(x) = x^p$) consider the object in $\mathcal{MF}^{[0,1]}(V_0[[x]])$ (as in [5]) with filtered basis m_0, m_1 (of degrees 0, 1) with $\Phi(m_0) = xm_0 + m_1$, $\Phi(m_1) = pm_0$. There exists a unique Frobenius-invariant connection trivial modulo x.

Over the ring

$$R = V_0[u, v]/(uv + p)$$

there exists (up to slight corrections due to differences in Frobenius-lifts) an isogeny

$$E(u - v^p) \to E(v - u^p)$$

which on the level of crystals can be described as follows: the induced map $M(E(v - u^p)) \to M(E(u - v^p))$ over R_{crys} sends m_0 to $am_0 + bm_1$ and m_1 to $cm_0 + dm_1$. Here $a = u$, $d = v$,

$$b = \prod_{n=1}^{\infty} \left(1 + \frac{(uv)^{p^n}}{p}\right)^{\pm 1} = \frac{(1 + (uv)^p/p)\cdots}{(1 + (uv)^{p^2}/p)\cdots},$$

$c = (p + uv)/b$. One checks that the map respects the F-filtration and Frobenius-maps, and thus is parallel for the connections ∇. By fully faithfulness the map on crystals is induced by an isogeny of p-divisible groups, of degree p. However this is the same as an isogeny of the elliptic curves over R defined by these crystals. As the Frobenius-lifts on $V_0[[x]]$ and $V_0[[y]]$ are not compatible with the maps $x = u - v^p$, $y = v - u^p$ they have to be modified as in [5].

We also consider Shimura varieties of Hodge type. More precisely let $p > 2$ and consider the Shimura variety associated to a modification $\mathrm{GSpin}(2n)$ of the spin-group $\mathrm{Spin}(2n)$. GSpin is obtained from Spin by pushout $\mu_2 \subset \mathbb{G}_m$, with μ_2 the kernel $\mathrm{Spin}(2n) \to \mathrm{SO}(2n)$. It contains a parabolic associated to a filtration $F^1 \subset F^0 = F^{1,\perp} \subset \mathbb{V}_{\not{k}}^{2n} = E$, where F^1 is an isotropic line (the Hodge-structure is most easily described by its realisation on the standard representation of $\mathrm{SO}(2n)$, and there it looks like the one above). GSpin embeds into the symplectic similitudes GSp (and the Shimura variety into the Siegel-space) via the spin-representation $S(E)$. Namely, $S(E)$ is a module over the Clifford-algebra $C(E)$. It is $\mathbb{Z}/(2)$-graded and admits a non-degenerate inner product, and $\mathbb{G}_m$ operates by scalars. Denote by $S^+(E)$ one of its irreducible components and by $S^-(E)$ the other. If n is even, both admit non-degenerate inner products, symmetric if n is divisible by 4 and antisymmetric otherwise. If n is odd, both spaces are dual to each other and so their sum admits an antisymmetric inner product. The image of Clifford-multiplication by F^1 is isotropic and defines Hodge-filtrations on $S(E)$, $S^{\pm}(E)$, and we get the Hodge structure like that associated to an abelian variety $A(E)$ if the inner product is symplectic. Otherwise we tensor with a symplectic module of rank two. The reflex field of the Shimura datum is $\mathbb{Q}$, thus the Shimura variety is a closed subscheme, defined over $\mathbb{Q}$, of the moduli-space of principally polarised abelian varieties with level structure.

It is defined by certain Tate-cycles, and already almost defined by the projection from the Clifford-algebra $C = C(E)$ to E. Over $\mathbb{Q}$ this projection uses the canonical isomorphism $\bigwedge E \to C(E)$ which maps $e_1 \wedge \cdots \wedge e_m$ to the alternating average of their products, in all possible orders. It is integral over $\mathbb{Z}[1/2]$: if $e_1, \ldots, e_{2n}$ form an orthogonal basis of E, map any Clifford-product of order $\neq 1$ of different e_i to 0. Once we have this projection, its image E is a quadratic space (use the product on C) whose Clifford-algebra is C. A suitable generator of $\det(E)$ identifies the two abelian varieties to the ones given by $S^{\pm}(E)$. Also their Hodge-filtrations define a Hodge-filtration on $C(E)$, with Hodge-degrees 0, ± 1, which induces the Hodge-filtration on E.

A. Vasiu has shown ([8, § 4, Theorem 1], [9, Theorem 5.7.1]) that for $p \geqslant 5$ the normalisation of the moduli-space of abelian varieties in the canonical model of this Shimura variety is smooth and admits an object in $\mathcal{MF}$ as described above (it is locally the deformation space of this object). More precisely the crystal defining $A(E)$ also defines a local $\mathbb{Z}_p$-system on the Shimura variety, and the crystal of the Lie-algebra $\mathfrak{gspin}(E)$ defines the local system given by the adjoint representation. This has been improved by M. Kisin ([7]) to $p \geqslant 3$. We claim that our theory of semistable $\mathcal{MF}$ structures allows us to extend this to certain level-structures, giving semistable models.

The method of proof goes as follows: we can define versal families by considering a fixed object (E, F^1, Φ) over an unramified discrete valuation ring V_0. That is, $E = V_0^{2n}$ with the standard inner product, $F \subset E$ is an isotropic line, and Φ a Frobenius semilinear isomorphism from $pE + F^{\perp} + p^{-1}F$ to E. We then obtain a versal family over a smooth local V_0-algebra R, with quotient $V_0 = R/I$, by fixing a Frobenius-lift ϕ on R with

$$\phi(z) - z^p \in pI^p$$

for any z in the augmentation ideal I of R. Define an object of $\mathcal{MF}(R)$ (or better a projective system of objects modulo p^r, for each r) by considering the constants (defined over V_0) E, F^1 but changing Φ to $g\Phi$ with $g \in \mathrm{SO}(E)(R)$ (or rather a lift to $\mathrm{GSpin}(2n)$). In [5] it is explained how to get a canonical connection ∇. Here g should lie in a smooth subscheme of $\mathrm{SO}(E)$ whose tangent space at the origin projects onto a complement of $F^0(\mathfrak{so}(E))$ (these smooth subschemes tend to lift to the Spin group and thus give families of Hodge-structures on $S^{\pm}(E)$). It is associated to a polarised p-divisible group which integrates to an abelian variety if this holds over the closed point. Then $A(E)$ is a family of abelian schemes over the completion of R at this closed point, with the desired higher Hodge cycles.

Now if for a discrete valuation ring V over V_0, with possible high index of ramification, we have a V-point in the normalisation of the moduli-scheme of abelian varieties in the canonical model of the Shimura variety, the abelian scheme $A(E)$ over it admits étale Tate cycles which define the subspace

$$E_{\mathrm{et}} \subset \mathrm{End}\big(H^1_{\mathrm{et}}(A(E))\big).$$

As the period-map for this object of $\mathcal{MF}$ as well as its inverse become integral after multiplication by t, the orthogonal projection from the Clifford-algebra (essentially the endomorphisms of $H^1_{\mathrm{cr}}(A(E))$) onto the crystalline version of E is integral if $p \geqslant 5$. (The Hodge-cycle corresponding to the Tate-cycle is an endomorphism of objects in $\mathcal{MF}$ of Hodge length two.) Thus we can write V as a quotient of $V_0[[u]]$ and denote by $\mathcal{R}$ the divided power-hull (of the ideal defining V), with the Frobenius-lift which sends u to u^p. The first crystalline cohomology of $A(E)$ over V extends to a Frobenius-crystal over $\mathcal{R}$ whose fibre over the common residue field k of V and V_0 lifts to V_0. Then our family over $\mathcal{R}$ is pushforward from the versal family over R, that is, filtrations and Frobenius match. As the connection is the unique one compatible with Frobenius, it also matches. Thus our object over V is induced from the versal family. The p-divisible group over R integrates to an abelian variety, with higher Hodge-cycles as prescribed by the Shimura datum, thus $\mathrm{Spec}(R)$ maps to the Siegel space and in characteristic zero over the Shimura subvariety as this holds at one point (given by V). Hence R is isomorphic to the completion of the local ring of the normalisation of the Shimura variety.

In [7, Cor. 1.3.5 and Cor. 1.4.3, assertions (2, 3)] it is shown that for all $p \geqslant 3$ the Hodge cycles define a reductive group isomorphic to $\mathrm{GSpin}(2n)$ acting on the crystalline cohomology, and again that this is induced from the example above.

In generalisation of the Deligne–Rapoport example we want to define a family of isogenies between two copies $A(E)$ and $A(E')$, over a subvariety of the product with the moduli-space with itself. That is, in the generic fibre we want to consider isogenies induced by the étale analogue of an inclusion $E' \subset E$ of the pre-image of a maximal isotropic subspace modulo p, and denote by $\mathcal{N}$ the normalisation of $\mathcal{M} \times \mathcal{M}$ in the corresponding scheme. Over it we have filtered Frobenius-crystals given by the first crystalline cohomologies of $A(E')$ and $A(E)$, as well as E', E, $\mathfrak{so}(E')$ and $\mathfrak{so}(E)$. That is, for any map

$$T_0 \to \mathcal{N}$$

and any DP-embedding

$$T_0 \subseteq T$$

of schemes on which p is nilpotent, we get such filtered objects over T, with functorial maps from their Frobenius pullbacks (the filtrations only matter on T_0). If they are compatible with the orthogonal structure they uniquely extend to T. There are maps

$$\mathfrak{so}(E') \leftrightarrows \mathfrak{so}(E)$$

with mutual compositions p. They étale locally induce

$$E' \leftrightarrows E$$

which however are unique only up to a scalar, and up to μ_2 if we require compatibility with the inner product. The induced maps

$$F^1(E') \leftrightarrows F^1(E)$$

(with composition p) exhibit $\mathcal{N}$ locally in the étale topology as a scheme over $V[u,v]/(uv+p)$.

If at a point of $\mathcal{N}$ one of the projections to $\mathcal{M}$ is smooth, then $\mathcal{N}$ itself is étale at this point, and the rest is rather easy. This happens if one of the maps

$F^1(E') \leftrightarrows F^1(E)$ is an isomorphism. We thus concentrate on points where this is not the case.

We consider a level p-structure associated to a maximal isotropic subspace $E_0 \subset E/pE$. For this denote by $E' \subset E$ the pre-image of E_0. It also has a non-degenerate quadratic form (divide by p). We lift E_0 to a maximally isotropic sublattice G and write

$$E = G \oplus H, \quad E' = pG \oplus H,$$

with a maximally isotropic complement H. If we have a V-point of $\mathcal{M} \times \mathcal{M}$ whose generic fibre ends up in the Shimura variety given by the level-structure, we get over V two abelian varieties $A(E)$, $A(E')$ with an isogeny between them, all given by Tate-cycles. Then the induced maps on crystalline cohomology are induced from a pair $E' \subset E$ as above.

For example, if $n = 2$, then

$$\mathrm{Spin}(4) = \mathrm{SL}(2) \times \mathrm{SL}(2), \quad E = S^+(E) \otimes S^-(E)$$

(both factors with symplectic products) and we chose for E_0 the pre-image of a line in $S^-(E)$. Then the Deligne–Rapoport construction applied to the factor $S^+(E)$ gives an object in $\mathcal{MF}(R)$ for $R = V[u,v]/(uv+p)$ (or better an inclusion of index p of two objects where the underlying space for the bigger one may be identified with $S^+(E)$, which we tensor with a constant structure on $S^-(E)$ with weights -1, 0). The relevant isotropic subspace is not constant but defined over R_{crys}, but we can apply an R_{crys}-linear automorphism to remedy this. Also for later use we note that the maps

$$pE \subset E' \subset E$$

induce modulo the maximal ideal of R_{crys} the zero maps on the Hodge lines F^1.

For general n we note that for perpendicular elements $g \in G$, $h \in H$ with one of them part of a basis we get a possible filtration F^1 generated by $f = (ug, h)$, $f' = (-pg, vh)$. Varying g and h (up to scalars) defines a smooth algebra R of relative dimension $2n-3$ over $V_0[u,v]/(uv+p)$. Two such possible pairs are mapped to each other by an element of $\mathrm{GL}(G)$ (acting on H by duality), and this exhausts all possible Hodge filtrations over R (the divided power-hull of V). Over $V_0[u,v]/(uv+p)$ we obtained such a pair by the Deligne–Rapoport construction. If we replace the Frobenius by $m\Phi$ (for m in a suitable smooth subscheme of $\mathrm{GL}(G)$ of dimension $2n-3$) the method of [5] gives a unique compatible connection ∇ (start with the old connection and subtract $(dm)m^{-1}$ and higher-order correction terms). Thus we get a family of isogenies over a base R smooth of relative dimension $2n-3$ over $V_0[u,v]/(uv+p)$. It is a versal deformation. Namely, any V-point of $\mathcal{N}$ such that both induced maps

$$F^1(E') \to F^1(E), \quad F^1(E) \to F^1(E')$$

are not isomorphisms is induced from our family (use the method from [5, Sect. 7, Th. 10]). The filtration is induced from the versal case. The two Frobenius morphisms differ by an element of G. Write it as a product of an automorphism respecting filtrations and an element in the parameter-space. We then obtain the next step, with the additional complication that in between we have to account for the difference in Frobenius-lifts ϕ). At the other points one of the projection of $\mathcal{N}$ to $\mathcal{M}$ is locally étale and the scheme is smooth. In short we apply the method from [5] where we use the Deligne–Rapoport example as basepoint, instead of V_0-points.

After that the proof proceeds as before (following [7], [8], [9]) and gives that the normalisation $\mathcal{N}$ of the moduli-space of abelian varieties in the generic fibre of the Shimura variety is locally $\mathrm{Spec}(R)$, that is, the completion of the local ring at a closed point in characteristic p is formally étale over R.

There exists a (possibly higher ramified) extension V of V_0 and a V-point of the normalisation lifting the given closed point in characteristic p. The two projections of $\mathcal{N}$ to $\mathcal{M}$ define two abelian varieties over V, linked by a pair of isogenies, and admitting Hodge- and Tate-cycles (essentially the projections from $C(E)$ to E) which correspond under the étale-crystalline comparison, and are respected by the isogenies. Over the divided power-hull of V the filtered Frobenius-crystals defined by the abelian varieties are determined (by pullback from $\mathcal{M}$) by triples (E, F, Φ) consisting of a quadratic space, an isotropic line, and a Frobenius. The isogenies are induced by inclusions among E's with composition p, and image maximal isotropic modulo p, because this holds on the étale side. Of course these inclusions also respect Frobenius-morphisms. By versality the crystalline object over the DP-hull of V is induced from R, and the Tate-modules of the two abelian varieties over V are the two Galois representations induced from the Galois representations defined by the universal object in $\mathcal{MF}(R)$. These two Galois representations are then associated to abelian varieties with the appropriate Hodge- and Tate-cycles, that is, $\mathrm{Spec}(R)$ maps to $\mathcal{M} \times \mathcal{M}$ and the two abelian varieties admit isogenies. Thus the map lifts to a map from $\mathrm{Spec}(R)$ to $\mathcal{N}$ which is a closed immersion, or better an unramified map. As R is normal and has the correct dimension it is a local isomorphism.

Remark. The referee has pointed out that [7, Prop. 2.3.5] may not hold in the semistable case (on $\mathcal{N}$). However it holds on the $\mathcal{M}$'s and follows for $\mathcal{N}$ by pullback.

It is a honour to dedicate this note to J.-P. Serre on the occasion of his 90th birthday. He has been at the forefront of mathematics for far more than two-thirds of this time span. I also thank the referee for his careful reading.

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