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Rational structure of $X(N)$ over $\mathbb{Q}$ and Explicit Galois action on CM points*

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Abstract In this note, we review a less-known rational structure on the Siegel modular variety $X(N) = \Gamma(N)\backslash\mathbb{H}_g$ over $\mathbb{Q}$ for integers $g, N \geq 1$. We then describe explicitly how Galois groups act on CM points on this variety. Finally, we give another proof of the Shimura reciprocity law using the result and the q-expansion principle.

Keywords Siegel modular variety, Galois action, explicit reciprocity law
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1 Introduction

Let $g \geq 1$ and $N \geq 1$ be positive integers, and let $\mathbb{H}_g$ be the Siegel upper half plane of genus g, i.e., the set of symmetric complex matrices τ of order g such that $\Im(\tau) > 0$. Let

$$\Gamma(N) = \{\gamma \in \mathrm{Sp}_g(\mathbb{Z}) : \gamma \equiv 1 \pmod N\}$$

be the main congruence subgroup and let $X(N) = \Gamma(N)\backslash\mathbb{H}_g$ be the complex manifold which turns out to be an algebraic variety. To construct cryptosystem using genus g ($g = 1, 2$) CM curves, it is important to compute a CM point in $X(N)$ and its Galois conjugates in $X(N)$ explicitly so that one can compute $f(\tau)$ explicitly for some explicit modular functions (invariants) f on $X(N)$. For this, one needs to interprets $X(N)$ in terms of moduli. There are two well-known moduli schemes: $\mathcal{X}_0$ over $\mathbb{Q}(\mu_N)$ whose $\mathbb{C}$-points give $X(N)$, and $\mathcal{X}$ over $\mathbb{Q}$ whose $\mathbb{C}$-points give $(\mathbb{Z}/N)^\times$-copies of $X(N)$ (see Section **??** for review), which is thus not connected. Here μ_N is the group of N-th root of unity. Neither one is handy for our purpose as the first one is only defined over $\mathbb{Q}(\mu_N)$ and the second one has extra $(\mathbb{Z}/N)^\times$ in addition to $X(N)$. There turns out to be a third non-standard moduli scheme $\mathcal{X}^*$ over $\mathbb{Q}$, whose $\mathbb{C}$-points also give $X(N)$, which is natural and good for our purpose. This is constructed as a quotient of $\mathcal{X}$ in Section **??**. This moduli interpretation is a special case of general Shimura variety construction although not explicitly appeared in the literature and should be of interest to publicize it. Using this interpretation, we give an explicit Galois action on a CM point in $X(N)$ in Section **??**. This is the main purpose of this note, inspired by my joint project with C. Castello, A. Deines-Shartz, and K. Lauter [**?**] on computing genus two curves with 2-torsion points. As a byproduct, We give in Section **??** a direct proof of the well-known Shimura

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reciprocity law, which Shimura developed in 1970s (see for example [?]), and its explicit version given by Streng recently [?].

As just mentioned, this work was motivated while working with Castello, Deines-Shartz, and Lauter during my Microsoft visit in Fall 2013. I thank them for the inspiration and the joint work. During that time, I did not realize the existence of Streng's excellent work on explicit Shimura reciprocity law in [?] (otherwise, this note would not have existed). My approach is very different from his. The last section of this work is inspired by his work. I thank B. Conrad, B. Howard, M. H. Nicole, M. Rapoport, and Xinyi Yuan for helpful discussion. This work is done during my visit to Microsoft in Fall 2013 and the MPIM-Bonn in Spring 2014. I thank both institutes for providing me excellent working condition. Finally, I thank the anonymous referee for his/her careful reading and suggestion/comments on the early version of this paper.

2 Open modular variety $X(N)$ over $\mathbb{Q}$

Let $G = \mathrm{GSp}_g$ be the generalized symplectic group (matrices of order $2g$) with similitude character μ, and let $G_0 = \mathrm{Sp}_g$ be the usual symplectic group, i.e., the kernel of μ:

$$1 \to \mathrm{Sp}_g \to \mathrm{GSp}_g \to \mathbb{G}_m \to 1.$$

There are two well-known moduli spaces associated to $X(N)$ which we now briefly review, and refer to [?] for more thorough review. Let μ_N be the group of N-th roots of unity in $\mathbb{C}$, and fix an isomorphism $\mu_N \cong \mathbb{Z}/N$ and identify them in this paper. Then any principally polarized abelian variety A over a field F (of character prime to N), the Weil pairing on the N-torsion $A[N]$ becomes a symplectic pairing

$$\langle\, ,\, \rangle_{\mathrm{we}} : A[N](F) \times A[N](F) \to \mathbb{Z}/N,$$

which is perfect if $A[N](F) = A[N]$.

Let $\mathcal{X}$ be the moduli space over $\mathbb{Z}[\frac{1}{N}]$ as follows. For a $\mathbb{Z}[\frac{1}{N}]$-scheme S, $\mathcal{X}(S)$ consists of isomorphism classes of the triplets (A, λ, ϕ), where

1. A is an abelian scheme over S,
2. $\lambda : A \to A^\vee$ is a principal polarization of A, and
3. $\phi : (\mathbb{Z}/N)^{2g} \to A[N](S)$ is locally a similitude symplectic isomorphism, i.e., $\langle \phi(x), \phi(y)\rangle_{\mathrm{we}} = d\langle x, y\rangle$ for some $d \in (\mathbb{Z}/N)^\times$ (both ϕ and d may vary depending on local connected components of S). Here we use the standard sympletic form on $(\mathbb{Z}/N)^{2g}$:

$$\langle x, y\rangle = \sum_{i=1}^{g} x_i y_{g+i} - \sum_{i=1}^{g} x_{g+i} y_i.$$

Notice that due to the freedom on $d \in (\mathbb{Z}/N)^\times$, the moduli problem does not depend on the choice of our identification $\mu_N \cong \mathbb{Z}/N$. It is well-known that this moduli space is represented by a smooth Deligne-Mumford stack, still denoted by $\mathcal{X}$, over $\mathbb{Z}[\frac{1}{N}]$. It is actually a smooth scheme when $N \geq 3$.

Let $\mathcal{X}_0$ be the moduli space over $\mathbb{Z}[\frac{1}{N}, \mu_N]$ as follows. For a $\mathbb{Z}[\frac{1}{N}, \zeta_N]$-scheme S, $\mathcal{X}(S)$ consists of isomorphism classes of the triplets (A, λ, ϕ), where

1. A is a abelian scheme over S,
2. $\lambda : A \to A^\vee$ is a principal polarization of A, and
3. $\phi : (\mathbb{Z}/N)^{2g} \to A[N](S)$ is a symplectic isomorphism, i.e., $\langle \phi(x), \phi(y)\rangle_{\mathrm{we}} = \langle x, y\rangle$.

It is also well-known that this moduli space is represented by a smooth Deligne-Mumford stack, still denoted by $\mathcal{X}_0$, over $\mathbb{Z}[\frac{1}{N}, \mu_N]$. It is again a smooth scheme when $N \ge 3$.

In terms of Shimura datum, one has the following. Let

$$K(N) = \{g \in G(\hat{\mathbb{Z}}) : g \equiv 1 \pmod N\}, \quad K_0(N) = K(N) \cap G_0(\hat{\mathbb{Z}}).$$

Then $\mathcal{X}$ is the Shimura variety associated to K, i.e.,

$$\mathcal{X}(\mathbb{C}) = G(\mathbb{Q})\backslash(\mathbb{H}_g^\pm \times G(\mathbb{A}_f)/K(N)) = (\Gamma(N)\backslash\mathbb{H}_g) \times (\mathbb{Z}/N)^\times.$$

Moreover,

$$\mathcal{X}_0(\mathbb{C}) = X(N) = \mathbb{G}_0(\mathbb{Q})\backslash(\mathbb{H}_g \times G_0(\mathbb{A}_f)/K_0(N))$$

is the connected component of $\mathcal{X}(\mathbb{C})$.

It turns out that there is a (less known) third Shimura variety $\mathcal{X}^*$ over $\mathbb{Z}[\frac{1}{N}]$ directly related to $X(N)$. It is associated to the compact open subgroup of G

$$K^*(N) = \{g \in G(\hat{\mathbb{Z}}) : g \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & * \end{smallmatrix}\right) \pmod N\}.$$

By the strong approximation theorem, one has

$$\mathcal{X}^*(\mathbb{C}) = G(\mathbb{Q})\backslash(\mathbb{H}_g^\pm \times G(\mathbb{A}_f)/K^*(N)) = X(N).$$

$$\mathcal{X}_0(\mathbb{C}) \hookrightarrow \mathcal{X}(\mathbb{C}) \twoheadrightarrow (\mathbb{Z}/N)^\times,$$

and

$$(\mathbb{Z}/N)^\times \circlearrowright \mathcal{X}(\mathbb{C}) \twoheadrightarrow \mathcal{X}^*(\mathbb{C}).$$

Here the action is given by $d \circ [z, g] = [z, gv(d)]$, the natural project from $\mathcal{X}(\mathbb{C})$ to $\mathcal{X}^*(\mathbb{C})$ has fiber $(\mathbb{Z}/N)^\times$. Here $v(d) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)$ with respect to the standard symplectic basis of $(\mathbb{Z}/N)^{2g}$.

To give the moduli problem for this variety, let $(\mathbb{Z}/N)^\times$ act on $\mathcal{X}$ as follows:

$$d \circ (A, \lambda, \phi) = (A, \lambda, \phi \circ v(d))$$

The action is free, so there is a quotient stack (scheme for $N \ge 3$) $\mathcal{X}^* = \mathcal{X}/(\mathbb{Z}/N)^\times$ which represents the following quotient moduli problem over $\mathbb{Z}[\frac{1}{N}]$. For a $\mathbb{Z}[\frac{1}{N}]$-scheme S, $\mathcal{X}^*(S)$ consists of the equivalence classes of the triples (A, λ, ϕ) as in $\mathcal{X}(S)$, but with the following equivalence relation: $(A_1, \lambda_1, \phi_1) \sim (A_2, \lambda_2, \phi_2)$ if and only if there is an S-isomorphism $f : A_1 \to A_2$ commuting with the polarizations λ_i and $\phi_2 = \phi_1 \circ v(d)$ for some $d \in (\mathbb{Z}/N)^\times$. Alternatively, $\mathcal{X}^*(S)$ is the equivalence classes of the triples $(A, \lambda, \vec{e})$ where (A, λ) is a principally polarized abelian scheme over S, and $\vec{e} = (e_1, \cdots, e_{2g})$ is locally an ordered similitude symplectic basis of $A[N](S)$ with respect to the Weil pairing, i.e., for $i \le j$

$$\langle e_i, e_j\rangle_{\mathrm{we}} = \begin{cases} d & \text{if } 1 \le i \le g, j = g + i, \\ 0 & \text{otherwise,} \end{cases}$$

for some $d \in (\mathbb{Z}/N)^\times$. A similitude symplectic basis is called a symplectic basis if $d=1$. Two such triples $(A,\lambda,\vec{e}) \sim (A', B', \vec{e}')$ if and only if there is an S-isomorphism $f:(A,\lambda)\to(A',\lambda')$ such that $f(e_i)=e_i'$ for $1\le i\le g$ and $f(e_i)=de_i'$ for all $g+1\le i\le 2g$ and locally some $d\in(\mathbb{Z}/N)^\times$.

Proposition 2.1. *One has over* $\mathbb{Z}[\frac{1}{N},\mu_N]$

$$\mathcal{X}^* = \mathcal{X}_0.$$

Proof. Let $\vec{e}=(e_1,e_2,\cdots,e_{2g})$ be the standard symplectic basis of $(\mathbb{Z}/N)^{2g}$. Given two triples (A,λ,ϕ) and (A',λ',ϕ') in $\mathcal{X}_0(S)$ for an $\mathbb{Z}[\frac{1}{N},\mu_N]$-scheme S. Suppose that they are equal in $\mathcal{X}^*(S)$, i.e., there is an S-isomorphism $f:(A,\lambda)\to(A',\lambda')$ and $d\in(\mathbb{Z}/N)^\times$ such that $\phi'=f\circ\phi\circ v(d)$. Since ϕ and ϕ' are symplectic isomorphisms, and f preserves the Weil pairing, one has

$$1=\langle\phi'(e_i),\phi'(e_{i+g})\rangle_{\rm we}=\langle\phi(v(d)e_i),\phi(v(d)e_{g+i})\rangle_{\rm we}=\langle e_i, de_{g+i}\rangle = d\in(\mathbb{Z}/N)^\times.$$

So $(A,\lambda,\phi)=(A',\lambda',\phi')$ in $\mathcal{X}_0(S)$. This gives an injection $\mathcal{X}_0\to\mathcal{X}^*$ over $\mathbb{Z}[\frac{1}{N},\mu_N]$. To verify the surjectivity, let $(A,\lambda,\phi)\in\mathcal{X}^*(S)$. Let $P_i=\phi(e_i)$, then there is $d\in(\mathbb{Z}/N)^\times$ such that

$$\langle P_i,P_j\rangle_{\rm we}=d\langle e_i,e_j\rangle.$$

Take $\phi'=\phi\circ v(d^{-1})$, then one sees that ϕ' is an symplectic isomorphism. So $(A,\lambda,\phi)=(A,\lambda,\phi')\in\mathcal{X}^*(S)$ is the image of $(A,\lambda,\phi')\in\mathcal{X}_0(S)$. □

Remark 2.2. There is another moduli interpretation for $X(N)$ over $\mathbb{Q}$ as follows. Let $\mathcal{X}'$ be the moduli space of the equivalence classes of the triplets (A,λ,ϕ) where (A,λ) are principally polarized abelian schemes as above, and $\phi:(\mathbb{Z}/N)^g\times(\mu_N)^g\to A[N]$ is a Galois equivariant map which respects the pairings. The equivalence is the usual one as in the moduli interpretation of $\mathcal{X}$. Here the pairing at the right hand side is the Weil pairing while the one at the left hand side is the obvious one

$$\langle(n,\xi),(\tilde{n},\tilde{\xi})\rangle=\sum_{i=1}^g\tilde{\xi}_i^{n_i}-\sum_{i=1}^g\xi_i^{\tilde{n}_i}.$$

The natural maps

$$\mathcal{X}'\to\mathcal{X}_0\to\mathcal{X}\to\mathcal{X}^*$$

are defined over $\mathbb{Q}(\mu_N)$. One can prove that the composition $\mathcal{X}'\to\mathcal{X}^*$ is actually isomorphism defined over $\mathbb{Q}$. This remark belongs to the anonymous referee.

Remark 2.3. The moduli variety $\mathcal{X}^*$ is quite natural both in terms of Shimura datum and in terms of moduli interpretation. It is curious and a little strange that it has not appeared in literatures to my best knowledge. For example, it could naturally have been in [**?**, Table(7.4.3)], its analogues for $\Gamma_1(N)$ and $\Gamma_0(N)$ are both there.

Remark 2.4. If we let N change and temporarily write $\mathcal{X}(N)$ for $\mathcal{X}$ and take inverse limit. Then the pro-Shimura variety $\mathcal{X}=\varprojlim\mathcal{X}(N)$ is a right $G(\mathbb{A}_f)$-module but far from connected. On the other hand $\mathcal{X}^*=\varprojlim\mathcal{X}^*(N)=\mathcal{X}/v(\hat{\mathbb{Z}}^\times)$ is a connected quotient of $\mathcal{X}$. However, only the normalizer of $v(\hat{\mathbb{Z}}^\times)$ in $G(\mathbb{A}_f)$, not the whole $G(\mathbb{A}_f)$, can act on $\mathcal{X}^*$.

3 Complex multiplication and Galois orbit of a CM point

Let (E, Φ) be a CM number field with CM type Φ, and let $(\tilde{E}, \tilde{\Phi})$ be the reflex CM field with reflex CM type. Let M be a Galois extension of $\mathbb{Q}$ containing both E and $\tilde{E}$. Recall the type norm on elements

$$\mathrm{N}_\Phi : E^\times \to \tilde{E}^\times, x \mapsto \prod_{\sigma\in\Phi} \sigma(x),$$

and on ideals

$$\mathrm{N}_\Phi(\mathfrak{a}) = (\prod_{\sigma\in\Phi} \sigma(\mathfrak{a})\mathcal{O}_M) \cap \mathcal{O}_{\tilde{E}}.$$

Here M is a (any) Galois extension of $\mathbb{Q}$ containing both E and $\tilde{E}$. For the convenience of the reader, we first recall the well-known main theorem of Shimura and Taniyama on complex multiplication (see [?], [?]). A CM abelian variety over a field $L \hookrightarrow \mathbb{C}$ of CM type (E, Φ) is in this paper a pair (A, ι), where A is an abelian variety over L of dimension $\frac{1}{2}[E : \mathbb{Q}]$, $\iota : \mathcal{O}_E \to \mathrm{End}_L(A)$ is an isomorphism, and there is a $\mathbb{C}$ basis $\{\omega_\sigma, \sigma \in \Phi\}$ on $\Omega_{A/\mathbb{C}}$ such $i(z)^*\omega_\sigma = \sigma(z)\omega_\sigma$. For a number field E, we denote E_f for the finite adeles of E, and denote $\hat{\mathcal{O}}_E$ for the ring of integers of E_f.

Theorem 3.1. *(Shimura-Taniyama) Let* $\mathbf{b} \in \tilde{E}_f^\times$ *and* $\sigma \in \mathrm{Aut}(\mathbb{C}/\tilde{E})$ *such that* $\sigma|_{\tilde{E}^{ab}} = \sigma_{\mathbf{b}^{-1}}$ *via the Class field theory (Artin map). Here* $\tilde{E}^{ab}$ *is the maximal abelian extension of* $\tilde{E}$*. Let* (A, ι) *be a CM abelian variety over* $\mathbb{C}$ *of CM type* (E, Φ)*. Then there is an isomorphism* $f : \mathbb{C}^g/\Phi(\mathfrak{a}) \cong A$ *for some fractional ideal* $\mathfrak{a}$ *of* E *over* $\mathbb{C}$*. Fix such an isomorphism* f *(and* $\mathfrak{a}$*), there is a unique isomorphism* $f' : \mathbb{C}^g/\Phi(\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}\,\mathbf{b}) \to A^\sigma$ *over* $\mathbb{C}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} E/\mathfrak{a} & \xrightarrow{f\circ\Phi} & A_{tor} \\ \downarrow{\cdot \mathrm{N}_{\tilde{\Phi}}(\mathbf{b})} & & \downarrow{\sigma} \\ E/\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathbf{b}) & \xrightarrow{\Phi\circ f'} & A^\sigma_{tor} \end{array}$$

Here the multiplication by the idele in the column makes sense via the canonical isomorphism $E/\mathfrak{a} = \oplus_{\mathfrak{p}} E_{\mathfrak{p}}/\mathfrak{a}_{\mathfrak{p}}$*. Here* $E_{\mathfrak{p}}$ *(resp.* $\mathfrak{a}_{\mathfrak{p}}$*) is the completion of* E *(resp.* $\mathfrak{a}$*) with respect to prime ideal* $\mathfrak{p}$*.*

A CM point of CM type (E, Φ) in $\mathcal{X}^*(L)$, for a field $L \subset \mathbb{C}$, is a tuple $(A, \iota, \lambda, \phi)$ where (A, ι) is a CM abelian variety of CM type (E, Φ) and $(A, \lambda, \phi) \in \mathcal{X}^*(L)$ such that the Rosati involution associated to λ induces the complex conjugation on E. Let $\mathrm{CM}(E, \Phi)$ be the set of CM points in $\mathcal{X}^*(\mathbb{C}) = X(N)$ of CM type (E, Φ).

Let R be a (communicative) ring, and let V be a free R-module of rank $2g$ with non-degenerate symplectic form $\langle\, ,\, \rangle$. A basis $\vec{a} = (a_1, \cdots, a_{2g})$ is called a similitude symplectic basis if the associated matrix

$$(\langle a_i, a_j\rangle) = \begin{pmatrix} 0 & dI_g \\ -I_g & 0 \end{pmatrix}$$

for some $d \in R^\times$. When $d = 1$, we call it a symplectic basis.

Proposition 3.2. *There are bijections among following sets.*

(1) *The set* $\mathrm{CM}(E, \Phi) \subset X(N)$ *of CM points of CM type* (E, Φ)*.*

(2) *The set of points* $[\tau] \in X(N)$ *such that* $\Lambda_\tau = \tau\mathbb{Z}^g + \mathbb{Z}^g \subset \mathbb{C}^g$ *is a (projective)* $\mathcal{O}_E$*-module via* $\Phi = \{\sigma_1, \cdots, \sigma_g\}$*, where* E *acts on* $\mathbb{C}^g$ *via* $\iota(z)x = \mathrm{diag}(\sigma_1(z), \cdots, \sigma_g(z))x$ *for* $z \in E$ *and* $x \in \mathbb{C}^g$*.*

(3) *The set of equivalence classes of* $(\mathfrak{a}, \xi, \vec{a})$*, where* $\mathfrak{a}$ *is a fractional ideal of* E*,* $\xi \in E^\times$ *such that* $\bar{\xi} = -\xi$ *such that* $\mathfrak{a}$ *is integral and self-dual with respect to the symplectic pairing (Riemann form)*

$$E_\xi : E \times E \to \mathbb{Q}, \quad E_\xi(x, y) = \mathrm{tr}_{E/\mathbb{Q}} \xi x \bar{y}, \tag{3.1}$$

i.e., $\xi\partial_E \mathfrak{a}\bar{\mathfrak{a}} = \mathcal{O}_E$*, where* ∂_E *is the different of* E*.* $\vec{a} = (a_1, \cdots, a_{2g})$ *is an ordered symplectic basis of* $\mathfrak{a}$ *with respect to* E_ξ*. Two triples* $(\mathfrak{a}, \xi, \vec{a})$ *and* $(\mathfrak{b}, \eta, \vec{b})$ *are equivalent if there is* $r \in E^\times$ *and* $\gamma \in \Gamma(N)$ *such that* $r\bar{r} \in \mathbb{Q}^\times$*,* $\mathfrak{a} = r\mathfrak{b}$*,* $\xi = (r\bar{r})^{-1}\eta$*, and* $\vec{a} = r\gamma\vec{b}$*.*

(3') *The set of equivalence classes of* $(\mathfrak{a}, \xi, \frac{1}{N}\vec{a})$*, where* $\mathfrak{a}$ *is a fractional ideal of* E*,* $\xi \in E^\times$ *such that* $\bar{\xi} = -\xi$ *such that* $\frac{1}{N}\vec{a}$ *is a symplectic basis for* $\frac{1}{N}\mathfrak{a}/\mathfrak{a}$ *with respect to the Weil pairing*

$$\langle \frac{x}{N}, \frac{y}{N} \rangle_{\mathrm{we}} = E_\xi(x, y) \pmod N.$$

Two triples $(\mathfrak{a}, \xi, \frac{1}{N}\vec{a})$ *and* $(\mathfrak{b}, \eta, \frac{1}{N}\vec{b})$ *are equivalent if there is* $r \in E^\times$ *such that* $r\bar{r} \in \mathbb{Q}^\times$*,* $\mathfrak{a} = r\mathfrak{b}$*,* $\xi = (r\bar{r})^{-1}\eta$*, and* $\frac{1}{N}\vec{a} = \frac{1}{N}r\vec{b}$ *(i.e.,* $\vec{a} \equiv r\vec{b} \pmod N$*).*

(4) *The set of equivalence classes of triples* $(A_\mathfrak{a}, E_\xi, \frac{1}{N}\vec{a})$ *where* $A_\mathfrak{a} = \mathbb{C}^g/\Phi(\mathfrak{a})$ *is a CM abelian variety of CM type* (E, Φ) *over* $\mathbb{C}$*,* E_ξ*, as defined in (**??**), is a Riemann form on* $A_\mathfrak{a}$*, which gives a principally polarization on* $A_\mathfrak{a}$*, and* $\frac{1}{N}\vec{a}$ *is a similitude symplectic basis of* $A_\mathfrak{a}[N] = \frac{1}{N}\mathfrak{a}/\mathfrak{a}$ *with respect to the Weil pairing:*

$$\langle \frac{x}{N}, \frac{y}{N} \rangle_{\mathrm{we}} = E_\xi(x, y) \pmod N.$$

Two triples $(A_\mathfrak{a}, E_\xi, \vec{a})$ *and* $(A_\mathfrak{b}, E_\eta, \vec{b})$ *are equivalent if there is an* $r \in E$ *such that* $r\bar{r} \in \mathbb{Q}^\times$*,* $\mathfrak{a} = r\mathfrak{b}$*,* $\xi = (r\bar{r})^{-1}\eta$*, and* $\frac{1}{N}\vec{a} = v(d)(r\vec{b})$ *in* $\frac{1}{N}\mathfrak{a}/\mathfrak{a}$ *for some* $d \in (\mathbb{Z}/N)^\times$*.*

Proof. (sketch) The bijection between (1) and (3) follows from $X(N) = \mathcal{X}_0(\mathbb{C})$ and Theorem **??**. The bijection between (1) and (4) follows from $X(N) = \mathcal{X}^*(\mathbb{C})$ and Theorem **??**. The bijection between (3) and (3′) is due to the fact that $\mathrm{SL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{Z}/N)$ is surjective. Now we describe the bijection between (2) and (3). Recall that $\tau \in X(N) = \mathcal{X}_0(\mathbb{C})$ gives the triple $(A_\tau, E_\tau, \frac{1}{N}\vec{e}_\tau)$, where $A_\tau = \mathbb{C}^g/\Lambda_\tau$ with $\Lambda_\tau = \tau\mathbb{Z}^g + \mathbb{Z}^g$ and principal polarization $E_\tau = \Im H_\tau$ where

$$H_\tau(x, y) = x^t \Im(\tau)^{-1}\bar{y}$$

is the associated positive definite Hermitian form on $\mathbb{C}^g$, and $\vec{e}_\tau = (e_i)_{1 \le i \le 2g}$ with

$$(e_1, e_2, \cdots, e_g) = \tau, \text{ and } (e_{g+1}, \cdots, e_{2g}) = I_g.$$

Notice that $\vec{e}_\tau$ is symplectic basis of Λ_τ with respect to E_τ, and that $H_\tau(x, y) = E_\tau(ix, y) + iE_\tau(x, y)$ (see for example [**?**] or [**?**])

Given a triple $(\mathfrak{a}, \xi, \vec{a})$ in (3), let $\tau = (\Phi(a_{g+1}), \cdots, \Phi(a_{2g}))^{-1}(\Phi(a_1), \cdots, \Phi(a_g))$, also denoted by $\tau(\mathfrak{a}, \xi, \vec{a})$. Then

$$f : A_\tau \cong A_\mathfrak{a}, \quad f(z) = (\Phi(a_{g+1}), \cdots, \Phi(a_{2g}))z,$$

which sends $\vec{e}_\tau$ to $\vec{a}$. So the $(A_\tau, E_\tau, \frac{1}{N}\vec{e}_\tau) = (A_\mathfrak{a}, E_\xi, \frac{1}{N}\vec{a}) \in \mathcal{X}_0(\mathbb{C}) = X(N)$. Via map f, $\Lambda_\tau \cong \Phi(\mathfrak{a})$ becomes an $\mathcal{O}_E$-module.

Conversely , if Λ_τ is an $\mathcal{O}_E$-module via Φ, then it is finitely generated and torsion free and thus projective of rank 1 (comparing the $\mathbb{Z}$-rank). So there is an fractional ideal $\mathfrak{a}$ of E and an $\mathcal{O}_E$-module isomorphism $f : \Phi(\mathfrak{a}) \cong \Lambda_\tau$, which extends to an isomorphism $f : A_\mathfrak{a} \cong A_\tau$. The Riemann form E_τ on Λ_τ gives an self-dual sympletic form on $\mathfrak{a}$. So there is ξ such that $\mathfrak{a}$ is E_ξ-self-dual, and that $\vec{a} = \Phi^{-1} f^{-1}(\vec{e}_\tau)$ is a symplectic basis of $\mathfrak{a}$. That is $\tau = \tau(\mathfrak{a}, \xi, \vec{a})$. This gives the bijection between (2) and (3). □

We will identify each set in Proposition **??** with $\mathrm{CM}(E, \Phi)$. Given $(\mathfrak{a}, \xi, \vec{a}) \in \mathrm{CM}(E, \Phi)$, we write the associated τ in $X(N)$ as $\tau = \tau(\mathfrak{a}, \xi, \vec{a})$. It is given by

$$\tau = (\Phi(a_{g+1}), \cdots, \Phi(a_{2g}))^{-1}(\Phi(a_1), \cdots, \Phi(a_g)). \tag{3.2}$$

View $\vec{a}$ as a $\mathbb{Q}$-basis of E, one obtains an embedding

$$\epsilon : E^\times \to \mathrm{GL}_{2g}(\mathbb{Q}), \quad \epsilon(z)a_i = za_i, \tag{3.3}$$

and a map

$$g = g(\mathfrak{a}, \xi, \vec{a}) : \tilde{E}^\times \to \mathrm{GSp}_g(\mathbb{Q})^+, \quad g(z) = \epsilon(\mathrm{N}_{\tilde{\Phi}}(z)). \tag{3.4}$$

The map is well-defined as

$$E_\xi(g(z)(a_i), g(z)(a_j)) = E_\xi(\mathrm{N}_{\tilde{\Phi}}(z)a_i, \mathrm{N}_{\tilde{\Phi}}(z)a_j) = \mathrm{N}_{\tilde{\Phi}}(z)\overline{\mathrm{N}_{\tilde{\Phi}}(z)}E_\xi(a_i, a_j) = \mathrm{N}(z)E_\xi(a_i, a_j).$$

One has further $\mu(g(z)) = \mathrm{N}(z)$. The maps g and ϵ depend on the point τ.

Let $\mathrm{Cl}(\tilde{\Phi}, N)$ be the type class group of modulus N, defined as the quotient of all fractional ideals of $\tilde{E}$ prime to N by the subgroup

$$P(\tilde{\Phi}, N) = \{\mathfrak{a} \subset \tilde{E} : \mathrm{N}_{\tilde{\Phi}}(\mathfrak{a}) = \mu\mathcal{O}_E, \text{ for some } \mu \equiv 1 \pmod{N}, \mu\bar{\mu} = \mathrm{N}(\mathfrak{a})\}.$$

Let $H(\tilde{\Phi}, N)$ be the associated type class field of $\tilde{E}$. For a number field E, we write E_f as its finite adeles and $\hat{\mathcal{O}}_E$ as the ring of integers of E_f. The following isomorphism is well-known

$$\mathrm{Cl}(\tilde{\Phi}, N) \cong \tilde{E}_f^\times / U(\tilde{\Phi}, N), \quad [\mathfrak{b}] \mapsto [\mathbf{b}], \tag{3.5}$$

where $\mathbf{b} \in \tilde{E}_f^\times$ satisfies $(\mathbf{b}) = \mathbf{b}\hat{\mathcal{O}}_{\tilde{E}} \cap \tilde{E} = \mathfrak{b}$ and $\mathbf{b}_\mathfrak{p} \equiv 1 \pmod{N}$ for all $\mathfrak{p}|N$. Here

$$U(\tilde{\Phi}, N) = \{x \in \tilde{E}_f^\times : \mathrm{N}_{\tilde{\Phi}}(x) \in E^\times((1 + N\hat{\mathcal{O}}_E) \cap \hat{\mathcal{O}}_E^\times)\}.$$

Proposition 3.3. *(1) For every CM point $\tau = \tau(\mathfrak{a}, \xi, \vec{a}) \in \mathcal{X}_0(\mathbb{C})$, its field of definition is the class field $H(\tilde{\Phi}, N)$.*

(2) For a CM point $\tau = \tau(\mathfrak{a}, \xi, \vec{a}) \in \mathcal{X}^*(\mathbb{C})$, its field of definition is the class field $H^*(\tilde{\Phi}, N)$ associated to the class group $\mathrm{Cl}^*(\tilde{\Phi}, N) = \tilde{E}_f^\times / U^*(\tilde{\Phi}, N)$, where

$$U^*(\tilde{\Phi}, N) = \{\mathbf{b} \in \tilde{E}_f^\times : \mathrm{N}_{\tilde{\Phi}}(\mathbf{b}) = \alpha u : \alpha \in E^\times, \alpha\bar{\alpha} = \mathrm{N}(\mathfrak{b}), \epsilon(u) \in K^*(N)\}.$$

Proof. This proposition is a direct consequence of Theorem **??** and we give a sketch of (2) for convenience. Let $\sigma \in \mathrm{Aut}(\mathbb{C})$ with $\sigma|_{H^*(\tilde{\Phi}, N)} = \sigma_{\mathbf{b}^{-1}}$ via the class field theory. Here we use the normalization in [**?**] for the Artin map, i.e., $\sigma_\mathfrak{p}(x) \equiv x^{\mathrm{N}(\mathfrak{p})} (\mod \mathfrak{p})$. Let $\mathfrak{b} = (\mathbf{b})$ be the ideal of $\mathbf{b}$. Assume $\tau^\sigma = \tau$, then $A_{\mathfrak{a}\mathrm{N}_{\tilde{\Phi}}(\mathbf{b})} \cong A_\mathfrak{a}$, and so $\mathrm{N}_{\tilde{\Phi}}\,\mathfrak{b} = \alpha\mathcal{O}_E$ for some $\alpha \in E^\times$. Write $\mathrm{N}_{\tilde{\Phi}}\,\mathbf{b} = \alpha u$ with $u \in \hat{\mathcal{O}}_E^\times$. So we have

$$\tau^\sigma = \tau(\mathfrak{a}, \xi\frac{\alpha\bar{\alpha}}{\mathrm{N}(\mathfrak{b})}, \frac{1}{N}u\vec{a}) = \tau(\mathfrak{a}, \xi, \frac{1}{N}\vec{a}).$$

This implies that we can change α properly to make $\alpha\bar{\alpha} = \mathrm{N}(\mathfrak{b})$. Since the two symplectic similitude bases $\frac{1}{N}u\vec{a}$ and $\frac{1}{N}\vec{a}$ of $\frac{1}{N}\mathfrak{a}/\mathfrak{a}$ with respect to the Weil pairing have to be equivalent, i.e., differing only by $v(d)$ for some $d \in (\mathbb{Z}/N)^\times$, one has $\epsilon(u) \in K^*(N)$. The other way is the same. □

Notice that $\mu_N \subset H(\tilde{\Phi}, N)$ and $\mathcal{X}_0 = \mathcal{X}^*_{\mathbb{Q}(\mu_N)}$, so one has that $H(\tilde{\Phi}, N) = H^*(\tilde{\Phi}, N)(\mu_N)$. We remark that the class field $H^*(\tilde{\Phi}, N)$ might depends on the map ϵ in (**??**), and thus the CM point τ. It is an interesting question whether and how $H^*(\tilde{\Phi}, N)$ really depends on τ. For example, do different Galois orbits in $\mathrm{CM}(E, \Phi)$ have the same cardinality? (or does the index $[H^*(\tilde{\Phi}, N) : \tilde{E}]$ depend on τ?)

Theorem 3.4. *Let* $\tau = \tau(\mathfrak{a}, \xi, \vec{a}) \in \mathrm{CM}(E, \Phi) \in X(N)(\mathbb{C})$. *Let* $\sigma \in \mathrm{Aut}(\mathbb{C}/\tilde{E})$ *and* $[\mathfrak{b}] \in \mathrm{Cl}(\tilde{\Phi}, N)$ *such that* $\sigma|_{H(\tilde{\Phi},N)} = \sigma_{\mathfrak{b}^{-1}}$ *via the class field theory. Choose an (ordered) symplectic basis* $\vec{c}$ *of* $\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathfrak{b})$ *with respect to the symplectic form* $E_{\xi\,\mathrm{N}(\mathfrak{b})^{-1}}$ *such that*

$$c_i \equiv a_i \pmod N, 1 \le i \le g, \quad c_i \equiv a_i\,\mathrm{N}(\mathfrak{b}) \pmod N, g+1 \le i \le 2g.$$

Then

$$\tau(\mathfrak{a}, \xi, \vec{a})^\sigma = \tau(\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathfrak{b}), \xi\,\mathrm{N}(\mathfrak{b})^{-1}, \vec{c}).$$

Proof. Choose $\mathbf{b} \in \tilde{E}_f^\times$ such that $(\mathbf{b}) = \mathfrak{b}$ and $\mathbf{b}_\mathfrak{p} = 1$ for all primes of $\tilde{E}$ above N, as in (**??**). We may assume that $\mathfrak{b}$ is integral. Then Theorem **??** implies

$$(A_\mathfrak{a}, E_\xi, \frac{1}{N}\vec{a})^\sigma = (A_{\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathfrak{b})}, E_{\xi\,\mathrm{N}(\mathfrak{b})^{-1}}, \frac{1}{N}\mathrm{N}_{\tilde{\Phi}}(\mathbf{b})\vec{a}).$$

Notice

$$\begin{aligned}\langle\frac{1}{N}\mathrm{N}_{\tilde{\Phi}}(\mathbf{b})a_i, \frac{1}{N}\mathrm{N}_{\tilde{\Phi}}(\mathbf{b})a_j\rangle_{\mathrm{we}} &\equiv \frac{\mathrm{N}(\mathbf{b})}{\mathrm{N}(\mathfrak{b})}E_\xi(a_i, a_j) \pmod N\\ &\equiv \frac{1}{\mathrm{N}(\mathfrak{b})}E_\xi(a_i, a_j) \pmod N.\end{aligned}$$

So one has in $(\frac{1}{N}\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathfrak{b}))/\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathfrak{b})$,

$$\frac{1}{N}c_i = \frac{1}{N}\mathrm{N}_{\tilde{\Phi}}(\mathbf{b})a_i, 1 \le i \le g,$$

and

$$\frac{1}{N}c_i = \frac{1}{N}\mathrm{N}_{\tilde{\Phi}}(\mathbf{b})\,\mathrm{N}(\mathfrak{b})a_i, g+1 \le i \le 2g.$$

So $\frac{1}{N}\vec{c} = v(\mathrm{N}(\mathfrak{b}))(\frac{1}{N}\mathrm{N}_{\tilde{\Phi}}(\mathbf{b})\vec{a})$. Indeed, for a prime $\mathfrak{p}$ of E above N, it is true by our choice of $\vec{c}$ and that $\mathrm{N}_{\tilde{\Phi}}\,\mathbf{b}_\mathfrak{p} = 1$. For $\mathfrak{p} \nmid N$, both sides are zero. Therefore,

$$(A_\mathfrak{a}, E_\xi, \frac{1}{N}\vec{a})^\sigma = (A_{\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathfrak{b})}, E_{\xi\,\mathrm{N}(\mathfrak{b})^{-1}}, \vec{c}),$$

i.e.,

$$\tau(\mathfrak{a}, \xi, \vec{a})^\sigma = \tau(\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathfrak{b}), \xi\,\mathrm{N}(\mathfrak{b})^{-1}, \vec{c}).$$

□

Let $f(\tau)$ be a memormorphic modular function on $\mathbb{H}_g$ for $\Gamma(N)$, viewed also as a rational function on $\mathcal{X}^*(\mathbb{C})$, and let

$$f(\tau) = \sum_{T \in \mathrm{Sym}_g(\mathbb{Z})^*} c(T) q_N^T$$

be the Fourier expansion of $f(\tau)$ with $c(n) \in \mathbb{C}$ and $q_N^T = e(\frac{1}{N} \operatorname{tr} T\tau)$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, f^σ, as a rational function on $\mathcal{X}^*(\mathbb{C})$, is defined to satisfy the following condition: for every $P \in \mathcal{X}^*(\mathbb{C})$, one has

$$f(P)^\sigma = f^\sigma(P^\sigma).$$

By the q-expansion principle, f^σ has the following Fourier expansion:

$$f^\sigma(\tau) = \sum_T c(T)^\sigma q_N^T.$$

Now the following explicit Galois action formula on CM values follows directly from Theorems **??** and **??**.

Corollary 3.5. *Let $f(\tau)$ be a memomorphic modular function on $\mathbb{H}_g$ for $\Gamma(N)$ (also momomorphic at cusps). Let $\tau = \tau(\mathfrak{a}, \xi, \vec{a}) \in \mathrm{CM}(E, \Phi)$ be a CM point on $X(N)$. Let $\sigma \in \mathrm{Aut}(\mathbb{C}/\tilde{E})$, and let $[\mathfrak{b}] \in \mathrm{Cl}(\tilde{\Phi}, N)$ such that $\sigma|_{\tilde{E}^{ab}} = \sigma_{\mathfrak{b}^{-1}}$. Then*

$$f(\tau)^\sigma = f^\sigma(\tau(\mathfrak{a} \, \mathrm{N}_{\tilde{\Phi}} \, \mathfrak{b}, \xi \, \mathrm{N}(\mathfrak{b})^{-1}, \vec{c})).$$

where $\tau(\mathfrak{a} \, \mathrm{N}_{\tilde{\Phi}} \, \mathfrak{b}, \xi \, \mathrm{N}(\mathfrak{b})^{-1}, \vec{c}) = \tau^\sigma$ is given as in Theorem **??**.

Proof. Let X be a toriodal compactification of $\mathcal{X}^*/\mathbb{Q}$ which is a projective algebraic variety. By our assumption, f is a rational function on X. So $f(\tau)^\sigma = f^\sigma(\tau^\sigma)$, and the first claim follows directly from Theorem **??**. □

The case $N = 2$ and $g = 2$ is used in [**?**] and was the initial motivation for this work.

Remark 3.6. It should be very interesting to work out the whole Galois orbit of a CM point under $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. It should be doable using Deligne and Langlands' generalization of Theorem **??** (see [**?**]).

Remark 3.7. There is another group acting on $\mathrm{CM}(E, \Phi)$. Let

$$C_0(E, N) = \frac{\{(\mathfrak{b}, \alpha \in \mathbb{Q}_{>0}, b \in \mathfrak{b}/N\mathfrak{b}) : \mathrm{N}_{E/F} \, \mathfrak{b} = \alpha \mathcal{O}_F, b\bar{b}\alpha^{-1} \equiv 1 \pmod{N}\}}{\{(\xi \mathcal{O}_E, \xi\bar{\xi}, \xi) : \xi \in E^\times, \xi \equiv 1 \pmod{N}\}}.$$

The action is given as follows.

$$(\mathfrak{b}, \alpha, b)(\mathfrak{a}, \xi, \frac{1}{N}\vec{a}) = (\mathfrak{a}\mathfrak{b}, \alpha^{-1}\xi, \frac{b}{N}\vec{a}).$$

4 Reciprocity law

In this section, we will use Corollary **??** to give another proof of Streng's explicit Shimura reciprocity law and the original Shimura reciprocity law. We need some notations before stating their theorems ([**?**], [**?**], [**?**]). We will mainly follow [**?**] in the review and refer to it for more detail. Let $\mathcal{F}_N$ be the field of meromorphic Siegel modular functions $\frac{g_1}{g_2}$ where g_i are holomorphic

Siegel modular forms of level N and equal weight with Fourier coefficients in $\mathbb{Q}(\mu_N)$, and $g_2 \neq 0$. By the q-expansion principle, one has $\mathcal{F}_N = \mathbb{Q}(\mu_N)(\mathcal{X}_0) = \mathbb{Q}(\mu_N)(\mathcal{X}^*)$. let $\mathcal{F}_\infty = \cup \mathcal{F}_N$. The following proposition is due to Shimura, see [**?**, Propositions 2.1, 3.1]. Let $G(\mathbb{R})^+$ be the subgroup of $G(\mathbb{R})$ with $\mu(g) > 0$, $G(\mathbb{A})^+ = G(\mathbb{A}_f) \times G(\mathbb{R})^+$ and $G(\mathbb{Q})^+ = G(\mathbb{R})^+ \cap G(\mathbb{Q})$. Recall $G = \mathrm{GSp}_g$ (GSp_{2g} in Streng's notation).

Proposition 4.1. *(a) There is a unique action of $G(\mathbb{A})^+$ on $\mathcal{F}_\infty$ satisfying the following condition:*

1. *For $\gamma \in G(\mathbb{Q})^+$, one has $f^\gamma(\tau) = f(\gamma\tau)$,*
2. *for $x \in \mathbb{A}^\times$, one has $f^{v(x)} = f^{\sigma_x}$. Here $\sigma_x \in \mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}$ is the Artin map image of x via the class field theory, $v(x) = \mathrm{diag}(I_g, xI_g)$, and $f^\sigma(\tau)$ is the new modular function with σ acts on the Fourier coefficients of f.*
3. *for any $N \geq 1$, the group $K(N) \times G(\mathbb{R})^\times$ acts on $\mathcal{F}_N$ trivially. Here we recall that $K(N)$ is the compact open subgroup of $G(\mathbb{A}_f)$ defining $\mathcal{X}$.*

(b) There is a unique action of $G(\mathbb{Z}/N)$ on $\mathcal{F}_N$ as follows:

1. *The action of $\mathrm{Sp}_g(\mathbb{Z}/N)$ on $\mathcal{F}_N$ is given by $f^{\gamma \pmod N} = f^\gamma$ for $\gamma \in \mathrm{Sp}_g(\mathbb{Z})$, where f^γ is given by (a)(1) above,*
2. *for any $x \in (\mathbb{Z}/N)^\times$, $f^{v(x)} = f^{\sigma_x}$.*

Now we are ready to give a direct proof of Streng's explicit reciprocity law (without using Shimura's reciprocity law). Please note that we only deal with the case of maximal order of E while Shimura and Streng deals with general case, although our method works in general too.

Theorem 4.2. *([**?**, Theorem 2.4]) Let $\tau = \tau(\mathfrak{a}, \xi, \vec{a}) \in X(N)$ be a CM point of CM type (E, Φ) as before. Let $\sigma = \sigma_{\mathfrak{b}^{-1}} \in \mathrm{Gal}(H(\tilde{\Phi}, N)/\tilde{E})$. Let $\vec{b}$ be a symplectic basis of $\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathfrak{b})$ with respect to $E_{\xi\,\mathrm{N}(\mathfrak{b})^{-1}}$. Let $M \in \mathrm{GSp}_g(\mathbb{Q})^+$ such that $M(\vec{a}) = \vec{b}$. Then M is N-integral and invertiable modulo N. Let $U = M^{-1} \pmod N \in \mathrm{GSp}_g(\mathbb{Z}/N)$. Then for any $f \in \mathcal{F}_N$, one has*

$$f(\tau)^\sigma = f^U(M\tau).$$

Proof. By Corollary **??**, one has

$$f(\tau)^\sigma = f^\sigma(\tau(\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathfrak{b}), \xi d, \vec{c}))$$

where $d = \mathrm{N}(\mathfrak{b})^{-1}$ and $\vec{c}$ is the symplectic basis of $\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathfrak{b})$ with respect to $E_{d\xi}$ given in Theorem **??**. Let $\mathrm{Cl}(\mathbb{Z}, N)$ be the ray class group of $\mathbb{Q}$ with modulus N, its associated class field is $\mathbb{Q}(\mu_N)$. Notice that the norm map maps $\mathrm{Cl}(\tilde{\Phi}, N)$ onto $\mathrm{Cl}(\mathbb{Z}, N)$ (this also explains $\mathbb{Q}(\mu_N) \subset H(\tilde{\Phi}, N)$. So by the class field theory, one has $\mathbb{Q}(\mu_N) \subset H(\tilde{\Phi}, N)$, and

$$\sigma_{\mathfrak{b}^{-1}}|_{\mathbb{Q}(\mu_N)} = \sigma_{\mathrm{N}(\mathfrak{b})^{-1}}|_{\mathbb{Q}(\mu_N)}.$$

So $f^\sigma = f^{\sigma_d} = f^{v(d)}$, and

$$f(\tau)^\sigma = f^{v(d)}(\tau(\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathfrak{b}), \xi d, \vec{c}))$$

Let $\gamma \in \mathrm{Sp}_g(\mathbb{Z})$ such that $\gamma(\vec{b}) = \vec{c}$. Then $\gamma M(\tau) = \tau(\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathfrak{b}), \xi d, \vec{c})$. On the other hand, $M\vec{a} = \vec{b}$ implies $\mu(M) = d^{-1}$ and thus $\mu(U) = d \pmod N$, and $U(\vec{b}) = \vec{a} \pmod N$. So

$$U = v(d)\gamma \pmod N.$$

Therefore

$$f^U(M\tau) = f^{v(d)}(\gamma M\tau) = f^{v(d)}(\tau(\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathfrak{b}), \xi d, \vec{c})) = f(\tau)^\sigma$$

as claimed. □

Finally, we derive Shimura's reciprocity law in its original adelic form ([?, Page 57]). Let $\tau = \tau(\mathfrak{a}, \xi, \vec{a}) \in \mathrm{CM}(E, \Phi) \in X(N)$ as before. Recall the maps ϵ and g in (??) and (??). The following is the Shimura's reciprocity law ([?, Page 57]) (see also [?, Theorem 3.4]).

Theorem 4.3. *(Shimura) Let $\tau = \tau(\mathfrak{a}, \xi, \vec{a}) \in \mathrm{CM}(E, \Phi) \in X(N)$ be a CM point of CM type (E, Φ), and let $g : \tilde{E}_{\mathbb{A}}^\times \to \mathrm{GSp}_g(\mathbb{A})^+$ be the adelization of the map g defined in (??). Then for any $f \in \mathcal{F}_\infty$ such that $f(\tau)$ is finite, and any $\mathbf{b} \in \tilde{E}_{\mathbb{A}}^\times$, we have*

$$f(\tau) \in \tilde{E}^{ab}, \text{ and } f(\tau)^{\sigma_{\mathbf{b}^{-1}}} = f^{g(\mathbf{b})}(\tau).$$

Proof. We can choose N big enough so that $f \in \mathcal{F}_N$, and view then τ as a CM point on $X(N)$. So $f(\tau) \in H(\tilde{\Phi}, N)$, and both sides of the identity depends only on the idele class $[\mathbf{b}] \in \tilde{E}^\times \backslash \tilde{E}_f^\times / U(\tilde{\Phi}, N)$. Therefore we may assume that $\mathbf{b}_\mathfrak{p} = 1$ for all primes of $\tilde{E}$ above N, and let $\mathfrak{b} = (\mathbf{b})$ be the fractional ideal of $\tilde{E}$ associated to $\mathbf{b}$. Let $\tau^\sigma = (\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathfrak{b}), \xi\,\mathrm{N}(\mathfrak{b})^{-1}, \vec{c})$ as in Theorem ??. We write $\hat{\mathfrak{a}} = \mathfrak{a} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, and $g = g(\mathbf{b})$. Then $\widehat{\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathfrak{b})}$ has two similitude symplectic $\hat{\mathbb{Z}}$-bases $g(\vec{a}) = \mathrm{N}_{\tilde{\Phi}}\,\mathbf{b}\vec{a}$ and $\vec{c}$ (with respect to E_ξ). So there is $\gamma \in \mathrm{GSp}_g(\hat{\mathbb{Z}})$ such that $\gamma^{-1}g(\vec{a}) = \vec{c}$ and $\mu(\gamma^{-1}) = \frac{\mathrm{N}(\mathfrak{b})}{\mathrm{N}(\mathbf{b})} \in \hat{\mathbb{Z}}^\times$. Let $M = \gamma^{-1}g \in \mathrm{GSp}_g(\mathbb{A}_f)$ with $\mu(M) = \mathrm{N}(\mathfrak{b})$. Since $\vec{a}$ and $\vec{c} = M(\vec{a})$ are both similitude symplectic $\mathbb{Q}$-bases of E (with respect to E_ξ), one has $M \in \mathrm{GSp}_g(\mathbb{Q})^+$. Write $\gamma = \gamma_1 v(\frac{\mathrm{N}(\mathbf{b})}{\mathrm{N}(\mathfrak{b})})$ with $\gamma_1 \in \mathrm{Sp}_g(\hat{\mathbb{Z}})$. Recall the condition on $\vec{c}$ in Theorem ?? and that $\mathbf{b}_\mathfrak{p} = 1$ for all $\mathfrak{p}|N$, one sees that γ_1 maps $\vec{a}$ to $\vec{a}$ modulo N. So $\gamma_1 \equiv 1 \pmod{N}$. Now $g = \gamma_1 v(\frac{\mathrm{N}(\mathbf{b})}{\mathrm{N}(\mathfrak{b})})M$ (since we write elements in G as maps in the proof, this order of decomposition is correct), one has by Proposition ?? that

$$f^g(\tau) = f^{v(\frac{\mathrm{N}(\mathbf{b})}{\mathrm{N}(\mathfrak{b})})}(M\tau) = f^{\sigma_{\mathrm{N}(\mathfrak{b})^{-1}}}(\tau(\mathfrak{a}\,\mathrm{N}_{\tilde{\Phi}}(\mathfrak{b}), \xi\,\mathrm{N}(\mathfrak{b})^{-1}, \vec{c})) = f(\tau)^\sigma$$

as claimed. □

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