

# Pivotal tricategories and a categorification of inner-product modules

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## Abstract

This article investigates duals for bimodule categories over finite tensor categories. We show that finite bimodule categories form a tricategory and discuss the dualities in this tricategory using inner homs. We consider inner-product bimodule categories over pivotal tensor categories with additional structure on the inner homs. Inner-product module categories are related to Frobenius algebras and lead to the notion of  $*$ -Morita equivalence for pivotal tensor categories. We show that inner-product bimodule categories form a tricategory with two duality operations and an additional pivotal structure. This work is motivated by defects in topological field theories.

## 1 Introduction

Tensor categories are intimately linked with low-dimensional topology via 3-dimensional topological field theory and 2-dimensional conformal field theories (TFT and CFT). There is ongoing work by many researchers to generalize these theories by constructing on the one hand non-semisimple theories [17, 24] and on the other to include defects in existing semisimple theories [15, 25]. It is expected by physics and topological reasons that defects of all codimension in these theories form a tricategory with certain dualities. In an oriented theory, the dualities should be equipped with additional structures. We propose the notion of a pivotal tricategory with duals as structure of oriented defects in 3-d TFT. It is expected that an important class of defects can be obtained from bimodule categories over tensor categories [16, 25]. We define the class of inner-product bimodule categories over pivotal finite tensor categories and show that these define a pivotal tricategory with duals. This naturally leads to a notion of  $*$ -Morita equivalence for pivotal finite tensor categories, in particular it allows to transport pivotal structures to the categories of endofunctors. We show that inner-product module categories are linked with Frobenius algebras in tensor categories.

**Defects in TFT** Defects arose first in the physics literature as lower-dimensional regions separating different phases of a statistical mechanical theory. The defects themselves can have defects of lower dimensions. In a 3-dimensional theory, a general defect is located on a surface, that itself could contain line defects which in turn might exhibit various point defects. If the theory is topological and the defects are topological, there should be a notion of a fusion of two defects of a given codimension to a defect of the same codimension. Furthermore one requires some sort of associativity and unital properties of the fusion. In the case of 2-dimensional theories [5] considers the bicategory of bimodules over algebras as defects. In three dimensions the relevant notion is that of a tricategory. If additionally the defects are oriented, orientation reversal corresponds to certain duality operations in the tricategory including a relation between the left and right duals of the defects. These structures are formalized in the

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notion of a pivotal tricategory with duals. In particular, in a pivotal tricategory with duals describing defects, the monoidal category of line defects from a surface defect to itself has duals and a pivotal structure.

The dualities in a pivotal tricategory with duals are weak duals in the sense that they are not equipped with fixed coherence structures. In [29] it is shown that a pivotal tricategory with duals can be strictified to a Gray category with duals [1], where all coherence structures are fixed such that they suit a 3-dimensional diagrammatic calculus.

It is a fundamental problem to identify for a given TFT and CFT algebraic input data for defects and to describe the corresponding tricategory. In the case of a semisimple TFT, it is argued in [16], that bimodule categories describe a large class of surface defects. In [16, 22], the close relationship between defects and Frobenius algebras is discussed. The oriented semisimple TFT of [2, 32] uses a spherical fusion category as algebraic input datum. It is natural to expect that the bimodule categories have to be compatible with the spherical structure in order to define surface defects. A possible compatibility that is intimately related to Frobenius algebras is provided by the notion of a (bi)module trace [30]. For non-semisimple tensor categories we propose a categorification of inner-product modules as algebraic datum for surface defects.

**Inner-product modules** Inner-product module were first defined by Kaplansky [21] in the setting of  $C^*$ -algebras and used by Rieffel [28] to define the notion of strong Morita equivalence for  $C^*$ - and  $W^*$ -algebras. The theory was developed in an algebraic setting in [4] for  $*$ -algebras  $C$  over  $\mathbb{C}$  and more general ordered rings, where  $*$  is an antilinear involutive antihomomorphism. The following is a slight modification of the definition in [4].

**Definition 1.1** *Let  $C$  be a  $*$ -algebra over  $\mathbb{C}$ . An inner-product module over  $C$  is a  $C$ -module  $M$  with a  $C$ -valued sesquilinear inner product  ${}_C\langle \cdot, \cdot \rangle^M : M \times M \rightarrow C$  that is non-degenerate,  $C$ -linear in the first argument, i.e.  ${}_C\langle c \triangleright m, \tilde{m} \rangle = c \cdot \langle m, \tilde{m} \rangle$  and satisfies  ${}_C\langle m, \tilde{m} \rangle^M = ({}_C\langle \tilde{m}, m \rangle^M)^*$ .*

*An inner product bimodule  ${}_D M_C$  for two  $*$ -algebras is a bimodule that is both a left  $D$  and right  $C$  inner product module, such that  ${}_C\langle m \triangleleft c, \tilde{m} \rangle^M = {}_C\langle m, \tilde{m} \triangleleft c^* \rangle^M$  and  $\langle d \triangleright m, \tilde{m} \rangle_D^M = \langle m, d^* \triangleright \tilde{m} \rangle_D^M$ .*

One motivation for this definition is that inner-product bimodules over  $*$ -algebras form a bicategory with objects  $*$ -algebras, 1-morphisms inner-product bimodules and 2-morphisms intertwiners. The  $D$ -valued inner-product of the relative tensor product  ${}_D M_C \otimes_{C,C} N_E$  of two bimodule categories is thereby defined by the so-called Rieffel-induction: For  $m \otimes n, \tilde{m} \otimes \tilde{n} \in M \otimes N$ ,

$${}_D\langle m \triangleleft (n, \tilde{n})^N, \tilde{m} \rangle^M \quad (1.1)$$

is  $C$ -balanced and induces a  $D$ -valued inner product on  ${}_D M_C \otimes_{C,C} N_E$ . Restricted to invertible inner-product bimodules, this bicategory leads to the  $*$ -Picard groupoid for  $*$ -algebras and to the notion of  $*$ -Morita equivalence [28].

We apply these notions to tensor categories using the following observation. If  $\mathcal{C}$  is a pivotal tensor category over  $\mathbb{C}$ , its complexified Grothendieck ring  $Gr(\mathcal{C})$  is naturally a  $*$ -algebra, where the  $*$ -structure is given by the (say left) duality operation on  $\mathcal{C}$  extended antilinearly to the complexification. The pivotal structure serves to guarantee the identity  $** = 1$  on  $Gr(\mathcal{C})$ . Furthermore, for bimodule categories  ${}_D \mathcal{M}_E$  over finite tensor categories, there exist inner hom functors, that provide the Grothendieck group  $Gr(\mathcal{M})$  with the structure of an  $(Gr(\mathcal{D}), Gr(\mathcal{C}))$ -bimodule with two algebra valued inner-products, apart from the condition  $\langle m, n \rangle^* = \langle n, m \rangle$ . Our definition of an inner-product bimodule category is such that its Grothendieck group satisfies also this relation.

While in Definition 1.1 the inner-products are additional structure on the bimodules, in our categorified version, the inner products are canonically given by the inner-hom for a bimodule category. It is just a coherent isomorphism  $I_{m,n} : \langle m, n \rangle^* \simeq \langle n, m \rangle$  for objects  $m, n \in \mathcal{M}$  that appears as additional structure. It might be interesting to consider also bimodule categories, where a different inner product is part of the structure.

**Results** Our first result considers bimodule categories over finite tensor categories.

**Theorem 1.2** *Finite tensor categories, finite bimodule categories, right exact bimodule functors and bimodule natural transformations form an algebraic tricategory  $\mathbf{Bimod}$  in the sense of [20].*

This result is obtained using the 2-functorial properties of the tensor product of bimodule categories in a systematic way. It extends the work of [19] and is built heavily on the results of [13, 14]. For the existence of the tensor product of bimodule categories we rely on [8] and the subsequent work [10]<sup>1</sup>.

Using the inner hom functors we describe duals for the bimodule categories in this tricategory. These dualities were obtained by different methods in [8] and independently in the semisimple case in [29]. The methods in this article are generalizations of the methods used in [29]. We show that for separable bimodule categories [8], the bimodule functors have both adjoints and thus the tricategory of separable bimodule categories is a tricategory with (two types of) duals [8, 14]. The calculus of the inner homs allows furthermore to characterize the Serre equivalences between the left and right duals of separable bimodule categories.

For a pivotal tricategory with duals we furthermore ask for a pivotal structure for the bimodule functors. To this end we define inner-product bimodule categories over pivotal finite tensor category and show the following.

**Theorem 1.3** *Inner-product bimodule categories over pivotal finite tensor categories form a pivotal tricategory with duals. In particular, the category of endofunctors for an inner-product module category is again a pivotal finite tensor category in a canonical way.*

It is shown that a version of the Rieffel-induction on inner homs holds for finite bimodule categories and this is used in a crucial way to induce the structure of an inner-product bimodule category on the tensor product of two inner-product bimodule categories. The invertible morphisms in this tricategory lead to notion of \*-Morita equivalence for pivotal finite tensor categories. Examples of inner-product module categories are obtained from Frobenius algebras and from module traces [30] in the semisimple case:

- Theorem 1.4**
- i) If  $A \in \mathcal{C}$  is a special symmetric Frobenius algebra in a pivotal finite tensor category  $\mathcal{C}$ , then the category of modules  $\mathbf{Mod}(\mathcal{C})_A$  is a  $\mathcal{C}$ -inner-product module category.*
  - ii) A semisimple bimodule category  ${}_D\mathcal{M}_C$  over spherical fusion categories  $\mathcal{C}, \mathcal{D}$  with bimodule trace is an inner-product bimodule category.*

TFTs with defects of all codimension are expected to define also fully-extended TFTs [26]. It is not surprising that some of the structures in this work appear also in the work [8] on fully dualizable 3-d TFTs. It is shown in [9] using different methods that finite bimodule categories over finite tensor categories form even a symmetric monoidal tricategory and in [8] that finite tensor categories are 2-dualizable in this symmetric monoidal tricategory. Furthermore it is argued that they should lead to a non-compact framed 3-d TFT. The tricategory  $\mathbf{Bimod}$  should then constitute defects for this TFT. It is natural to expect that inner-product bimodule categories are related to a homotopy fixed point for passing from this framed to an oriented theory.

Our description of the duals in  $\mathbf{Bimod}$  uses mainly tools from enriched category theory and might be interesting for other higher categories as well.

**Structure of the article** In Section 2 we recall the basic notions of bimodule categories, bimodule functors, bimodule natural transformations and balanced functors. We then investigate the tensor product of module categories in Section 3. In the remainder of this section we develop the theory of multi-module categories that serves as an important tool in the proof that bimodule categories over finite tensor categories form a tricategory. In Section 4 we define tricategories with duals and pivotal tricategories. Next we enhance the existing calculus of the inner hom and use it to define the duals for bimodule categories. Furthermore we

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<sup>1</sup>We are grateful to the authors for communicating early drafts of [9, 10].

discuss the Serre bimodule functors between the left and right duals of a separable bimodule category. In the last section we define inner-product bimodule categories and show that they form a pivotal tricategory. The example of Frobenius algebras and the relation with bimodule traces for semisimple module categories is discussed. The appendix contains definitions and conventions for duals in monoidal categories, bicategories and the definition of an algebraic tricategory.

Part of this work, especially some results of Section 3 appeared in the authors PhD thesis [29].

## 2 Preliminaries on (bi)module categories

We summarize definitions and known results about module categories over finite tensor categories. Let  $\mathbb{k}$  be a field. Throughout this work, all categories are assumed to be  $\mathbb{k}$ -linear and abelian and all functors are requested to be linear unless stated otherwise.

**Module categories, functors and natural transformations** The definition of a finite tensor category is recalled in Definition A.3. The systematic investigation of module categories over finite tensor categories, was initiated in [14].

**Definition 2.1** ([27], [3]) *Let  $\mathcal{C}$  be a finite tensor category. A (left)  $\mathcal{C}$ -module category is a finite  $\mathbb{k}$ -linear abelian category  $\mathcal{M}$ , together with a bilinear exact functor  $\triangleright : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$ , called the action of  $\mathcal{C}$  on  $\mathcal{M}$ , and natural isomorphisms*

$$\mu_{x,y,m}^{\mathcal{M}} : (x \otimes y) \triangleright m \rightarrow x \triangleright (y \triangleright m), \quad \lambda_m^{\mathcal{M}} : \mathbf{1}_{\mathcal{C}} \triangleright m \rightarrow m, \quad (2.1)$$

for all  $x, y \in \mathcal{C}$ ,  $m \in \mathcal{M}$ , called the module constraints, such that the diagrams

$$\begin{array}{ccc} & ((x \otimes y) \otimes z) \triangleright m & \\ \swarrow \omega_{x,y,z} \triangleright \text{id}_m & & \searrow \mu_{x \otimes y, z, m}^{\mathcal{M}} \\ (x \otimes (y \otimes z)) \triangleright m & & (x \otimes y) \triangleright (z \triangleright m) \\ \downarrow \mu_{x, y \otimes z, m}^{\mathcal{M}} & & \downarrow \mu_{x, y, z \triangleright m}^{\mathcal{M}} \\ x \triangleright ((y \otimes z) \triangleright m) & \xrightarrow{\text{id}_x \triangleright \mu_{y, z, m}^{\mathcal{M}}} & x \triangleright (y \triangleright (z \triangleright m)), \end{array} \quad (2.2)$$

and

$$\begin{array}{ccc} (x \otimes \mathbf{1}) \triangleright m & \xrightarrow{\mu_{x, \mathbf{1}, m}^{\mathcal{M}}} & x \triangleright (\mathbf{1} \triangleright m) \\ & \searrow \rho_x \triangleright m & \swarrow \mathbf{1}_x \triangleright \lambda_m^{\mathcal{M}} \\ & x \triangleright m & \end{array} \quad (2.3)$$

commute for all objects  $x, y, z \in \mathcal{C}$  and  $m \in \mathcal{M}$ , where the isomorphisms  $\omega_{x,y,z} : (x \otimes y) \otimes z \rightarrow x \otimes (y \otimes z)$  and  $\rho_x : x \otimes \mathbf{1} \rightarrow x$  are the constraint morphisms of  $\mathcal{C}$  as a monoidal category. To emphasize that  $\mathcal{M}$  is a left  $\mathcal{C}$ -module category, we denote it  ${}_e\mathcal{M}$ . Whenever this is unambiguous, we denote the constraints of  $\mathcal{M}$  just by  $\mu$  and  $\lambda$ .

The definition of a right  $\mathcal{C}$ -module category  $\mathcal{M}_e$  is analogously given in terms of a bilinear exact functor  $\triangleleft : \mathcal{M}_e \times \mathcal{C} \rightarrow \mathcal{M}_e$ . We denote the constraint for the unit of a right module category by  $\rho_m^{\mathcal{M}} : m \triangleleft \mathbf{1}_{\mathcal{C}} \rightarrow m$  and where it is otherwise ambiguous, we denote a left module action on a category  $\mathcal{M}$  by  $\mu^{\mathcal{M}, l}$  or just  $\mu^l$  and the right module action by  $\mu^{\mathcal{M}, r}$  or just  $\mu^r$ .

It is clear, that a  $\mathcal{D}$ -module category structure on  $\mathcal{M}$  is the same as a tensor functor  $L_- : \mathcal{D} \rightarrow \text{Fun}(\mathcal{M}, \mathcal{M})$ , where  $L_d(m) = d \triangleright m$  for  $d \in \mathcal{D}$  and  $m \in \mathcal{M}$ .

The following is an important subclass of module categories, that is investigated in detail in [14].

**Definition 2.2** ([14, Def. 3.1]) *A module category  ${}_e\mathcal{M}$  is called exact, if for any projective object  $P \in \mathcal{C}$  and any object  $m \in \mathcal{M}$ , the object  $P \triangleright m$  is projective in  $\mathcal{M}$ .*

If  $\mathcal{C}$  is semisimple, a module category  ${}_e\mathcal{M}$  is exact if and only if it is semisimple [14].

We denote by  $\mathcal{C}^{\text{rev}}$  the category  $\mathcal{C}$  with the reversed monoidal product, but the same source and target map for the morphisms. This has to be distinguished from  $\mathcal{C}^{\text{op}}$  which is the category  $\mathcal{C}$  with reversed order of the arrows but with the same monoidal product as  $\mathcal{C}$ . It follows directly from the definitions, that a  $\mathcal{C}$ -right module category is the same as a  $\mathcal{C}^{\text{rev}}$ -left module category.

**Example 2.3** We consider some examples of module categories over  $\mathcal{C}$ .

- i) Let  $\mathbf{Vect}$  denote the category of finite dimensional  $\mathbb{k}$ -vector spaces regarded as semisimple tensor category. Every finite linear category  $\mathcal{M}$  is a  $\mathbf{Vect}$ -module category with action determined by  $\mathbf{Hom}_{\mathcal{M}}(\tilde{m}, V \otimes m) = \mathbf{Hom}_{\mathcal{M}}(\tilde{m}, m) \otimes_{\mathbb{k}} V$  for  $V \in \mathbf{Vect}$ ,  $m, \tilde{m} \in \mathcal{M}$ .
- ii) The category  $\mathcal{C}$  itself is a left  $\mathcal{C}$ - and right  $\mathcal{C}$ -module category with actions given by the tensor product. It is exact as left  $\mathcal{C}$ - and right  $\mathcal{C}$ -module category.
- iii) Let  $A \in \mathcal{C}$  be an algebra object, then the category  $\text{Mod}(\mathcal{C})_A$  of  $A$ -right modules in  $\mathcal{C}$  is naturally a left  $\mathcal{C}$ -module category with module action given by the tensor product.

**Remark 2.4** Let  ${}_e\mathcal{M}$  be a  $\mathcal{C}$ -module category and  $\mathcal{N}$  any finite category. It is clear that the functor  $\triangleright \times 1_{\mathcal{N}} : \mathcal{C} \times \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{M} \times \mathcal{N}$  satisfies the properties (2.2) and (2.3) of a  $\mathcal{C}$ -module action on  $\mathcal{M} \times \mathcal{N}$ . We will thus abuse notation and call the category  ${}_e\mathcal{M} \times \mathcal{N}$  also a  $\mathcal{C}$ -module category, although the functor  $\triangleright \times 1_{\mathcal{N}}$  is of course only bilinear with respect to the first argument but not bilinear as a functor  $\mathcal{C} \times (\mathcal{M} \times \mathcal{N}) \rightarrow \mathcal{M} \times \mathcal{N}$ .

Module functors between left  $\mathcal{C}$ -module categories  $({}_e\mathcal{M}, \mu^{\mathcal{M}}, l^{\mathcal{M}})$  and  $({}_e\mathcal{N}, \mu^{\mathcal{N}}, l^{\mathcal{N}})$  are functors with additional constraint isomorphisms that relate the two module actions.

**Definition 2.5** ([27]) A  $\mathcal{C}$ -module functor  $F : {}_e\mathcal{M} \rightarrow {}_e\mathcal{N}$  is a linear functor  $F$  together with natural isomorphisms  $\phi_{x,m}^F : F(x \triangleright m) \rightarrow x \triangleright F(m)$ , such that the diagrams

$$\begin{array}{ccc}
 & F((x \otimes y) \triangleright m) & \\
 & \swarrow \quad \searrow & \\
 F(x \triangleright (y \triangleright m)) & & (x \otimes y) \triangleright F(m) \\
 \downarrow \phi_{x,y \triangleright m}^F & & \downarrow \mu_{x,y,F(m)}^{\mathcal{N}} \\
 x \triangleright F(y \triangleright m) & \xrightarrow{\text{id}_x \triangleright \phi_{y,m}^F} & x \triangleright (y \triangleright F(m))
 \end{array} \quad (2.4)$$

and

$$\begin{array}{ccc}
 & F(\mathbf{1}_e \triangleright m) & \\
 & \swarrow \quad \searrow & \\
 \mathbf{1}_e \triangleright F(m) & & F(m) \\
 & \xrightarrow{\lambda_{F(m)}^{\mathcal{N}}} & \\
 \phi_{\mathbf{1}_e, m}^F & & F(\lambda_m^{\mathcal{M}})
 \end{array} \quad (2.5)$$

commute for all  $x, y \in \mathcal{C}$  and  $m \in \mathcal{M}$ . We sometimes write  $(F, \phi^F)$  for a module functor and call  $\phi^F$  a left module constraint for  $F$ . Whenever this is unambiguous, we denote the constraint just by  $\phi$ . There is the analogous definition for module functors between right  $\mathcal{C}$ -module categories.

Natural transformations between module functors are required to be compatible in the following way.

**Definition 2.6** ([27]) Let  $(F, \phi^F) : {}_e\mathcal{M} \rightarrow {}_e\mathcal{N}$  and  $(G, \phi^G) : {}_e\mathcal{M} \rightarrow {}_e\mathcal{N}$  be module functors. A module natural transformation  $\eta : F \rightarrow G$  is a natural transformation such that the diagram

$$\begin{array}{ccc}
 F(x \triangleright m) & \xrightarrow{\eta_{x \triangleright m}} & G(x \triangleright m) \\
 \downarrow \phi_{x,m}^F & & \downarrow \phi_{x,m}^G \\
 x \triangleright F(m) & \xrightarrow{\text{id}_x \triangleright \eta_m} & x \triangleright G(m),
 \end{array} \quad (2.6)$$

commutes for all  $x \in \mathcal{C}$  and  $m \in \mathcal{M}$ .

It is easy to see that the composite of module natural transformations is again a module natural transformation. Hence, for module categories  ${}_e\mathcal{M}$  and  ${}_e\mathcal{N}$ , the module functors and module natural transformations from  ${}_e\mathcal{M}$  to  ${}_e\mathcal{N}$  form a category that is denoted  $\text{Fun}_e({}_e\mathcal{M}, {}_e\mathcal{N})$ .

**Bimodule categories** When combined, the notions of left and right module categories lead to the notion of bimodule categories. First we present a compact definition of a bimodule category that uses the notion of the Deligne product [6] of abelian categories. For finite tensor categories  $\mathcal{C}$  and  $\mathcal{D}$ , the category  $\mathcal{C} \boxtimes \mathcal{D}$  is again a finite tensor category, see also [11, Section 1.46] for more details.

**Definition 2.7** *A  $(\mathcal{D}, \mathcal{C})$ -bimodule category  ${}_D\mathcal{M}_e$  is a left  $\mathcal{D} \boxtimes \mathcal{C}^{\text{rev}}$ -module category. A bimodule category  ${}_D\mathcal{M}_e$  is called exact if it is exact as a  $\mathcal{D} \boxtimes \mathcal{C}^{\text{rev}}$ -module category.*

If one unpacks this definition, one sees that a  $(\mathcal{D}, \mathcal{C})$ -bimodule category  ${}_D\mathcal{M}_e$  is the same as a left  $\mathcal{D}$ - and right  $\mathcal{C}$ -module category  ${}_D\mathcal{M}_e$  together with a family of natural isomorphisms  $\gamma_{d,m,c} : (d \triangleright m) \triangleleft c \rightarrow d \triangleright (m \triangleleft c)$ , for  $d \in \mathcal{D}$ ,  $c \in \mathcal{C}$  and  $m \in \mathcal{M}$ , that satisfies pentagon diagrams with respect to the action of  $\mathcal{C}$ ,  $\mathcal{D}$  and the triangle diagram with respect to the units, see e.g. [19, Proof of Prop. 2.12]. This second view on bimodule categories allows for the notion of biexact bimodule categories.

**Definition 2.8** *Let  $\mathcal{C}, \mathcal{D}$  be finite tensor categories. A biexact  $(\mathcal{D}, \mathcal{C})$ -bimodule category  ${}_D\mathcal{M}_e$  is a finite bimodule category that is exact both as a left  $\mathcal{C}$ -module and as right  $\mathcal{D}$ -module category.*

This definition differs from the weaker notion of an exact bimodule category in Definition 2.7. To see the difference consider  ${}_e\mathcal{C}$  as  $(\mathcal{C}, \text{Vect})$ -bimodule category. It is an exact left  $\mathcal{C} \boxtimes \text{Vect} \simeq \mathcal{C}$ -module category, hence an exact bimodule category in the sense of Definition 2.7. However it is only an exact right  $\text{Vect}$ -module category if  $\mathcal{C}$  is semisimple, hence in general not a biexact bimodule category. However the converse statement holds in general.

**Lemma 2.9** *A biexact  $(\mathcal{D}, \mathcal{C})$  bimodule category  ${}_D\mathcal{M}_e$  is an exact  $\mathcal{D} \boxtimes \mathcal{C}^{\text{rev}}$ -module category.*

Proof. According to [8, Lemma 3.3.6], exactness of a module category  $\mathcal{M}_e$  is equivalent to the property that for a set of generating projective objects  $P = \{p_\alpha\}$ ,  $p_\alpha \triangleright m$  is projective for all  $m \in \mathcal{M}$  and all  $\alpha$ . If  $\{p_\alpha\}$  and  $\{q_\beta\}$  are sets of generating projective objects for  $\mathcal{C}$  and  $\mathcal{D}$ , then  $\{p_\alpha \boxtimes q_\beta\}$  is a set of generating projective objects of  $\mathcal{C} \boxtimes \mathcal{D}$ . This can be seen most easily if we choose  $\text{Vect}$ -algebras  $A$  and  $B$ , such that  $\mathcal{C} \simeq \text{Mod}_A$ ,  $\mathcal{D} \simeq \text{Mod}_B$  as linear categories. Then  $\{A\}$  and  $\{B\}$  are generating projective objects of  $\mathcal{C}$  and  $\mathcal{D}$ , while  $\{A \otimes B\}$  is generating projective for  $\mathcal{C} \boxtimes \mathcal{D} \simeq \text{Mod}_{A \otimes B}$ . Assume now that  ${}_D\mathcal{M}_e$  is a biexact bimodule category and  $\{p_\alpha\}$  and  $\{q_\beta\}$  are sets of generating projective objects for  $\mathcal{C}$  and  $\mathcal{D}$ . Then  $(p_\alpha \boxtimes q_\beta) \triangleright m = p_\alpha \triangleright m \triangleleft q_\beta$  is projective, hence  ${}_D\mathcal{M}_e$  is an exact bimodule category.  $\square$  Clearly,  $\mathcal{C}$  is a  $(\mathcal{C}, \mathcal{C})$ -bimodule category with actions given by the tensor product, see Example 2.3 ii). This bimodule category is denoted  ${}_e\mathcal{C}_e$  and called the unit bimodule category.

The compact definition of a bimodule category also directly defines bimodule functors and bimodule natural transformations between  $(\mathcal{D}, \mathcal{C})$ -bimodule categories as  $\mathcal{D} \boxtimes \mathcal{C}^{\text{rev}}$ -module functors and  $\mathcal{D} \boxtimes \mathcal{C}^{\text{rev}}$ -module natural transformation. The following gives a more explicit characterization of bimodule functors, see [19].

**Lemma 2.10** *A bimodule functor  $F : {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_{\mathcal{D}}\mathcal{N}_e$  is the same as a left  $\mathcal{D}$ - and right  $\mathcal{C}$ -module functor with module constraints  $\phi^l$  and  $\phi^r$ , respectively, such that*

$$\begin{array}{ccc}
& F((x \triangleright m) \triangleleft y) & \\
& \swarrow \quad \searrow & \\
F(x \triangleright (m \triangleleft y)) & \xrightarrow{F(\gamma_{x,m,y})} & F(x \triangleright m) \triangleleft y \\
\downarrow \phi_{x,m \triangleleft y}^l & & \downarrow \phi_{x,m \triangleleft y}^l \\
x \triangleright F(m \triangleleft y) & \xrightarrow{1_x \triangleright \phi_{m,y}^r} & (x \triangleright F(m)) \triangleleft y \\
& \searrow \quad \swarrow & \\
& x \triangleright (F(m) \triangleleft y) &
\end{array} \tag{2.7}$$

*commutes for all possible objects.*

A bimodule natural transformations in turn, is the same as a left and right module natural transformation, see [29, Lemma 2.3.6] for details.

The category of bimodule functors and bimodule natural transformations between two bimodule categories  ${}_{\mathcal{D}}\mathcal{M}_e$  and  ${}_{\mathcal{D}}\mathcal{N}_e$  is denoted  $\text{Fun}_{\mathcal{D},e}({}_{\mathcal{D}}\mathcal{M}_e, {}_{\mathcal{D}}\mathcal{N}_e)$ . It is straightforward to see that for two finite tensor categories  $\mathcal{C}$  and  $\mathcal{D}$ , the  $(\mathcal{D}, \mathcal{C})$ -bimodule categories  ${}_{\mathcal{D}}\mathcal{M}_e, {}_{\mathcal{D}}\mathcal{N}_e$ ,  $(\mathcal{D}, \mathcal{C})$ -bimodule functors  $F, G : {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_{\mathcal{D}}\mathcal{N}_e$  and  $(\mathcal{D}, \mathcal{C})$ -bimodule natural transformations  $\eta : F \rightarrow G$  form a 2-category called  $\text{Bimod}(\mathcal{D}, \mathcal{C})$  with the composition of functors as horizontal composition and the composition of natural transformations as vertical composition.

**Proposition 2.11** *Let  ${}_{e}\mathcal{M}_{\mathcal{D}}$  and  ${}_{e}\mathcal{N}_{\mathcal{D}}$  be bimodule categories.*

- i) A right (left) exact bimodule functor  $F : {}_{e}\mathcal{M}_{\mathcal{D}} \rightarrow {}_{e}\mathcal{N}_{\mathcal{D}}$  has a right (left) adjoint that is naturally bimodule functors from  ${}_{e}\mathcal{N}_{\mathcal{D}}$  to  ${}_{e}\mathcal{M}_{\mathcal{D}}$  such that the adjunctions consist of bimodule natural isomorphisms.*
- ii) If  ${}_{e}\mathcal{M}_{\mathcal{D}}$  and  ${}_{e}\mathcal{N}_{\mathcal{D}}$  are exact bimodule categories, every bimodule functor  $F : {}_{e}\mathcal{M}_{\mathcal{D}} \rightarrow {}_{e}\mathcal{N}_{\mathcal{D}}$  is exact, in particular the monoidal category  $\text{Fun}_{\mathcal{D},e}({}_{\mathcal{D}}\mathcal{M}_e, {}_{\mathcal{D}}\mathcal{M}_e)$  has left and right duals.*

*Proof.* It is a well known fact that a functor between finite linear categories has a right (left) adjoint if and only it is right (left) exact, see [10] for a detailed discussion. There is a unique way to equip the adjoint of a bimodule functor with the structure of a module functor, such that the adjunctions consist of bimodule natural isomorphisms, see e.g. [14, Sec. 3.3]. The second statement is shown in [14, Lemma 3.21]. The duals in  $\text{Fun}_{\mathcal{D},e}({}_{\mathcal{D}}\mathcal{M}_e, {}_{\mathcal{D}}\mathcal{M}_e)$  are thus given by the the left and right adjoint functors  $F^r$ , and  $F^l$ .  $\square$

Moreover, for each bimodule natural transformation  $\eta : F \rightarrow G$  between exact bimodule functors, there are canonical bimodule natural transformations  $\eta^l : G^l \rightarrow F^l$  and  $\eta^r : G^r \rightarrow F^r$ . It is shown in [14], that for exact module categories  ${}_{e}\mathcal{M}$  and  ${}_{e}\mathcal{N}$ , all module functors in  $\text{Fun}_e({}_{e}\mathcal{M}, {}_{e}\mathcal{N})$  are exact and thus  $\text{Fun}_e({}_{e}\mathcal{M}, {}_{e}\mathcal{M})$  has left and right duals, moreover it is a again a finite tensor category that is also denoted  $\mathcal{C}_{\mathcal{M}}^*$ .

**The dual bimodule categories** Let  ${}_{\mathcal{D}}\mathcal{M}_e$  be a  $(\mathcal{D}, \mathcal{C})$ -bimodule category. We then define two  $(\mathcal{C}, \mathcal{D})$ -bimodule categories  ${}_{e}\mathcal{M}_{\mathcal{D}}^{\#}$  and  ${}_{e}^{\#}\mathcal{M}_{\mathcal{D}}$  as follows: As categories, they are both  $\mathcal{M}^{\text{op}}$ , with actions

$$c \triangleright^{\#} m \triangleleft^{\#} d = {}^*d \triangleright m \triangleleft {}^*c, \tag{2.8}$$

for  $m \in {}_{e}\mathcal{M}_{\mathcal{D}}^{\#}$ , and

$$c^{\#} \triangleright m^{\#} \triangleleft d = d^* \triangleright m \triangleleft c^* \tag{2.9}$$

for  $m \in {}_{e}^{\#}\mathcal{M}_{\mathcal{D}}$ . These conventions agree with [8, Def. 3.4.4].

**Balanced functors** In the sequel we require, in addition to module functors, another notion of compatibility of functors with module structures. The following definition is taken from [13, Def 3.1] with the minor change of adding the obvious compatibility axiom with the units.

**Definition 2.12** Let  $\mathcal{A}$  be a linear category.

- i) A bilinear functor  $F : \mathcal{M}_e \times {}_e\mathcal{N} \rightarrow \mathcal{A}$  is called  $\mathcal{C}$ -balanced with balancing constraint  $\beta^F$ , if it is equipped with a family of natural isomorphisms

$$\beta_{m,c,n}^F : F(m \triangleleft c \times n) \rightarrow F(m \times c \triangleright n), \quad (2.10)$$

such that the pentagon diagram

$$\begin{array}{ccc} & F(m \triangleleft (x \otimes y) \times n) & \\ & \swarrow \quad \quad \searrow & \\ F((m \triangleleft x) \triangleleft y \times n) & & F(m \times (x \otimes y) \triangleright n) \\ \downarrow \beta_{m \triangleleft x, y, n}^F & & \downarrow F(1_m \times \mu_{x, y, n}^{\mathcal{N}}) \\ F(m \triangleleft x \times y \triangleright n) & \xrightarrow{\beta_{m, x, y \triangleright n}^F} & F(m \times x \triangleright (y \triangleright n)) \end{array} \quad (2.11)$$

and the triangle diagram

$$\begin{array}{ccc} F(m \triangleleft 1_e \times n) & \xrightarrow{\beta_{m, 1, n}^F} & F(m \times 1_e \triangleright n) \\ F(\rho_m^{\mathcal{M}} \times 1_n) \downarrow & & \swarrow F(1_m \times \lambda_n^{\mathcal{N}}) \\ & & F(m \times n) \end{array} \quad (2.12)$$

commute for all possible objects. We denote the balancing constraint  $\beta^F$  simply by  $\beta$  if this is unambiguous.

- ii) Let  $F, G : \mathcal{M}_e \times {}_e\mathcal{N} \rightarrow \mathcal{A}$  be balanced functors. A balanced natural transformation  $\eta : F \rightarrow G$  is a natural transformation  $\eta : F \rightarrow G$ , such that the diagrams

$$\begin{array}{ccc} F(m \triangleleft c \times n) & \xrightarrow{\eta_{m \triangleleft c \times n}} & G(m \triangleleft c \times n) \\ \beta_{m, c, n}^F \downarrow & & \downarrow \beta_{m, c, n}^G \\ F(m \times c \triangleright n) & \xrightarrow{\eta_{m \times c \triangleright n}} & G(m \times c \triangleright n) \end{array} \quad (2.13)$$

commute for all possible objects.

It is clear that the identity natural transformation  $1_F : F \rightarrow F$  for a balanced functor  $F$  is balanced and that the composition of balanced natural transformations yields a balanced natural transformation. Hence the balanced functors and balanced natural transformations from  $\mathcal{M}_e \times {}_e\mathcal{N}$  to  $\mathcal{A}$  form a category that is denoted  $\text{Fun}^{\text{bal}}(\mathcal{M}_e \times {}_e\mathcal{N}, \mathcal{A})$ .

As example for a balanced functor, consider a module category  ${}_e\mathcal{M}$ . It is straightforward to see that the action  $\triangleright : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$  is a balanced functor. The following statement follows directly from the definitions.

**Lemma 2.13** The composition of functors and natural transformations defines functors

- i)  $\text{Fun}^{\text{bal}}(\mathcal{M}'_e \times {}_e\mathcal{N}', \mathcal{A}) \times \text{Fun}_{e,e}(\mathcal{M}_e \times {}_e\mathcal{N}, \mathcal{M}'_e \times {}_e\mathcal{N}') \rightarrow \text{Fun}^{\text{bal}}(\mathcal{M}_e \times {}_e\mathcal{N}, \mathcal{A})$ ,  
ii)  $\text{Fun}(\mathcal{A}, \mathcal{B}) \times \text{Fun}^{\text{bal}}(\mathcal{M}_e \times {}_e\mathcal{N}, \mathcal{A}) \rightarrow \text{Fun}^{\text{bal}}(\mathcal{M}_e \times {}_e\mathcal{N}, \mathcal{B})$ .

**Balanced module functors** The following combines the notion of module functor with the notion of a balanced functor.

**Definition 2.14** A balanced (left)  $\mathcal{D}$ -module functor  $F : {}_{\mathcal{D}}\mathcal{M}_e \times {}_e\mathcal{N} \rightarrow {}_{\mathcal{D}}\mathcal{Y}$  is a balanced functor with balancing structure  $b^F$  that is also a module functor with module structure  $\phi^F$



such that the diagram

$$\begin{array}{ccc}
F((d \triangleright m) \triangleleft c \times n) & \xrightarrow{\beta_{d \triangleright m, c, n}^F} & F(d \triangleright m \times c \triangleright n) \\
\downarrow F(\gamma_{d, m, c} \times n) & & \downarrow \phi_{d, m \times c \triangleright m}^F \\
F(d \triangleright (m \triangleleft c) \times n) & & d \triangleright F(m \times c \triangleright n) \\
\downarrow \phi_{d, m \triangleleft c \times n}^F & \nearrow d \triangleright \beta_{m, c, n}^F & \\
d \triangleright F(m \triangleleft c \times n) & & 
\end{array} \tag{2.14}$$

commutes for all objects  $c \in \mathcal{C}$ ,  $d \in \mathcal{D}$ ,  $m \in \mathcal{M}$  and  $n \in \mathcal{N}$ . Balanced right module functor are defined analogously.

A balanced bimodule functor is a bimodule functor that is a balanced left- and a balanced right module functor. A balanced module natural transformation between balanced module functors is a natural transformation that is balanced and a module natural transformation.

It is clear that the balanced bimodule functors from  ${}_{\mathcal{D}}\mathcal{M}_e \times {}_e\mathcal{N}_e$  to  ${}_{\mathcal{D}}\mathcal{Y}_e$  together with balanced bimodule natural transformations form a category  $\text{Fun}_{\mathcal{D}, e}^{\text{bal}}({}_{\mathcal{D}}\mathcal{M}_e \times {}_e\mathcal{N}_e, {}_{\mathcal{D}}\mathcal{Y}_e)$ .

For example, if  ${}_{\mathcal{D}}\mathcal{M}_e$  is a bimodule category, the actions  $\triangleright : \mathcal{D} \times {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_{\mathcal{D}}\mathcal{M}_e$  and  $\triangleleft : {}_{\mathcal{D}}\mathcal{M}_e \times \mathcal{C} \rightarrow {}_{\mathcal{D}}\mathcal{M}_e$  are balanced bimodule functors, when we consider  $\mathcal{C}$  and  $\mathcal{D}$  as bimodule categories. The following statements follow directly from the definitions.

**Lemma 2.15** *i) The left action of  $\mathcal{C}$  on  ${}_e\mathcal{M}_{\mathcal{D}} \times {}_{\mathcal{D}}\mathcal{N}$  is given by  $\mathcal{D}$ -balanced module functors  $L_c : {}_e\mathcal{M}_{\mathcal{D}} \times {}_{\mathcal{D}}\mathcal{N} \rightarrow {}_e\mathcal{M}_{\mathcal{D}} \times {}_{\mathcal{D}}\mathcal{N}$  for all  $c \in \mathcal{C}$ .*

*ii) A left  $\mathcal{C}$ -module functor  $F : {}_e\mathcal{M}_{\mathcal{D}} \times {}_{\mathcal{D}}\mathcal{N} \rightarrow {}_e\mathcal{Y}$  that is also  $\mathcal{D}$ -balanced is a balanced module functor if and only if the left module constraints  $\phi_c^F : F \circ L_c^{\mathcal{M} \times \mathcal{N}} \rightarrow L_c^{\mathcal{Y}} \circ G$  are balanced natural isomorphisms for all  $c \in \mathcal{C}$ .*

**Inner hom objects** An important tool in the theory of module categories is the inner hom. In this work, the inner hom will play a dominant role in the construction of duals for bimodule categories. Let  ${}_{\mathcal{D}}\mathcal{M}$  be a left  $\mathcal{D}$ -module category. For  $m, \tilde{m} \in \mathcal{M}$ , the functor  $\text{Hom}_{\mathcal{M}}((-) \triangleright m, \tilde{m}) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Vect}$  is left exact and thus representable [14, Sec. 3.2]. The following is a change in convention with regard to the definition used in [27] as will be explained in detail in Lemma 2.17.

**Definition 2.16** *Let  ${}_{\mathcal{D}}\mathcal{M}$  be a left  $\mathcal{D}$ -module category. An inner hom for  $\mathcal{M}$  is an object  ${}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}} \in \mathcal{D}$  for all  $m, \tilde{m} \in \mathcal{M}$  together with natural isomorphisms*

$$\alpha_{d, m, \tilde{m}}^{\mathcal{M}} : \text{Hom}_{\mathcal{M}}(m, d \triangleright \tilde{m}) \simeq \text{Hom}_{\mathcal{D}}({}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}}, d), \tag{2.15}$$

for all  $d \in \mathcal{D}$  and  $m, \tilde{m} \in \mathcal{M}$ .

We write  ${}_{\mathcal{D}}\langle m, \tilde{m} \rangle$  for the inner hom objects and omit the labels of  $\alpha$ , when the relevant module category  $\mathcal{M}$  is clear from the context. If they exist inner homs are unique up to a unique isomorphism and determine a bilinear functor

$$\mathcal{M} \times \mathcal{M}^{\text{op}} \ni (m \times \tilde{m}) \mapsto {}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}} \in \mathcal{D}, \tag{2.16}$$

called the inner hom functor. Analogously, a right  $\mathcal{C}$ -module category  $\mathcal{N}_e$  gives rise to an inner hom with natural isomorphisms

$$\alpha_{\tilde{n}, n, c}^{\mathcal{N}} : \text{Hom}_e(\langle \tilde{n}, n \rangle_e^{\mathcal{N}}, c) \simeq \text{Hom}_{\mathcal{N}}(n, \tilde{n} \triangleleft c), \tag{2.17}$$

that yield a functor

$$\mathcal{N}^{\text{op}} \times \mathcal{N} \ni (\tilde{n} \times n) \mapsto \langle \tilde{n}, n \rangle_e^{\mathcal{N}} \in \mathcal{C}. \tag{2.18}$$

Next we show that inner hom objects exist in our setting and compare their definition with the usual definition of inner hom in the literature.

**Lemma 2.17** *Let  ${}_{\mathcal{D}}\mathcal{M}_e$  be a module category.*

i) There exists a left exact functor

$$\underline{\mathbf{Hom}}_{\mathcal{D}} : \mathcal{M} \times \mathcal{M}^{\text{op}} \ni (m, \tilde{m}) \mapsto \underline{\mathbf{Hom}}_{\mathcal{D}}(m, \tilde{m}) \in \mathcal{D}$$

together with natural isomorphisms

$$\mathbf{Hom}_{\mathcal{M}}(d \triangleright \tilde{m}, m) \simeq \mathbf{Hom}_{\mathcal{D}}(d, \underline{\mathbf{Hom}}(\tilde{m}, m)). \quad (2.19)$$

ii) Similarly, there exists a left exact functor

$$\underline{\mathbf{Hom}}_{\mathcal{C}} : \mathcal{M}^{\text{op}} \times \mathcal{M} \ni (\tilde{m}, m) \mapsto \underline{\mathbf{Hom}}_{\mathcal{C}}(\tilde{m}, m) \in \mathcal{C}$$

with natural isomorphisms

$$\mathbf{Hom}_{\mathcal{M}}(m \triangleleft c, \tilde{m}) \simeq \mathbf{Hom}_{\mathcal{C}}(c, \underline{\mathbf{Hom}}_{\mathcal{C}}(m, \tilde{m})). \quad (2.20)$$

iii) There are natural isomorphisms

$${}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}} \simeq {}^*\underline{\mathbf{Hom}}_{\mathcal{D}}(m, \tilde{m}) \quad \text{and} \quad \langle \tilde{m}, m \rangle_e^{\mathcal{M}} \simeq \underline{\mathbf{Hom}}(m, \tilde{m})^*.$$

In particular, inner hom objects exist for finite tensor categories. Moreover, the functors  ${}_{\mathcal{D}}\langle -, - \rangle^{\mathcal{M}}$  and  $\langle -, - \rangle_e^{\mathcal{M}}$  are right exact in each argument.

Proof. The objects  $\underline{\mathbf{Hom}}(m, \tilde{m})$  in the first two parts are what are more commonly called inner hom objects, see [14, Sec. 3.2]. The existence and the left exactness of  $\underline{\mathbf{Hom}}(-, -)$  follows directly from the left exactness of the  $\mathbf{Hom}$ -functor. For the last part we compute using the duality in  $\mathcal{D}$

$$\begin{aligned} \mathbf{Hom}({}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}}, d) &\simeq \mathbf{Hom}(m, d \triangleright \tilde{m}) \simeq \mathbf{Hom}(d^* \triangleright m, \tilde{m}) \\ &\simeq \mathbf{Hom}(d^*, \underline{\mathbf{Hom}}_{\mathcal{D}}(m, \tilde{m})) \simeq \mathbf{Hom}({}^*\underline{\mathbf{Hom}}(m, \tilde{m}), d). \end{aligned} \quad (2.21)$$

All isomorphisms are natural in all arguments and induce a natural isomorphism  ${}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}} \simeq {}^*\underline{\mathbf{Hom}}_{\mathcal{D}}(m, \tilde{m})$  by the Yoneda-Lemma. The second isomorphism is obtained similarly. Since the duality functor of a finite tensor category is an exact functor  $(-)^* : \mathcal{D} \rightarrow \mathcal{D}^{\text{op}}$ , it follows that the inner hom functors from Definition 2.16 and Equation (2.17) are right exact.  $\square$

The inner hom functors are compatible with the module structures in the following way.

**Proposition 2.18** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be finite tensor categories and  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$  a bimodule category.*

i) *The  $\mathcal{D}$ -valued inner hom is a  $\mathcal{C}$ -balanced bimodule functor*

$${}_{\mathcal{D}}\langle -, - \rangle^{\mathcal{M}} : {}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}} \times {}_{\mathcal{C}}^{\#}\mathcal{M}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}},$$

*i.e. there are coherent natural isomorphisms*

$${}_{\mathcal{D}}\langle d \triangleright m, \tilde{m} \rangle^{\mathcal{M}} \simeq d \otimes {}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}}, \quad {}_{\mathcal{D}}\langle m, \tilde{m} \# \triangleleft d \rangle^{\mathcal{M}} \simeq {}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}} \otimes d \quad (2.22)$$

*and*

$${}_{\mathcal{D}}\langle m \triangleleft c, \tilde{m} \rangle^{\mathcal{M}} \simeq {}_{\mathcal{D}}\langle m, c \# \triangleright \tilde{m} \rangle^{\mathcal{M}}. \quad (2.23)$$

ii) *The  $\mathcal{C}$ -valued inner hom is a  $\mathcal{D}$ -balanced bimodule functor*

$$\langle -, - \rangle_e^{\mathcal{M}} : {}_{\mathcal{C}}^{\#}\mathcal{M}_{\mathcal{D}} \times {}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}} \rightarrow {}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}},$$

*i.e. there are coherent natural isomorphisms*

$$\langle \tilde{m}, m \triangleleft c \rangle_e^{\mathcal{M}} \simeq \langle \tilde{m}, m \rangle_e^{\mathcal{M}} \otimes c, \quad \langle c \triangleright \# \tilde{m}, m \rangle_e^{\mathcal{M}} \simeq c \otimes \langle \tilde{m}, m \rangle_e^{\mathcal{M}} \quad (2.24)$$

*and*

$$\langle \tilde{m}, d \triangleright m \rangle_e^{\mathcal{M}} \simeq \langle \tilde{m} \triangleleft \# d, m \rangle_e^{\mathcal{M}}. \quad (2.25)$$

Proof. Most of these natural isomorphisms are defined in [27, Lemma 5]. All natural isomorphisms are obtained directly from the definitions of the inner hom and the isomorphisms induced by the dualities in the tensor categories.  $\square$

For the unit bimodule category  ${}_e\mathcal{C}_e$ , the inner homs are for example given by  ${}_e\langle x, \tilde{x} \rangle = x \otimes {}^*\tilde{x}$  and  $\langle \tilde{x}, x \rangle_e = \tilde{x}^* \otimes x$ .

If we pass from a module category  ${}_D\mathcal{M}$  to the Grothendieck ring  $Gr(\mathcal{M})$ , Proposition 2.18 shows that the inner hom satisfies all requirements of a  $Gr(\mathcal{D})$ -valued inner product except the compatibility with the  $*$ -involution. This compatibility will be considered in Section 5.

The inner hom allows to prove the following theorem.

**Theorem 2.19** ([14]) *Let  ${}_e\mathcal{M}$  be an module category over  $\mathcal{C}$ . Then there exists an algebra object  $A \in \mathcal{C}$ , such that  ${}_e\mathcal{M}$  is equivalent to  $\text{Mod}(\mathcal{C})_A$ , the category of  $A$ -right modules in  $\mathcal{D}$  with  $\mathcal{D}$ -left action given by the tensor product.*

Next we recall the inner hom for the module categories obtained from algebras.

**Example 2.20** We summarize the computation of the inner hom objects in the case of  ${}_e\mathcal{N} = \text{Mod}(\mathcal{C})_B$  and the  $\mathcal{C}_M^*$ -valued inner hom for  $\mathcal{M}_e = {}_A\text{Mod}(\mathcal{C})$  in [14, Example 3.19]. Note that for a right  $B$ -module in  $\text{Mod}(\mathcal{C})_B$ ,  ${}^*n$  is naturally a left  $B$ -module, while for  $m \in {}_A\text{Mod}(\mathcal{C})$ ,  $m^*$  is a right  $A$ -module. Using the tensor product over the algebra  $B$  in  $\mathcal{C}$ , see [11, Def. 2.9.22], we compute

$$\text{Hom}_e({}_e\langle n, \tilde{n} \rangle^N, c) \simeq \text{Hom}_N(n, c \otimes \tilde{n}) = \text{Hom}_e(n \otimes_B {}^*n, c).$$

It follows that  ${}_e\langle n, \tilde{n} \rangle^N = n \otimes_B {}^*n$ . It is shown in [11, Prop. 2.12.2], that

$$\mathcal{C}_M^* = \text{Fun}_e({}_A\text{Mod}(\mathcal{C}), {}_A\text{Mod}(\mathcal{C})) = {}_A\text{Mod}(\mathcal{C})_A,$$

where  ${}_A\text{Mod}(\mathcal{C})_A$  is the category of  $(A, A)$ -bimodules in  $\mathcal{C}$  with the tensor product  $\otimes_A$  over  $A$  as monoidal structure. We compute the inner hom of  ${}_A\text{Mod}(\mathcal{C})$  regarded as left  ${}_A\text{Mod}(\mathcal{C})_A$ -module category. Let  $x \in {}_A\text{Mod}(\mathcal{C})_A$ , then

$$\begin{aligned} \text{Hom}_{{}_A\text{Mod}(\mathcal{C})_A}({}_e\langle m, \tilde{m} \rangle^M, x) &= \text{Hom}_M(m, x \otimes_A \tilde{m}) \simeq \text{Hom}_{{}_A\text{Mod}(\mathcal{C})_A}(A, x \otimes_A \tilde{m} \otimes m^*) \\ &= \text{Hom}_{{}_A\text{Mod}(\mathcal{C})_A}({}^*(\tilde{m} \otimes m^*), x), \end{aligned} \quad (2.26)$$

where the left dual in the last expression is taken in  ${}_A\text{Mod}(\mathcal{C})_A$ . It follows that  ${}_e\langle m, \tilde{m} \rangle^M = {}^*(\tilde{m} \otimes m^*)$ .

### 3 The tricategory of bimodule categories

The goal of this section is to show that finite tensor categories, bimodule categories, right exact bimodule functors and bimodule natural transformation form an algebraic tricategory [20], see also Definition B.1. The idea of the proof is a direct generalization of the analogous statement in [29] in the semisimple case. To simplify notation, we make the following assumptions. *In this section we assume that all tensor categories are finite and all module categories are finite unless specified otherwise. Furthermore, all functors are assumed to be right exact.*

The results in this section are a generalization of the results in [29, Sec. 3] to the non-semisimple case.

#### 3.1 The tensor product of module categories

In this subsection we first recall the definition of the tensor product of module categories. The tensor product is defined by an universal property with respect to right exact balanced functors. Then we show that the tensor product naturally defines a 2-functor from a suitable 2-category into the 2-category of abelian categories.

Let  $\mathcal{M}_e$  and  ${}_e\mathcal{N}$  be left and right  $\mathcal{C}$ -module categories, respectively. A tensor product  $\mathcal{M}_e \square {}_e\mathcal{N}$  of  $\mathcal{M}_e$  and  ${}_e\mathcal{N}$  is a linear abelian category that is defined -up to equivalence of categories- by a universal property that can be regarded as the analogue of the universal property of the tensor product of modules over a ring. The definition uses the category

$\text{Fun}^{\text{bal}}(\mathcal{M}_e \times {}_e\mathcal{N}, \mathcal{A})$  of (right exact) balanced functors and balanced natural transformations from  $\mathcal{M}_e \times {}_e\mathcal{N}$  to a linear category  $\mathcal{A}$ . The following definition is an extension of [13, Definition 3.3] in the sense that we require a fixed adjoint equivalence as part of the data of a tensor product.

**Definition 3.1** *A tensor product  $(\mathcal{M}_e \square {}_e\mathcal{N}, \mathbf{B}_{\mathcal{M}, \mathcal{N}}, \Psi_{\mathcal{M}, \mathcal{N}}, \varphi_{\mathcal{M}, \mathcal{N}}, \kappa_{\mathcal{M}, \mathcal{N}})$  of a right  $\mathcal{C}$ -module category  $\mathcal{M}_e$  with a left  $\mathcal{C}$ -module category  ${}_e\mathcal{N}$  is a finite linear abelian category  $\mathcal{M}_e \square {}_e\mathcal{N}$  together with*

i) *a  $\mathcal{C}$ -balanced functor  $\mathbf{B}_{\mathcal{M}, \mathcal{N}} : \mathcal{M}_e \times {}_e\mathcal{N} \rightarrow \mathcal{M}_e \square {}_e\mathcal{N}$ , such that the functor*

$$\begin{aligned} \Phi_{\mathcal{M}, \mathcal{N}} : \text{Fun}(\mathcal{M}_e \square {}_e\mathcal{N}, \mathcal{A}) &\rightarrow \text{Fun}^{\text{bal}}(\mathcal{M} \times \mathcal{N}, \mathcal{A}) \\ \mathbf{G} &\mapsto \mathbf{G} \circ \mathbf{B}_{\mathcal{M}, \mathcal{N}} \end{aligned} \quad (3.1)$$

*is an equivalence of categories. Here  $\text{Fun}(\mathcal{M}_e \square {}_e\mathcal{N}, \mathcal{A})$  denotes the category of (right exact) functors  $\mathcal{M}_e \square {}_e\mathcal{N} \rightarrow \mathcal{A}$  to some linear abelian category  $\mathcal{A}$ .*

ii) *a choice of a functor*

$$\Psi_{\mathcal{M}, \mathcal{N}} : \text{Fun}^{\text{bal}}(\mathcal{M}_e \times {}_e\mathcal{N}, \mathcal{A}) \rightarrow \text{Fun}(\mathcal{M}_e \square {}_e\mathcal{N}, \mathcal{A}) \quad (3.2)$$

*together with a specified adjoint equivalence  $\varphi_{\mathcal{M}, \mathcal{N}} : 1 \rightarrow \Phi_{\mathcal{M}, \mathcal{N}} \Psi_{\mathcal{M}, \mathcal{N}}$  and  $\kappa_{\mathcal{M}, \mathcal{N}} : 1 \rightarrow \Psi_{\mathcal{M}, \mathcal{N}} \Phi_{\mathcal{M}, \mathcal{N}}$  between  $\Phi_{\mathcal{M}, \mathcal{N}}$  and  $\Psi_{\mathcal{M}, \mathcal{N}}$ .*

For simplicity we sometimes write  $m \square n$  instead of  $\mathbf{B}(m \times n)$  for  $m \times n \in \mathcal{M}_e \times {}_e\mathcal{N}$ . We record the existence of the tensor product from the literature.

**Theorem 3.2** *The tensor product  $\mathcal{M}_e \square {}_e\mathcal{N}$  of finite module categories  $\mathcal{M}_e, {}_e\mathcal{N}$  exists. In particular it has the following descriptions.*

i) *The tensor product is equivalent to the following functor categories*

$$\mathcal{M}_e \square {}_e\mathcal{N} \simeq \text{Fun}_e({}_e\mathcal{M}^\#, {}_e\mathcal{N}) \simeq \text{Fun}_e({}_e\mathcal{N}, \mathcal{M}_e).$$

ii) *If we choose algebras  $A, B \in \mathcal{C}$  such that  $\mathcal{M}_e \simeq {}_A\text{Mod}(\mathcal{C})$ ,  ${}_e\mathcal{N} \simeq \text{Mod}(\mathcal{C})_B$ , then  $\mathcal{M}_e \square {}_e\mathcal{N} \simeq {}_A\text{Mod}(\mathcal{C})_B$ , the category of  $(A, B)$ -bimodules in  $\mathcal{C}$ . In this case, the universal balancing functor is given by the tensor product  $\otimes : {}_A\text{Mod}(\mathcal{C}) \times \text{Mod}(\mathcal{C})_B \rightarrow {}_A\text{Mod}(\mathcal{C})_B$ .*

*Proof.* The first description of the tensor product is shown in [8, Cor. 3.4.11] in general and in [13] in the semisimple case. The second statement is shown in [10], see [8, Thm. 3.2.17].  $\square$  Concretely, for every balanced functor  $F : \mathcal{M}_e \times {}_e\mathcal{N} \rightarrow \mathcal{A}$ , the tensor product yields a functor  $\widehat{F} = \Psi_{\mathcal{M}, \mathcal{N}}(F) : \mathcal{M} \square \mathcal{N} \rightarrow \mathcal{A}$ , that is unique up to unique natural isomorphism. The following lemma is a direct consequence of the properties of the adjoint equivalence in the definition of the tensor product.

**Lemma 3.3** *Let  $F, G : \mathcal{M}_e \times {}_e\mathcal{N} \rightarrow \mathcal{A}$  be balanced functors. For every balanced natural transformation  $\rho : F \rightarrow G$  there exists a unique natural transformation  $\widehat{\rho} : \widehat{F} \rightarrow \widehat{G}$ , such that*

$$(\widehat{\rho} \circ \mathbf{B}) \cdot \varphi(F) = \varphi(G) \cdot \rho. \quad (3.3)$$

Next we consider the 2-functorial properties of the tensor product. Denote by  $\text{Cat}^{\text{lin}}$  the 2-category of linear abelian categories, (right exact) linear functors and linear natural transformations.

**Proposition 3.4** *The tensor product defines a 2-functor*

$$\square : \text{Mod}^r(\mathcal{C}) \times \text{Mod}^l(\mathcal{C}) \rightarrow \text{Cat}^{\text{lin}}, \quad (3.4)$$

*where  $\text{Mod}^r(\mathcal{C})$ ,  $\text{Mod}^l(\mathcal{C})$  denote the 2-categories of right and left  $\mathcal{C}$ -module categories, respectively. This amounts for the following structures: Let  $\mathcal{M}_e, \mathcal{M}'_e$  and  ${}_e\mathcal{N}, {}_e\mathcal{N}'$  be  $\mathcal{C}$ -left- and right module categories, respectively.*

- i) For every bimodule functor  $F : \mathcal{M}_e \times {}_e\mathcal{N} \rightarrow \mathcal{M}'_e \times {}_e\mathcal{N}'$ , the tensor product of module categories defines a functor  $\Psi(\mathbf{B}_{\mathcal{M}',\mathcal{N}'}F) : \mathcal{M}_e \square {}_e\mathcal{N} \rightarrow \mathcal{M}'_e \square {}_e\mathcal{N}'$ , called  $\widehat{F}$  in the sequel, and a balanced natural isomorphism

$$\begin{array}{ccc} \mathcal{M} \times \mathcal{N} & \xrightarrow{F} & \mathcal{M}' \times \mathcal{N}' \\ \downarrow \mathbf{B}_{\mathcal{M},\mathcal{N}} \swarrow \varphi_F & & \downarrow \mathbf{B}_{\mathcal{M}',\mathcal{N}'} \\ \mathcal{M} \square \mathcal{N} & \xrightarrow{\widehat{F}} & \mathcal{M}' \square \mathcal{N}' . \end{array} \quad (3.5)$$

- ii) For every pair of bimodule functors  $F, G : \mathcal{M}_e \times {}_e\mathcal{N} \rightarrow \mathcal{M}'_e \times {}_e\mathcal{N}'$  and every bimodule natural transformation  $\rho : F \rightarrow G$ , there is a unique natural transformation  $\widehat{\rho} : \widehat{F} \rightarrow \widehat{G}$ , such that

$$(\widehat{\rho} \circ \mathbf{B}_{\mathcal{M},\mathcal{N}}) \cdot \mathbf{B}_F = \mathbf{B}_G \cdot \rho. \quad (3.6)$$

This is equivalent to imposing the following condition on the associated diagrams

$$\begin{array}{ccc} \begin{array}{ccc} \mathcal{M} \times \mathcal{N} & \begin{array}{c} \xrightarrow{F} \\ \Downarrow \rho \\ \xrightarrow{G} \end{array} & \mathcal{M}' \times \mathcal{N}' \\ \downarrow \mathbf{B}_{\mathcal{M},\mathcal{N}} \swarrow \varphi_G & & \downarrow \mathbf{B}_{\mathcal{M}',\mathcal{N}'} \\ \mathcal{M} \square \mathcal{N} & \xrightarrow{\widehat{G}} & \mathcal{M}' \square \mathcal{N}' , \end{array} & = & \begin{array}{ccc} \mathcal{M} \times \mathcal{N} & \xrightarrow{F} & \mathcal{M}' \times \mathcal{N}' \\ \downarrow \mathbf{B}_{\mathcal{M},\mathcal{N}} \swarrow \varphi_F & & \downarrow \mathbf{B}_{\mathcal{M}',\mathcal{N}'} \\ \mathcal{M} \square \mathcal{N} & \begin{array}{c} \xrightarrow{\widehat{F}} \\ \Downarrow \widehat{\rho} \\ \xrightarrow{\widehat{G}} \end{array} & \mathcal{M}' \square \mathcal{N}' . \end{array} \end{array} \quad (3.7)$$

- iii) For any two composable bimodule functors  $F : \mathcal{M}_e \times {}_e\mathcal{N} \rightarrow \mathcal{M}'_e \times {}_e\mathcal{N}'$ ,  $G : \mathcal{M}'_e \times {}_e\mathcal{N}' \rightarrow \mathcal{M}''_e \times {}_e\mathcal{N}''$ , there is a unique natural isomorphism  $\phi_{G,F} : \widehat{G}\widehat{F} \rightarrow \widehat{G}\widehat{F}$  such that the following diagram of natural isomorphisms commutes:

$$\begin{array}{ccc} \mathbf{B}_{\mathcal{M}'',\mathcal{N}''} \mathbf{G}\mathbf{F} & \xrightarrow{\varphi_{G,F}} & \widehat{G}\mathbf{B}_{\mathcal{M}',\mathcal{N}'}\mathbf{F} \\ \downarrow \varphi_{G,F} & & \downarrow \widehat{G}\varphi_F \\ \widehat{G}\widehat{F}\mathbf{B}_{\mathcal{M},\mathcal{N}} & \xleftarrow{\phi_{G,F}\mathbf{B}_{\mathcal{M},\mathcal{N}}} & \widehat{G}\widehat{F}\mathbf{B}_{\mathcal{M},\mathcal{N}} . \end{array} \quad (3.8)$$

- iv) For three composable bimodule functors  $F : \mathcal{M}_e \times {}_e\mathcal{N} \rightarrow \mathcal{M}'_e \times {}_e\mathcal{N}'$ ,  $G : \mathcal{M}'_e \times {}_e\mathcal{N}' \rightarrow \mathcal{M}''_e \times {}_e\mathcal{N}''$ ,  $H : \mathcal{M}''_e \times {}_e\mathcal{N}'' \rightarrow \mathcal{M}'''_e \times {}_e\mathcal{N}'''$ , the following diagram of natural isomorphisms commutes

$$\begin{array}{ccc} \widehat{H}\widehat{G}\widehat{F} & \xrightarrow{\phi_{H,G}\widehat{F}} & \widehat{H}\widehat{G}\widehat{F} \\ \downarrow \widehat{H}\phi_{G,F} & & \downarrow \phi_{H,G,F} \\ \widehat{H}\widehat{G}\widehat{F} & \xrightarrow{\phi_{H,G,F}} & \widehat{H}\widehat{G}\widehat{F} . \end{array} \quad (3.9)$$

- v) The natural transformation  $\kappa_{\mathcal{M},\mathcal{N}}$  from Definition 3.1 defines a natural isomorphism

$$\kappa_{\mathcal{M},\mathcal{N}}(\widehat{1_{\mathcal{M}} \square \mathcal{N}}) : \widehat{1_{\mathcal{M}} \square \mathcal{N}} \rightarrow \widehat{1_{\mathcal{M}} \square \mathcal{N}}, \quad (3.10)$$

such that for all bimodule functors  $F : \mathcal{M}_e \times {}_e\mathcal{N} \rightarrow \mathcal{M}'_e \times {}_e\mathcal{N}'$  the following diagrams commute

$$\begin{array}{ccc} \widehat{F}\widehat{1_{\mathcal{M}} \square \mathcal{N}} & \xrightarrow{\phi_{F,1}} & \widehat{F} \\ \downarrow \widehat{F}\kappa_{\mathcal{M},\mathcal{N}} & \swarrow \text{id} & \downarrow \kappa_{\mathcal{M}'_e \square \mathcal{N}'} \\ \widehat{F}\widehat{1_{\mathcal{M}} \square \mathcal{N}} & & \widehat{F} \end{array} \quad \begin{array}{ccc} \widehat{1_{\mathcal{M}} \square \mathcal{N}}\widehat{F} & \xrightarrow{\phi_{1,F}} & \widehat{F} \\ \downarrow \kappa_{\mathcal{M}'_e \square \mathcal{N}'} & \swarrow \text{id} & \downarrow \kappa_{\mathcal{M}'_e \square \mathcal{N}'} \\ \widehat{1_{\mathcal{M}'_e} \square \mathcal{N}'}\widehat{F} & & \widehat{F} . \end{array} \quad (3.11)$$

Proof. The functor  $\mathbf{B}F$  in the first part is balanced and hence the functor  $\Psi(\mathbf{B}F) = \widehat{F}$  is well defined. From the natural isomorphism  $\varphi : 1 \rightarrow \Phi\Psi$  in Definition 3.1, we obtain the balanced natural isomorphism  $\varphi_F : \mathbf{B}F \rightarrow \widehat{F}\mathbf{B}$ . This shows the first part. Part ii) follows directly by

applying Lemma 3.3 to the natural transformation  $\Psi(\mathbf{B}\rho)$ , which is denoted  $\widehat{\rho}$  in the sequel. To show statement *iii*), note that from the first part we obtain balanced natural isomorphisms

$$\widehat{\mathbf{G}}\mathbf{F}\mathbf{B} \xrightarrow{\widehat{\mathbf{G}}\varphi_{\mathbf{F}}^{-1}} \widehat{\mathbf{G}}\mathbf{B}\mathbf{F} \xrightarrow{\varphi_{\mathbf{G}}^{-1}\mathbf{F}} \mathbf{B}\mathbf{G}\mathbf{F} \xrightarrow{\varphi_{\mathbf{G}\mathbf{F}}} \widehat{\mathbf{G}}\mathbf{F}\mathbf{B}, \quad (3.12)$$

which compose to a balanced natural isomorphism from  $\widehat{\mathbf{G}}\mathbf{F}\mathbf{B}$  to  $\widehat{\mathbf{G}}\mathbf{F}\mathbf{B}$ . The natural isomorphism  $\phi_{\mathbf{G},\mathbf{F}} : \widehat{\mathbf{G}}\mathbf{F} \rightarrow \widehat{\mathbf{G}}\mathbf{F}$  is then defined as

$$\phi_{\mathbf{G},\mathbf{F}} = \kappa^{-1}(\widehat{\mathbf{G}}\mathbf{F}) \cdot \Psi\left(\varphi_{\mathbf{G}\mathbf{F}} \cdot \varphi_{\mathbf{G}}^{-1}\mathbf{F} \cdot \widehat{\mathbf{G}}\varphi_{\mathbf{F}}^{-1}\right) \cdot \kappa(\widehat{\mathbf{G}}\mathbf{F}).$$

This proves the existence and uniqueness of the natural isomorphism  $\phi_{\mathbf{G},\mathbf{F}}$ , such that (3.8) commutes. Hence the third part follows. To show the fourth part, note that by definition of  $\phi_{\mathbf{F},\mathbf{G}}$  and by the interchange law for 2-categories, the following diagram commutes

$$\begin{array}{ccccc} & & \widehat{\mathbf{H}}\mathbf{B}\mathbf{G}\mathbf{F} & \xrightarrow{\widehat{\mathbf{H}}\varphi_{\mathbf{G}\mathbf{F}}} & \widehat{\mathbf{H}}\widehat{\mathbf{G}}\mathbf{B}\mathbf{F} & & \\ & \nearrow \varphi_{\mathbf{H}\mathbf{G}\mathbf{F}} & & & & \searrow \widehat{\mathbf{H}}\widehat{\mathbf{G}}\varphi_{\mathbf{F}} & \\ \mathbf{B}\mathbf{H}\mathbf{G}\mathbf{F} & \xrightarrow{\varphi_{\mathbf{H}\mathbf{G}\mathbf{F}}} & \widehat{\mathbf{H}}\mathbf{G}\mathbf{B}\mathbf{F} & & \widehat{\mathbf{H}}\mathbf{G}\mathbf{F}\mathbf{B} & & \\ & \searrow \varphi_{\mathbf{H}\mathbf{G}\mathbf{F}} & & & & \swarrow \phi_{\mathbf{H},\widehat{\mathbf{G}}\mathbf{F}\mathbf{B}} & \\ & & \widehat{\mathbf{H}}\mathbf{G}\mathbf{F}\mathbf{B} & \xleftarrow{\phi_{\mathbf{H}\mathbf{G},\mathbf{F}\mathbf{B}}} & \widehat{\mathbf{H}}\widehat{\mathbf{G}}\mathbf{F}\mathbf{B} & & \end{array} \quad (3.13)$$

Here the interchange law is used to establish the commutativity of the parallelogram on the right, and part *iii*) shows the commutativity of the two parallelograms on the left in (3.13). It also follows from *iii*) that the following diagram commutes

$$\begin{array}{ccccc} & & \widehat{\mathbf{H}}\mathbf{B}\mathbf{G}\mathbf{F} & \xrightarrow{\widehat{\mathbf{H}}\varphi_{\mathbf{G}\mathbf{F}}} & \widehat{\mathbf{H}}\widehat{\mathbf{G}}\mathbf{B}\mathbf{F} & & \\ & \nearrow \varphi_{\mathbf{H}\mathbf{G}\mathbf{F}} & & & & \searrow \widehat{\mathbf{H}}\widehat{\mathbf{G}}\varphi_{\mathbf{F}} & \\ \mathbf{B}\mathbf{H}\mathbf{G}\mathbf{F} & \xrightarrow{\varphi_{\mathbf{H}\mathbf{G}\mathbf{F}}} & \widehat{\mathbf{H}}\mathbf{G}\mathbf{B}\mathbf{F} & & \widehat{\mathbf{H}}\mathbf{G}\mathbf{F}\mathbf{B} & & \\ & \searrow \varphi_{\mathbf{H}\mathbf{G}\mathbf{F}} & & & & \swarrow \widehat{\mathbf{H}}\phi_{\mathbf{G},\mathbf{F}\mathbf{B}} & \\ & & \widehat{\mathbf{H}}\mathbf{G}\mathbf{F}\mathbf{B} & \xleftarrow{\phi_{\mathbf{H}\mathbf{G},\mathbf{F}\mathbf{B}}} & \widehat{\mathbf{H}}\widehat{\mathbf{G}}\mathbf{F}\mathbf{B} & & \end{array} \quad (3.14)$$

Since all outer arrows in the diagrams (3.13) and (3.14) that do not contain  $\phi$  agree and all arrows are labeled by natural isomorphisms, it follows that the diagram

$$\begin{array}{ccc} \widehat{\mathbf{H}}\widehat{\mathbf{G}}\mathbf{F}\mathbf{B} & \xrightarrow{\phi_{\mathbf{H},\widehat{\mathbf{G}}\mathbf{F}\mathbf{B}}} & \widehat{\mathbf{H}}\widehat{\mathbf{G}}\mathbf{F}\mathbf{B} \\ \downarrow \widehat{\mathbf{H}}\phi_{\mathbf{G},\mathbf{F}\mathbf{B}} & & \downarrow \phi_{\mathbf{H}\mathbf{G},\mathbf{F}\mathbf{B}} \\ \widehat{\mathbf{H}}\mathbf{G}\mathbf{F}\mathbf{B} & \xrightarrow{\phi_{\mathbf{H}\mathbf{G},\mathbf{F}\mathbf{B}}} & \widehat{\mathbf{H}}\mathbf{G}\mathbf{F}\mathbf{B} \end{array}$$

commutes. As the functor  $\Phi$  is fully faithful, this shows that (3.9) commutes. For the last statement, note that the natural isomorphism  $\kappa_{\mathbf{M},\mathbf{N}} : \mathbf{1} \rightarrow \Psi_{\mathbf{M},\mathbf{N}}\widehat{\Phi}_{\mathbf{M},\mathbf{N}}$  from Definition 3.1 provides a natural isomorphism

$$\widehat{\mathbf{1}_{\mathbf{M} \times \mathbf{N}}} = \Psi(\mathbf{B} \circ \mathbf{1}_{\mathbf{M} \times \mathbf{N}}) = \Psi(\mathbf{1}_{\mathbf{M} \square \mathbf{N}} \circ \mathbf{B}) \xrightarrow{\kappa_{\mathbf{M} \square \mathbf{N}}(\mathbf{1}_{\mathbf{M} \square \mathbf{N}})} \mathbf{1}_{\mathbf{M} \square \mathbf{N}}. \quad (3.15)$$

The snake identity (A.3) for the adjoint equivalence then implies that the diagram

$$\begin{array}{ccc} \Phi(\mathbf{1}_{\mathbf{M} \times \mathbf{N}}) = \mathbf{B} & \xrightarrow{\Phi\varphi} & \Phi\Psi\Phi(\mathbf{1}_{\mathbf{M} \times \mathbf{N}}) = \widehat{\mathbf{1}_{\mathbf{M} \times \mathbf{N}}}\mathbf{B} \\ & \searrow \mathbf{1} & \downarrow \kappa\Phi \\ & & \Phi(\mathbf{1}_{\mathbf{M} \times \mathbf{N}}) \end{array} \quad (3.16)$$

commutes. Hence the diagram

$$\begin{array}{ccc}
\widehat{F}B & \xrightarrow{\varphi_F} & BF \\
\widehat{F}\varphi(B) \downarrow & \searrow \text{id} & \downarrow \varphi_F \\
\widehat{F}\widehat{1}B & \xrightarrow{\widehat{F}\kappa_B} & \widehat{F}B
\end{array} \tag{3.17}$$

commutes. By the unique characterization of the natural isomorphism  $\phi_{\widehat{F}, \widehat{1}} : \widehat{F}\widehat{1} \rightarrow \widehat{F}$  from part *iii*), we deduce that  $\phi_{\widehat{F}, \widehat{1}} = \widehat{F}\kappa(1_{\mathcal{M} \square \mathcal{N}})$ . The remaining identity is proven analogously using the unique characterization of  $\phi_{\widehat{1}, \widehat{F}}$  from part *iii*).  $\square$

Note that the notation  $\widehat{F}$  was already used for image of  $\Psi$  on balanced functors. It should be clear from the context, whether a functor is balanced or a module functor, hence the notation is unambiguous. Moreover, we next unify the notions of balanced functors and module functors, and regard them as 1-morphisms in a certain 2-category. This provides further justification for the notation  $\widehat{F}$ .

The map  $F \mapsto \widehat{F}$  from the previous proposition for balanced functors  $F$  is compatible with the composition of bimodule functors and balanced functors. To make this statement precise, we define the following 2-category, that combines balanced functors and bimodule functors into a single 2-category.

**Proposition 3.5** *The following data defines a 2-category  $\text{Mod}_e^{\text{bal}}$  for a finite tensor category  $\mathcal{C}$ .*

- i) The objects of  $\text{Mod}_e^{\text{bal}}$  are  $(\mathcal{C}, \mathcal{C})$ -bimodule categories  $\mathcal{M}_e \times_e \mathcal{N}$  and linear categories  $\mathcal{A}$ .*
- ii) The categories of 1- and 2-morphisms between the objects are given by:*
  - (a) For bimodule categories  $\mathcal{M}_e \times_e \mathcal{N}$  and  $\mathcal{M}'_e \times_e \mathcal{N}'$ ,  $\text{Mod}_e^{\text{bal}}(\mathcal{M}_e \times_e \mathcal{N}, \mathcal{M}'_e \times_e \mathcal{N}')$  is the category  $\text{Bimod}(\mathcal{M}_e \times_e \mathcal{N}, \mathcal{M}'_e \times_e \mathcal{N}')$  of bimodule functors and bimodule natural transformation between them.*
  - (b) For a bimodule category  $\mathcal{M}_e \times_e \mathcal{N}$  and a category  $\mathcal{A}$ ,  $\text{Mod}_e^{\text{bal}}(\mathcal{M}_e \times_e \mathcal{N}, \mathcal{A})$  is the category  $\text{Fun}^{\text{bal}}(\mathcal{M}_e \times_e \mathcal{N}, \mathcal{A})$  of balanced functors and balanced natural transformations between them.*
  - (c) For two categories  $\mathcal{A}$ , and  $\mathcal{B}$ ,  $\text{Mod}_e^{\text{bal}}(\mathcal{A}, \mathcal{B})$  is the category  $\text{Fun}(\mathcal{A}, \mathcal{B})$  of functors and natural transformations between them*
  - (d) There is just the zero morphism from a category  $\mathcal{A}$  to a bimodule category  $\mathcal{M}_e \times_e \mathcal{N}$ .*
- iii) The compositions are induced by the horizontal composition of functors and the vertical composition of natural transformations.*

Proof. It follows from Lemma 2.13 that the various compositions of 1- and 2-morphisms are well defined. It follows by a direct computation that all compositions are strictly associative and strictly compatible with the units.  $\square$

**Proposition 3.6** *The tensor product of module categories defines a 2-functor*

$$\widehat{(-)} : \text{Mod}_e^{\text{bal}} \rightarrow \text{Cat}^{\text{lin}}. \tag{3.18}$$

Proof. On objects,  $\widehat{(-)}$  is defined by  $\widehat{\mathcal{M}_e \times_e \mathcal{N}} = \mathcal{M}_e \square_e \mathcal{N}$  and  $\widehat{\mathcal{A}} = \mathcal{A}$ . On 1-morphisms,  $\widehat{(-)}$  is defined as follows. For a bimodule functor  $F : \mathcal{M}_e \times_e \mathcal{N} \rightarrow \mathcal{M}'_e \times_e \mathcal{N}'$ , and for a balanced functor  $G : \mathcal{M}_e \times_e \mathcal{N} \rightarrow \mathcal{A}$ , the functors  $\widehat{F} : \mathcal{M}_e \square_e \mathcal{N} \rightarrow \mathcal{M}'_e \square_e \mathcal{N}'$  and  $\widehat{G} : \mathcal{M}_e \square_e \mathcal{N} \rightarrow \mathcal{A}$  are already defined in Definition 3.1 and in Proposition 3.4, *i*), respectively. For a functor  $H : \mathcal{A} \rightarrow \mathcal{B}$ , we define  $\widehat{H} = H$ . On 2-morphisms,  $\widehat{(-)}$  is already defined for bimodule natural transformations and balanced natural transformations. For a natural transformation  $\eta : H \rightarrow H'$  between functors  $H, H' : \mathcal{A} \rightarrow \mathcal{B}$ , we define  $\widehat{\eta} = \eta$ .

The coherence structures of  $\widehat{(-)}$  are the following.

- i) For all bimodule categories  $\mathcal{M}_e \times_e \mathcal{N}$ , the coherence isomorphism  $\widehat{1_{\mathcal{M} \times \mathcal{N}}} \rightarrow 1_{\mathcal{M} \square \mathcal{N}}$  is defined by  $\kappa_{\mathcal{M}, \mathcal{N}}(1_{\mathcal{M} \square \mathcal{N}})$ , as in Proposition 3.4, *v*).*
- ii) For composable bimodule functors  $F, G$ , there is a natural isomorphism  $\phi_{G, F} : \widehat{G}\widehat{F} \rightarrow \widehat{GF}$  that is defined by Proposition 3.4, *iii*).*

- iii) For composable functors  $H : \mathcal{A} \rightarrow \mathcal{B}$  and  $K : \mathcal{B} \rightarrow \mathcal{C}$ , we define  $\phi_{K,H} = \text{id}_{KH}$ .
- iv) For a balanced functor  $F : \mathcal{M}_e \times_e \mathcal{N} \rightarrow \mathcal{A}$  and a bimodule functor  $G : \mathcal{M}'_e \times_e \mathcal{N}' \rightarrow \mathcal{M}_e \times_e \mathcal{N}$ , it follows from Lemma 3.3, that there exists a unique balanced natural isomorphism  $\phi_{F,G} : \widehat{FG} \rightarrow \widehat{FG}$ , such that the following diagram commutes

$$\begin{array}{ccc}
FG & \xrightarrow{\varphi(F)} & \widehat{FG} \\
\downarrow \varphi(FG) & & \downarrow \widehat{F}\varphi_G \\
\widehat{FG} & \xleftarrow{\phi_{F,G}} & \widehat{FG}.
\end{array} \tag{3.19}$$

- v) For a balanced functor  $F : \mathcal{M}_e \times_e \mathcal{N} \rightarrow \mathcal{A}$  and a functor  $H : \mathcal{A} \rightarrow \mathcal{B}$  it follows from Lemma 3.3, that there exists a unique balanced natural isomorphism  $\phi_{H,F} : \widehat{HF} \rightarrow \widehat{HF}$ , such that the following diagram commutes

$$\begin{array}{ccc}
HF & \xrightarrow{\text{id}} & HF \\
\downarrow \varphi(HF) & & \downarrow H\varphi(F) \\
\widehat{HF} & \xleftarrow{\phi_{H,F}} & \widehat{HF}.
\end{array} \tag{3.20}$$

The proof that for three composable 1-morphisms, the diagram (A.11) commutes is analogous to the proof of Proposition 3.4 iv), while the compatibility of  $\widehat{(-)}$  with the unit from axiom (A.10) follows analogously to the proof of Proposition 3.4 v).  $\square$

## 3.2 The tensor product of bimodule categories

Next we show that the tensor product of module categories naturally extends to a tensor product of bimodule categories. Furthermore we consider the corresponding extension of the tensor product as a 2-functor.

**Proposition 3.7** *Let  ${}_{\mathcal{D}}\mathcal{M}_e$  and  ${}_e\mathcal{N}_{\mathcal{E}}$  be bimodule categories. The tensor product  ${}_{\mathcal{D}}\mathcal{M}_e \square {}_e\mathcal{N}_{\mathcal{E}}$  has a canonical structure of a  $(\mathcal{D}, \mathcal{E})$ -bimodule category, such that*

$$\mathbf{B} : {}_{\mathcal{D}}\mathcal{M}_e \times_e \mathcal{N}_{\mathcal{E}} \rightarrow {}_{\mathcal{D}}\mathcal{M}_e \square {}_e\mathcal{N}_{\mathcal{E}} \tag{3.21}$$

is a balanced bimodule functor and for all bimodule categories  ${}_{\mathcal{D}}\mathcal{A}_{\mathcal{E}}$  the adjoint equivalence from Definition 3.1 restricts to an adjoint equivalence

$$\begin{aligned}
\Phi &: \text{Fun}_{\mathcal{D}, \mathcal{E}}({}_{\mathcal{D}}\mathcal{M}_e \square {}_e\mathcal{N}_{\mathcal{E}}, {}_{\mathcal{D}}\mathcal{A}_{\mathcal{E}}) \rightarrow \text{Fun}_{\mathcal{D}, \mathcal{E}}^{\text{bal}}({}_{\mathcal{D}}\mathcal{M}_e \times_e \mathcal{N}_{\mathcal{E}}, {}_{\mathcal{D}}\mathcal{A}_{\mathcal{E}}), \\
\Psi &: \text{Fun}_{\mathcal{D}, \mathcal{E}}^{\text{bal}}({}_{\mathcal{D}}\mathcal{M}_e \times_e \mathcal{N}_{\mathcal{E}}, {}_{\mathcal{D}}\mathcal{A}_{\mathcal{E}}) \rightarrow \text{Fun}_{\mathcal{D}, \mathcal{E}}({}_{\mathcal{D}}\mathcal{M}_e \square {}_e\mathcal{N}_{\mathcal{E}}, {}_{\mathcal{D}}\mathcal{A}_{\mathcal{E}}),
\end{aligned} \tag{3.22}$$

where  $\text{Fun}_{\mathcal{D}, \mathcal{E}}^{\text{bal}}({}_{\mathcal{D}}\mathcal{M}_e \times_e \mathcal{N}_{\mathcal{E}}, {}_{\mathcal{D}}\mathcal{A}_{\mathcal{E}})$  is the category of balanced bimodule functors from Definition 2.14.

*Proof.* To define the left  $\mathcal{D}$ -module structure on  $\mathcal{M} \square \mathcal{N}$ , note that for all  $d \in \mathcal{D}$ , the functors  $L_d : \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{M} \times \mathcal{N}$  provided by the action of  $d \in \mathcal{D}$  are  $(\mathcal{C}, \mathcal{C})$ -bimodule functors and the module constraint for the left action of  $\mathcal{D}$  consists of  $(\mathcal{C}, \mathcal{C})$ -bimodule natural isomorphisms  $\mu_{d,d'} : L_d \circ L_{d'} \rightarrow L_{d \otimes d'}$  for all  $d, d' \in \mathcal{D}$ . Hence we can apply the 2-functor  $\widehat{(-)}$  from Proposition 3.6 and we obtain for all  $d \in \mathcal{D}$  functors  $\widehat{L}_d : \mathcal{M} \square \mathcal{N} \rightarrow \mathcal{M} \square \mathcal{N}$  and natural isomorphisms  $\widehat{\mu}_{d,d'} : \widehat{L}_d \circ \widehat{L}_{d'} \rightarrow \widehat{L}_{d \otimes d'}$  for all  $d, d' \in \mathcal{D}$ . The module constraint (2.2) for these natural isomorphisms is obtained by applying the 2-functor  $\widehat{(-)}$  to the corresponding module constraint for  ${}_e\mathcal{M}$ .

The right  $\mathcal{E}$ -module structure on  ${}_e\mathcal{M}_{\mathcal{D}} \square {}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}}$  is defined analogously by considering the  $(\mathcal{C}, \mathcal{C})$ -bimodule functors  $R_e : \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{M} \times \mathcal{N}$  for all  $e \in \mathcal{E}$ .

It follows that  $\mathcal{M} \square \mathcal{N}$  is a  $(\mathcal{D}, \mathcal{E})$ -bimodule category since the bimodule constraints follow directly by applying the 2-functor  $\widehat{(-)}$  to the corresponding diagrams for  ${}_{\mathcal{D}}\mathcal{M}_e \times_e \mathcal{N}_{\mathcal{E}}$ .

Next we show that  $\mathbf{B} : {}_{\mathcal{D}}\mathcal{M}_e \times_e \mathcal{N}_{\mathcal{E}} \rightarrow {}_{\mathcal{D}}\mathcal{M}_e \square {}_e\mathcal{N}_{\mathcal{E}}$  is a balanced bimodule functor. By definition of the left  $\mathcal{D}$ -module structure on  ${}_{\mathcal{D}}\mathcal{M} \square \mathcal{N}$ , we obtain balanced natural isomorphisms



$\varphi_d : \mathbf{B}L_d \rightarrow \widehat{L_d}\mathbf{B}$  for all  $d \in \mathcal{D}$ , that are compatible with the compositions  $\widehat{L_d} \circ \widehat{L_{d'}}$  according to Proposition 3.4 *iii*). This shows that  $\mathbf{B}$  is a left  $\mathcal{D}$ -module functor, and by the analogous argument, a right  $\mathcal{E}$ -module functor. The compatibility between these two module functor structures follows from Proposition 3.4, since  $L_d R_e = R_e L_d$  as functors  $\mathcal{M} \times \mathcal{N} \rightarrow \mathcal{M} \times \mathcal{N}$  for objects  $d \in \mathcal{D}$  and  $e \in \mathcal{E}$ . Hence  $\mathbf{B}$  is a bimodule functor, and since the module constraints are balanced natural isomorphisms, it is also a balanced bimodule functor according to Lemma 2.15.

In the next step we show that the functor  $\Psi$  from Definition 3.1 restricts to a functor

$$\Psi : \text{Fun}_{\mathcal{D}, \mathcal{E}}^{\text{bal}}(\mathcal{M}_e \times \mathcal{N}_e, \mathcal{A}_e) \rightarrow \text{Fun}_{\mathcal{D}, \mathcal{E}}(\mathcal{M}_e \square \mathcal{N}_e, \mathcal{A}_e).$$

Let  $\mathbf{G} \in \text{Fun}_{\mathcal{D}, \mathcal{E}}^{\text{bal}}(\mathcal{M}_e \times \mathcal{N}_e, \mathcal{A}_e)$  be a balanced bimodule functor. The left  $\mathcal{D}$ -module functor structure on  $\mathbf{G}$  is given by  $\mathcal{C}$ -balanced natural isomorphisms  $\phi_d^{\mathbf{G}} : \mathbf{G} \circ L_d^{\mathcal{M} \times \mathcal{N}} \rightarrow L_d^{\mathcal{A}} \circ \mathbf{G}$  for all  $d \in \mathcal{D}$  according to Lemma 2.15. Hence we can apply the 2-functor  $\widehat{(-)}$  and obtain natural isomorphisms  $\widehat{\phi}_d^{\mathbf{G}} : \widehat{\mathbf{G}}L_d^{\mathcal{M} \times \mathcal{N}} \rightarrow L_d^{\mathcal{A}}\widehat{\mathbf{G}}$ . Furthermore, applying  $\widehat{(-)}$  to the module constraint diagram for  $\mathbf{G}$  yields the module constraint diagram for  $\widehat{\mathbf{G}}$ . Hence we deduce that  $\widehat{\mathbf{G}}$  is a left  $\mathcal{D}$ -module functor. The proof that  $\widehat{\mathbf{G}}$  is a right  $\mathcal{E}$ -module functor is analogous. The compatibility between left and right module actions of  $\widehat{\mathbf{G}}$  follows by applying the functor  $\widehat{(-)}$  to the corresponding compatibility diagram of  $\mathbf{G}$ . Hence  $\widehat{\mathbf{G}}$  is a bimodule functor.

If  $\eta : \mathbf{G} \rightarrow \mathbf{F}$  is a balanced bimodule natural transformation between balanced bimodule functors  $\mathbf{F}$  and  $\mathbf{G}$ , it follows again by applying the 2-functor  $\widehat{(-)}$  that  $\widehat{\eta} : \widehat{\mathbf{G}} \rightarrow \widehat{\mathbf{F}}$  is a bimodule natural transformation.

It remains to show that for all balanced bimodule functors  $\mathbf{F} \in \text{Fun}_{\mathcal{D}, \mathcal{E}}^{\text{bal}}(\mathcal{M}_e \times \mathcal{N}_e, \mathcal{A}_e)$ , the natural isomorphism  $\varphi(\mathbf{G}) : \mathbf{G} \rightarrow \widehat{\mathbf{G}}\mathbf{B}$  is a balanced bimodule natural isomorphism and for all bimodule functors  $\mathbf{G} : \mathcal{M} \square \mathcal{N} \rightarrow \mathcal{A}$ , the natural isomorphism  $\kappa(\mathbf{G}) : \mathbf{G} \rightarrow \widehat{\mathbf{G}}\mathbf{B}$  is a bimodule natural isomorphism. The first statement follows directly from the definition of the bimodule structure of  $\widehat{\mathbf{G}}$ . For the second statement, we show that the lower rectangle in the diagram

$$\begin{array}{ccc}
 \widehat{\mathbf{G}}L_d\mathbf{B} & \xrightarrow{\phi_d^{\mathbf{G}}\mathbf{B}} & L_d\widehat{\mathbf{G}}\mathbf{B} \\
 \downarrow 1\varphi_{L_d}^{-1} & & \downarrow 1\varphi\mathbf{G}\mathbf{B} \\
 \mathbf{G}\mathbf{B}L_d & & \\
 \downarrow \varphi(\mathbf{G}\mathbf{B})1 & & \\
 \widehat{\mathbf{G}}\mathbf{B}\mathbf{B}L_d & & \\
 \downarrow 1\varphi_{L_d} & & \downarrow \widehat{\phi}_d^{\mathbf{G}}1 \\
 \widehat{\mathbf{G}}\mathbf{B}L_d\mathbf{B} & \xrightarrow{\widehat{\phi}_d^{\mathbf{G}}1} & L_d\widehat{\mathbf{G}}\mathbf{B}\mathbf{B} \\
 \downarrow \kappa^{-1}(\mathbf{G})1 & & \downarrow 1\kappa^{-1}(\mathbf{G})1 \\
 \mathbf{G}L_d\mathbf{B} & \xrightarrow{\phi_d^{\mathbf{G}}1} & L_d\mathbf{G}\mathbf{B}
 \end{array} \quad (3.23)$$

commutes. Because  $\Phi$  is fully faithful,  $\kappa(\mathbf{G})$  is then a bimodule natural isomorphism. The big diagram in the middle commutes by definition of  $\widehat{\phi}_d^{\mathbf{G}}$ . The diagram on the right commutes since  $\kappa$  and  $\varphi$  satisfy the snake identity. The diagram on the left commutes also by the snake identity for  $\kappa$  and  $\varphi$  after applying once the interchange law for functors and natural transformations.  $\square$

We further generalize the results of the previous section. First we unify balanced bimodule functors and bimodule functors in one 2-category. The next statement follows directly from the obvious version of Lemma 2.13 for balanced bimodule functors.

**Proposition 3.8** *For every finite tensor category  $\mathcal{C}$  and every pair of finite tensor categories  $(\mathcal{D}, \mathcal{E})$ , the following data define a 2-category  $\text{Bimod}_{\mathcal{C}}^{\text{bal}}(\mathcal{D}, \mathcal{E})$ .*

- i) The objects of  $\text{Bimod}_{\mathcal{C}}^{\text{bal}}(\mathcal{D}, \mathcal{E})$  are  $(\mathcal{D}, \mathcal{E})$ -bimodule categories  $\mathcal{M}_e \times \mathcal{N}_e$  and  $(\mathcal{D}, \mathcal{E})$ -bimodule categories  $\mathcal{A}_e$ .*
- ii) The following defines the categories of 1- and 2-morphisms between the objects:*

- (a) The category  $\mathbf{Bimod}_e^{\text{bal}}({}_{\mathcal{D}}\mathcal{M}_e \times {}_e\mathcal{N}_\varepsilon, {}_{\mathcal{D}}\mathcal{M}'_e \times {}_e\mathcal{N}'_\varepsilon)$  for two  $(\mathcal{D}, \varepsilon)$ -bimodule categories  ${}_{\mathcal{D}}\mathcal{M}_e \times {}_e\mathcal{N}_\varepsilon$  and  ${}_{\mathcal{D}}\mathcal{M}'_e \times {}_e\mathcal{N}'_\varepsilon$  is the category  $\text{Fun}_{\mathcal{D}, e}^{\mathcal{C}, \varepsilon}({}_{\mathcal{D}}\mathcal{M}_e \times {}_e\mathcal{N}_\varepsilon, {}_{\mathcal{D}}\mathcal{M}'_e \times {}_e\mathcal{N}'_\varepsilon)$  of  $(\mathcal{D} \times \mathcal{C}^{\text{rev}} \times \mathcal{C} \times \varepsilon)$ -module functors and  $(\mathcal{D} \times \mathcal{C}^{\text{rev}} \times \mathcal{C} \times \varepsilon)$ -module natural transformations between them.
  - (b) The category  $\mathbf{Bimod}_e^{\text{bal}}({}_{\mathcal{D}}\mathcal{M}_e \times {}_e\mathcal{N}_\varepsilon, {}_{\mathcal{D}}\mathcal{A}_\varepsilon)$  for bimodule categories  ${}_{\mathcal{D}}\mathcal{M}_e \times {}_e\mathcal{N}_\varepsilon$  and  ${}_{\mathcal{D}}\mathcal{A}_\varepsilon$  is the category  $\text{Fun}_{\mathcal{D}, \varepsilon}^{\text{bal}}({}_{\mathcal{D}}\mathcal{M}_e \times {}_e\mathcal{N}_\varepsilon, {}_{\mathcal{D}}\mathcal{A}_\varepsilon)$  of balanced bimodule functors and balanced bimodule natural transformations between them.
  - (c) The category  $\mathbf{Bimod}_e^{\text{bal}}({}_{\mathcal{D}}\mathcal{A}_\varepsilon, {}_{\mathcal{D}}\mathcal{B}_\varepsilon)$  for two bimodule categories  ${}_{\mathcal{D}}\mathcal{A}_\varepsilon$ , and  ${}_{\mathcal{D}}\mathcal{B}_\varepsilon$  is the category  $\mathbf{Bimod}({}_{\mathcal{D}}\mathcal{A}_\varepsilon, {}_{\mathcal{D}}\mathcal{B}_\varepsilon)$  of bimodule functors and bimodule natural transformations between them.
  - (d) There is just the zero morphism from a bimodule category  ${}_{\mathcal{D}}\mathcal{A}_\varepsilon$  to a bimodule category  ${}_{\mathcal{D}}\mathcal{M}_e \times {}_e\mathcal{N}_\varepsilon$ .
- iii) The compositions are induced by the horizontal composition of functors and the vertical composition of natural transformations.

If we restrict to the case where  $\mathcal{D} = \varepsilon = \mathbf{Vect}$ , we recover the 2-category from Proposition 3.5, i.e.  $\mathbf{Bimod}_e^{\text{bal}}(\mathbf{Vect}, \mathbf{Vect}) = \mathbf{Mod}_e^{\text{bal}}$ .

**Proposition 3.9** *The tensor product of bimodule categories defines a 2-functor*

$$\widehat{(-)} : \mathbf{Bimod}_e^{\text{bal}}(\mathcal{D}, \varepsilon) \rightarrow \mathbf{Bimod}(\mathcal{D}, \varepsilon). \quad (3.24)$$

In particular, it induces a 2-functor

$$\square : \mathbf{Bimod}(\mathcal{D}, \varepsilon) \times \mathbf{Bimod}(\mathcal{C}, \mathcal{D}) \rightarrow \mathbf{Bimod}(\mathcal{C}, \varepsilon). \quad (3.25)$$

Proof. Proposition 3.7 shows that the functors  $\Phi, \Psi$  and the natural transformations  $\varphi$  and  $\kappa$  that appear in the definition of the tensor product, are compatible with the  $(\mathcal{C}, \varepsilon)$  bimodule structure of a bimodule category  ${}_e\mathcal{M}_{\mathcal{D}} \times {}_{\mathcal{D}}\mathcal{N}_\varepsilon$ . It is straightforward to see that the analogue of Proposition 3.4 and Proposition 3.6 hold for bimodule categories. In particular, all coherence structures of the 2-functor  $\widehat{(-)}$  from Proposition 3.6 are bimodule natural isomorphisms.  $\square$

### 3.3 Multi-module categories

In the following we consider also multiple tensor products of the form  $({}_{\mathcal{K}}\mathcal{K}_\varepsilon \square {}_\varepsilon\mathcal{N}_{\mathcal{D}}) \square {}_{\mathcal{D}}\mathcal{M}_e$ . This requires an extension of the notion of balanced functors to so-called multi-balanced functors from  $(\mathcal{K}_\varepsilon \times {}_\varepsilon\mathcal{N}_{\mathcal{D}}) \times {}_{\mathcal{D}}\mathcal{M}$  to a linear category  $\mathcal{A}$ . An example is the functor

$$({}_{\mathcal{K}}\mathcal{K}_\varepsilon \times {}_\varepsilon\mathcal{N}_{\mathcal{D}}) \times {}_{\mathcal{D}}\mathcal{M}_e \xrightarrow{\mathbf{B}_{\mathcal{K}, \mathcal{N}} \times 1} ({}_{\mathcal{K}}\mathcal{K}_\varepsilon \square {}_\varepsilon\mathcal{N}_{\mathcal{D}}) \times {}_{\mathcal{D}}\mathcal{M}_e \xrightarrow{\mathbf{B}_{\mathcal{K} \square \mathcal{N}, \mathcal{M}}} ({}_{\mathcal{K}}\mathcal{K}_\varepsilon \square {}_\varepsilon\mathcal{N}_{\mathcal{D}}) \square {}_{\mathcal{D}}\mathcal{M}_e. \quad (3.26)$$

We then group these multi-balanced functors into a suitable bicategory, such that (3.26) is a composition in this bicategory. Note, however, that the functor  $\mathbf{B}_{\mathcal{K}, \mathcal{N}} \times 1$  is balanced with respect to the first two categories, but it is a bimodule functor (the identity) with respect to the third. Therefore we need to extend the notion of multi-balanced functors even further to so-called multi-balanced module functors, in order to guarantee that the functor  $\mathbf{B}_{\mathcal{K}, \mathcal{N}} \times 1$  is in this bicategory. The multi-balanced module functors will play an essential role in the proof that bimodule categories forms a tricategory.

In order to define the associator in this tricategory, we will be careful and distinguish the two categories  $(\mathcal{M} \times \mathcal{N}) \times \mathcal{K}$  and  $\mathcal{M} \times (\mathcal{N} \times \mathcal{K})$  for three categories  $\mathcal{M}, \mathcal{N}$  and  $\mathcal{K}$ . The relation between these categories will then finally lead to the associator in the tricategory of bimodule categories. Hence we say that a *bracketing*  $b$  of a string  $\underline{X} = (X_1, \dots, X_n)$  of letters  $X_i$  is a choice of parenthesis that uniquely specifies a sequence of pairings like e.g.  $(X_1(X_2X_3))X_4$ .

For a functor  $F : (\mathcal{M} \times \mathcal{N}) \times \mathcal{K} \rightarrow \mathcal{A}$ , we denote the functor on objects just by  $F(m \times n \times k)$ , if the bracketing is clear from the context. Recall from Remark 2.4, that for a module category  ${}_e\mathcal{M}$  and a finite linear category  $\mathcal{N}$ , we consider the category  ${}_e\mathcal{M} \times \mathcal{N}$  again as module category with  $\mathcal{C}$ -module action  $\triangleright \times \text{id}_{\mathcal{N}}$ . In the following it is always understood that the Cartesian product of module categories is equipped with this module action. We call two bimodule categories  $\mathcal{M}$  and  $\mathcal{N}$  composable, if the category that acts from the left on  $\mathcal{N}$  coincides with the category that acts from the right on  $\mathcal{M}$ .

**Definition 3.10** ([19, Def 3.4]) *i)* A multi-module category  $(\underline{\mathcal{M}}, b)$  from  $\mathcal{C}$  to  $\mathcal{D}$  is a finite string of composable bimodule categories  $\mathcal{M}^j$  for  $j \in \{1, \dots, n\}$  with  $n \in \mathbb{N}$ , where  $\mathcal{M}^n$  is a  $\mathcal{C}$ -right module category and  $\mathcal{M}^1$  is a  $\mathcal{D}$ -left module category, together with a bracketing  $b$  of  $\underline{\mathcal{M}}$ . We denote by  $ev(\underline{\mathcal{M}}, b)$  the Cartesian product of the categories  $\mathcal{M}_j$ , in the order that corresponds to the bracketing  $b$ .

*ii)* A multi-balanced functor  $F : (\underline{\mathcal{M}}, b) \rightarrow \mathcal{A}$ , from a  $(\mathbf{Vect}, \mathbf{Vect})$  multi-module category  $(\underline{\mathcal{M}}, b)$  to a linear category  $\mathcal{A}$  is a functor  $F : ev(\underline{\mathcal{M}}, b) \rightarrow \mathcal{A}$ , that is balanced in each argument, i.e. it is equipped with with natural isomorphisms

$$b_{m_1, \dots, m_i, c, m_{i+1}, \dots, m_n}^F : F(m_1 \times \dots \times m_i \triangleleft c \times m_{i+1} \times \dots \times m_n) \rightarrow F(m_1 \times \dots \times m_i \times c \triangleright m_{i+1} \times \dots \times m_n),$$

for each string of objects  $\underline{m} \in ev(\underline{\mathcal{M}}, b)$ , each  $i \in J$  and  $c \in \mathcal{C}$ , such that the natural isomorphisms  $b^F$  satisfy the diagram (2.11) in each entry  $i \in J$ . In the sequel we will abbreviate  $b_{m_1, \dots, m_i, c, m_{i+1}, \dots, m_n}^F$  by  $b_{\dots, m_i, c, m_{i+1}, \dots}$  whenever it is unambiguous.

Additionally, these isomorphisms are required to be compatible with the bimodule category structures, i.e. the diagram

$$\begin{array}{ccc} F(\dots m_{i-1} \triangleleft c \times m_i \triangleleft d \times m_{i+1} \dots) & \xrightarrow{b_{\dots, m_i, d, m_{i+1}, \dots}} & F(\dots m_{i-1} \triangleleft c \times m_i \times d \triangleright m_{i+1} \dots) \\ \downarrow b_{\dots, m_{i-1}, c, m_i \triangleleft d, \dots} & & \downarrow b_{\dots, m_{i-1}, c, m_i, \dots} \\ F(\dots m_{i-1} \times c \triangleright (m_i \triangleleft d) \times m_{i+1} \dots) & & \\ \downarrow \gamma_{c, m_i, d}^{-1} & & \\ F(\dots m_{i-1} \times (c \triangleright m_i) \triangleleft d \times m_{i+1} \dots) & \xrightarrow{b_{\dots, c, d \triangleright m_i, m_{i+1}, \dots}} & F(\dots m_{i-1} \times c \triangleright m_i \times d \triangleright m_{i+1} \dots), \end{array}$$

commutes for each possible entry  $i \in J$  and for all possible objects. Here the argument of the functor  $F$  is abbreviated and only the relevant part of the string  $\underline{m}$  is shown.

*iii)* A multi-balanced natural transformation  $\eta : F \rightarrow G$  between multi-balanced functors  $F, G : (\underline{\mathcal{M}}, b) \rightarrow \mathcal{A}$  is a natural transformation  $\eta$  that is balanced in each entry, i.e. it satisfies diagram (2.13) for all entries of a string of objects  $\underline{m}$  in  $\underline{\mathcal{M}}$ .

*iv)* For every multi-module category  $(\underline{\mathcal{M}}, b)$ , there is a corresponding string of finite tensor categories  $S(\underline{\mathcal{M}}, b)$ , that is given by the finite tensor categories acting on the bimodule categories in  $(\underline{\mathcal{M}}, b)$  such that for the string  $({}_{\mathcal{A}}\mathcal{K}_{\mathcal{E}}, {}_{\mathcal{E}}\mathcal{N}_{\mathcal{F}}, \dots, {}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}})$ , the corresponding string of finite tensor categories is  $S({}_{\mathcal{A}}\mathcal{K}_{\mathcal{E}}, {}_{\mathcal{E}}\mathcal{N}_{\mathcal{F}}, \dots, {}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}) = (\mathcal{H}, \mathcal{E}, \mathcal{F}, \dots, \mathcal{D}, \mathcal{C})$ . Note that by definition  $S(\underline{\mathcal{M}}, b) = S(\underline{\mathcal{M}}, b')$  is independent of the bracketing  $b$  and just called  $S(\underline{\mathcal{M}})$  in the sequel.

It is clear, that for each linear category  $\mathcal{A}$ , the multi-balanced functors and multi-balanced natural transformations from  $(\underline{\mathcal{M}}, b)$  to  $\mathcal{A}$  form a category denoted  $\mathbf{Fun}^{\text{bal}}((\underline{\mathcal{M}}, b), \mathcal{A})$ .

Next we consider multi-module functors.

**Definition 3.11** *i)* A multi-module functor  $F : (\underline{\mathcal{M}}, b) \rightarrow (\underline{\mathcal{M}}', b')$  between multi-module categories  $(\underline{\mathcal{M}}, b)$  and  $(\underline{\mathcal{M}}', b')$  with  $S(\underline{\mathcal{M}}) = S(\underline{\mathcal{M}}')$  is a functor  $F : ev(\underline{\mathcal{M}}, b) \rightarrow ev(\underline{\mathcal{M}}', b')$  together with a family of natural isomorphisms

$$\Phi_{m_1, \dots, m_i, d, m_{i+1}, \dots, m_n}^{F, l, i} : F(m_1 \times \dots \times m_i \triangleleft d \times m_{i+1} \times \dots \times m_n) \rightarrow F(m_1 \times \dots \times m_i \times m_{i+1} \times \dots \times m_n) \triangleleft^i d,$$

for each  $\underline{m} \in \underline{\mathcal{M}}$  and each  $i \in J$ , where  $\triangleleft^i : \underline{\mathcal{M}} \times \mathcal{D} \rightarrow \underline{\mathcal{M}}$  denotes the action of  $\mathcal{D}$  on  ${}_e\mathcal{M}_{\mathcal{D}}^i$ . Similarly we require that there exists a family of natural isomorphisms

$$\Phi_{m_1, \dots, c, m_i, \dots, m_n}^{F, r, i} : F(m_1 \times \dots \times c \triangleright m_i \times \dots \times m_n) \simeq c \triangleright^i F(m_1 \times \dots \times m_i \times \dots \times m_n),$$

where  $\triangleright^i : \mathcal{C} \times \underline{\mathcal{M}} \rightarrow \underline{\mathcal{M}}$  is induced by the left action of  $\mathcal{C}$  on  ${}_e\mathcal{M}_{\mathcal{D}}^i$ . The isomorphisms  $\Phi^{F, l, i}$  and  $\Phi^{F, r, i}$  are required to satisfy the bimodule constraint (2.7) for each  $i \in J$ .

*ii)* A multi-module natural transformation  $\eta : F \rightarrow G$  between multi-module functors  $F$  and  $G$  is a natural transformation that satisfies equation (2.6) in each entry.

**Example 3.12** *i)* For two bimodule functors  $F : {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_{\mathcal{D}}\mathcal{M}'_e$  and  $G : {}_e\mathcal{N}_e \rightarrow {}_e\mathcal{N}'_e$ , the functor  $G \times F : \mathcal{N} \times \mathcal{M} \rightarrow \mathcal{N}' \times \mathcal{M}'$  is a multi-module functor.

*ii)* For three bimodule categories a multi-module functor  $\alpha : ({}_{\mathcal{K}}\mathcal{K}_e \times {}_e\mathcal{N}_{\mathcal{D}}) \times {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_{\mathcal{K}}\mathcal{K}_e \times ({}_e\mathcal{N}_{\mathcal{D}} \times {}_{\mathcal{D}}\mathcal{M}_e)$  is given by  $\alpha((h \times n) \times m) = h \times (n \times m)$  on objects and morphisms  $((h \times n) \times m)$  in  $({}_{\mathcal{K}}\mathcal{K}_e \times {}_e\mathcal{N}_{\mathcal{D}}) \times {}_{\mathcal{D}}\mathcal{M}_e$ .

Next we consider multi-balanced module functors.

**Definition 3.13** *i)* For a string  $(X_1, \dots, X_n)$ , a reduced string  $(X'_1, \dots, X'_m)$  is a string that is obtained from  $(X_1, \dots, X_n)$  by erasing entries as follows. It is required that there exists an injective map  $f : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$  with  $f(1) = 1$  and  $f(m) = n$  and  $f(j) > f(i)$  for all  $j > i$  in  $\{1, \dots, m\}$  and  $X'_i = X_{f(i)}$  for all  $i \in \{1, \dots, m\}$ . An entry  $X_j$  is called erased in  $(X_1, \dots, X_n)$  if  $j \in \{1, \dots, n\}$  is not in the image of  $f$ .

*ii)* Let  $(\underline{\mathcal{M}}, b)$  and  $(\underline{\mathcal{M}}', b')$  be multi-module categories, such that the string  $S(\underline{\mathcal{M}}', b')$  of finite tensor categories is obtained by reducing the string  $S(\underline{\mathcal{M}}, b)$ . A multi-balanced module functor  $F : (\underline{\mathcal{M}}, b) \rightarrow (\underline{\mathcal{M}}', b')$  is a functor  $F : ev(\underline{\mathcal{M}}, b) \rightarrow ev(\underline{\mathcal{M}}', b')$  that is balanced in each erased entry of  $S(\underline{\mathcal{M}}, b)$  and is a multi-module functor in each other entry. We furthermore require that at entries where  $F$  is balanced, it is compatible with the bimodule category structures. That means that with each entry next to it,  $F$  satisfies either the diagram of Definition 3.10 ii), if  $F$  is also balanced at the neighboring entry, or the diagram (2.14), if  $F$  if the neighboring entry is not erased.

*iii)* A multi-balanced module natural transformation  $\eta : F \rightarrow G$  between multi-balanced module functors  $F$  and  $G$  is a natural transformation  $\eta : F \rightarrow G$  that is balanced in each erased entry in the target of  $F$  and  $G$  and a bimodule natural transformation in all other entries.

**Example 3.14** *i)* Every multi-module functor and every multi-balanced functor is also a multi-balanced module functor.

*ii)* For three composable bimodule categories, the functor  $(B_{\mathcal{K}, \mathcal{N}} \times 1) : (\mathcal{K} \times \mathcal{N}) \times \mathcal{M} \rightarrow (\mathcal{K} \square \mathcal{N}) \times \mathcal{M}$  is a multi-balanced module functor.

It follows directly from the definitions, that if a string of multi-module categories  $(\underline{\mathcal{M}}'', b'')$  is reduced from a string  $(\underline{\mathcal{M}}', b')$  and  $(\underline{\mathcal{M}}', b')$  is reduced from  $(\underline{\mathcal{M}}, b)$ , then the composite GF of multi-balanced module functors  $F : (\underline{\mathcal{M}}, b) \rightarrow (\underline{\mathcal{M}}', b')$  and  $G : (\underline{\mathcal{M}}', b') \rightarrow (\underline{\mathcal{M}}'', b'')$  is a multi-balanced module functor. We can therefore generalize Proposition 3.8 and follows.

**Proposition 3.15** For every pair of finite tensor categories  $(\mathcal{C}, \mathcal{D})$ , the following data define a 2-category  $\text{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{D})$ .

- i)* Objects are multi-module categories  $(\underline{\mathcal{M}}, b)$ ,  $(\underline{\mathcal{M}}', b')$  from  $\mathcal{C}$  to  $\mathcal{D}$ .
- ii)* 1-morphisms between objects  $(\underline{\mathcal{M}}, b)$  and  $(\underline{\mathcal{M}}', b')$  are multi-balanced module functors  $F : (\underline{\mathcal{M}}, b) \rightarrow (\underline{\mathcal{M}}', b')$  if the string  $(\underline{\mathcal{M}}', b')$  is reduced from  $(\underline{\mathcal{M}}, b)$ . Otherwise, the set of 1-morphisms from  $(\underline{\mathcal{M}}, b)$  to  $(\underline{\mathcal{M}}', b')$  contains just the zero morphism.
- iii)* 2-morphisms between multi-balanced module functors  $F, G : (\underline{\mathcal{M}}, b) \rightarrow (\underline{\mathcal{M}}', b')$  are multi-balanced module natural transformations  $\eta : F \rightarrow G$ .
- iv)* The compositions are induced by the horizontal composition of functors and the vertical composition of natural transformations.

**Remark 3.16** For all finite tensor categories  $\mathcal{C}$  the 2-categories  $\text{Bimod}_e^{\text{bal}}(\mathcal{E}, \mathcal{D})$  from Proposition 3.8 are full 2-subcategories of  $\text{Bimod}^{\text{multi}}(\mathcal{E}, \mathcal{D})$  whose objects are bimodule categories  ${}_{\mathcal{D}}\mathcal{M}_e \times {}_e\mathcal{N}_e$  and  ${}_{\mathcal{D}}\mathcal{A}_e$ .

For a multi-module category  $(\underline{\mathcal{M}}, b)$  we already defined the category  $ev(\underline{\mathcal{M}}, b)$  that is obtained from the Cartesian product of the elements in the string. Now, let  $ev_{\square}(\underline{\mathcal{M}}, b)$  denote the category that is obtained by the tensor product of the bimodule categories in the string  $(\underline{\mathcal{M}}, b)$  in the order that corresponds to the bracketing  $b$ . We call  $ev_{\square}(\underline{\mathcal{M}}, b)$  the tensor product of the multi-module category  $(\underline{\mathcal{M}}, b)$ .

**Lemma 3.17** *Let  $(\underline{\mathcal{M}}, b)$  be a multi-module category from  $\mathcal{C}$  to  $\mathcal{D}$ . Then the tensor product  $ev_{\square}(\underline{\mathcal{M}}, b)$  of  $(\underline{\mathcal{M}}, b)$  is a  $(\mathcal{D}, \mathcal{C})$ -bimodule category and it is equipped with*

- i) a multi-balanced  $(\mathcal{D}, \mathcal{C})$ -bimodule functor  $\underline{\mathcal{B}}_{\underline{\mathcal{M}}} : (\underline{\mathcal{M}}, b) \rightarrow ev_{\square}(\underline{\mathcal{M}}, b)$ ,*
- ii) for every bimodule category  ${}_{\mathcal{D}}\mathcal{A}_e$  with a functor*

$$\underline{\Psi}_{\underline{\mathcal{M}}} : \mathbf{Bimod}^{\text{multi}}((\underline{\mathcal{M}}, b), {}_{\mathcal{D}}\mathcal{A}_e) \rightarrow \text{Fun}_{\mathcal{D}, e}(ev_{\square}(\underline{\mathcal{M}}, b), {}_{\mathcal{D}}\mathcal{A}_e), \quad (3.27)$$

- iii) an adjoint equivalence between the functor  $\underline{\Psi}_{\underline{\mathcal{M}}}$  and the functor*

$$\begin{aligned} \underline{\Phi}_{\underline{\mathcal{M}}} : \text{Fun}_{\mathcal{D}, e}(ev_{\square}(\underline{\mathcal{M}}, b), {}_{\mathcal{D}}\mathcal{A}_e) &\rightarrow \mathbf{Bimod}^{\text{multi}}((\underline{\mathcal{M}}, b), {}_{\mathcal{D}}\mathcal{A}_e) \\ \mathbf{G} &\mapsto \mathbf{G} \circ \underline{B}_{\underline{\mathcal{M}}}. \end{aligned} \quad (3.28)$$

Proof. It follows by repeated use of Proposition 3.7, starting with the inner most bracketing of  $(\underline{\mathcal{M}}, b)$ , that  $ev_{\square}(\underline{\mathcal{M}}, b)$  is a  $(\mathcal{D}, \mathcal{C})$ -bimodule category. The functor  $\underline{\mathcal{B}}_{\underline{\mathcal{M}}} : (\underline{\mathcal{M}}, b) \rightarrow ev_{\square}(\underline{\mathcal{M}}, b)$  is defined iteratively as indicated in equation (3.26) for a string of three bimodule categories. It is shown in Proposition 3.7 that  $\underline{\mathcal{B}} : {}_{\mathcal{D}}\mathcal{M}_e \times_e \mathcal{N}_e \rightarrow {}_{\mathcal{D}}\mathcal{M}_e \square_e \mathcal{N}_e$  is a multi-balanced module functor and hence  $\underline{\mathcal{B}}_{\underline{\mathcal{M}}}$  is a multi-balanced module functor as it is the composition of multi-balanced module functors. Hence the first part is proven. To show the second statement, let  $\underline{\mathcal{F}} : (\underline{\mathcal{M}}, b) \rightarrow {}_{\mathcal{D}}\mathcal{A}_e$  be a multi-balanced module functor from the multi-module category  $(\underline{\mathcal{M}}, b) = (\mathcal{M}_1, \dots, (\mathcal{M}_i, \mathcal{M}_{i+1}), \dots, \mathcal{M}_n)$  to a bimodule category  ${}_{\mathcal{D}}\mathcal{A}_e$ . Assume that  $\mathcal{M}_i$  and  $\mathcal{M}_{i+1}$  are  $\mathcal{E}$ -left, respectively right, module categories.  $\underline{\mathcal{F}}$  is clearly a  $\mathcal{E}$ -balanced bimodule functor and hence induces a bimodule functor  $\underline{\mathcal{F}}_1 : (\mathcal{M}_1, \dots, \mathcal{M}_{i-1}, \mathcal{M}_i \square \mathcal{M}_{i+1}, \dots, \mathcal{M}_n) \rightarrow {}_{\mathcal{D}}\mathcal{A}_e$ , by Proposition 3.9. It is straightforward to see that  $\underline{\mathcal{F}}$  is again a multi-balanced module functor and we continue iteratively to obtain a bimodule functor  $\widehat{\underline{\mathcal{F}}} : ev_{\square}(\underline{\mathcal{M}}, b) \rightarrow {}_{\mathcal{D}}\mathcal{A}_e$ . Defined in the analogous way for multi-balanced module natural transformations, this yields the functor  $\underline{\Psi}_{\underline{\mathcal{M}}}$ . Furthermore, it is clear that by construction,  $\underline{\Psi}_{\underline{\mathcal{M}}}$  and  $\underline{\Phi}_{\underline{\mathcal{M}}}$  form an adjoint equivalence since both functors are obtained by composing the corresponding functors from the constituents of the string  $(\underline{\mathcal{M}}, b)$  which from adjoint equivalences by Proposition 3.7.  $\square$

By using this lemma and by repeated use of the 2-functor  $\widehat{(-)}$  from Proposition 3.9, we can extend the tensor product to a 2-functor as follows.

**Proposition 3.18** *For all pairs of finite tensor categories  $\mathcal{C}$  and  $\mathcal{D}$ , the tensor product defines a 2-functor*

$$\widehat{(-)} : \mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{D}) \rightarrow \mathbf{Bimod}(\mathcal{C}, \mathcal{D}). \quad (3.29)$$

We are going to apply this 2-functor to diagrams of (horizontally and vertically) composable 2-morphisms. Such diagrams are called pasting diagrams and are defined with more precision in [3], see also [23].

**Corollary 3.19** *For every pasting diagram  $D$  in  $\mathbf{Bimod}^{\text{multi}}$ , the 2-functor  $\widehat{(-)}$  yields a pasting diagram  $\widehat{D}$  with the same underlying graph in which all 1-morphisms  $\underline{\mathcal{F}}$  are replaced by  $\widehat{\underline{\mathcal{F}}}$  and all 2-morphisms  $\rho$  are replaced by a composite of  $\widehat{\rho}$  with coherence morphisms of the 2-functor  $\widehat{(-)}$ . If two pasting diagrams  $D, D'$  in  $\mathbf{Bimod}^{\text{multi}}$  with the same 1-morphisms one the outer arrows evaluate to the same 2-morphism, then also  $\widehat{D}$  and  $\widehat{D}'$  evaluate to the same 2-morphisms.*

Proof. These statements hold for general 2-functors. Assume that  $\underline{\mathcal{H}} : \mathcal{B} \rightarrow \mathcal{R}$  is a strict 2-functor between strict 2-categories. Then it is clear that  $\underline{\mathcal{H}}$  applied to a pasting diagrams in  $\mathcal{B}$  yields a pasting diagram in  $\mathcal{R}$ . By the strictification result for general 2-functors, see e.g. [20, Chapter 2], any 2-functor  $\underline{\mathcal{H}} : \mathcal{B} \rightarrow \mathcal{R}$  between (not necessarily strict) bicategories applied to a pasting diagrams in  $\mathcal{B}$  yields a pasting diagram in  $\mathcal{R}$ . The last statement follows directly for strict 2-functors and hence again for general 2-functors as wells.  $\square$

Note however, that if  $D$  is a commutative diagram of 1-morphisms in  $\mathbf{Bimod}^{\text{multi}}$ , the corresponding diagram  $\widehat{D}$  commutes in general only up to natural 2-isomorphisms, that is built from the coherence structure of the 2-functor  $\widehat{(-)}$ .

We now consider structures in the collection of the bicategories  $\mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{D})$  for different  $\mathcal{C}, \mathcal{D}$ . These structures are the main tool in the construction of the tricategory of bimodule categories.

**Proposition 3.20** *The family of bicategories  $\mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{D})$  for finite tensor categories  $\mathcal{C}$  and  $\mathcal{D}$  is equipped with the following additional structures.*

i) *The Cartesian product of module categories defines 2-functors*

$$\times^{\text{multi}} : \mathbf{Bimod}^{\text{multi}}(\mathcal{D}, \mathcal{E}) \times \mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{D}) \rightarrow \mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{E}). \quad (3.30)$$

ii) *The tensor product of module categories defines 2-functors*

$$\square : \mathbf{Bimod}^{\text{multi}}(\mathcal{D}, \mathcal{E}) \times \mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{D}) \rightarrow \mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{E}). \quad (3.31)$$

iii) *The universal balanced functors in the definition of a tensor product of module categories yield a pseudo-natural transformation  $\mathbf{B} : \times^{\text{multi}} \rightarrow \square$*

iv) *The canonical bimodule category  ${}_e\mathcal{C}_e$  defines the (strict) unit 2-functors  $I_e : I \rightarrow \mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{C})$ , where  $I$  denotes the unit 2-category.*

v) *For four finite tensor categories  $\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}$  there is an adjoint equivalence, where we abbreviated  $\mathbf{Bimod}^{\text{multi}}$  with  $\mathbf{B}^m$*

$$\begin{array}{ccc} \mathbf{B}^m(\mathcal{E}, \mathcal{F}) \times \mathbf{B}^m(\mathcal{D}, \mathcal{E}) \times \mathbf{B}^m(\mathcal{C}, \mathcal{D}) & \xrightarrow{(\times^{\text{multi}}) \times 1} & \mathbf{B}^m(\mathcal{D}, \mathcal{F}) \times \mathbf{B}^m(\mathcal{C}, \mathcal{D}) \\ \downarrow 1 \times \times^{\text{multi}} & \Downarrow \alpha & \downarrow \times^{\text{multi}} \\ \mathbf{B}^m(\mathcal{E}, \mathcal{F}) \times \mathbf{B}^m(\mathcal{C}, \mathcal{E}) & \xrightarrow{\times^{\text{multi}}} & \mathbf{B}^m(\mathcal{C}, \mathcal{F}), \end{array} \quad (3.32)$$

more precisely,  $\alpha : \times^{\text{multi}} \circ (\times^{\text{multi}} \times 1) \rightarrow \times^{\text{multi}} \circ (1 \times \times^{\text{multi}})$  is a pseudo-natural transformation and there exists a pseudo-natural transformation  $\alpha^- : \times^{\text{multi}} \circ (1 \times \times^{\text{multi}}) \rightarrow \times^{\text{multi}} \circ (\times^{\text{multi}} \times 1)$ , such that  $\alpha$  and  $\alpha^-$  form an adjoint equivalence.

vi) *For finite tensor categories  $\mathcal{C}, \mathcal{D}$  there are pseudo-natural transformations*

$$\begin{array}{ccc} & \mathbf{Bimod}^{\text{multi}}(\mathcal{D}, \mathcal{D}) \times \mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{D}) & \\ I_{\mathcal{D}} \times 1 \nearrow & \Downarrow \triangleright & \searrow \times^{\text{multi}} \\ \mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{D}) & \xrightarrow{1} & \mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{D}) \end{array} \quad (3.33)$$

and

$$\begin{array}{ccc} & \mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{D}) \times \mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{C}) & \\ 1 \times I_e \nearrow & \Downarrow \triangleleft & \searrow \times^{\text{multi}} \\ \mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{D}) & \xrightarrow{1} & \mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{D}). \end{array} \quad (3.34)$$

vii) *For all bimodule categories  ${}_{\mathcal{D}}\mathcal{M}_e$  and  ${}_e\mathcal{N}_e$ , the balancing constraint of  $\mathbf{B}$  defines an invertible modification  $\beta$  with components*

$$\begin{array}{ccc} (\mathcal{M} \times \mathcal{C}) \times \mathcal{N} & \xrightarrow{\alpha} & \mathcal{M} \times (\mathcal{C} \times \mathcal{N}) \\ \downarrow \triangleleft_{\mathcal{M}} \times 1 & & \downarrow 1 \times \triangleright_{\mathcal{N}} \\ \mathcal{M} \times \mathcal{N} & \xrightarrow{\beta} & \mathcal{M} \times \mathcal{N} \\ & \searrow \mathbf{B} & \swarrow \mathbf{B} \\ & \mathcal{M} \square \mathcal{N}. & \end{array} \quad (3.35)$$

viii) *For all bimodule categories  ${}_{\mathcal{D}}\mathcal{M}_e$  and  ${}_e\mathcal{N}_e$ , the following diagrams of pseudo-natural transformations commute*

$$\begin{array}{ccc} (\mathcal{C} \times \mathcal{M}) \times \mathcal{N} & \xrightarrow{\triangleright_{\mathcal{M}} \times 1} & \mathcal{M} \times \mathcal{N} \\ \searrow \alpha & & \swarrow \triangleright_{\mathcal{M} \times \mathcal{N}} \\ & \mathcal{C} \times (\mathcal{M} \times \mathcal{N}), & \end{array} \quad (3.36)$$



$$\begin{array}{c}
\begin{array}{ccc}
\mathcal{M}((\mathcal{N}\mathcal{D})\mathcal{K}) & \xrightarrow{1\alpha} & \mathcal{M}(\mathcal{N}(\mathcal{D}\mathcal{K})) \\
\downarrow 1(\triangleleft 1) & \swarrow \alpha & \downarrow 1(1\triangleright) \\
\mathcal{M}(\mathcal{N}\mathcal{K}) & \begin{array}{c} (\mathcal{M}(\mathcal{N}\mathcal{D}))\mathcal{K} \\ \circlearrowleft \\ ((\mathcal{M}\mathcal{N})\mathcal{D})\mathcal{K} \\ \circlearrowleft \\ (\mathcal{M}\mathcal{N})\mathcal{K} \end{array} & \begin{array}{c} (\mathcal{M}\mathcal{N})(\mathcal{D}\mathcal{K}) \\ \circlearrowleft \\ (\mathcal{M}\mathcal{N})\mathcal{K} \end{array} \\
\swarrow \alpha & \downarrow (1\triangleleft 1) & \downarrow (11)\triangleright \\
& (\mathcal{M}\mathcal{N})\mathcal{K} & (\mathcal{M}\mathcal{N})\mathcal{K} \\
& \downarrow \triangleleft 1 & \leftarrow \beta \\
& (\mathcal{M}\mathcal{N})\mathcal{K} & \\
& \downarrow \mathbf{B} & \downarrow \mathbf{B} \\
& (\mathcal{M}\mathcal{N})\square\mathcal{K} & 
\end{array} \\
\parallel \\
\begin{array}{ccc}
\mathcal{M}((\mathcal{N}\mathcal{D})\mathcal{K}) & \xrightarrow{1\alpha} & \mathcal{M}(\mathcal{N}(\mathcal{D}\mathcal{K})) \\
\downarrow 1(\triangleleft 1) & & \downarrow 1(1\triangleright) \\
\mathcal{M}(\mathcal{N}\mathcal{K}) & \begin{array}{c} \leftarrow \beta \\ (\mathcal{M}\mathcal{N})\mathcal{K} \end{array} & \mathcal{M}(\mathcal{N}\mathcal{K}) \\
\swarrow \alpha & & \swarrow \alpha \\
& (\mathcal{M}\mathcal{N})\mathcal{K} & (\mathcal{M}\mathcal{N})\mathcal{K} \\
& \downarrow \mathbf{B} & \downarrow \mathbf{B} \\
& (\mathcal{M}\mathcal{N})\square\mathcal{K} & 
\end{array}
\end{array} \tag{3.40}$$

Proof. The first part follows from the definitions, the second part is shown in Proposition 3.18. For the third part we first show that  $\mathbf{B}$  defines a pseudo-natural transformation between the 2-functors  $\times, \square : \mathbf{Bimod}(\mathcal{D}, \mathcal{E}) \times \mathbf{Bimod}(\mathcal{C}, \mathcal{D}) \rightarrow \mathbf{Bimod}(\mathcal{C}, \mathcal{E})$ . By Proposition 3.9, the bimodule natural isomorphisms  $\varphi_{F \times G} : \mathbf{B}(F \times G) \rightarrow (F \square G)\mathbf{B}$  for bimodule functors  $F \times G : {}_{\varepsilon}\mathcal{N}_{\mathcal{D}} \times {}_{\mathcal{D}}\mathcal{M}_{\varepsilon} \rightarrow {}_{\varepsilon}\mathcal{N}'_{\mathcal{D}} \times {}_{\mathcal{D}}\mathcal{M}'_{\varepsilon}$  are compatible with the composition of bimodule functors. The compatibility of the natural isomorphisms  $\varphi_{F \times G}$  with bimodule natural transformations follows also directly from the 2-functorial properties of the tensor product. Since we used only the 2-functoriality of the tensor product, this argument extends first to bimodule categories and then by repeated application also to multi-module categories. This shows the third part. Parts *iv)* and *v)* are clear. The properties of a pseudo-natural transformation for the module action in part *vi)* follow from the compatibility conditions between module actions and bimodule functors and bimodule natural transformations.

Parts *vii)- ix)* are clear from the definitions. The axioms in equations (3.39) and (3.40) follow directly from the properties of the Cartesian product of categories.  $\square$

### 3.4 The tricategory of bimodule categories

We finally show that bimodule categories define an algebraic tricategory according to Definition B.1, that is a slight modification of [20, Def. 3.1.2].

**Theorem 3.21** *Finite tensor categories, finite bimodule categories, right exact bimodule functors and bimodule natural transformations from an algebraic tricategory  $\mathbf{Bimod}$  in the sense of Definition B.1. The composition  $\square$  is given by the tensor product of bimodule categories, the*



horizontal composition  $\circ$  is given by the composition of functors and the vertical composition is defined by the vertical composition of natural transformations.

By unpacking Definition B.1, one finds that the claim of the theorem follows from the following.

- i)  $\mathbf{Bimod}(\mathcal{C}, \mathcal{D})$  is a strict 2-category with the composition of functors as horizontal composition  $\circ$  and the composition of natural transformations as vertical composition  $\cdot$ .
- ii) For any three finite tensor categories  $\mathcal{C}, \mathcal{D}, \mathcal{E}$ , the tensor product of module categories defines a 2-functors

$$\square : \mathbf{Bimod}(\mathcal{D}, \mathcal{E}) \times \mathbf{Bimod}(\mathcal{C}, \mathcal{D}) \rightarrow \mathbf{Bimod}(\mathcal{C}, \mathcal{E}). \quad (3.41)$$

- iii) The bimodule category  ${}_e\mathcal{C}_e$  defines for each object  $\mathcal{C}$  a (strict) unit 2-functor  $I_{\mathcal{C}} : I \rightarrow \mathbf{Bimod}(\mathcal{C}, \mathcal{C})$ , where  $I$  denotes the unit 2-category with one object 1, one 1-morphism  $1_1$  and one 2-morphism  $1_{11}$ .
- iv) For any four objects  $\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}$  there is an adjoint equivalence  $a : \square(\square \times 1) \Rightarrow \square(1 \times \square)$ , called associator in the following. More precisely,  $a$  consists of a pseudo-natural transformation

$$\begin{array}{ccc} \mathbf{Bimod}(\mathcal{E}, \mathcal{F}) \times \mathbf{Bimod}(\mathcal{D}, \mathcal{E}) \times \mathbf{Bimod}(\mathcal{C}, \mathcal{D}) & \xrightarrow{\square \times 1} & \mathbf{Bimod}(\mathcal{D}, \mathcal{F}) \times \mathbf{Bimod}(\mathcal{C}, \mathcal{D}) \\ \downarrow 1 \times \square & & \downarrow \square \\ \mathbf{Bimod}(\mathcal{E}, \mathcal{F}) \times \mathbf{Bimod}(\mathcal{C}, \mathcal{E}) & \xrightarrow{\square} & \mathbf{Bimod}(\mathcal{C}, \mathcal{F}), \end{array} \quad (3.42)$$

and there is a pseudo-natural transformation  $a^- : \square(1 \times \square) \rightarrow \square(\square \times 1)$ , such that  $a$  and  $a^-$  form an adjoint equivalence, see Definition A.5.

- v) For any two objects  $\mathcal{C}, \mathcal{D}$ , there are adjoint equivalences  $l : \square(I_{\mathcal{D}} \times 1) \Rightarrow 1$  and  $r : \square(1 \times I_{\mathcal{C}}) \Rightarrow 1$ , called the unit 2-morphisms,

$$\begin{array}{ccc} & \mathbf{Bimod}(\mathcal{D}, \mathcal{D}) \times \mathbf{Bimod}(\mathcal{C}, \mathcal{D}) & \\ I_{\mathcal{D}} \times 1 \nearrow & \Downarrow l & \searrow \square \\ \mathbf{Bimod}(\mathcal{C}, \mathcal{D}) & \xrightarrow{1} & \mathbf{Bimod}(\mathcal{C}, \mathcal{D}) \end{array} \quad (3.43)$$

and

$$\begin{array}{ccc} & \mathbf{Bimod}(\mathcal{C}, \mathcal{D}) \times \mathbf{Bimod}(\mathcal{C}, \mathcal{C}) & \\ 1 \times I_{\mathcal{C}} \nearrow & \Downarrow r & \searrow \square \\ \mathbf{Bimod}(\mathcal{C}, \mathcal{D}) & \xrightarrow{1} & \mathbf{Bimod}(\mathcal{C}, \mathcal{D}). \end{array} \quad (3.44)$$

By definition of an adjoint equivalence,  $l$  and  $r$  are pseudo-natural transformations. Furthermore there are corresponding pseudo-natural transformations  $l^- : 1 \Rightarrow \square(I_{\mathcal{D}} \times 1)$  and  $r^- : 1 \Rightarrow \square(1 \times I_{\mathcal{C}})$ .

- vi) For all bimodule categories  ${}_d\mathcal{M}_e$  and  ${}_e\mathcal{N}_e$  there is an invertible modification  $\mu$  with component 3-morphisms

$$\begin{array}{ccc} (\mathcal{M} \square \mathcal{C}) \square \mathcal{N} & \xrightarrow{a} & \mathcal{M} \square (\mathcal{C} \square \mathcal{N}) \\ & \Downarrow \mu & \\ r_{\mathcal{M}} \square 1 \searrow & & \swarrow 1 \square l_{\mathcal{N}} \\ & \mathcal{M} \square \mathcal{N}. & \end{array} \quad (3.45)$$

- vii) For all bimodule categories  ${}_e\mathcal{M}_d$  and  ${}_d\mathcal{N}_e$  there is an invertible modification  $\lambda$  with component 3-morphisms

$$\begin{array}{ccc} (\mathcal{C} \square \mathcal{M}) \square \mathcal{N} & \xrightarrow{l_{\mathcal{M}} \square 1} & \mathcal{M} \square \mathcal{N} \\ & \Downarrow \lambda & \\ a \searrow & & \swarrow l_{\mathcal{M}} \square \mathcal{N} \\ & \mathcal{C} \square (\mathcal{M} \square \mathcal{N}). & \end{array} \quad (3.46)$$

viii) For all bimodule categories  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{E}}$  and  ${}_{\mathcal{E}}\mathcal{N}_{\mathcal{E}}$  there is an invertible modification  $\rho$  with component 3-morphisms

$$\begin{array}{ccc}
 \mathcal{M} \square (\mathcal{N} \square \mathcal{E}) & \xrightarrow{1 \square r_{\mathcal{N}}} & \mathcal{M} \square \mathcal{N} \\
 \searrow a & \rho \Downarrow & \nearrow r_{\mathcal{M} \square \mathcal{N}} \\
 & (\mathcal{M} \square \mathcal{N}) \square \mathcal{E} &
 \end{array} \tag{3.47}$$

ix) For all composable bimodule categories  $\mathcal{K}, \mathcal{N}, \mathcal{M}$  and  $\mathcal{L}$ , there is an invertible modification  $\pi$  with component 3-morphisms

$$\begin{array}{ccccc}
 & & ((\mathcal{K} \square \mathcal{N}) \square \mathcal{M}) \square \mathcal{L} & & \\
 & \swarrow a & & \searrow a \square 1 & \\
 (\mathcal{K} \square \mathcal{N}) \square \mathcal{M} \square \mathcal{L} & & & & (\mathcal{K} \square (\mathcal{N} \square \mathcal{M})) \square \mathcal{L} \\
 \downarrow a & & \leftarrow \pi & & \downarrow a \\
 \mathcal{K} \square (\mathcal{N} \square (\mathcal{M} \square \mathcal{L})) & \xleftarrow{1 \square a} & & & \mathcal{K} \square ((\mathcal{N} \square \mathcal{M}) \square \mathcal{L})
 \end{array} \tag{3.48}$$

x) The following three axioms are satisfied. In the first axiom, the unmarked isomorphisms are isomorphisms from the naturality of  $a$ .

$$\begin{array}{c}
\begin{array}{c}
((\mathcal{K}(\mathcal{L}\mathcal{M})\mathcal{N})\mathcal{R}) \xrightarrow{1a} (\mathcal{K}((\mathcal{L}\mathcal{M})\mathcal{N}))\mathcal{R} \xrightarrow{(1a)1} (\mathcal{K}(\mathcal{L}(\mathcal{M}\mathcal{N})))\mathcal{R} \\
\downarrow \pi_1 \quad \searrow a_1 \quad \downarrow a \quad \searrow 1a \\
((\mathcal{K}\mathcal{L})(\mathcal{M}\mathcal{N}))\mathcal{R} \xrightarrow{a} (\mathcal{K}\mathcal{L})(\mathcal{M}\mathcal{N})\mathcal{R} \xrightarrow{a} (\mathcal{K}\mathcal{L})(\mathcal{M}\mathcal{N})\mathcal{R} \\
\downarrow \pi \quad \downarrow \pi \quad \downarrow \pi \\
((\mathcal{K}\mathcal{L})\mathcal{M})(\mathcal{N}\mathcal{R}) \xrightarrow{a} (\mathcal{K}\mathcal{L})(\mathcal{M}(\mathcal{N}\mathcal{R}))
\end{array} \\
\parallel \\
\begin{array}{c}
((\mathcal{K}(\mathcal{L}\mathcal{M})\mathcal{N})\mathcal{R}) \xrightarrow{a_1} (\mathcal{K}((\mathcal{L}\mathcal{M})\mathcal{N}))\mathcal{R} \xrightarrow{(1a)1} (\mathcal{K}(\mathcal{L}(\mathcal{M}\mathcal{N})))\mathcal{R} \\
\downarrow \pi \quad \searrow a \quad \downarrow a \quad \searrow 1a \\
((\mathcal{K}\mathcal{L})\mathcal{M})(\mathcal{N}\mathcal{R}) \xrightarrow{a} (\mathcal{K}(\mathcal{L}\mathcal{M}))(\mathcal{N}\mathcal{R}) \xrightarrow{1a} (\mathcal{K}((\mathcal{L}\mathcal{M}))\mathcal{N})\mathcal{R} \xrightarrow{1a} (\mathcal{K}(\mathcal{L}(\mathcal{M}\mathcal{N})))\mathcal{R} \\
\downarrow \pi \quad \downarrow \pi \quad \downarrow 1\pi \\
((\mathcal{K}\mathcal{L})\mathcal{M})(\mathcal{N}\mathcal{R}) \xrightarrow{a} (\mathcal{K}\mathcal{L})(\mathcal{M}(\mathcal{N}\mathcal{R}))
\end{array}
\end{array}$$

(3.49)

$$\begin{array}{ccc}
((\mathcal{M} \mathcal{E}) \mathcal{N}) \mathcal{K} & \xrightarrow{a1} & (\mathcal{M}(\mathcal{E} \mathcal{N})) \mathcal{K} \\
\downarrow (r1)1 & \searrow a & \downarrow \pi \\
(\mathcal{M} \mathcal{E})(\mathcal{N} \mathcal{K}) & \xrightarrow{a} & \mathcal{M}((\mathcal{E} \mathcal{N}) \mathcal{K}) \\
\downarrow (r1)1 & \swarrow a & \downarrow 1a \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}(\mathcal{E}(\mathcal{N} \mathcal{K})) \\
\downarrow a & \swarrow r(11) & \downarrow 1l \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}(\mathcal{N} \mathcal{K}) \\
\downarrow a & \swarrow \mu & \downarrow 1\lambda \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}(\mathcal{N} \mathcal{K}) \\
\downarrow a & \swarrow 1(l1) & \downarrow a \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}(\mathcal{N} \mathcal{K}) \\
\downarrow a & \swarrow a & \downarrow a \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}(\mathcal{N} \mathcal{K})
\end{array}$$

$\parallel$

$$\begin{array}{ccc}
((\mathcal{M} \mathcal{E}) \mathcal{N}) \mathcal{K} & \xrightarrow{a1} & (\mathcal{M}(\mathcal{E} \mathcal{N})) \mathcal{K} \\
\downarrow (r1)1 & \swarrow \Rightarrow \mu 1 & \downarrow (1l)1 \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}(\mathcal{N} \mathcal{K}) \\
\downarrow a & \swarrow \simeq & \downarrow a \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}(\mathcal{N} \mathcal{K}) \\
\downarrow a & \swarrow (1l)1 & \downarrow a \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}(\mathcal{N} \mathcal{K})
\end{array}$$

(3.50)

$$\begin{array}{ccc}
\mathcal{M}((\mathcal{N} \mathcal{D}) \mathcal{K}) & \xrightarrow{1a} & \mathcal{M}(\mathcal{N}(\mathcal{D} \mathcal{K})) \\
\downarrow 1(r1) & \searrow a & \downarrow \pi \\
(\mathcal{M}(\mathcal{N} \mathcal{D})) \mathcal{K} & \xrightarrow{a} & (\mathcal{M} \mathcal{N})(\mathcal{D} \mathcal{K}) \\
\downarrow 1(r1) & \swarrow a & \downarrow 1a \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}((\mathcal{N} \mathcal{D}) \mathcal{K}) \\
\downarrow a & \swarrow (1r)1 & \downarrow r1 \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}(\mathcal{N} \mathcal{K}) \\
\downarrow a & \swarrow \rho 1 & \downarrow (11)l \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}(\mathcal{N} \mathcal{K}) \\
\downarrow a & \swarrow \Rightarrow \mu & \downarrow a \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}(\mathcal{N} \mathcal{K}) \\
\downarrow a & \swarrow (1l)1 & \downarrow a \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}(\mathcal{N} \mathcal{K}) \\
\downarrow a & \swarrow a & \downarrow a \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}(\mathcal{N} \mathcal{K})
\end{array}$$

$\parallel$

$$\begin{array}{ccc}
\mathcal{M}((\mathcal{N} \mathcal{D}) \mathcal{K}) & \xrightarrow{1a} & \mathcal{M}(\mathcal{N}(\mathcal{D} \mathcal{K})) \\
\downarrow 1(1r) & \swarrow \Rightarrow 1\mu & \downarrow 1(1l) \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}(\mathcal{N} \mathcal{K}) \\
\downarrow a & \swarrow \simeq & \downarrow a \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}(\mathcal{N} \mathcal{K}) \\
\downarrow a & \swarrow 1(1r) & \downarrow a \\
(\mathcal{M} \mathcal{N}) \mathcal{K} & \xrightarrow{a} & \mathcal{M}(\mathcal{N} \mathcal{K})
\end{array}$$

(3.51)

Proof. Note first that our conventions regarding an algebraic tricategory differ slightly from the conventions in [20, Definition 3.12], see Remark B.2. The reason for our convention will become clear from the construction of the pseudo-natural transformations  $l$  and  $r$  in the proof.

The remainder of this section is concerned with the proof of Theorem 3.21. The basic idea is to apply the 2-functor  $\widehat{(-)}$  from Proposition 3.18 to the structures and axioms of  $\mathbf{Bimod}^{\text{multi}}$  in Proposition 3.20 to obtain the corresponding structures and axioms for  $\mathbf{Bimod}$ .

- i) We already remarked in Section 2, that  $\mathbf{Bimod}(\mathcal{C}, \mathcal{D})$  is a strict 2-category.
- ii) The tensor product defines a 2-functor  $\square : \mathbf{Bimod}(\mathcal{D}, \mathcal{E}) \times \mathbf{Bimod}(\mathcal{C}, \mathcal{D}) \rightarrow \mathbf{Bimod}(\mathcal{C}, \mathcal{E})$  according to Proposition 3.9.
- iii) The unit bimodule categories  ${}_e\mathcal{C}_e$ , the identity bimodule functor  $\text{id}_e : {}_e\mathcal{C}_e \rightarrow {}_e\mathcal{C}_e$  and the identity bimodule natural transformation  $\text{id}_{\text{id}_e} : \text{id}_e \rightarrow \text{id}_e$  define the strict 2-functor  $I_e : 1 \rightarrow \mathbf{Bimod}(\mathcal{C}, \mathcal{C})$  from iii), where 1 denotes the unit bicategory.
- iv) We now define the structures in iv). Let  $\mathcal{M}, \mathcal{N}$  and  $\mathcal{K}$  be composable bimodule categories. The 2-functor  $\widehat{(-)}$  applied to  $\alpha : (\mathcal{M} \times \mathcal{N}) \times \mathcal{K} \rightarrow \mathcal{M} \times (\mathcal{N} \times \mathcal{K})$  from Proposition 3.20, defines a functor

$$a = \widehat{\alpha} : (\mathcal{M} \square \mathcal{N}) \square \mathcal{K} \rightarrow \mathcal{M} \square (\mathcal{N} \square \mathcal{K}). \quad (3.52)$$

Since  $a$  is the composite of a 2-functor with the pseudo-natural transformation  $\alpha$ ,  $a$  is also a pseudo-natural transformation. Analogously, the multi-module functor  $\alpha^- : \mathcal{M} \times (\mathcal{N} \times \mathcal{K}) \rightarrow (\mathcal{M} \times \mathcal{N}) \times \mathcal{K}$  defines a 2-transformation  $a^-$ , and it follows from Lemma A.9 that  $a$  and  $a^-$  form an adjoint equivalence.

- v) We construct the adjoint equivalence of bimodule categories  ${}_D\mathcal{D}_D \square {}_D\mathcal{M}_e \rightarrow {}_D\mathcal{M}_e$ . Let  ${}_D\mathcal{M}_e$  be a bimodule category. Recall that the action  $\triangleright : \mathcal{D} \times \mathcal{M} \rightarrow \mathcal{M}$  is a balanced bimodule functor.

**Lemma 3.22** *The bimodule functor  $l_{\mathcal{M}} = \widehat{\triangleright} : \mathcal{D} \square \mathcal{M} \rightarrow \mathcal{M}$  induced by the balanced bimodule functor  $\triangleright : \mathcal{D} \times {}_D\mathcal{M}_e \rightarrow {}_D\mathcal{M}_e$  defines a pseudo-natural transformation*

$$\begin{array}{ccc} & \mathbf{Bimod}(\mathcal{D}, \mathcal{D}) \times \mathbf{Bimod}(\mathcal{C}, \mathcal{D}) & \\ & \nearrow I_{\mathcal{D}} \times 1 & \searrow \square \\ \mathbf{Bimod}(\mathcal{C}, \mathcal{D}) & \xrightarrow{1} & \mathbf{Bimod}(\mathcal{C}, \mathcal{D}) \end{array} \quad (3.53)$$

Proof. Let  $F : {}_D\mathcal{M}_e \rightarrow {}_D\mathcal{M}'_e$  be a module functor. Then the module constraint  $\phi^F$  yields the diagram

$$\begin{array}{ccc} \mathcal{D} \times \mathcal{M} & \xrightarrow{1 \times F} & \mathcal{D} \times \mathcal{M}' \\ \downarrow \triangleright & \Downarrow \phi^F & \downarrow \triangleright \\ \mathcal{M} & \xrightarrow{F} & \mathcal{M}' \end{array} \quad (3.54)$$

This defines a bimodule natural isomorphism, where we use the abbreviation  $\mathcal{D} \square F = 1_{\mathcal{D}} \square F$ ,

$$l_F : l_{\mathcal{M}'}(\mathcal{D} \square F) \rightarrow Fl_{\mathcal{M}} \quad (3.55)$$

between the bimodule functors  $l_{\mathcal{M}'}(\mathcal{D} \square F)$  and  $Fl_{\mathcal{M}}$ . We show that the isomorphisms  $l_F$  are natural in  $F$ . If  $G : {}_D\mathcal{M}_e \rightarrow {}_D\mathcal{M}'_e$  is another bimodule functor and  $\rho : F \rightarrow G$  is a bimodule natural transformation, we have to prove that the following natural transformations are equal:

$$\begin{array}{ccc} \begin{array}{ccc} \mathcal{D} \square \mathcal{M} & \xrightarrow{\mathcal{D} \square F} & \mathcal{D} \square \mathcal{N} \\ \downarrow l_{\mathcal{M}} & \Downarrow \mathcal{D} \square \rho & \downarrow l_{\mathcal{N}} \\ \mathcal{M} & \xrightarrow{F} & \mathcal{M}' \\ \downarrow l_{\mathcal{M}} & \Downarrow \rho & \downarrow l_{\mathcal{N}} \\ \mathcal{M} & \xrightarrow{G} & \mathcal{N} \end{array} & = & \begin{array}{ccc} \mathcal{D} \square \mathcal{M} & \xrightarrow{\mathcal{D} \square F} & \mathcal{D} \square \mathcal{N} \\ \downarrow l_{\mathcal{M}} & \Downarrow l_F & \downarrow l_{\mathcal{N}} \\ \mathcal{M} & \xrightarrow{F} & \mathcal{M}' \\ \downarrow l_{\mathcal{M}} & \Downarrow \rho & \downarrow l_{\mathcal{N}} \\ \mathcal{M} & \xrightarrow{G} & \mathcal{N} \end{array} \end{array} \quad (3.56)$$

Since  $\rho$  is a bimodule natural transformation, one has

$$\begin{array}{ccc}
\begin{array}{ccc}
\mathcal{D} \times \mathcal{M} & \xrightarrow{\mathcal{D} \times \mathbf{F}} & \mathcal{D} \times \mathcal{N} \\
\downarrow \triangleright_{\mathcal{M}} & \Downarrow \mathcal{D} \times \rho & \downarrow \triangleright_{\mathcal{N}} \\
\mathcal{M} & \xrightarrow{\mathbf{G}} & \mathcal{N}
\end{array} & = & \begin{array}{ccc}
\mathcal{D} \times \mathcal{M} & \xrightarrow{\mathcal{D} \times \mathbf{F}} & \mathcal{D} \times \mathcal{N} \\
\downarrow \triangleright_{\mathcal{M}} & \Downarrow \Phi^{\mathbf{F}} & \downarrow \triangleright_{\mathcal{N}} \\
\mathcal{M} & \xrightarrow{\mathbf{G}} & \mathcal{N}
\end{array}
\end{array} \quad (3.57)$$

By applying the 2-functor  $\widehat{(-)} : \mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{D}) \rightarrow \mathbf{Bimod}^{\text{multi}}(\mathcal{C}, \mathcal{D})$  to (3.57), one obtains (3.56). This proves that  $l_{\mathcal{M}} : \mathcal{D} \square \mathcal{M} \rightarrow \mathcal{M}$  is a pseudo-natural transformation.  $\square$

To define the bimodule functor  $l_{\mathcal{M}}^- : \mathcal{M} \rightarrow \mathcal{D} \square \mathcal{M}$ , denote by  $\iota : \mathcal{M} \rightarrow \mathcal{D} \times \mathcal{M}$  the canonical embedding functor that is defined by  $\iota(x) = 1 \times x$  for objects and morphisms  $x$  in  $\mathcal{M}$ . Clearly,  $\iota$  is a right  $\mathcal{C}$ -module functor. We define:

$$l_{\mathcal{M}}^- = \mathbf{B}_{\mathcal{D}, \mathcal{M}} \circ \iota : {}_{\mathcal{D}}\mathcal{M}_{\mathcal{e}} \rightarrow {}_{\mathcal{D}}\mathcal{D} \square {}_{\mathcal{D}}\mathcal{M}_{\mathcal{e}}. \quad (3.58)$$

Then  $l_{\mathcal{M}}^-$  inherits a left module functor structure from the balancing constraint of  $\mathbf{B}_{\mathcal{D}, \mathcal{M}}$  according to Proposition 3.7 and we have the following result.

**Proposition 3.23** *The functor  $l_{\mathcal{M}}^-$  defines a pseudo-natural transformation and together with the functor  $l_{\mathcal{M}}$ , it forms an adjoint equivalence of the bimodule categories  ${}_{\mathcal{D}}\mathcal{D} \square {}_{\mathcal{D}}\mathcal{M}_{\mathcal{e}}$  and  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{e}}$ .*

Proof. Let  $\mathbf{F} : {}_{\mathcal{D}}\mathcal{M}'_{\mathcal{e}} \rightarrow {}_{\mathcal{D}}\mathcal{M}_{\mathcal{e}}$  be a bimodule functor. Then the diagram

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\mathbf{F}} & \mathcal{N} \\
\downarrow \iota & & \downarrow \iota \\
\mathcal{D} \times \mathcal{M} & \xrightarrow{\mathcal{D} \times \mathbf{F}} & \mathcal{D} \times \mathcal{N} \\
\downarrow \mathbf{B} & \Downarrow \varphi_{\mathbf{F}} & \downarrow \mathbf{B} \\
\mathcal{D} \square \mathcal{M} & \xrightarrow{\mathcal{D} \square \mathbf{F}} & \mathcal{D} \square \mathcal{N}
\end{array} \quad (3.59)$$

defines the bimodule natural transformations  $l_{\mathbf{F}}^- : l_{\mathcal{N}}^- \circ \mathbf{F} \rightarrow (\mathcal{D} \square \mathbf{F}) \circ l_{\mathcal{M}}^-$ . It follows directly from the properties of the natural isomorphisms  $\varphi_{\mathbf{F}}$ , that  $l_{\mathbf{F}}^-$  is natural in  $\mathbf{F}$  and compatible with the composition of bimodule functors. Hence  $l_{\mathcal{M}}^-$  is a pseudo-natural transformation.

We now show that  $l_{\mathcal{M}}$  and  $l_{\mathcal{M}}^-$  form an adjoint equivalence. For all bimodule categories  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{e}}$  there exists a natural isomorphism  $\alpha^l : l_{\mathcal{M}} \circ l_{\mathcal{M}}^- \rightarrow 1_{\mathcal{M}}$  defined as the composite

$$\alpha^l_{\mathcal{M}} : l_{\mathcal{M}} \circ l_{\mathcal{M}}^- = l_{\mathcal{M}} \circ \mathbf{B} \circ \iota \xrightarrow{\varphi_{\iota}^{-1} \circ \iota} \triangleright \circ \iota \xrightarrow{\lambda^{\mathcal{M}}} 1_{\mathcal{M}}, \quad (3.60)$$

where  $\varphi_{\iota}^{-1}$  is the bimodule natural transformation from Definition 3.1 and  $\lambda^{\mathcal{M}}$  is the natural isomorphism from Definition 2.1 with component morphisms  $\lambda_m^{\mathcal{M}} : \mathbf{1}_{\mathcal{C}} \triangleright m \rightarrow m$ . If we equip the functor  $\triangleright \circ \iota : \mathcal{M} \rightarrow \mathcal{M}$  with the canonical bimodule functor structure, it follows from the axioms of a module category, that  $\lambda^{\mathcal{M}}$  is a bimodule natural isomorphism. Hence the natural transformations  $\alpha^l_{\mathcal{M}}$  are bimodule natural isomorphisms. Next we show that they define a modification  $\alpha^l$ . Consider a bimodule functor  $\mathbf{F} : {}_{\mathcal{D}}\mathcal{M}'_{\mathcal{e}} \rightarrow {}_{\mathcal{D}}\mathcal{N}_{\mathcal{e}}$ . We have to show that the following two diagrams are equal

$$\begin{array}{ccc}
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{l_{\mathcal{M}}^-} & \mathcal{D} \square \mathcal{M} \xrightarrow{l_{\mathcal{M}}} \mathcal{M} \\
\downarrow \mathbf{F} & \Downarrow l_{\mathbf{F}}^- & \downarrow \mathcal{D} \square \mathbf{F} \quad \Downarrow l_{\mathbf{F}} \\
\mathcal{N} & \xrightarrow{l_{\mathcal{N}}^-} & \mathcal{D} \square \mathcal{N} \xrightarrow{l_{\mathcal{N}}} \mathcal{N}
\end{array} & = & \begin{array}{ccc}
\mathcal{M} & \xrightarrow{l_{\mathcal{N}}^-} & \mathcal{D} \square \mathcal{M} \xrightarrow{l_{\mathcal{M}}} \mathcal{M} \\
\downarrow \mathbf{F} & \searrow & \downarrow \mathbf{F} \\
\mathcal{N} & \xrightarrow{1} & \mathcal{N}
\end{array}
\end{array} \quad (3.61)$$

If we insert the corresponding definition of the arrows in these diagrams, it is easy to see that equation (3.61) is equivalent to the equation

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\iota} \mathcal{D} \times \mathcal{M} & \xrightarrow{\triangleright} \mathcal{M} \\
\downarrow \mathbf{F} & \downarrow \mathcal{D} \times \mathbf{F} \Downarrow \phi^{\mathbf{F}} & \downarrow \mathbf{F} \\
\mathcal{N} & \xrightarrow{\iota} \mathcal{D} \times \mathcal{N} & \xrightarrow{\triangleright} \mathcal{N} \\
& \downarrow \alpha & \\
& \mathbf{1} & 
\end{array}
=
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\iota} \mathcal{D} \times \mathcal{M} & \xrightarrow{\triangleright} \mathcal{M} \\
\downarrow \mathbf{F} & \downarrow \lambda^{\mathcal{M}} & \downarrow \mathbf{F} \\
\mathcal{N} & \xrightarrow{\iota} \mathcal{D} \times \mathcal{N} & \xrightarrow{\triangleright} \mathcal{N} \\
& \downarrow \mathbf{1} & \\
& \mathbf{1} & 
\end{array}, \quad (3.62)$$

where  $\phi^{\mathbf{F}}$  is the module functor constraint of the functor  $\mathbf{F}$ . The commutativity of this diagram corresponds directly to the identity (2.5) for the module functor  $\mathbf{F}$ .

To define bimodule natural isomorphisms

$$(\beta_{\mathcal{M}}^{\iota})^{-1} : \iota_{\mathcal{M}}^{-} \circ \iota_{\mathcal{M}} \rightarrow \mathbf{1}_{\mathcal{D} \square \mathcal{M}}, \quad (3.63)$$

note that the balancing structure of  $\mathbf{B}$  provides a natural balanced isomorphism  $\mathbf{B} \circ \iota \circ \triangleright \rightarrow \mathbf{B}$  for the two balanced module functors  $\mathbf{B} \circ \iota \circ \triangleright, \mathbf{B} : \mathcal{D} \times \mathcal{M} \rightarrow \mathcal{D} \square \mathcal{M}$ . By applying the 2-functor  $(-)$  we obtain the bimodule natural isomorphism  $(\beta_{\mathcal{M}}^{\iota})^{-1}$ . To show that these natural isomorphisms define a modification  $(\beta^{\iota})^{-1}$ , we have to prove the equation

$$\begin{array}{ccc}
\mathcal{D} \square \mathcal{M} & \xrightarrow{\iota_{\mathcal{M}}} \mathcal{M} & \xrightarrow{\iota_{\mathcal{M}}^{-}} \mathcal{D} \square \mathcal{M} \\
\downarrow \mathcal{D} \square \mathbf{F} & \downarrow \mathbf{F} \Downarrow \iota_{\mathcal{M}}^{-} & \downarrow \mathcal{D} \square \mathbf{F} \\
\mathcal{D} \square \mathcal{N} & \xrightarrow{\iota_{\mathcal{N}}} \mathcal{N} & \xrightarrow{\iota_{\mathcal{N}}^{-}} \mathcal{D} \square \mathcal{N} \\
& \downarrow \beta^{-1} & \\
& \mathbf{1} & 
\end{array}
=
\begin{array}{ccc}
\mathcal{D} \square \mathcal{M} & \xrightarrow{\iota_{\mathcal{N}}} \mathcal{M} & \xrightarrow{\iota_{\mathcal{M}}^{-}} \mathcal{D} \square \mathcal{M} \\
\downarrow \mathcal{D} \square \mathbf{F} & \downarrow \beta^{-1} & \downarrow \mathcal{D} \square \mathbf{F} \\
\mathcal{D} \square \mathcal{N} & \xrightarrow{\iota_{\mathcal{N}}} \mathcal{N} & \xrightarrow{\iota_{\mathcal{N}}^{-}} \mathcal{D} \square \mathcal{N} \\
& \downarrow \mathbf{1} & \\
& \mathbf{1} & 
\end{array} \quad (3.64)$$

for all bimodule functors  $\mathbf{F} : {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_{\mathcal{D}}\mathcal{N}_e$ . Inserting the definitions, one finds that this is equivalent to the condition that the following two diagrams are equal.

$$\begin{array}{ccc}
\mathcal{D} \times \mathcal{M} & \xrightarrow{\triangleright} \mathcal{M} & \xrightarrow{\iota} \mathcal{D} \times \mathcal{M} & \xrightarrow{\mathbf{B}} \mathcal{D} \square \mathcal{M} \\
\downarrow \mathcal{D} \times \mathbf{F} & \downarrow \mathcal{D} \times \mathbf{F} \Downarrow \phi^{\mathbf{F}} & \downarrow \mathcal{D} \times \mathbf{F} & \downarrow \mathcal{D} \times \mathbf{F} \Downarrow \varphi^{\mathcal{D} \times \mathbf{F}} & \downarrow \mathcal{D} \square \mathbf{F} \\
\mathcal{D} \times \mathcal{N} & \xrightarrow{\triangleright} \mathcal{N} & \xrightarrow{\iota} \mathcal{D} \times \mathcal{N} & \xrightarrow{\mathbf{B}} \mathcal{D} \square \mathcal{N}, \\
& & \downarrow \beta & \\
& & \mathbf{B} & 
\end{array} \quad (3.65)$$

$$\begin{array}{ccc}
\mathcal{D} \times \mathcal{M} & \xrightarrow{\triangleright} \mathcal{M} & \xrightarrow{\iota} \mathcal{D} \times \mathcal{M} & \xrightarrow{\mathbf{B}} \mathcal{D} \square \mathcal{M} \\
\downarrow \mathcal{D} \times \mathbf{F} & \downarrow \mathcal{D} \times \mathbf{F} & \downarrow \mathcal{D} \times \mathbf{F} & \downarrow \mathcal{D} \square \mathbf{F} \\
\mathcal{D} \times \mathcal{N} & \xrightarrow{\triangleright} \mathcal{N} & \xrightarrow{\iota} \mathcal{D} \times \mathcal{N} & \xrightarrow{\mathbf{B}} \mathcal{D} \square \mathcal{N}. \\
& & \downarrow \varphi^{\mathcal{D} \times \mathbf{F}} & \\
& & \mathbf{B} & 
\end{array} \quad (3.66)$$

We compute both sides on objects. When evaluated on objects  $d \in \mathcal{D}$  and  $m \in \mathcal{M}$ , the first diagram yields the morphism

$$(\mathcal{D} \square \mathbf{F})\mathbf{B}(\mathbf{1} \times d \triangleright m) \xrightarrow{\varphi^{\mathcal{D} \times \mathbf{F}}(\mathbf{1} \times d \triangleright m)} \mathbf{B}(\mathbf{1} \times \mathbf{F}(d \triangleright m)) \xrightarrow{\phi_{d,m}^{\mathbf{F}}} \mathbf{B}(\mathbf{1} \times d \triangleright \mathbf{F}(m)) \xrightarrow{\beta} \mathbf{B}(d \times \mathbf{F}(m)), \quad (3.67)$$

while the other diagram corresponds to

$$(\mathcal{D} \square \mathbf{F})\mathbf{B}(\mathbf{1} \times d \triangleright m) \xrightarrow{b} (\mathcal{D} \square \mathbf{F})\mathbf{B}(d \times m) \xrightarrow{\varphi^{\mathcal{D} \times \mathbf{F}}(d \times m)} \mathbf{B}(d \times \mathbf{F}(m)). \quad (3.68)$$

These two morphisms are equal since  $\varphi_{\mathcal{D} \times \mathcal{F}}$  is a balanced natural isomorphism.

It remains to prove that the natural isomorphisms  $\alpha^l$  and  $\beta^l$  define an adjoint equivalence according to Definition A.5. We have to show that the composites

$$l_{\mathcal{M}} \xrightarrow{l_{\mathcal{M}}\beta^l} l_{\mathcal{M}}l_{\mathcal{M}}^-l_{\mathcal{M}} \xrightarrow{\alpha^l l_{\mathcal{M}}} l_{\mathcal{M}}, \quad l_{\mathcal{M}}^- \xrightarrow{\beta^l l_{\mathcal{M}}^-} l_{\mathcal{M}}^-l_{\mathcal{M}}l_{\mathcal{M}}^-l_{\mathcal{M}}^- \alpha^l \longrightarrow l_{\mathcal{M}}^- \quad (3.69)$$

are the respective identities. In the first case this is equivalent to the commutativity of the diagram

$$l_{\mathcal{M}}l_{\mathcal{M}}^-l_{\mathcal{M}} = l_{\mathcal{M}}\mathbf{B}ul_{\mathcal{M}} \xrightarrow{\varphi^l} \triangleright ul_{\mathcal{M}} \xrightarrow{\lambda^{\mathcal{M}}} 1l_{\mathcal{M}} = l_{\mathcal{M}}. \quad (3.70)$$

$\alpha^l l_{\mathcal{M}}$

By definition of  $\alpha^l$ , this is equivalent to the commutativity of the diagram

$$l_{\mathcal{M}}\mathbf{B} \xrightarrow{b} l_{\mathcal{M}}\mathbf{B}l_{\mathcal{M}} \xrightarrow{\varphi^l} \triangleright l_{\mathcal{M}} \xrightarrow{\lambda^{\mathcal{M}}} \triangleright \quad (3.71)$$

$\varphi^l$

Evaluated on objects, this diagram takes the form

$$l_{\mathcal{M}}\mathbf{B}(d \times m) \xrightarrow{b} l_{\mathcal{M}}\mathbf{B}(\mathbf{1} \times d \triangleright m) \xrightarrow{\varphi^l(\mathbf{1} \times d \triangleright m)} \mathbf{1} \triangleright (d \triangleright m) \xrightarrow{\lambda_{d \triangleright m}^{\mathcal{M}}} d \triangleright m. \quad (3.72)$$

$\varphi_l(d \times m)$

This last diagram commutes since  $\varphi$  is a balanced natural isomorphism.

In the second case, the requirement that morphism (3.73) is the identity is equivalent to the condition that

$$\mathbf{B}l \xrightarrow{(\beta^l)^{-1}} \mathbf{B}l_{\triangleright l} \xrightarrow{\mathbf{B}l\lambda^{\mathcal{M}}} \mathbf{B}l \quad (3.73)$$

is the identity natural transformation on  $\mathbf{B}l$ . Evaluated on objects, this yields

$$\mathbf{B}(\mathbf{1} \times m) \xrightarrow{\beta} \mathbf{B}(\mathbf{1} \times \mathbf{1} \triangleright m) \xrightarrow{\mathbf{B}l\lambda^{\mathcal{M}}} \mathbf{B}(\mathbf{1} \times m), \quad (3.74)$$

which is the identity on the object  $\mathbf{B}(\mathbf{1} \times m)$ , by equation (2.12).  $\square$

The bimodule functors  $r_{\mathcal{M}} : {}_{\mathcal{D}}\mathcal{M}_e \square_e \mathcal{C}_e \rightarrow {}_{\mathcal{D}}\mathcal{M}_e$  and  $r_{\mathcal{M}}^- : {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_{\mathcal{D}}\mathcal{M}_e \square_e \mathcal{C}_e$  are defined analogously using the right action of  $\mathcal{C}$  on  ${}_{\mathcal{D}}\mathcal{M}_e$  and the proof that they define an adjoint equivalence is similar.

- vi*) The modification  $\mu$ , from 3.21, *vi*) is defined by applying the functor  $\widehat{(-)}$  to the diagram (3.35) in  $\mathbf{Bimod}^{\text{multi}}$ . It follows directly that  $\mu$  is a modification, since it is the composite of a 2-functor with a modification.
- vii*) The modification  $\lambda$  is obtained by applying  $\widehat{(-)}$  to the diagrams (3.36).
- viii*) The modification  $\rho$  is obtained analogously by applying  $\widehat{(-)}$  to the diagram (3.37).
- ix*) Applying  $\widehat{(-)}$  to the diagram (3.38) defines the modification  $\pi$ .
- x*) To complete the proof that  $\mathbf{Bimod}$  is a tricategory, it remains to verify the three axioms in Definition B.1. All the structures of  $\mathbf{Bimod}$  are defined in terms of structures in  $\mathbf{Bimod}^{\text{multi}}$  and every axiom for  $\mathbf{Bimod}$  is a pasting diagram that is obtained from a pasting diagram in  $\mathbf{Bimod}^{\text{multi}}$  according to Corollary 3.19. Hence Corollary 3.19 reduces the proof of the axioms to the commutativity of the corresponding pasting diagrams in  $\mathbf{Bimod}^{\text{multi}}$ . The first axiom in Definition B.1 is the so-called Stasheff 5-polytope, the higher analogue of the pentagon axiom for monoidal categories. This axiom is trivial in  $\mathbf{Bimod}^{\text{multi}}$ , since the associator  $\alpha$  in  $\mathbf{Bimod}^{\text{multi}}$  already satisfies the pentagon axiom and hence the corresponding modification  $\pi$  is the identity. The remaining axioms follow by applying the 2-functor  $\widehat{(-)}$  to the diagram (3.39) and to diagram (3.40).  $\square$



## 4 Tricategories with duals and the example of Bimod

In this section we develop the notions of a tricategory with duals and of a pivotal tricategory. We provide useful techniques for computations involving the inner homs for bimodule categories and use these to construct duals for bimodule categories in the tricategory  $\mathbf{Bimod}$ . This is achieved by constructing for each bimodule category  ${}_{\mathcal{D}}\mathcal{M}_e$  2-morphisms in the tricategory  $\mathbf{Bimod}$  (recall the dual categories  ${}^{\#}_e\mathcal{M}_{\mathcal{D}}$  and  ${}_e\mathcal{M}_{\mathcal{D}}^{\#}$  from Section 2)

$$\text{coev}_{\mathcal{M}} : {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{M}_e \square {}^{\#}_e\mathcal{M}_{\mathcal{D}} \quad \text{and} \quad \text{ev}_{\mathcal{M}} : {}^{\#}_e\mathcal{M}_{\mathcal{D}} \square {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_e\mathcal{C}_e, \quad (4.1)$$

such that the snake identities (A.1) and (A.2) are satisfied up to a 3-isomorphism in  $\mathbf{Bimod}$ .

**The example of Vect-bimodule categories** To illustrate the idea behind the construction of the dualities for bimodule categories, we first consider the example of finite semisimple categories regarded as  $\mathbf{Vect}$ -bimodule categories. Let  $\mathcal{M}$  be a finite semisimple category. Choose a finite set  $\{m_i\}_{i \in I}$  of representatives of the simple objects of  $\mathcal{M}$ .

- i) The object  $R^{\mathcal{M}} = \bigoplus_{i \in I} m_i \boxtimes m_i \in \mathcal{M} \boxtimes \mathcal{M}^{\text{op}}$  represents the  $\mathbf{Hom}$ -functor, i.e. there is a natural isomorphism

$$\mathbf{Hom}_{\mathcal{M} \boxtimes \mathcal{M}^{\text{op}}}(m \boxtimes m', R^{\mathcal{M}}) \simeq \bigoplus_i \mathbf{Hom}_{\mathcal{M}}(m, m_i) \otimes \mathbf{Hom}_{\mathcal{M}^{\text{op}}}(m', m_i) \rightarrow \mathbf{Hom}_{\mathcal{M}}(m, m'), \quad (4.2)$$

using the semisimplicity of  $\mathcal{M}$ . Clearly the object  $R^{\mathcal{M}}$  defines a  $\mathbf{Vect}$ -bimodule functor

$$\text{coev}_{\mathcal{M}}^{\mathbf{Vect}} : \mathbf{Vect} \ni V \mapsto V \triangleright R^{\mathcal{M}} \in \mathcal{M} \boxtimes \mathcal{M}^{\text{op}}.$$

- ii) The  $\mathbf{Hom}$ -functor of  $\mathcal{M}$  defines a  $\mathbf{Vect}$ -bimodule functor

$$\text{ev}_{\mathcal{M}}^{\mathbf{Vect}} : \mathcal{M}^{\text{op}} \boxtimes \mathcal{M} \ni m' \boxtimes m \mapsto \mathbf{Hom}_{\mathcal{M}}(m', m) \in \mathbf{Vect}. \quad (4.3)$$

- iii) By composing we obtain the  $\mathbf{Vect}$ -bimodule functor

$$\Phi_{\mathcal{M}}^{\mathbf{Vect}} = r_{\mathcal{M}} \circ (\mathcal{M} \boxtimes \text{ev}_{\mathcal{M}}^{\mathbf{Vect}}) \circ (\text{coev}_{\mathcal{M}}^{\mathbf{Vect}} \boxtimes \mathcal{M}) \circ l_{\mathcal{M}}^- : \mathcal{M} \rightarrow \mathcal{M}.$$

**Proposition 4.1** *There exists a canonical  $\mathbf{Vect}$ -bimodule natural isomorphism*

$$T_{\mathcal{M}}^{\mathbf{Vect}} : \Phi_{\mathcal{M}}^{\mathbf{Vect}} \rightarrow 1_{\mathcal{M}}. \quad (4.4)$$

Proof. Inserting the definitions, it follows immediately, that  $\Phi_{\mathcal{M}}^{\mathbf{Vect}}(m) = \bigoplus_i m_i \otimes \mathbf{Hom}(m_i, m)$  on objects  $m \in \mathcal{M}$ . For all  $n \in \mathcal{M}$ , the defining property of  $R$  thus yields a natural isomorphism

$$\begin{aligned} \mathbf{Hom}(n, \Phi_{\mathcal{M}}^{\mathbf{Vect}}(m)) &= \mathbf{Hom}(n, \bigoplus_i m_i \otimes \mathbf{Hom}(m_i, m)) = \bigoplus_i \mathbf{Hom}(n, m_i) \otimes \mathbf{Hom}(m_i, m) \\ &\simeq \mathbf{Hom}(n, m). \end{aligned} \quad (4.5)$$

By the Yoneda-lemma, this defines an natural isomorphism  $T_{\mathcal{M}}^{\mathbf{Vect}} : \Phi_{\mathcal{M}}^{\mathbf{Vect}} \rightarrow 1_{\mathcal{M}}$ .  $\square$

In the next subsections we will show that analogous statements hold in the case of general finite bimodule category. We will thereby use the inner hom instead of the  $\mathbf{Hom}$ -spaces in  $\mathbf{Vect}$ . Therefore we require module versions of the Yoneda lemma and of representations of module functors.

### 4.1 Tricategories with duals and pivotal structure

The definition of duals in tricategories that we present in this subsection is inspired by the duals in higher categories in [26] and makes use of the notion of duals in bicategories, see Definition A.10. To this end we briefly recall the delooping procedure to obtain a  $(n-1)$ -category  $h\mathcal{X}$  from a  $n$ -category  $\mathcal{X}$ . We will use the following statement only for  $n \leq 3$ , but it is expected to hold for all reasonable models of higher categories.

**Lemma 4.2** *Let  $\mathcal{X}$  be a  $n$ -category. The following defines a  $(n-1)$ -category  $h\mathcal{X}$ . The objects and  $i$ -morphisms for  $i = 1, \dots, (n-2)$  of  $h\mathcal{X}$  are the same as in  $\mathcal{X}$ . The  $(n-1)$ -morphisms of  $h\mathcal{X}$  are the isomorphism classes of  $(n-1)$ -morphisms in  $\mathcal{X}$ . The compositions and coherence structures in  $h\mathcal{X}$  are induced from the compositions and coherence structures in  $\mathcal{X}$ .*

For a tricategory  $\mathcal{T}$ , the delooping is thus a bicategory  $h\mathcal{T}$ .

**Definition 4.3** *Let  $\mathcal{T}$  be a tricategory.*

- i) *We say that  $\mathcal{T}$  has  $*$ -duals, if for objects  $b, c \in \mathcal{T}$ , the bicategory  $\mathcal{T}(b, c)$  has both left and right duals according to Definition A.10.*
- ii)  *$\mathcal{T}$  has  $\#$ -duals if the bicategory  $h\mathcal{T}$  is a bicategory with left and right duals.*
- iii)  *$\mathcal{T}$  is called a tricategory with duals if it has  $*$ -duals and  $\#$ -duals.*

This definition means that for every 2-morphism  $\varphi : F \Rightarrow G$  in a tricategory with  $*$ -duals, there exists a 2-morphism  $\varphi^* : G \Rightarrow F$  and duality 3-morphisms  $\text{ev}_\varphi : \varphi^* \circ \varphi \Rightarrow 1_F$  and  $\text{coev}_\varphi : 1_G \Rightarrow \varphi \circ \varphi^*$ , that satisfy the snake identities (A.1) and (A.2).

The duality on  $h\mathcal{T}$  for a tricategory with duals is denoted  $\#$ , hence for every 1-morphism  $M : a \rightarrow b$  in a tricategory  $\mathcal{T}$  with  $\#$ -duals, there exists a 1-morphism  $M^\# : b \rightarrow a$  in  $\mathcal{T}$  together with 2-morphisms  $\text{ev}_M : M^\# \square M \Rightarrow 1_a$  and  $\text{coev}_M : 1_b \Rightarrow M \square M^\#$ , such that the snake identity holds in  $h\mathcal{T}$ .

The following is shown in [26, Rem. 3.4.22], [8, Lemma 2.4.4].

**Proposition 4.4** *Let  $\mathcal{T}$  be a tricategory with  $*$ -duals such that the bicategory  $h\mathcal{T}$  has right duals. Then the right duals in  $h\mathcal{T}$  are also left duals and thus  $\mathcal{T}$  is a tricategory with duals. In particular, left and right  $\#$ -duals in a tricategory with duals are equivalent.*

Next we turn to pivotal structure on tricategories. The notions of pivotal structures for bicategories and pivotal 2-functors is given in Definitions A.12 and A.13.

**Definition 4.5** *Let  $\mathcal{T}$  be a tricategory with  $*$ -duals.*

- i) *A pivotal structure for  $\mathcal{T}$  consists of a pivotal structure in the bicategory  $\mathcal{T}(b, c)$  such that for all 1-morphisms  $M : c \rightarrow d$ , the 2-functors*

$$M \square - : \mathcal{T}(b, c) \rightarrow \mathcal{T}(b, d) \quad \text{and} \quad - \square M : \mathcal{T}(d, e) \rightarrow \mathcal{T}(c, e), \quad (4.6)$$

*are pivotal 2-functors for all objects  $c, d, e$ . A tricategory with  $*$ -duals together with a pivotal structure is called a pivotal tricategory.*

- ii)  *$\mathcal{T}$  is a pivotal tricategory with duals, if it is a pivotal tricategory and the bicategory  $h\mathcal{T}$  is a bicategory with right duals.*

Concretely, in a pivotal tricategory for every pair of objects  $b, c \in \mathcal{T}$ ,  $\mathcal{T}(c, b)$  is a pivotal bicategory with duality  $*$  and pivotal structure  $a$ . The pivotal structure defines invertible 3-morphisms  $a_\varphi : \varphi \Rightarrow \varphi^{**}$  for all 2-morphisms  $\varphi$ .

**Remark 4.6** In [29] it is shown that the notion of a pivotal tricategory with duals (this structure is called tricategory with weak duals in [29]) is stable under triequivalences of tricategories, i.e. if  $\mathcal{T} \simeq \mathcal{G}$  as tricategories and  $\mathcal{T}$  is a pivotal tricategory with duals, then  $\mathcal{G}$  is canonically a pivotal tricategory. Using this result it is possible to strictify a pivotal tricategory with duals to a Gray category with duals in the sense of [1], see [29, Thm. 7.21].

## 4.2 The calculus with the inner hom and the dual categories

In this section we provide the technical tools that will be used to construct  $\#$ -duals in the tricategory  $\text{Bimod}$ . Therefore we first consider the dual bimodule categories and functors between them in more detail. Next we discuss various compatibilities between the inner homs, the dual categories and the tensor product of bimodule categories. Most importantly we prove a Rieffel-induction type formula that allows to compute the inner homs of a tensor product  ${}_b\mathcal{M}_c \square {}_e\mathcal{N}_d$  in terms of the inner homs of  $\mathcal{M}$  and  $\mathcal{N}$ . Finally we discuss module versions of the Yoneda-lemma and of the notion of representations of functors.

**Dual categories** Recall from Section 2, that for every bimodule category  ${}_{\mathcal{D}}\mathcal{M}_e$ , there are bimodule categories  ${}_e\mathcal{M}_{\mathcal{D}}^{\#}$  and  ${}^{\#}\mathcal{M}_{\mathcal{D}}$ . These so-called dual categories are compatible with the tensor product as follows.

**Lemma 4.7** *Let  ${}_{\mathcal{D}}\mathcal{M}_e$  and  ${}_e\mathcal{N}_{\mathcal{E}}$  be bimodule categories. There are canonical equivalences of bimodule categories*

$$i) \quad \#({}_e\mathcal{M}_{\mathcal{D}}^{\#}) \simeq {}_{\mathcal{D}}\mathcal{M}_e, \quad ({}^{\#}\mathcal{M}_{\mathcal{D}})^{\#} \simeq {}_{\mathcal{D}}\mathcal{M}_e. \quad (4.7)$$

$$ii) \quad ({}_{\mathcal{D}}\mathcal{M}_e \square {}_e\mathcal{N}_{\mathcal{E}})^{\#} \simeq {}_e\mathcal{N}_{\mathcal{E}}^{\#} \square {}_{\mathcal{D}}\mathcal{M}_e^{\#}, \quad \#({}_{\mathcal{D}}\mathcal{M}_e \square {}_e\mathcal{N}_{\mathcal{E}}) \simeq {}^{\#}\mathcal{N}_{\mathcal{E}} \square {}^{\#}\mathcal{M}_{\mathcal{D}}. \quad (4.8)$$

*Proof.* The equivalences for the first part are obtained directly from the identifications  $(\mathcal{M}^{\text{op}})^{\text{op}} \simeq \mathcal{M}$  and  ${}^*(c^*) \simeq ({}^*c)^* \simeq c$ ,  ${}^*(d^*) \simeq ({}^*d)^* \simeq d$ , for  $c \in \mathcal{C}$  and  $d \in \mathcal{D}$ . For the second part we define the functors  $\tau : \mathcal{N}^{\#} \boxtimes \mathcal{M}^{\#} \rightarrow (\mathcal{M} \boxtimes \mathcal{N})^{\#} \rightarrow (\mathcal{M} \square \mathcal{N})^{\#}$  by  $\tau(n \boxtimes m) = B(m \boxtimes n)$  for  $n \boxtimes m \in \mathcal{N}^{\#} \boxtimes \mathcal{M}^{\#}$ . It is straightforward to see that  $\tau$  is a balanced bimodule functor and moreover, that  $(\mathcal{M} \square \mathcal{N})^{\#}$  together with  $\tau$  is a tensor product of  $\mathcal{N}^{\#}$  and  $\mathcal{M}^{\#}$ . Thus by universality of the tensor product,  $\tau$  induces an equivalence of bimodule categories  $({}_{\mathcal{D}}\mathcal{M}_e \square {}_e\mathcal{N}_{\mathcal{E}})^{\#} \simeq {}_e\mathcal{N}_{\mathcal{E}}^{\#} \square {}_{\mathcal{D}}\mathcal{M}_e^{\#}$ . The analogous argument applies to the left duals.  $\square$

The dual bimodule categories extend to functors as follows. For each bimodule functor  $F : {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_{\mathcal{D}}\mathcal{N}_e$ , there are corresponding bimodule functors  $F^{\#} : {}_e\mathcal{M}_{\mathcal{D}}^{\#} \rightarrow {}_e\mathcal{N}_{\mathcal{D}}^{\#}$ , and  ${}^{\#}F : {}^{\#}\mathcal{M}_{\mathcal{D}} \rightarrow {}^{\#}\mathcal{N}_{\mathcal{D}}$  that are just the obvious functors  $F^{\text{op}} : \mathcal{M}^{\text{op}} \rightarrow \mathcal{N}^{\text{op}}$  as linear functors, with bimodule structures induced from the bimodule structures of  $F$ . Furthermore, each bimodule natural transformation  $\eta : F \rightarrow G$  between two such bimodule functors defines canonical bimodule natural transformations  $\eta^{\#} : F^{\#} \rightarrow G^{\#}$  and  ${}^{\#}\eta : {}^{\#}F \rightarrow {}^{\#}G$ . In total, we obtain 2-functors  $(-)^{\#}, {}^{\#}(-) : \text{Bimod}(\mathcal{D}, \mathcal{C}) \rightarrow \text{Bimod}(\mathcal{C}, \mathcal{D})$ .

Next we consider the duals of the unit bimodule categories. The following statement follows directly from the definitions.

**Lemma 4.8** *i) The right dual functor is an equivalence of bimodule categories  $(-)^* : {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}^{\#} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$  with quasi-inverse  ${}^*(-) : {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}^{\#}$ .*

*ii) The left dual functor is an equivalence of bimodule categories  ${}^*(-) : {}^{\#}\mathcal{D}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$  with quasi-inverse  $(-)^* : {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}} \rightarrow {}^{\#}\mathcal{D}_{\mathcal{D}}$ .*

Let  $F : {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}$  and  $G : {}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$  be bimodule functors. The equivalences from Lemma 4.8 induce bimodule functors

$$\begin{aligned} \tilde{\#}F : {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}} &\simeq {}^{\#}\mathcal{D}_{\mathcal{D}} \xrightarrow{\#F} {}^{\#}\mathcal{M}_{\mathcal{D}}, & F^{\tilde{\#}} : {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}} &\simeq {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}^{\#} \xrightarrow{F^{\#}} {}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}^{\#} \\ \tilde{\#}G : {}^{\#}\mathcal{M}_{\mathcal{D}} &\xrightarrow{\#G} {}^{\#}\mathcal{D}_{\mathcal{D}} \simeq {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}, & G^{\tilde{\#}} : {}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}^{\#} &\xrightarrow{G^{\#}} {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}^{\#} \simeq {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}, \end{aligned} \quad (4.9)$$

Hence on objects  $d \in \mathcal{D}$  and  $m \in \mathcal{M}$ , these functors take the following values:

$$\tilde{\#}F(d) = F(x^*), \quad F^{\tilde{\#}}(d) = F(*d), \quad \tilde{\#}(G)(m) = *G(m), \quad G^{\tilde{\#}}(m) = G(m)^*. \quad (4.10)$$

It follows from Lemma 4.7 that  $(\tilde{\#}F)^{\tilde{\#}} \simeq F \simeq \tilde{\#}(F^{\tilde{\#}})$  and similarly for  $G$ . These constructions will be needed for the coevaluation functor for bimodule categories.

**Representation of module functors** Our next goal is to obtain a module version of the Yoneda-Lemma and a notion of representation of module functors. First we extend the notion of balanced functors such that it includes the **Hom**-functor of a bimodule category.

**Definition 4.9** *A functor  $F : {}_{\mathcal{D}}\mathcal{M} \boxtimes \mathcal{N}_{\mathcal{D}} \rightarrow \mathcal{A}$  between the product of two module categories and a linear category  $\mathcal{A}$  is called  $\mathcal{D}$ -balanced, if it is equipped with natural isomorphisms*

$$F(d \triangleright m \boxtimes n) \simeq F(m \boxtimes n \triangleleft d^{**}), \quad (4.11)$$

for all  $d \in \mathcal{D}$ ,  $m \in \mathcal{M}$  and  $n \in \mathcal{N}$ , that satisfy the usual pentagon axiom with respect to the tensor product of  $\mathcal{D}$  and the triangle axiom with respect to the unit of  $\mathcal{D}$ . Balanced natural

transformation between two balanced functors of this type are defined as natural transformation, such that the obvious analogue of the diagram (2.13) commutes.

We say that a functor  $F : {}_{\mathcal{D}}\mathcal{M}_e \boxtimes_e \mathcal{N}_{\mathcal{D}} \rightarrow \mathcal{A}$  is multi-balanced if it is  $\mathcal{C}$ -balanced,  $\mathcal{D}$ -balanced in the given sense and furthermore both structures are compatible as in Definition 3.10.

It turns out that the  $\mathbf{Hom}$ -functor is balanced in this sense:

**Example 4.10** The functor  $\mathbf{Hom}_{\mathcal{M}} : {}_e^{\#}\mathcal{M}_{\mathcal{D}} \boxtimes_{\mathcal{D}} \mathcal{M}_e \rightarrow \mathbf{Vect}$  is multi-balanced. It is clearly  $\mathcal{C}$ -balanced, and the  $\mathcal{D}$ -balancing structure is obtained from the isomorphisms

$$\mathbf{Hom}_{\mathcal{M}}(e^{\#} \triangleright \tilde{m}, m) = \mathbf{Hom}_{\mathcal{M}}(\tilde{m} \triangleleft e^*, m) \simeq \mathbf{Hom}_{\mathcal{M}}(\tilde{m}, m \triangleleft e^{**}).$$

It is easy to see that both balancing structures are compatible. If we compose the  $\mathbf{Hom}$ -functor with the inner hom functor  ${}_{\mathcal{D}}\langle -, - \rangle^{\mathcal{M}} : {}_{\mathcal{D}}\mathcal{M}_e \boxtimes_e {}_e^{\#}\mathcal{M}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$ , we see that  $\mathbf{Hom}_{\mathcal{D}}(m, d \triangleright \tilde{m})$  and  $\mathbf{Hom}_{\mathcal{M}}({}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}}, d)$  define multi-balanced functors  ${}_{\mathcal{D}}^{\#}\mathcal{M}_e \boxtimes_e {}_e^{\#}\mathcal{M}_{\mathcal{D}} \boxtimes_{\mathcal{D}} \mathcal{D}_{\mathcal{D}} \rightarrow \mathbf{Vect}$ .

Moreover, the inner hom defines a multi-balanced natural isomorphism  $\mathbf{Hom}_{\mathcal{D}}(m, d \triangleright \tilde{m}) \simeq \mathbf{Hom}_{\mathcal{M}}({}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}}, d)$ .

Now we turn to the module version of the Yoneda-Lemma.

**Lemma 4.11** *i) Let  $F, G : {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_{\mathcal{D}}\mathcal{N}_e$  be bimodule functors. The set of bimodule natural transformations from  $F$  to  $G$  is in bijection with the set of multi-balanced natural transformations*

$$\mathbf{Hom}_{\mathcal{N}}(-, F(-)) \rightarrow \mathbf{Hom}_{\mathcal{N}}(-, G(-)), \quad (4.12)$$

*between multi-balanced functors  ${}_e^{\#}\mathcal{N}_{\mathcal{D}} \boxtimes_{\mathcal{D}} \mathcal{M}_e \rightarrow \mathbf{Vect}$ .*

*ii) Let  $F, G : {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_{\mathcal{D}}\mathcal{N}_e$  be bimodule functors. The set of bimodule natural transformations from  $F$  to  $G$  is in bijection with the set of balanced bimodule transformations*

$${}_{\mathcal{D}}\langle F(-), - \rangle^{\mathcal{N}} \rightarrow {}_{\mathcal{D}}\langle G(-), - \rangle^{\mathcal{N}} \quad (4.13)$$

*between balanced bimodule functors  ${}_{\mathcal{D}}\mathcal{M}_e \boxtimes_e {}_e^{\#}\mathcal{N}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$ .*

*There are the analogue statements switching the entries of the  $\mathbf{Hom}$ -functors and replacing covariant with contravariant.*

*Proof.* Let  $\Phi_{n,m} : \mathbf{Hom}_{\mathcal{N}}(n, F(m)) \rightarrow \mathbf{Hom}_{\mathcal{N}}(n, G(m))$  be a multi-balanced natural transformation. As in the usual Yoneda-lemma we define the corresponding natural transformation  $\eta : F \rightarrow G$  by  $\eta_m = \Phi_{F(m),m}(\mathrm{id}_{F(m)}) : F(m) \rightarrow G(m)$ . The following diagram commutes since  $\Phi$  is  $\mathcal{D}$ -balanced:

$$\begin{array}{ccc} \mathbf{Hom}(d^* \triangleright d \triangleright F(m), F(m)) \simeq & \xrightarrow{\Phi} & \mathbf{Hom}(d^* \triangleright d \triangleright F(m), G(m)) \simeq \\ \mathbf{Hom}(d \triangleright F(m), d \triangleright F(m)) & & \mathbf{Hom}(d \triangleright F(m), d \triangleright G(m)) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathbf{Hom}(F(d \triangleright m), F(d \triangleright m)) & \xrightarrow{\Phi} & \mathbf{Hom}(F(d \triangleright m), G(d \triangleright m)) \end{array} \quad (4.14)$$

If we consider the identity in the upper left  $\mathbf{Hom}$ -space, it gets mapped to  $d \triangleright \eta_m$  by the upper arrow and to  $\eta_{d \triangleright m}$  in the lower left  $\mathbf{Hom}$ -space by the lower arrow. Hence it follows that  $\eta$  is  $\mathcal{D}$ -balanced. The proof that  $\eta$  is  $\mathcal{C}$ -balanced is analogous. The converse and the contravariant versions of the statements follow analogously.

For the second part, assume that we are given a balanced bimodule natural transformation  $\Phi : {}_{\mathcal{D}}\langle F(-), - \rangle^{\mathcal{N}} \rightarrow {}_{\mathcal{D}}\langle G(-), - \rangle^{\mathcal{N}}$ . It is straightforward to see that  $\Phi$  induces a multi-balanced module natural transformation  $\mathbf{Hom}_{\mathcal{D}}({}_{\mathcal{D}}\langle F(m), n \rangle^{\mathcal{M}}, d) \simeq \mathbf{Hom}_{\mathcal{D}}({}_{\mathcal{D}}\langle G(m), n \rangle^{\mathcal{M}}, d)$  of multi-balanced functors  ${}_{\mathcal{D}}^{\#}\mathcal{M}_e \boxtimes_e {}_e^{\#}\mathcal{M}_{\mathcal{D}} \boxtimes_{\mathcal{D}} \mathcal{D}_{\mathcal{D}} \rightarrow \mathbf{Vect}$ . Restricting to  $d = \mathbf{1}_{\mathcal{D}}$ , this yields a balanced natural transformation  $\mathbf{Hom}(F(m), n) \rightarrow \mathbf{Hom}(G(m), n)$ , which defines a bimodule natural transformation  $F \rightarrow G$  by the contravariant version of the first part. The remaining statements follow analogously.  $\square$

Next we consider the module version of representable functors.

**Proposition 4.12** *i) Every right exact module functor  $F : {}_{\mathcal{D}}\mathcal{M} \rightarrow {}_{\mathcal{D}}\mathcal{D}$  is (left)  $\mathcal{D}$ -representable by  $F^r(\mathbf{1}_{\mathcal{D}})$ , i.e. there exists a module natural isomorphism*

$$F(x) \simeq {}_{\mathcal{D}}\langle x, F^r(\mathbf{1}_{\mathcal{D}}) \rangle^{\mathcal{M}}. \quad (4.15)$$

*This isomorphism extends to an equivalence of bimodule functors*

$${}_{\mathcal{D}}\langle -, \tilde{\#}(F^r)(-) \rangle^{\mathcal{M}} \simeq F(-) \otimes (-), \quad (4.16)$$

*from  ${}_{\mathcal{D}}\mathcal{M} \boxtimes {}_{\mathcal{D}}\mathcal{D}$  to  ${}_{\mathcal{D}}\mathcal{D}$ .*

*ii) Let  $F : {}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$  be a right exact bimodule functor. Then the equivalence (4.16) is an equivalence of balanced bimodule functors  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}} \boxtimes {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$ . Furthermore,  $F$  is right  $\mathcal{D}$ -representable by  $(F^r)^{\tilde{\#}}$ , i.e. there is a bimodule natural isomorphism*

$$\langle (F^r)^{\tilde{\#}}(-), - \rangle_{\mathcal{D}}^{\mathcal{M}} \simeq (-) \otimes F(-) \quad (4.17)$$

*between balanced bimodule functors  ${}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}} \boxtimes {}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$ .*

Proof. Since  $F$  is right exact, it admits a right adjoint, see Proposition 2.11. For the first part we first compute for all  $m \in \mathcal{M}$ ,  $x, d \in \mathcal{D}$

$$\begin{aligned} \text{Hom}_{\mathcal{D}}(\langle m, (\tilde{\#}F^r)(d) \rangle^{\mathcal{M}}, x) &\simeq \text{Hom}_{\mathcal{M}}(m, x \triangleright F^r(d^*)) \simeq \text{Hom}_{\mathcal{M}}(m, F^r(x \otimes d^*)) \\ &\simeq \text{Hom}_{\mathcal{D}}(F(m), x \otimes d^*) \simeq \text{Hom}(F(m) \otimes d, x). \end{aligned} \quad (4.18)$$

All isomorphisms are multi-balanced natural isomorphisms, hence the statement follows from the module Yoneda-Lemma 4.11 *i)*. If  $F$  is a bimodule functor as in the second statement, it follows that  $F^r : {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}$  is also a bimodule functor. Then equation (4.18) consists of multi-balanced natural isomorphisms and thus (4.16) is a bimodule natural isomorphism. We additionally have a natural multi-balanced isomorphism

$$\text{Hom}_{\mathcal{D}}(\langle (F^r)^{\tilde{\#}}(d), m \rangle_{\mathcal{D}}^{\mathcal{M}}, x) \simeq \text{Hom}_{\mathcal{M}}(d \otimes F(m), x), \quad (4.19)$$

that is constructed in a similar way to (4.18). This yields by the module Yoneda-Lemma, the claimed second bimodule natural isomorphism.  $\square$

The next statement follows using the same techniques as in the previous proposition.

**Lemma 4.13** *Let  $F : {}_{\mathcal{D}}\mathcal{M}_{\mathcal{E}} \rightarrow {}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}}$  be a right exact bimodule functor. Then there exist balanced bimodule natural isomorphisms*

$${}_{\mathcal{D}}\langle F(m), n \rangle^{\mathcal{N}} \simeq {}_{\mathcal{D}}\langle m, F^r(n) \rangle^{\mathcal{M}} \quad \text{and} \quad \langle n, F(m) \rangle_{\mathcal{E}}^{\mathcal{N}} \simeq \langle F^r(n), m \rangle_{\mathcal{E}}^{\mathcal{M}} \quad (4.20)$$

*for  $m \in \mathcal{M}$  and  $n \in \mathcal{N}$ . The analogous statement holds for left exact bimodule functors and the left adjoint functor.*

**Compatibilities inner hom, dual categories and tensor product** The inner hom and the dual categories are compatible in the following sense. Here and in the sequel we use the equivalences from Lemma 4.7 without mentioning. Recall the operation  $\tilde{\#}$  from equation (4.10).

**Lemma 4.14** *i) There is a canonical balanced bimodule natural isomorphism between the balanced bimodule functors*

$${}_{\mathcal{D}}\langle -, - \rangle^{\mathcal{M}}, \langle -, - \rangle_{\mathcal{D}}^{\# \mathcal{M}_{\mathcal{D}}} : {}_{\mathcal{D}}\mathcal{M}_{\mathcal{E}} \boxtimes_{\mathcal{E}}^{\#} \mathcal{M}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}. \quad (4.21)$$

*as well as between the balanced bimodule functors*

$$\langle -, - \rangle_{\mathcal{E}}^{\mathcal{M}}, {}_{\mathcal{E}}\langle -, - \rangle^{\mathcal{M}_{\mathcal{D}}} : {}_{\mathcal{E}}\mathcal{M}_{\mathcal{D}}^{\#} \boxtimes {}_{\mathcal{D}}\mathcal{M}_{\mathcal{E}} \rightarrow {}_{\mathcal{E}}\mathcal{C}_{\mathcal{E}}. \quad (4.22)$$

ii) There are canonical balanced bimodule natural isomorphism between the balanced bimodule functors

$$({}_{\mathcal{D}}\langle -, - \rangle^{\mathcal{M}})^{\# \#}, \langle -, - \rangle_{\mathcal{D}}^{\mathcal{M} \#} : {}_{\mathcal{D}}\mathcal{M}_e^{\# \#} \boxtimes_e \mathcal{M}_{\mathcal{D}}^{\#} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}. \quad (4.23)$$

and

$$\tilde{\# \#}(\langle -, - \rangle_e^{\mathcal{M}}, e\langle -, - \rangle^{\#e \mathcal{M}} : {}_e\mathcal{M}_{\mathcal{D}} \boxtimes^{\# \#} {}_{\mathcal{D}}\mathcal{M}_e. \quad (4.24)$$

Proof. Using the definitions of the inner homs we compute

$$\begin{aligned} \text{Hom}_{\mathcal{D}}(\langle m, \tilde{m} \rangle_{\mathcal{D}}^{\# \mathcal{M}_{\mathcal{D}}}, d) &\simeq \text{Hom}_{\# \mathcal{M}_{\mathcal{D}}}(\tilde{m}, m^{\# \triangleleft} d) = \text{Hom}_{\mathcal{M}}(d^* \triangleright m, \tilde{m}) \\ &\simeq \text{Hom}_{\mathcal{M}}(m, d \triangleright \tilde{m}) = \text{Hom}_{\mathcal{D}}({}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}}, d). \end{aligned}$$

Lemma 4.11 thus yields a balanced bimodule natural isomorphism  ${}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}} \simeq \langle m, \tilde{m} \rangle_{\mathcal{D}}^{\# \mathcal{M}_{\mathcal{D}}}$ . The argument for the  $\mathcal{C}$ -valued inner hom follows directly from the first statement and Lemma 4.7 i). For the second part, let  $d \in \mathcal{D}$ ,  $m, \tilde{m} \in \mathcal{M}$ . By the definition of the inner hom and the duality in  $\mathcal{D}$ , we have the following chain of natural isomorphisms.

$$\begin{aligned} \text{Hom}_{\mathcal{D}}(\langle \tilde{m}, m \rangle_{\mathcal{D}}^{\mathcal{M} \#}, d) &\simeq \text{Hom}_{\mathcal{M} \#}(m, \tilde{m} \triangleleft^{\#} d) \simeq \text{Hom}_{\mathcal{M}}(*d \triangleright \tilde{m}, m) \\ &\simeq \text{Hom}_{\mathcal{M}}(\tilde{m}, **d \triangleright m) \simeq \text{Hom}_{\mathcal{D}}({}_{\mathcal{D}}\langle \tilde{m}, m \rangle^{\mathcal{M}}, **d) \\ &\simeq \text{Hom}_{\mathcal{D}}(({}_{\mathcal{D}}\langle \tilde{m}, m \rangle^{\mathcal{M}})^{**}, d). \end{aligned} \quad (4.25)$$

All isomorphisms are multi-balanced bimodule natural isomorphisms and induce the required balanced bimodule natural isomorphism. The last statement follows again from the previous one and Lemma 4.7 i).  $\square$

Finally we discuss the compatibility of the inner hom and the tensor product of module categories.

**Proposition 4.15** i) The functor

$$\begin{aligned} \Lambda : {}_{\mathcal{D}}\mathcal{M}_e \boxtimes_e \mathcal{N}_e \boxtimes_e^{\#} \mathcal{N}_e \boxtimes_e^{\#} \mathcal{M}_{\mathcal{D}} &\rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}} \\ m \boxtimes n \boxtimes \tilde{n} \boxtimes \tilde{m} &\rightarrow {}_{\mathcal{D}}\langle m \triangleleft e\langle n, \tilde{n} \rangle^{\mathcal{N}}, \tilde{m} \rangle^{\mathcal{M}} \end{aligned} \quad (4.26)$$

is a multi-balanced  $(\mathcal{D}, \mathcal{D})$ -bimodule functor.

ii) The following diagram of multi-balanced module functors commutes up to a canonical multi-balanced module natural isomorphism

$$\begin{array}{ccc} {}_{\mathcal{D}}\mathcal{M}_e \boxtimes_e \mathcal{N}_e \boxtimes_e^{\#} \mathcal{N}_e \boxtimes_e^{\#} \mathcal{M}_{\mathcal{D}} & & \\ \downarrow \mathbf{B} \boxtimes \# \mathbf{B} & \searrow \Lambda & \\ {}_{\mathcal{D}}\mathcal{M}_e \square_e \mathcal{N}_e \boxtimes_e^{\#} \mathcal{N}_e \square_e^{\#} \mathcal{M}_{\mathcal{D}} & \xrightarrow{{}_{\mathcal{D}}\langle -, - \rangle^{\mathcal{M} \square \mathcal{N}}} & {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}. \end{array} \quad (4.27)$$

iii) Analogously, there exists a multi-balanced module natural isomorphism

$$\langle \tilde{m} \square \tilde{n}, m \square n \rangle_e^{\mathcal{M} \square \mathcal{N}} \simeq \langle \tilde{n}, \langle \tilde{m}, m \rangle_e^{\mathcal{N}} \triangleright n \rangle_e^{\mathcal{M}}. \quad (4.28)$$

Proof. It follows directly from the properties of the inner hom functors, see Proposition 2.18, that the functor  $\Lambda$  has the structure of a multi-balanced bimodule functor. For the second statement, note that the functor  $\mathbf{B} : \mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{M} \square \mathcal{N}$  and the functor  ${}_{\mathcal{D}}\langle -, - \rangle^{\mathcal{M} \square \mathcal{N}}$  are multi-balanced, hence the composite of the functors on the lower arrows in (4.27) defines as well a multi-balanced bimodule functor. To construct the multi-balanced natural isomorphism between these two functors, we proceed in two steps. First we proof that it suffices to show the result for  $\mathcal{D} = \mathcal{C}_{\mathcal{M}}^* = \text{Fun}_e(\mathcal{M}_e, \mathcal{M}_e)$ . Let  $\mathbf{F} : \mathcal{D} \rightarrow \mathcal{C}_{\mathcal{M}}^*$  be the tensor functor given by the action of  $\mathcal{D}$  on  $\mathcal{M}$ , i.e.  $d \triangleright m = \mathbf{F}(m)$ . This functor is exact, since the functor  $\triangleright : \mathcal{D} \times \mathcal{M} \rightarrow \mathcal{M}$  is biexact. Hence the adjoint functors to  $\mathbf{F}$  exist. The equation

$$\text{Hom}_{\mathcal{D}}({}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}}, d) = \text{Hom}_{\mathcal{M}}(m, d \triangleright \tilde{m}) = \text{Hom}_{e_{\mathcal{M}}^*}(e_{\mathcal{M}}^*\langle m, \tilde{m} \rangle^{\mathcal{M}}, \mathbf{F}(d))$$

shows that  ${}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}} = \mathbf{F}^!(e_{\mathcal{M}}^*\langle m, \tilde{m} \rangle^{\mathcal{M}})$ .

Assume now that the statement is proven for  $\mathcal{C}_{\mathcal{M}}^*$ , then

$$\begin{aligned} {}_{\mathcal{D}}\langle m \square n, \tilde{m} \square \tilde{n} \rangle^{\mathcal{M} \square \mathcal{N}} &= \mathbf{F}^l({}_{e_{\mathcal{M}}}^* \langle m \square n, \tilde{m} \square n \rangle^{\mathcal{M} \square \mathcal{N}}) \simeq \mathbf{F}^l({}_{e_{\mathcal{M}}}^* \langle m \triangleleft e \langle n, \tilde{n} \rangle^{\mathcal{N}}, \tilde{m} \rangle^{\mathcal{M}}) \\ &= {}_{\mathcal{D}}\langle m \triangleleft e \langle n, \tilde{n} \rangle^{\mathcal{N}}, \tilde{m} \rangle^{\mathcal{M}} \end{aligned} \quad (4.29)$$

shows that the statement follows for  $\mathcal{D}$ .

Next we prove the assertion for  $\mathcal{D} = \mathcal{C}_{\mathcal{M}}^*$  in the case that  $\mathcal{M}_e = {}_A \mathbf{Mod}(\mathcal{C})$  and  ${}_e \mathcal{N} = \mathbf{Mod}(\mathcal{C})_B$  for algebras  $A, B \in \mathcal{C}$ . Recall from Example 2.20 that in this case  $\mathcal{C}_{\mathcal{M}}^* = {}_A \mathbf{Mod}(\mathcal{C})_A$ . By Theorem 3.2, the tensor product  $\mathcal{M}_e \square_e \mathcal{N}$  is given by the category of  $(A, B)$ -bimodules in  $\mathcal{C}$  with universal balancing functor given by the tensor product in  $\mathcal{C}$ . Let  $m, \tilde{m} \in {}_A \mathbf{Mod}(\mathcal{C})$  and  $n, \tilde{n} \in \mathbf{Mod}(\mathcal{C})_B$  and  $x \in {}_A \mathbf{Mod}(\mathcal{C})_A$ .

$$\begin{aligned} \mathbf{Hom}_{{}_A \mathbf{Mod}(\mathcal{C})_B}({}_{e_{\mathcal{M}}}^* \langle m \otimes n, \tilde{m} \otimes \tilde{n} \rangle^{\mathcal{M} \square \mathcal{N}}, x) &= \mathbf{Hom}_{{}_A \mathbf{Mod}(\mathcal{C})_B}(m \otimes n, (x \otimes_A \tilde{m}) \otimes \tilde{n}) \\ &= \mathbf{Hom}_{{}_A \mathbf{Mod}(\mathcal{C})}(m \otimes (n \otimes_B {}^* \tilde{n}), x \otimes_A \tilde{m}) \\ &= \mathbf{Hom}_{{}_A \mathbf{Mod}(\mathcal{C})_A}(A, x \otimes_A \tilde{m} \otimes (m \otimes (n \otimes_B {}^* \tilde{n})))^* \\ &= \mathbf{Hom}_{{}_A \mathbf{Mod}(\mathcal{C})_A}({}^*(\tilde{m} \otimes (m \otimes (n \otimes_B {}^* n)))^*, x) \end{aligned} \quad (4.30)$$

The description of the inner hom objects from Example 2.20 shows that on the other side

$${}_{e_{\mathcal{M}}}^* \langle m \triangleleft e \langle n, \tilde{n} \rangle^{\mathcal{N}}, \tilde{m} \rangle^{\mathcal{M}} = {}_{e_{\mathcal{M}}}^* \langle m \triangleleft (n \otimes_B {}^* n), \tilde{m} \rangle^{\mathcal{M}} = {}^*(\tilde{m} \otimes (m \otimes (n \otimes_B {}^* n)))^* \quad (4.31)$$

This completes the second part in the case that  $\mathcal{M}_e = {}_A \mathbf{Mod}(\mathcal{C})$ . Since every module category is equivalent to one of this type, see Theorem 2.19, the statement holds in general. The third statement follows directly by applying the second part to the  $\mathcal{E}$ -valued inner hom of  $({}_{\mathcal{D}} \mathcal{M}_e \square_e \mathcal{N}_{\mathcal{E}})^{\#}$  and using Lemma 4.14.  $\square$

As we remarked in Section 2, the inner hom can be regarded as a categorification of an algebra valued inner product except one compatibility with the  $*$ -involution. In view of this analogy, the functor (4.27) can be regarded as a categorification of the Rieffel induction formula (1.1).

### 4.3 $\#$ -duals for bimodule categories using inner homs

We finally show using the inner hom functors, that the dual categories are indeed  $\#$ -duals in the tricategory  $\mathbf{Bimod}$ .

**The evaluation functors** Recall from Proposition 2.18, that the inner hom functors for a bimodule category  ${}_{\mathcal{D}} \mathcal{M}_e$  are right exact balanced bimodule functors

$${}_{\mathcal{D}}\langle -, - \rangle^{\mathcal{M}} : {}_{\mathcal{D}} \mathcal{M}_e \times_e^{\#} \mathcal{M}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}} \mathcal{D}_{\mathcal{D}}, \quad \langle -, - \rangle_e^{\mathcal{M}} : {}_e \mathcal{M}_{\mathcal{D}}^{\#} \times {}_{\mathcal{D}} \mathcal{M}_e \rightarrow {}_e \mathcal{C}_e. \quad (4.32)$$

Using the universal property of the tensor product of bimodule categories we obtain the following functors.

**Definition 4.16** *Let  ${}_{\mathcal{D}} \mathcal{M}_e$  be a bimodule category. The inner hom functors induce bimodule functors*

$$\mathbf{ev}_{{}_{\mathcal{D}} \mathcal{M}_e} : {}_{\mathcal{D}} \mathcal{M}_e \square_e^{\#} \mathcal{M}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}} \mathcal{D}_{\mathcal{D}}, \quad \mathbf{ev}_{\mathcal{M}_e} : {}_e \mathcal{M}_{\mathcal{D}}^{\#} \square_{{}_{\mathcal{D}} \mathcal{M}_e} \rightarrow {}_e \mathcal{C}_e, \quad (4.33)$$

*that are called the left, respectively right evaluation functors.*

From the definition of the tensor product it follows that the evaluation functors are right exact.

By applying the universal property of the tensor product to Lemma 4.14 and Proposition 4.15, we obtain the following compatibilities of the evaluation functors with the dual categories and the tensor product.

**Corollary 4.17** *Let  ${}_{\mathcal{D}} \mathcal{M}_e$  and  ${}_e \mathcal{N}_{\mathcal{E}}$  be two bimodule categories.*

*i) There are bimodule natural isomorphisms between the bimodule functors*

$$\mathbf{ev}_{{}_{\mathcal{D}} \mathcal{M}_e}, \mathbf{ev}_{\# \mathcal{M}_{\mathcal{D}}} : {}_{\mathcal{D}} \mathcal{M}_e \square_e^{\#} \mathcal{M}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}} \mathcal{D}_{\mathcal{D}} \quad \text{and} \quad \mathbf{ev}_{\mathcal{M}_e}, \mathbf{ev}_{e \mathcal{M}^{\#}} : {}_e \mathcal{M}_{\mathcal{D}}^{\#} \square_{{}_{\mathcal{D}} \mathcal{M}_e} \rightarrow {}_e \mathcal{C}_e. \quad (4.34)$$

ii) The bimodule functor  $\text{ev}_{\mathcal{D}} : {}_{\mathcal{D}}\mathcal{M}_e \square_e \mathcal{N}_\varepsilon \square^\# ({}_{\mathcal{D}}\mathcal{M}_e \square_e \mathcal{N}_\varepsilon) \rightarrow {}_{\mathcal{D}}\mathcal{D}$  is equivalent as a bimodule functor to the composite

$$\mathcal{M} \square \mathcal{N} \square^\# (\mathcal{M} \square \mathcal{N}) \simeq \mathcal{M} \square \mathcal{N} \square^\# \mathcal{N} \square^\# \mathcal{M} \xrightarrow{1 \square \text{ev}_{\mathcal{N}} \square 1} \mathcal{M} \square \mathcal{C} \square^\# \mathcal{M} \xrightarrow{r_{\mathcal{M}} \square 1} \mathcal{M} \square^\# \mathcal{M} \xrightarrow{\text{ev}_{\mathcal{D}} \square \mathcal{M}} \mathcal{D}. \quad (4.35)$$

iii) The bimodule functor  $\text{ev}_{\mathcal{M}} : (\mathcal{M} \square \mathcal{N})^\# \square \mathcal{M} \square \mathcal{N} \rightarrow \mathcal{E}$  is equivalent as a bimodule functor to the composite

$$(\mathcal{M} \square \mathcal{N})^\# \square \mathcal{M} \square \mathcal{N} \simeq \mathcal{N}^\# \square \mathcal{M}^\# \square \mathcal{M} \square \mathcal{N} \xrightarrow{1 \square \text{ev}_{\mathcal{M}} \square 1} \mathcal{N}^\# \square \mathcal{C} \square \mathcal{N} \xrightarrow{1 \square l_{\mathcal{N}}} \mathcal{N}^\# \square \mathcal{N} \xrightarrow{\text{ev}_{\mathcal{N}_\varepsilon}} \mathcal{E}. \quad (4.36)$$

In these formulas we suppressed the associativity functors of the tricategory  $\text{Bimod}$ .

Furthermore we have the following version of the Yoneda-lemma and the representations of module functors by applying the tensor product to Lemma 4.11 ii) and to Proposition 4.12 ii).

**Lemma 4.18** *Let  ${}_{\mathcal{D}}\mathcal{M}_e$ ,  ${}_{\mathcal{D}}\mathcal{N}_e$  and  ${}_{\mathcal{D}}\mathcal{K}_{\mathcal{D}}$  be bimodule categories.*

i) For bimodule functors  $F, G : {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_{\mathcal{D}}\mathcal{N}_e$  the set of bimodule natural transformations from  $F$  to  $G$  is in bijection with the set of balanced bimodule transformations

$$\text{ev}_{\mathcal{D}} \circ (F \square 1) \rightarrow \text{ev}_{\mathcal{D}} \circ (G \square 1) \quad \text{between functors} \quad {}_{\mathcal{D}}\mathcal{M}_e \square_e^\# \mathcal{N}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}, \quad (4.37)$$

ii) Let  $F : {}_{\mathcal{D}}\mathcal{K}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$  be a right exact bimodule functor. Then  $F$  is equivalent as a bimodule functor to the composites

$${}_{\mathcal{D}}\mathcal{K}_{\mathcal{D}} \simeq {}_{\mathcal{D}}\mathcal{K}_{\mathcal{D}} \square {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}} \xrightarrow{(F^r)^\#} {}_{\mathcal{D}}\mathcal{K}_{\mathcal{D}} \square^\# {}_{\mathcal{D}}\mathcal{K}_{\mathcal{D}} \xrightarrow{\text{ev}_{\mathcal{D}} \square \mathcal{D}} {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}, \quad (4.38)$$

as well as to

$${}_{\mathcal{D}}\mathcal{K}_{\mathcal{D}} \simeq {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}} \square {}_{\mathcal{D}}\mathcal{K}_{\mathcal{D}} \xrightarrow{\#(F^r)} {}_{\mathcal{D}}\mathcal{K}_{\mathcal{D}}^\# \square {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}} \xrightarrow{\text{ev}_{\mathcal{D}} \square \mathcal{D}} {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}. \quad (4.39)$$

**The coevaluation functors** In analogy to the case of module categories over  $\text{Vect}$  in the introduction to this section, we define the coevaluation functor using a representing object for the inner hom functors. Recall that the evaluation functors are right exact, hence there exists a right adjoint.

**Definition 4.19** *Let  ${}_{\mathcal{D}}\mathcal{M}_e$  be an bimodule category. The left coevaluation functor is defined by*

$$\text{coev}_{\mathcal{D}\mathcal{M}} = (\text{ev}_{\mathcal{D}\mathcal{M}}^r)^\# : {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}} \rightarrow ({}_{\mathcal{D}}\mathcal{M}_e \square_e^\# \mathcal{M}_{\mathcal{D}})^\# \simeq {}_{\mathcal{D}}\mathcal{M}_e \square_e \mathcal{M}_{\mathcal{D}}^\#, \quad (4.40)$$

while the right coevaluation functor is defined by

$$\text{coev}_{\mathcal{M}_e} = \#(\text{ev}_{\mathcal{M}_e}^r) : {}_e\mathcal{C}_e \rightarrow \#({}_e\mathcal{M}_{\mathcal{D}}^\# \square {}_{\mathcal{D}}\mathcal{M}_e) \simeq {}_e^\# \mathcal{M}_{\mathcal{D}} \square {}_{\mathcal{D}}\mathcal{M}_e. \quad (4.41)$$

In these formulas we used the equivalences from Lemma 4.7.

First we clarify the compatibility of the coevaluation functors and the dual bimodule categories.

**Lemma 4.20** *For a bimodule category  ${}_{\mathcal{D}}\mathcal{M}_e$  the bimodule functors*

$$\text{coev}_{\mathcal{D}\mathcal{M}}, \text{coev}_{\mathcal{M}_{\mathcal{D}}^\#} : \mathcal{D} \rightarrow {}_{\mathcal{D}}\mathcal{M} \square \mathcal{M}_{\mathcal{D}}^\#, \quad \text{and} \quad \text{coev}_{\mathcal{M}_e}, \text{coev}_{{}_e^\# \mathcal{M}} : \mathcal{C} \rightarrow {}_e^\# \mathcal{M} \square \mathcal{M}_e$$

are equivalent as bimodule functors.



Proof. To construct the first natural isomorphism it is sufficient to show that  $(\text{ev}_{\mathcal{D}\mathcal{M}}^r)^{\tilde{\#}\tilde{\#}} \simeq (\text{ev}_{\mathcal{M}_{\mathcal{D}}^{\#}})^r$  as bimodule functors. It is straightforward to see that taking the right adjoint and the operation  $(-)^{\tilde{\#}\tilde{\#}}$  commute up to a natural bimodule isomorphism. Hence it suffices to show that  $(\text{ev}_{\mathcal{D}\mathcal{M}})^{\tilde{\#}\tilde{\#}} \simeq \text{ev}_{\mathcal{M}_{\mathcal{D}}^{\#}}$ . This follows from Lemma 4.14 *i*). The remaining isomorphism for the  $\mathbb{C}$ -valued inner hom follows directly from the first statement using the equivalence  $(\# \mathcal{M})^{\#} \simeq \mathcal{M}_e$ .  $\square$

**Lemma 4.21** *i)* The coevaluation functors represent the inner hom functors, i.e. there are natural bimodule isomorphism

$${}_{\mathcal{D}}\langle -, \text{coev}_{\mathcal{D}\mathcal{M}}(\mathbf{1}) \rangle^{\mathcal{M} \square \# \mathcal{M}} \simeq \text{ev}_{\mathcal{D}\mathcal{M}}(-) \simeq \langle \text{coev}_{\mathcal{D}\mathcal{M}}(\mathbf{1}), - \rangle_{\mathcal{D}}^{\mathcal{M} \square \# \mathcal{M}}, \quad (4.42)$$

between bimodule functors  ${}_{\mathcal{D}}\mathcal{M}_e \square_e^{\#} \mathcal{M}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$ . Furthermore there are natural bimodule isomorphisms

$$e\langle -, \text{coev}_{\mathcal{M}_e}(\mathbf{1}) \rangle^{\mathcal{M}^{\#} \square \mathcal{M}} \simeq \text{ev}_{\mathcal{M}_e} \simeq \langle \text{coev}_{\mathcal{M}_e}(\mathbf{1}), - \rangle_e^{\mathcal{M}^{\#} \square \mathcal{M}} \quad (4.43)$$

between bimodule functors  ${}_{\mathcal{D}}\mathcal{M}_e \square_e^{\#} \mathcal{M}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$ .

*ii)* The composite bimodule functor

$${}_{\mathcal{D}}\mathcal{M}_e \square_e^{\#} \mathcal{M}_{\mathcal{D}} \simeq \mathcal{D} \square \mathcal{M} \square \# \mathcal{M} \xrightarrow{\text{coev}_{\mathcal{D}\mathcal{M}} \square \mathbf{1}} (\mathcal{M} \square \# \mathcal{M})^{\#} \square \mathcal{M} \square \# \mathcal{M} \xrightarrow{\text{ev}_{\mathcal{M} \square \# \mathcal{M}_{\mathcal{D}}}} \mathcal{D} \quad (4.44)$$

is equivalent to the evaluation functor  $\text{ev}_{\mathcal{D}\mathcal{M}}$  as bimodule functor. The composite bimodule functor

$$e\mathcal{M}_{\mathcal{D}}^{\#} \square_{\mathcal{D}} \mathcal{M}_e \simeq \mathcal{M}^{\#} \square \mathcal{M} \square \mathbb{C} \xrightarrow{\mathbf{1} \square \text{coev}_{\mathcal{M}_e}} \mathcal{M}^{\#} \square \mathcal{M} \square \# (\mathcal{M}^{\#} \square \mathcal{M}) \xrightarrow{\text{ev}_{\mathcal{M}^{\#} \square \mathcal{M}}} \mathbb{C} \quad (4.45)$$

is equivalent to  $\text{ev}_{\mathcal{M}_e}$  as a bimodule functor.

Proof. The first part follows directly from the definition of the coevaluation functor and from Proposition 4.12. The second part follows from applying Lemma 4.18 *ii*).  $\square$

**The triangulators** We finally show that the evaluation and coevaluation functors satisfy the snake identities up to a bimodule natural isomorphism. This is shown in [8, Proposition 4.2.1] using the description of the tensor product as functor category. Our method of proof is a generalization of the semisimple case considered in [29].

**Proposition 4.22** Let  ${}_{\mathcal{D}}\mathcal{M}_e$  be a bimodule category. There exists a bimodule natural isomorphism between the composite

$$\Phi_{\mathcal{M}} : \mathcal{M} \simeq \mathcal{D} \square \mathcal{M} \xrightarrow{\text{coev}_{\mathcal{D}\mathcal{M}} \square \mathbf{1}} \mathcal{M} \square \mathcal{M}^{\#} \square \mathcal{M} \xrightarrow{\mathbf{1} \square \text{ev}_{\mathcal{M}_e}} \mathcal{M} \square \mathbb{C} \simeq \mathcal{M}, \quad (4.46)$$

and the identity functor on  ${}_{\mathcal{D}}\mathcal{M}_e$ . Similarly, the bimodule functor

$$\mathcal{M} \simeq \mathcal{M} \square \mathbb{C} \xrightarrow{\mathbf{1} \square \text{coev}_{\mathcal{M}_e}} \mathcal{M} \square \# \mathcal{M} \square \mathcal{M} \xrightarrow{\text{ev}_{\mathcal{D}\mathcal{M}} \square \mathbf{1}} \mathcal{D} \square \mathcal{M} \simeq \mathcal{M}, \quad (4.47)$$

is equivalent to the identity bimodule functor  ${}_{\mathcal{D}}\mathcal{M}_e$ . These natural isomorphisms are called left and right triangulators, respectively.

Proof. We show that there exists a bimodule natural isomorphisms between  $\text{ev}_{\mathcal{D}\mathcal{M}} \circ (\Phi_{\mathcal{M}} \square \mathbf{1})$  and  $\text{ev}_{\mathcal{D}\mathcal{M}}$  as bimodule functors from  ${}_{\mathcal{D}}\mathcal{M}_e \square_e^{\#} \mathcal{M}_{\mathcal{D}}$  to  ${}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$ . Then the statement will follow by Lemma 4.18 *i*). According to Corollary 4.17, the functor  $\text{ev}_{\mathcal{M} \square \# \mathcal{M}_{\mathcal{D}}} : \mathcal{M} \square \mathcal{M}^{\#} \square \mathcal{M} \square \mathcal{M}^{\#} \rightarrow \mathcal{D}$  is isomorphic as bimodule functor to  $\text{ev}_{\# \mathcal{M}_{\mathcal{D}}} \circ (\mathbf{1} \square \text{ev}_{\mathcal{M}_e} \square \mathbf{1})$ , and hence according to equation (4.34) to  $\text{ev}_{\mathcal{D}\mathcal{M}} \circ (\mathbf{1} \square \text{ev}_{\mathcal{M}_e} \square \mathbf{1})$ . In these formulas we suppressed the unit bimodule functors for simplicity. This implies that Lemma 4.21 *ii*) gives a bimodule natural isomorphism  $\text{ev}_{\mathcal{D}\mathcal{M}} \circ (\Phi_{\mathcal{M}} \square \mathbf{1}) \rightarrow \text{ev}_{\mathcal{D}\mathcal{M}}$ . The second natural bimodule isomorphism is constructed analogously.  $\square$

With the help of the evaluation and coevaluation functors we obtain  $\#$ -duals for Bimod.

**Theorem 4.23** *Bimodule categories over finite tensor categories form a tricategory with #-duals.*

Proof. By Proposition 4.22, the dual bimodule categories with the evaluation and coevaluation functors satisfy the first snake identity (A.1) for the #-duals up to the triangulator. The remaining triangulator for  ${}_{\mathcal{D}}\mathcal{M}^{\#}$  is constructed as follows. By Lemma 4.20 and Corollary 4.17 i) the functor

$${}_{\mathcal{D}}\mathcal{M}^{\#} \simeq \mathcal{M}^{\#} \square \mathcal{D} \xrightarrow{1 \square \text{coev}_{\mathcal{D}} \mathcal{M}} \mathcal{M}^{\#} \square \mathcal{M} \square \mathcal{M}^{\#} \xrightarrow{\text{ev}_{\mathcal{M}} \square 1} \mathcal{C} \square \mathcal{M}^{\#} \simeq {}_{\mathcal{D}}\mathcal{M}^{\#}, \quad (4.48)$$

is equivalent as bimodule functor to the composite

$${}_{\mathcal{D}}\mathcal{M}^{\#} \simeq \mathcal{M}^{\#} \square \mathcal{D} \xrightarrow{1 \square \text{coev}_{\mathcal{D}} \mathcal{M}^{\#}} \mathcal{M}^{\#} \square \mathcal{M} \square \mathcal{M}^{\#} \xrightarrow{\text{ev}_{\mathcal{M}} \square 1} \mathcal{C} \square \mathcal{M}^{\#} \simeq {}_{\mathcal{D}}\mathcal{M}^{\#}. \quad (4.49)$$

Equation (4.47) in Proposition 4.22 applied to  ${}_{\mathcal{D}}\mathcal{M}^{\#}$  shows that this bimodule functor is equivalent to the identity.  $\square$

## 4.4 Separable bimodule categories and the Serre equivalence

First we show that separable bimodule categories form a tricategory with duals. This implies that the left and right dual of a bimodule category are equivalent by Proposition 4.4. We characterize these so called Serre equivalences using the inner homs. This might be important for applications to TFTs, since the Serre equivalences encode the homotopy action corresponding to the framing change in a framed TFT [8, 26].

**The tricategory of separable bimodule categories** It follows almost directly from results in [8] that separable bimodule categories form a tricategory with duals. We just need to prove that separable bimodule categories are biexact and hence the adjoints of all module functors exists.

**Definition 4.24** *Let  $\mathcal{C}, \mathcal{D}$  be finite tensor categories.*

- i) *An algebra  $A$  in  $\mathcal{C}$  is called separable, if the multiplication  $m : A \otimes A \rightarrow A$  splits as a map of  $(A, A)$ -bimodules, i.e. if there exists a bimodule morphism  $s : A \rightarrow A \otimes A$  such that  $m \circ s = 1_A$ .*
- ii) *A module category  ${}_{\mathcal{D}}\mathcal{M}$  over  $\mathcal{D}$  is called separable if there exists a separable algebra  $A \in \mathcal{D}$  such that  ${}_{\mathcal{D}}\mathcal{M} \simeq \mathbf{Mod}(\mathcal{D})_A$  as module categories.*
- iii) *A bimodule category  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$  is called separable if it is separable as  $\mathcal{D}$ -left and also as  $\mathcal{C}$ -right module category.*

**Lemma 4.25** *If  ${}_{\mathcal{D}}\mathcal{M}$  is a separable module category then it is also an exact module category.*

Proof. First we choose a separable algebra  $A$  and an equivalence  ${}_{\mathcal{D}}\mathcal{M} \simeq \mathbf{Mod}(\mathcal{D})_A$  as module categories. To show that  $\mathbf{Mod}(\mathcal{D})_A$  is exact we need to show that for all projective  $P \in \mathcal{D}$  and all  $m \in \mathbf{Mod}(\mathcal{D})_A$ , the object  $P \otimes m$  is projective in  $\mathbf{Mod}(\mathcal{D})_A$ . Clearly  $P \otimes m$  is projective in  $\mathcal{D}$ . Now consider the following morphism in  $\mathbf{Mod}(\mathcal{D})_A$ .

$$P \otimes m \simeq P \otimes m \otimes_A A \xrightarrow{1 \otimes \eta^s} P \otimes m \otimes_A (A \otimes A) \simeq P \otimes m \otimes A. \quad (4.50)$$

This establishes  $P \otimes m$  as retract of  $P \otimes m \otimes A$ . Since the object  $P \otimes m \otimes A$  is projective in  $\mathbf{Mod}(\mathcal{D})_A$ , it follows that also  $P \otimes m$  is projective in  $\mathbf{Mod}(\mathcal{D})_A$ .  $\square$

**Proposition 4.26** *Let  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$  and  ${}_{\mathcal{E}}\mathcal{N}_{\mathcal{D}}$  be separable bimodule categories. Then the  $(\mathcal{D}, \mathcal{E})$ -bimodule category  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}} \square {}_{\mathcal{E}}\mathcal{N}_{\mathcal{D}}$  is separable.*

Proof. This is shown in [8, Thm. 3.5.5] in the case that the tensor categories  $\mathcal{D}, \mathcal{C}$  and  $\mathcal{E}$  are semisimple. The proof does not require this assumption and thus the result follows in the general case in exactly the same way.  $\square$

**Theorem 4.27** *i) The following defines a tricategory  $\mathbf{Bimod}^{sep}$ . Objects are finite tensor categories, 1-morphisms are separable bimodule categories, 2-morphisms are bimodule functors, 3-morphisms are bimodule natural transformations. The tricategory structures are induced from  $\mathbf{Bimod}$ .*

*ii)  $\mathbf{Bimod}^{sep}$  is a tricategory with duals, where the duals of a 1-morphism  ${}_D\mathcal{M}_e$  are the bimodule categories  ${}_e\mathcal{M}_D^\#$  and  ${}^\#_e\mathcal{M}_D$ . The duals of bimodule functors are the left and right adjoint functors.*

*Proof.* We have to show that the tricategory structures of  $\mathbf{Bimod}$  are well defined on  $\mathbf{Bimod}^{sep}$ . The unit bimodule category  ${}_e\mathcal{C}_e$  is a separable bimodule category, since we can choose the tensor unit as separable algebra for the left and right module structures. According to Proposition 4.26, the tensor product is well defined for  $\mathbf{Bimod}^{sep}$ . Hence Theorem 3.21 implies that  $\mathbf{Bimod}^{sep}$  is a tricategory. For the second part note that if  $A$  is a separable algebra in a finite tensor category  $\mathcal{D}$ , then also  $A^{**}$  and  ${}^{**}A$  are canonically separable algebras. According to [8, Cor. 3.4.14], if  ${}_D\mathcal{M}$  is equivalent to  $\mathbf{Mod}(\mathcal{D})_A$ , then  ${}_D\mathcal{M}^\# \simeq {}_{A^{**}}\mathbf{Mod}(\mathcal{D})$  and  ${}^\#_D\mathcal{M} \simeq {}_A\mathbf{Mod}(\mathcal{D})$  and analogous for right module categories. It thus follows that  ${}_e\mathcal{M}_D^\#$  and  ${}^\#_e\mathcal{M}_D$  are separable bimodule categories if  ${}_D\mathcal{M}_e$  is. Thus it follows from Theorem 4.23 that  $\mathbf{Bimod}^{sep}$  is a tricategory with  $\#$ -duals. From Lemma 4.25 and it follows that a separable bimodule category is biexact and hence exact according to Lemma 2.9. Thus every module functor between separable bimodule categories is exact and has a left and right adjoint.  $\square$

**The Serre equivalence** It follows from Theorem 4.23 and the general Proposition 4.4 that for a separable bimodule category  ${}_D\mathcal{M}_e$ , there is an equivalence  $S_M : {}_e\mathcal{M}_D^\# \rightarrow {}^\#_e\mathcal{M}_D$  of bimodule categories. We apply the calculus of the inner hom to construct this equivalence more explicitly.

First we consider the duals of the inner hom and their properties. By applying the operation  $\tilde{\#}$  from Equation (4.9) to the inner hom functor  ${}_D\langle -, - \rangle^M$ , we obtain balanced bimodule functors

$$\begin{aligned} ({}_D\langle -, - \rangle^M)^{\tilde{\#}} : {}_D\mathcal{M}_e \boxtimes {}_e\mathcal{M}_D^\# \ni m \boxtimes \tilde{m} &\mapsto {}_D\langle \tilde{m}, m \rangle^* \in {}_D\mathcal{D}_D, \\ \tilde{\#}(\langle -, - \rangle_e^M) : {}^\#_e\mathcal{M}_D \boxtimes {}_D\mathcal{M}_e \ni \tilde{m} \boxtimes m &\mapsto {}^*\langle m, \tilde{m} \rangle_e \in {}_e\mathcal{C}_e. \end{aligned} \quad (4.51)$$

The following is the analogue of Proposition 4.12.

**Proposition 4.28** *i) Let  $F : {}_D\mathcal{M} \rightarrow {}_D\mathcal{D}$  be an left exact module functor. There exists a module natural isomorphism*

$${}_D\langle F^l(\mathbf{1}_D), x \rangle^M \simeq {}^*F(x), \quad (4.52)$$

*for all  $x \in {}_D\mathcal{M}$ . This extends to an equivalence of bimodule functors*

$${}_D\langle F^l(-), - \rangle^M \simeq (-) \otimes^{\tilde{\#}} F : {}_D\mathcal{D} \boxtimes {}^\#_D\mathcal{M}_D \rightarrow {}_D\mathcal{D}_D. \quad (4.53)$$

*ii) Let  $F : {}_D\mathcal{M}_D \rightarrow {}_D\mathcal{D}_D$  be a left exact bimodule functor. The equivalence from equation (4.53) is an equivalence of bimodule functors  ${}_D\mathcal{D}_D \boxtimes {}^\#_D\mathcal{M}_D \rightarrow {}_D\mathcal{D}_D$ . Furthermore, there is an equivalence of balanced bimodule functors*

$$\langle -, F^l(-) \rangle_D^M \simeq F^{\tilde{\#}}(-) \otimes (-) : {}_D\mathcal{M}_D^\# \boxtimes {}_D\mathcal{D}_D \rightarrow {}_D\mathcal{D}_D. \quad (4.54)$$

*Proof.* We compute for  $d, x \in \mathcal{D}$  and  $\tilde{m} \in \mathcal{M}_D^\#$

$$\mathbf{Hom}_D({}_D\langle F^l(d), \tilde{m} \rangle^M, x) \simeq \mathbf{Hom}_M(F^l(d), x \triangleright \tilde{m}) \simeq \mathbf{Hom}_D(d \otimes^{\tilde{\#}} F(\tilde{m}), x). \quad (4.55)$$

All isomorphisms are balanced natural isomorphisms between functors and hence the chain of isomorphisms induces the claimed module natural isomorphism by the module Yoneda lemma 4.11. The second part is shown analogously.  $\square$

According to Theorem 4.27 *ii)*, the left adjoint of the evaluation functor exists for separable bimodule categories.

**Definition 4.29** Let  ${}_{\mathcal{D}}\mathcal{M}_e$  be an separable bimodule category. The  $\mathcal{D}$ -Serre bimodule functor  $S_{\mathcal{D}\mathcal{M}} : {}_e\mathcal{M}_{\mathcal{D}}^{\#} \rightarrow {}^{\#}_e\mathcal{M}_{\mathcal{D}}$  is defined as the composite

$$S_{\mathcal{D}\mathcal{M}} : {}_e\mathcal{M}_{\mathcal{D}}^{\#} \simeq \mathcal{M}^{\#} \square \mathcal{D} \xrightarrow{1 \square \text{ev}_{\mathcal{D}\mathcal{M}}^l} \mathcal{M}^{\#} \square \mathcal{M} \square \# \mathcal{M} \xrightarrow{\text{ev}_{\mathcal{M}_e} \square 1} \mathcal{C} \square \# \mathcal{M} \simeq {}^{\#}_e\mathcal{M}_{\mathcal{D}}. \quad (4.56)$$

The  $\mathcal{C}$ -Serre bimodule functor  $S_{\mathcal{M}_e} : {}^{\#}_e\mathcal{M}_{\mathcal{D}} \rightarrow {}_e\mathcal{M}_{\mathcal{D}}^{\#}$  is the composite

$$S_{\mathcal{M}_e} : {}^{\#}_e\mathcal{M}_{\mathcal{D}} \simeq \mathcal{C} \square \# \mathcal{M} \xrightarrow{\text{ev}_{\mathcal{M}_e}^l \square 1} \mathcal{M}^{\#} \square \mathcal{M} \square \# \mathcal{M} \xrightarrow{1 \square \text{ev}_{\mathcal{D}\mathcal{M}}} \mathcal{M}^{\#} \square \mathcal{D} \simeq {}_e\mathcal{M}_{\mathcal{D}}^{\#}. \quad (4.57)$$

**Proposition 4.30** *i) The Serre bimodule functors from Definition 4.29 are equivalences of bimodule categories that are defined up to unique bimodule natural isomorphisms by the following properties.*

*ii) For  $m \boxtimes \tilde{m} \in {}_{\mathcal{D}}\mathcal{M}_e \boxtimes {}_e\mathcal{M}_{\mathcal{D}}^{\#}$  there is a canonical balanced bimodule natural isomorphism*

$${}_{\mathcal{D}}\langle \tilde{m}, m \rangle^* \simeq {}_{\mathcal{D}}\langle m, S_{\mathcal{D}\mathcal{M}}(\tilde{m}) \rangle^{\mathcal{M}}, \quad (4.58)$$

*between balanced bimodule functors  ${}_{\mathcal{D}}\mathcal{M}_e \boxtimes {}_e\mathcal{M}_{\mathcal{D}}^{\#} \rightarrow {}_{\mathcal{D}}\mathcal{D}$ .*

*iii) There exists a balanced bimodule natural isomorphism*

$${}^* \langle m, \tilde{m} \rangle_e \simeq \langle S_{\mathcal{M}_e}(\tilde{m}), m \rangle_e^{\mathcal{M}} \quad (4.59)$$

*between balanced bimodule functors  ${}^{\#}_e\mathcal{M}_{\mathcal{D}} \boxtimes {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_e\mathcal{C}_e$ .*

Proof. To show that the Serre functors are equivalences, we argue that the quasi-inverse of the  $\mathcal{D}$ -Serre functor is given by the composite

$${}^{\#}_e\mathcal{M}_{\mathcal{D}} \simeq \# \mathcal{M} \square \mathcal{D} \xrightarrow{1 \square \text{coev}_{\mathcal{D}\mathcal{M}}^l} \# \mathcal{M} \square \mathcal{M} \square \# \mathcal{M} \xrightarrow{\text{coev}_{\mathcal{M}_e}^l \square 1} \mathcal{C} \square \# \mathcal{M} \simeq {}_e\mathcal{M}_{\mathcal{D}}^{\#}. \quad (4.60)$$

To see that this functor is quasi-inverse to  $S_{\mathcal{D}\mathcal{M}}$ , one uses first the coherence structure in the tricategory  $\mathbf{Bimod}$ , then twice the triangulator from Proposition 4.22. The quasi-inverse of  $S_{\mathcal{M}_e}$  is constructed similarly. The proof of the remaining statements is analogous to the proof of Proposition 4.22: To show equation (4.58), first note that the composite

$${}_{\mathcal{D}}\mathcal{M}_e \square {}_e\mathcal{M}_{\mathcal{D}}^{\#} \simeq \mathcal{M} \square \# \mathcal{M} \square \mathcal{D} \xrightarrow{1 \square \text{ev}_{\mathcal{D}\mathcal{M}}^l} (\mathcal{M} \square \# \mathcal{M})^{\#} \square \mathcal{M} \square \# \mathcal{M} \xrightarrow{\text{ev}_{\mathcal{M}_e} \square \# \mathcal{M}} \mathcal{D} \quad (4.61)$$

is equivalent to the bimodule functor  $(\text{ev}_{\mathcal{D}\mathcal{M}})^{\#}$  by Proposition 4.28. This induces an equivalence of bimodule functors  $\text{ev}_{\mathcal{D}\mathcal{M}} \circ (1 \square S_{\mathcal{M}}) \simeq (\text{ev}_{\mathcal{D}\mathcal{M}})^{\#}$ . Thus the statement follows by using the universality of the tensor product. The proof of (4.59) is analogous. By the module Yoneda-lemma, these properties characterize the Serre functors up to a unique equivalence of bimodule functors.  $\square$

## 5 Inner-product bimodule categories as pivotal tri-category

For the rest of this article we consider bimodule categories over pivotal finite tensor categories. We develop the notion of inner-product bimodule category over pivotal finite tensor categories and show that it is compatible with the tensor product and the duality operations  $\#$ . In this way we obtain a tricategory  $\mathbf{Bimod}^{\theta}$  of inner-product bimodule categories with objects pivotal finite tensor categories, 1-morphisms inner-product bimodule categories, 2- and 3-morphisms as in  $\mathbf{Bimod}$ . Furthermore the structure of inner-product bimodule categories induces pivotal structures on the categories of functors  $\mathbf{Bimod}^{\theta}$ . This implies that inner-product bimodule categories over pivotal finite tensor categories form a pivotal tricategory and thus exhibit structures that we expect from defects for oriented TFTs.

## 5.1 Inner-product bimodule categories and the tensor product

We define inner-product bimodule categories over pivotal finite tensor categories, investigate the interaction with the tensor product and show that this structure is essentially unique for indecomposable module categories if it exists. Furthermore we relate this structure to the Serre bimodule functors.

First we consider the dual bimodule categories for bimodule categories over pivotal tensor categories  $\mathcal{C}$  and  $\mathcal{D}$ . Recall that a pivotal structure on a tensor category is a monoidal natural isomorphism  $a : \text{id} \rightarrow (-)^{**}$ , see Definition A.2. For a bimodule category  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ , the pivotal structures of  $\mathcal{C}$  and  $\mathcal{D}$  define canonical equivalences of bimodule categories  $A_{\mathcal{M}} : {}^{\#}\mathcal{M}_{\mathcal{D}} \rightarrow {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}^{\#}$  and  $A_{\mathcal{M}}^{-} : {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}^{\#} \rightarrow {}^{\#}\mathcal{M}_{\mathcal{D}}$  as follows. As linear functors,  $A_{\mathcal{M}}$  and  $A_{\mathcal{M}}^{-}$  are the identities on  $\mathcal{M}^{\text{op}}$ . The module structures are constructed in the obvious way from the pivotal structures. Similarly, there are equivalences of bimodule categories  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}^{\#\#} \simeq {}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$  and  ${}^{\#\#}\mathcal{M}_{\mathcal{C}} \simeq {}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$ . Precomposing these equivalences  $A_{\mathcal{M}}$  and  $A_{\mathcal{M}}^{-}$  with the bimodule functors in equation (4.51), we obtain the following.

**Lemma 5.1** *Let  $\mathcal{C}$ ,  $\mathcal{D}$  be pivotal finite tensor categories and  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$  a bimodule category. The functors*

$$\begin{aligned} {}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}} \boxtimes {}^{\#}\mathcal{M}_{\mathcal{D}} \ni m \boxtimes \tilde{m} &\mapsto {}_{\mathcal{D}}\langle \tilde{m}, m \rangle^* \in \mathcal{D}, \\ {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}^{\#} \boxtimes {}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}} \ni \tilde{m} \boxtimes m &\mapsto {}^*\langle m, \tilde{m} \rangle_{\mathcal{C}} \in \mathcal{C}, \end{aligned} \quad (5.1)$$

are balanced bimodule functors with respect to the pivotal structures of  $\mathcal{C}$  and  $\mathcal{D}$ .

**Definition 5.2** *Let  $\mathcal{C}$ ,  $\mathcal{D}$  be pivotal finite tensor categories.*

- i) *An inner-product module category over  $\mathcal{D}$  is a module category  ${}_{\mathcal{D}}\mathcal{M}$  together with a bimodule natural isomorphism*

$$I_{m, \tilde{m}}^{\mathcal{M}} : {}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}} \simeq {}_{\mathcal{D}}\langle \tilde{m}, m \rangle^*, \quad (5.2)$$

of bimodule functors  ${}_{\mathcal{D}}\mathcal{M} \boxtimes {}^{\#}\mathcal{M}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}$ .

- ii) *A  $\mathcal{D}$ -inner-product bimodule category is a bimodule category  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$  together with a  $\mathcal{C}$ -balanced  $\mathcal{D}$ -bimodule natural isomorphism*

$$I_{m, \tilde{m}}^{\mathcal{D}\mathcal{M}} : {}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}} \simeq {}_{\mathcal{D}}\langle \tilde{m}, m \rangle^* \quad (5.3)$$

of  $\mathcal{C}$ -balanced  $\mathcal{D}$ -bimodule functors  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}} \boxtimes {}^{\#}\mathcal{M}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}$ .

- iii) *A  $\mathcal{C}$ -inner-product bimodule category is a bimodule category  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$  together with a  $\mathcal{D}$ -balanced  $\mathcal{C}$ -bimodule natural isomorphism*

$$I_{\tilde{m}, m}^{\mathcal{M}_{\mathcal{C}}} : \langle \tilde{m}, m \rangle_{\mathcal{C}}^{\mathcal{M}} \simeq {}^*\langle m, \tilde{m} \rangle_{\mathcal{C}}, \quad (5.4)$$

of  $\mathcal{D}$ -balanced  $\mathcal{C}$ -bimodule functors  ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}^{\#} \boxtimes {}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}} \rightarrow {}_{\mathcal{C}}\mathcal{C}$ .

- iv) *An inner-product  $(\mathcal{D}, \mathcal{C})$ -bimodule category is a bimodule category  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$  together with the structures of a  $\mathcal{D}$ - and  $\mathcal{C}$ -inner-product bimodule category.*

**Remark 5.3** Let  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$  be a separable bimodule category.

- i) By the module Yoneda-Lemma 4.11 it follows that the structure of a  $\mathcal{D}$ -inner-product bimodule category on  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$  is the same as a bimodule natural isomorphism from  $A_{\mathcal{M}} \circ S_{\mathcal{D}, \mathcal{M}} : {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}^{\#} \rightarrow {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}^{\#}$  to the identity functor on  ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}^{\#}$ . Similarly, the structure of a  $\mathcal{C}$ -inner-product bimodule category on  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$  is the same as a bimodule natural isomorphism from the bimodule functor  $A_{\mathcal{M}} \circ S_{\mathcal{M}, \mathcal{C}}$  to the identity on  ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}^{\#}$ .
- ii) It is clear that for an inner-product module category  ${}_{\mathcal{D}}\mathcal{M}$  the inner hom functor is exact. In the proof of [14, Prop. 3.16] it is shown that a module category is exact if and only if the inner hom functor is exact. Thus an inner-product module category is necessarily exact.

It follows directly from the definition, that by passing to the Grothendieck group, an inner-product bimodule category defines an inner-product bimodule over the Grothendieck ring of the tensor categories in the sense of Definition 1.1.

**Example 5.4** *i)* Let  $\mathcal{C}$  be a pivotal finite tensor category with pivotal structure  $a : \text{id}_e \rightarrow (-)^{**}$ . Then the bimodule category  ${}_e\mathcal{C}_e$  has the structure of an inner-product bimodule category induced by the pivotal structure. Indeed, for  $\tilde{x}, x \in \mathcal{C}$  the left inner-product structure is defined by

$${}_e\langle \tilde{x}, x \rangle^* = x \otimes \tilde{x}^* \stackrel{1 \otimes^* a_{\tilde{x}}}{\simeq} x \otimes {}^*\tilde{x} = {}_e\langle x, \tilde{x} \rangle,$$

while the right inner-product structure is induced by

$${}^*\langle x, \tilde{x} \rangle_e = {}^*\tilde{x} \otimes x \stackrel{a_{\tilde{x}} \otimes 1}{\simeq} \tilde{x}^* \otimes x = \langle \tilde{x}, x \rangle_e.$$

Moreover, if  $a$  and  $b$  are two pivotal structures for  $\mathcal{C}$ , it is easy to see that  $({}_{(e,a)}\mathcal{C}_{(e,b)})$  has a structure of and inner-product bimodule category if and only if  $a = b$ .

*ii)* Let  ${}_D\mathcal{M}_e$  be an inner-product bimodule category. Then the dual categories  ${}^{\#}_e\mathcal{M}_D$  and  ${}_e\mathcal{M}_D^{\#}$  have a natural structure of inner-product bimodule categories: It is shown in Lemma 4.14 that the  $\mathcal{C}$ -valued inner hom of  ${}_e\mathcal{M}_D^{\#}$  and the  $\mathcal{D}$ -valued inner hom of  ${}^{\#}_e\mathcal{M}_D$  are given by inner homs of  ${}_D\mathcal{M}_e$ . Using the pivotal structure of  $\mathcal{D}$ , every bimodule functor  $F : {}_D\mathcal{M}_D \rightarrow {}_D\mathcal{D}_D$  is equivalent to the composite

$${}_D\mathcal{M}_D \simeq {}_D\mathcal{M}_D^{\#\#} \stackrel{F^{\#\#}}{\simeq} {}_D\mathcal{D}_D. \quad (5.5)$$

It thus follows from Lemma 4.14 that the  $\mathcal{D}$ -valued inner hom of  ${}_e\mathcal{M}_D^{\#}$  is equivalent as balanced bimodule functor to the composite

$${}_D\mathcal{M}_e^{\#\#} \boxtimes {}_e\mathcal{M}_D^{\#} \simeq {}_D\mathcal{M}_e \boxtimes {}^{\#}_e\mathcal{M}_D \stackrel{{}_D\langle \cdot, \cdot \rangle^{\mathcal{M}}}{\longrightarrow} {}_D\mathcal{D}_D. \quad (5.6)$$

Analogously, the  $\mathcal{C}$ -valued inner hom of  ${}^{\#}_e\mathcal{M}_D$  can be expressed by the  $\mathcal{C}$ -valued inner hom of  ${}_D\mathcal{M}_e$ .

Thus it follows that the inner-product bimodule category structure of  ${}_D\mathcal{M}_e$  induces the structures of inner-product bimodule categories on  ${}^{\#}_e\mathcal{M}_D$  and  ${}_e\mathcal{M}_D^{\#}$ .

The following shows that the structure of an inner-product (bi)-module category is essentially unique if it exists.

**Proposition 5.5** *Let  ${}_D\mathcal{M}$  be an indecomposable exact module category over a pivotal finite tensor category  $\mathcal{D}$  with and inner-product module structure  $I$ . For any other balanced natural isomorphism  $I' : {}_D\langle m, \tilde{m} \rangle^{\mathcal{M}} \simeq {}_D\langle \tilde{m}, m \rangle^*$ , there exists a scalar  $\lambda \in \mathbb{k}^\times$  such that  $I' = \lambda \cdot I$ .*

*Proof.* Assume  $I'$  provides another inner-product module structure on  ${}_D\mathcal{M}$ . Then the natural isomorphism

$${}_D\langle m, \tilde{m} \rangle^{\mathcal{M}} \stackrel{I'}{\simeq} {}_D\langle \tilde{m}, m \rangle^* \stackrel{I^{-1}}{\simeq} {}_D\langle m, \tilde{m} \rangle^{\mathcal{M}}, \quad (5.7)$$

is balanced and defines by Lemma 4.11 a module natural isomorphism  $1_{\mathcal{M}} \rightarrow 1_{\mathcal{M}}$ , which is an endomorphism of the tensor unit in the category  $\mathcal{D}_{\mathcal{M}}^* = \text{Fun}_D({}_D\mathcal{M}, {}_D\mathcal{M})$ . According to [14, Lemma 3.24], this category is a finite tensor category. In particular the unit object is absolutely simple. Hence the module natural isomorphism  $1_{\mathcal{M}} \rightarrow 1_{\mathcal{M}}$  is a multiple of the identity and thus the statement follows.  $\square$

For indecomposable bimodule categories, the structure of an inner-product bimodule category is thus unique up to simultaneous scaling of both balanced natural isomorphisms with independent scalars.

Next we consider the compatibility of inner-product bimodule categories with the tensor product.

**Proposition 5.6** *Let  $\mathcal{C}, \mathcal{D}, \mathcal{E}$  be pivotal finite tensor categories and  ${}_D\mathcal{M}_e, {}_e\mathcal{N}_{\mathcal{E}}$  inner-product bimodule categories. There is an induced structure of an inner-product bimodule category on  ${}_D\mathcal{M}_e \square_e \mathcal{N}_{\mathcal{E}}$ .*

Proof. In order to show that  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}} \square_{\mathcal{E}} \mathcal{N}_{\mathcal{E}}$  has the structure of an  $\mathcal{D}$ -inner-product bimodule category, we construct for  $m, \tilde{m} \in \mathcal{M}$  and  $n, \tilde{n} \in \mathcal{N}$  a multi-balanced  $\mathcal{D}$ -bimodule natural isomorphism

$$({}_{\mathcal{D}}\langle B(\tilde{m} \boxtimes \tilde{n}), B(m \boxtimes n) \rangle^{\mathcal{M} \square \mathcal{N}})^* \simeq {}_{\mathcal{D}}\langle B(m \boxtimes n), B(\tilde{m} \boxtimes \tilde{n}) \rangle^{\mathcal{M} \square \mathcal{N}}$$

of multi-balanced bimodule functors  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}} \boxtimes_{\mathcal{E}} \mathcal{N}_{\mathcal{E}} \boxtimes_{\mathcal{E}}^{\#} \mathcal{N}_{\mathcal{E}} \boxtimes_{\mathcal{E}}^{\#} \mathcal{M}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$ . Note that both functors are  $\mathcal{C}$ -balanced in two arguments and  $\mathcal{E}$ -balanced in the middle argument. By universality of the tensor product, such a multi-balanced isomorphism induces the structure of a  $\mathcal{D}$ -inner-product bimodule category on  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}} \square_{\mathcal{E}} \mathcal{N}_{\mathcal{E}}$ . The claimed isomorphism is defined as the composite

$$\begin{aligned} {}_{\mathcal{D}}\langle B(\tilde{m} \boxtimes \tilde{n}), B(m \boxtimes n) \rangle^* &\simeq {}_{\mathcal{D}}\langle \tilde{m} \triangleleft_{\mathcal{E}} \langle \tilde{n}, n \rangle^{\mathcal{N}}, m \rangle^* \\ &\stackrel{I^{\mathcal{E}\mathcal{M}}}{\simeq} {}_{\mathcal{D}}\langle m, \tilde{m} \triangleleft_{\mathcal{E}} \langle \tilde{n}, n \rangle^{\mathcal{N}} \rangle^{\mathcal{M}} \\ &\simeq {}_{\mathcal{D}}\langle m \triangleleft^* \langle \tilde{n}, n \rangle_{\mathcal{E}}, \tilde{m} \rangle^{\mathcal{M}} \tag{5.8} \\ &\stackrel{I^{\mathcal{E}\mathcal{N}}}{\simeq} {}_{\mathcal{D}}\langle m \triangleleft_{\mathcal{E}} \langle n, \tilde{n} \rangle^{\mathcal{N}}, \tilde{m} \rangle^{\mathcal{M}} \\ &\simeq {}_{\mathcal{D}}\langle B(m \boxtimes n), B(\tilde{m} \boxtimes \tilde{n}) \rangle^{\mathcal{M} \square \mathcal{N}}. \end{aligned}$$

By definition, the isomorphisms in step two and four are multi-balanced bimodule natural isomorphism. The first and the last isomorphism are multi-balanced bimodule isomorphism obtained from Proposition 4.15 *ii*). The remaining isomorphism in the third step is induced by the duality of  $\mathcal{C}$  and clearly is a multi-balanced bimodule isomorphism.

The proof that  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}} \square_{\mathcal{E}} \mathcal{N}_{\mathcal{E}}$  is an  $\mathcal{E}$ -inner-product bimodule category is analogous.  $\square$

## 5.2 The pivotal tricategory of inner-product bimodule categories

We finally prove that inner-product bimodule categories yield a pivotal tricategory with duals. To this end we first show that there is a natural way to identify a bimodule functor between inner-product bimodule categories with its double left adjoint bimodule functor. These natural bimodule isomorphisms are then shown to constitute a pivotal structure on the following 2-categories.

**Definition 5.7** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be pivotal finite tensor categories. The 2-categories of  $(\mathcal{C}, \mathcal{D})$  inner-product bimodule categories together with bimodule functors and bimodule natural transformations are denoted  $\mathbf{Bimod}^{\theta}(\mathcal{C}, \mathcal{D})$ .*

A pivotal structure on the 2-categories  $\mathbf{Bimod}^{\theta}(\mathcal{C}, \mathcal{D})$  is constructed by generalizing [30, Thm. 4.5] in the semisimple case.

**Theorem 5.8** *Let  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}, {}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}}, {}_{\mathcal{D}}\mathcal{Y}_{\mathcal{E}} \in \mathbf{Bimod}^{\theta}$ . For all module functors  $F : {}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}} \rightarrow {}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}}$ , the  $\mathcal{D}$ -inner product structures on  $\mathcal{M}$  and  $\mathcal{N}$  induce a bimodule natural isomorphism  $a_F : F \rightarrow F^{ll}$  from  $F$  to the double left adjoint module functor of  $F$ .*

- i) The natural isomorphisms  $a_F$  are natural with respect to bimodule natural transformations, i.e. for any bimodule functor  $G : {}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}} \rightarrow {}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}}$  and any bimodule natural transformation  $\rho : F \rightarrow G$ , the following diagram commutes*

$$\begin{array}{ccc} F & \xrightarrow{a_F} & F^{ll} \\ \rho \downarrow & & \downarrow \rho^{ll} \\ G & \xrightarrow{a_G} & G^{ll} \end{array}$$

- ii) For all bimodule functors  $F : {}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}} \rightarrow {}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}}$  and  $K : {}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}} \rightarrow {}_{\mathcal{D}}\mathcal{Y}_{\mathcal{E}}$ ,*

$$a_{KF} = a_K \circ a_F : K \circ F \rightarrow (K \circ F)^{ll}. \tag{5.9}$$

- iii) For the identity bimodule functor  $1_M : {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_{\mathcal{D}}\mathcal{M}_e$ , the natural isomorphism is given by  $a_{1_M} = \text{id}_{1_M}$ .
- iv) The 2-categories  $\text{Bimod}^\theta(\mathcal{C}, \mathcal{D})$  equipped with these natural isomorphisms are pivotal 2-categories.

Proof. For a bimodule functor  $F : {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_{\mathcal{D}}\mathcal{N}_e$ , consider the following composite of balanced bimodule natural isomorphism of functors  ${}_{\mathcal{D}}\mathcal{M}_e \boxtimes_e^{\#} \mathcal{M}_{\mathcal{D}} \rightarrow {}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$ :

$${}_{\mathcal{D}}\langle F(m), n \rangle^{\mathcal{N}} \xrightarrow{I^{\mathcal{D}^{\mathcal{N}}}} {}_{\mathcal{D}}\langle n, F(m) \rangle^* \simeq {}_{\mathcal{D}}\langle F^l(n), m \rangle^* \xrightarrow{(I^{\mathcal{D}^{\mathcal{M}}})^{-1}} {}_{\mathcal{D}}\langle m, F^l(n) \rangle^{\mathcal{M}} \simeq {}_{\mathcal{D}}\langle F^{ll}(m), n \rangle^{\mathcal{N}}. \quad (5.10)$$

Here the first and third natural isomorphisms are obtained from Lemma 4.13. By Lemma 4.11 ii), the composite induces a bimodule natural isomorphism  $a_F : F \rightarrow F^{ll}$ . Let  $\rho : F \rightarrow G$  be a bimodule natural isomorphism between bimodule functors from  ${}_{\mathcal{D}}\mathcal{M}_e$  to  ${}_{\mathcal{D}}\mathcal{N}_e$  in  $\text{Bimod}^\theta(\mathcal{C}, \mathcal{D})$ . Then consider the following diagram:

$$\begin{array}{ccc}
 \langle Fm, n \rangle & \xrightarrow{\langle \rho m, n \rangle} & \langle Gm, n \rangle \\
 \downarrow I^{\mathcal{N}} & & \downarrow I^{\mathcal{N}} \\
 \langle n, Fm \rangle^* & \xrightarrow{\langle n, \rho m \rangle^*} & \langle n, Gm \rangle^* \\
 \downarrow \simeq & & \downarrow \simeq \\
 \langle F^l n, m \rangle^* & \xrightarrow{\langle \rho^l m, n \rangle^*} & \langle G^l n, m \rangle^* \\
 \downarrow (I^{\mathcal{M}})^{-1} & & \downarrow (I^{\mathcal{M}})^{-1} \\
 \langle m, F^l n \rangle & \xrightarrow{\langle n, \rho^l m \rangle} & \langle m, G^l n \rangle \\
 \downarrow \simeq & & \downarrow \simeq \\
 \langle F^{ll} m, n \rangle & \xrightarrow{\langle \rho^{ll} m, n \rangle} & \langle G^{ll} m, n \rangle.
 \end{array} \quad (5.11)$$

$\langle a_F, 1 \rangle$  (left curved arrow),  $\langle a_G, 1 \rangle$  (right curved arrow)

It is straightforward to see that all small diagrams in this diagram commute. Thus the outer diagram commutes as well which shows part i). For the second part consider additionally a bimodule functor  $K : {}_{\mathcal{D}}\mathcal{N}_e \rightarrow {}_{\mathcal{D}}\mathcal{Y}_e$  in  $\text{Bimod}^\theta(\mathcal{C}, \mathcal{D})$ . It is enough to prove that the following diagram commutes:

$$\begin{array}{ccc}
 \langle K F m, y \rangle & & \\
 \downarrow I^{\mathcal{Y}} & & \\
 \langle y, K F m \rangle^* & & \\
 \downarrow \simeq & & \\
 \langle K^l y, F m \rangle^* & \xrightarrow{\simeq} \langle F^l K^l y, m \rangle^* \xrightarrow{(I^{\mathcal{M}})^{-1}} \langle m, F^l K^l y \rangle \xrightarrow{\simeq} \langle K^{ll} F^{ll} m, y \rangle & \\
 \downarrow (I^{\mathcal{N}})^{-1} & & \\
 \langle F m, K^l y \rangle & & \\
 \downarrow \simeq & & \\
 \langle K^{ll} F m, y \rangle & & 
 \end{array} \quad (5.12)$$

$\langle a_{KF}, 1 \rangle$  (top curved arrow),  $\langle K^{ll} a_F, 1 \rangle$  (bottom curved arrow)

The upper triangle and the lower subdiagram commute due to the definition of  $a_{KF}$  and  $a_K$ , respectively. The remaining diagram commutes due to the naturality of the adjunctions. The



part *iv*) is just the collection of the previous statements according to Definition A.12 of a pivotal 2-category.  $\square$

**Remark 5.9** If we consider  ${}_e\mathcal{C}$  as an inner-product module category according to Example 5.4 *i*), the induced pivotal structure on  $\text{Fun}_e({}_e\mathcal{C}, {}_e\mathcal{C}) \simeq \mathcal{C}$  coincides with the pivotal structure of  $\mathcal{C}$ .

By replacing the  $\mathcal{D}$ -valued with the  $\mathcal{C}$ -valued inner product, the analogue of equation (5.10) defines another pivotal structure on  $\text{Bimod}^\theta(\mathcal{C}, \mathcal{D})$  that will be in general different than the one considered, in particular since we can scale each of the two isomorphisms  $I^{\mathcal{D}\mathcal{M}}$  and  $I^{\mathcal{M}e}$  independently.

The following statement follows directly from Theorem 5.8 and the explicit description of the induced pivotal structure in equation (5.10).

**Corollary 5.10** *Let  ${}_{\mathcal{D}}\mathcal{M}$  be an inner-product module category over a pivotal finite tensor category. By Theorem 5.8, the (multi-)tensor category  $\mathcal{D}_{\mathcal{M}}^*$  acquires a pivotal structure. With respect to this pivotal structure, the natural isomorphism  $I^{\mathcal{M}}$  from  ${}_{\mathcal{D}}\mathcal{M}$  is  $(\mathcal{D}, \mathcal{D}_{\mathcal{M}}^*)$ -balanced.*

Next we consider the compatibility of these pivotal structures with the tensor product of inner-product bimodule categories.

**Lemma 5.11** *Let  $F : {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_{\mathcal{D}}\mathcal{M}'_e$  and  $G : {}_e\mathcal{N}_e \rightarrow {}_e\mathcal{N}'_e$  be bimodule functors. The left inner hom induces a bimodule natural isomorphism  $\xi_{F,G} : (F \square G)^l \rightarrow F^l \square G^l$ .*

Proof. Consider the following composite of balanced bimodule natural isomorphisms

$$\begin{aligned}
{}_{\mathcal{D}}\langle (F \square G)^l(m' \square n'), m \square n \rangle^{\mathcal{M} \square \mathcal{N}} &\simeq {}_{\mathcal{D}}\langle m' \square n', F(m) \square G(n) \rangle^{\mathcal{M}' \square \mathcal{N}'} \\
&\simeq {}_{\mathcal{D}}\langle m' \triangleleft {}_{\mathcal{D}}\langle n', G(n) \rangle^{\mathcal{N}'}, F(m) \rangle^{\mathcal{M}'} \\
&\simeq {}_{\mathcal{D}}\langle F^l(m') \triangleleft {}_{\mathcal{D}}\langle G^l(n'), n \rangle^{\mathcal{N}}, m \rangle^{\mathcal{M}} \\
&\simeq {}_{\mathcal{D}}\langle (F^l \square G^l)(m' \square n'), m \square n \rangle^{\mathcal{M} \square \mathcal{N}}.
\end{aligned} \tag{5.13}$$

By Lemma 4.11 this induces the claimed bimodule natural isomorphism.  $\square$

**Proposition 5.12** *Let  $F : {}_{\mathcal{D}}\mathcal{M}_e \rightarrow {}_{\mathcal{D}}\mathcal{M}'_e$  and  $G : {}_e\mathcal{N}_e \rightarrow {}_e\mathcal{N}'_e$  be bimodule functors between inner-product bimodule categories. The following diagram of bimodule natural isomorphisms commutes*

$$\begin{array}{ccc}
F \square G & \xrightarrow{a_{F \square G}} & (F \square G)^l \\
a_F \square a_G \downarrow & & \uparrow (\xi_{F,G})^l \\
F^l \square G^l & \xleftarrow{\xi_{F^l, G^l}} & (F^l \square G^l)^l.
\end{array} \tag{5.14}$$

Proof. It is sufficient to show that the corresponding diagram inside the inner homs commutes:

$$\begin{array}{ccc}
{}_{\mathcal{D}}\langle F \square G(m \square n), m' \square n' \rangle^{\mathcal{M}' \square \mathcal{N}'} & \xrightarrow{a_{F \square G}} & {}_{\mathcal{D}}\langle (F \square G)^l(m \square n), m' \square n' \rangle^{\mathcal{M}' \square \mathcal{N}'} \\
a_F \square a_G \downarrow & & \uparrow (\xi_{F,G})^l \\
{}_{\mathcal{D}}\langle F^l \square G^l(m \square n), m' \square n' \rangle^{\mathcal{M}' \square \mathcal{N}'} & \xleftarrow{\xi_{F^l, G^l}} & {}_{\mathcal{D}}\langle (F^l \square G^l)^l(m \square n), m' \square n' \rangle^{\mathcal{M}' \square \mathcal{N}'} .
\end{array} \tag{5.15}$$

This follows from a lengthy but straightforward computation using the definitions of the pivotal structure in Theorem 5.8 and Lemma 5.11 as well as Proposition 4.15 and the Definition of the inner-product structure on  $\mathcal{M} \square \mathcal{N}$  by Proposition 5.6.  $\square$

Combining the results of the previous subsections together we obtain the following.

**Theorem 5.13** *Finite pivotal tensor categories, inner-product bimodule categories, bimodule functors and bimodule natural transformations form a tricategory  $\mathbf{Bimod}^\theta$ . With the pivotal structure induced by Theorem 5.8, the tricategory  $\mathbf{Bimod}^\theta$  is a pivotal tricategory duals according to Definition 4.5.*

Proof. It is shown in Example 5.4 *i*), that for a pivotal tensor category  $\mathcal{C}$  the unit bimodule category  ${}_e\mathcal{C}_e$  is in  $\mathbf{Bimod}^\theta$ . Furthermore, the tensor product of two inner-product bimodule categories is again an inner-product bimodule category due to Proposition 5.6. Hence  $\mathbf{Bimod}^\theta$  forms a tricategory with the tricategory structure induced from  $\mathbf{Bimod}$ .

Inner-product module categories are exact, see Remark 5.3, hence the adjoints of bimodule functors exist and provide  $*$ -duals for  $\mathbf{Bimod}^\theta$ . The pivotal structure on the bicategories  $\mathbf{Bimod}^\theta(\mathcal{C}, \mathcal{D})$  is defined in Theorem 5.8. It follows directly from Proposition 5.12 by restricting to the case  $F = \text{id}$ , that the 2-functors

$${}_e\mathcal{K}_{\mathcal{D}} \square - : \mathbf{Bimod}^\theta(\mathcal{C}, \mathcal{D}) \rightarrow \mathbf{Bimod}^\theta(\mathcal{C}, \mathcal{E})$$

are pivotal 2-functors for all finite pivotal tensor categories  $\mathcal{C}, \mathcal{D}, \mathcal{E}$  and all  ${}_e\mathcal{K}_{\mathcal{D}} \in \mathbf{Bimod}^\theta$ . Analogously it follows that  $-\square {}_e\mathcal{K}_{\mathcal{D}}$  are pivotal 2-functors. Hence  $\mathbf{Bimod}^\theta$  is a pivotal tricategory. The  $\#$ -duals in  $\mathbf{Bimod}^\theta$  are defined in Theorem 4.23. The duals of inner-product bimodule categories are again in  $\mathbf{Bimod}^\theta$  according to Example 5.4 *ii*).  $\square$

**Remark 5.14** One might also consider a generalization of inner-product bimodule categories to bimodule categories with two tensor category-valued inner products that are not necessarily given by the inner homs. If the inner products have properties analogous to the inner homs, i.e. right exact balanced bimodule functors with balanced bimodule natural isomorphisms as in (5.3), (5.4), then the Rieffel-formula (4.26) defines inner products on the same type on the tensor product of two such generalized inner-product bimodule categories. This leads to a tricategory of generalized inner-product bimodule categories. However, this is not investigated further in this article.

For finite tensor categories, the 3-groupoid that is induced from  $\mathbf{Bimod}$  is called Brauer-Picard 3-groupoid. In the semisimple case this has been investigated in [13]. As a more refined 3-groupoid for the theory of pivotal finite tensor categories we propose the following. Recall the notion of invertible bimodule category from [13, Sec.4].

**Definition 5.15** *The  $*$ -Brauer-Picard 3-groupoid  $\underline{\mathbf{BrPic}}^*$  of pivotal finite tensor categories is the following 3-groupoid. Objects are finite pivotal tensor categories, 1-morphisms invertible inner-product bimodule categories, 2-morphisms equivalences of bimodule categories and 3-morphisms bimodule natural isomorphisms.*

This defines the notion of  $*$ -Morita equivalence for pivotal finite tensor categories:  $\mathcal{C}$  and  $\mathcal{D}$  are called  $*$ -Morita equivalent if there exists an invertible inner-product bimodule category  ${}_d\mathcal{M}_e$ . This notion of equivalence allows to distinguish different pivotal structures:

**Corollary 5.16** *If  ${}_d\mathcal{M}_e$  is a 1-morphism in  $\underline{\mathbf{BrPic}}^*$ , then the pivotal structure of  $\mathcal{C}$  is uniquely determined by the pivotal structure of  $\mathcal{D}$  and the  $\mathcal{D}$ -valued inner-product module structure of  ${}_d\mathcal{M}$ .*

Proof. Note that Formula (5.10) defines a pivotal structure for  $\mathcal{C} \simeq \text{Fun}_{\mathcal{D}}({}_d\mathcal{M}, {}_d\mathcal{M})$ . Now the balancing of the isomorphism  $I^{\mathcal{D}\mathcal{M}}$  in the inner-product module structure of  ${}_d\mathcal{M}$  demands that this pivotal structure agrees with the pivotal structure of  $\mathcal{C}$ .  $\square$

### 5.3 Inner-product module categories from Frobenius algebras

We show that special symmetric Frobenius algebras in pivotal finite tensor categories provide examples of inner-product module categories.

First recall the definition of a special symmetric normalized Frobenius algebra. In the following we use the graphical calculus for tensor categories, where objects are presented by strings, tensor product is presented by juxtaposition and diagrams are read from up to down.

**Definition 5.17** Let  $\mathcal{C}$  be a pivotal tensor category.

i) A Frobenius algebra  $A \in \mathcal{C}$  is a algebra  $A$  that is also a coalgebra with

$$\text{multiplication } \begin{array}{c} \cup \\ | \end{array}, \quad \text{unit } \begin{array}{c} \bullet \\ | \\ \circ \end{array}, \quad \text{comultiplication } \begin{array}{c} | \\ \cap \end{array}, \quad \text{counit morphism } \begin{array}{c} | \\ \circ \end{array}, \quad (5.16)$$

such that

$$\begin{array}{c} \cup \\ | \\ \cap \end{array} = \begin{array}{c} \cup \\ | \\ \cup \\ | \\ \cap \end{array} = \begin{array}{c} \cup \\ | \\ \cap \\ | \\ \cup \end{array}. \quad (5.17)$$

ii) A Frobenius algebra  $A \in \mathcal{C}$  is called special if there exist  $\beta_1, \beta_A \in \mathbb{k} \setminus \{0\}$  such that

$$\begin{array}{c} \bullet \\ | \\ \circ \end{array} = \beta_1, \quad \text{and} \quad \begin{array}{c} | \\ \circ \\ | \end{array} = \beta_A. \quad (5.18)$$

A special Frobenius algebra  $A$  is called normalized if  $\beta_A = 1$ .

iii) A Frobenius algebra  $A$  is called symmetric, if

$$A^* \begin{array}{c} \cup \\ | \\ \cap \end{array} = \begin{array}{c} \cup \\ | \\ \cap \end{array} \begin{array}{c} A^* \\ \square \\ *a_A \\ *A \end{array}. \quad (5.19)$$

A special Frobenius algebra can always be normalized by an appropriate scaling of  $\Delta$  and  $\epsilon$ . If  $A$  is a special symmetric normalized Frobenius algebra, there exists a projector onto  $m \otimes_A {}^* \tilde{m}$  for all  $m, \tilde{m} \in \text{Mod}(\mathcal{C})_A$ , that has the following graphical description, where we use the obvious picture to present the module multiplication  $m \otimes A \rightarrow m$ :

$$P_{m, \tilde{m}} = m \begin{array}{c} | \\ \cup \\ | \\ \cap \end{array} \begin{array}{c} \bullet \\ | \\ \circ \end{array} \begin{array}{c} | \\ \cap \\ | \\ \cup \end{array} {}^* \tilde{m}. \quad (5.20)$$

**Lemma 5.18** Let  $A \in \mathcal{C}$  be a special symmetric normalized Frobenius algebra in a pivotal finite tensor category. Then for every  $m \in \text{Mod}(\mathcal{C})_A$ , the following two morphisms in  $\mathcal{C}$  agree.

$$\begin{array}{c} m^* \\ | \\ \cup \\ | \\ \cap \\ | \\ \cup \\ | \\ \cap \\ | \\ A \\ \square \\ *a_m \\ *m \end{array} = \begin{array}{c} m^* \\ | \\ \cup \\ | \\ \cap \\ | \\ \cup \\ | \\ \cap \\ | \\ A \\ \square \\ *a_m \\ *m \end{array}. \quad (5.21)$$

Proof. This follows from a straightforward computation using the diagrammatic calculus: First use equation (5.19) on the left hand side, then the pivotal structure to transform the resulting right dual of a morphism in  $\mathcal{C}$  to the left dual.  $\square$

Using this lemma, the following proposition is straightforward to show using the diagrammatic calculus.

**Proposition 5.19** *Let  $\mathcal{C}$  be a pivotal finite tensor category and  $A \in \mathcal{C}$  a special symmetric normalized Frobenius algebra. The following diagram of morphisms in  $\mathcal{C}$  commutes*

$$\begin{array}{ccc}
(\tilde{m} \otimes {}^*m)^* & \xrightarrow{P_{\tilde{m},m}^*} & (\tilde{m} \otimes {}^*m)^* \\
\simeq m \otimes \tilde{m}^* & & \simeq m \otimes \tilde{m}^* \\
\downarrow 1 \otimes {}^*a_{\tilde{m}} & & \downarrow 1 \otimes {}^*a_{\tilde{m}} \\
m \otimes {}^*\tilde{m} & \xrightarrow{P_{m,\tilde{m}}} & m \otimes {}^*\tilde{m}.
\end{array} \tag{5.22}$$

This result allows to construct inner-product module categories from Frobenius algebras.

**Theorem 5.20** *Let  $\mathcal{C}$  be a pivotal finite tensor category and  $A \in \mathcal{C}$  a special symmetric normalized Frobenius algebra. Then the pivotal structure of  $\mathcal{C}$  induces the structure of an inner-product module category on  ${}_e\mathcal{M} = \mathbf{Mod}(\mathcal{C})_A$ .*

Proof. According to Example 2.20, the inner hom of  ${}_e\mathcal{M} = \mathbf{Mod}(\mathcal{C})_A$  is given by  ${}_e\langle m, \tilde{m} \rangle^{\mathcal{M}} = m \otimes_A {}^*\tilde{m}$ . Thus we have to construct a natural isomorphism

$${}_e\langle \tilde{m}, m \rangle^* = (\tilde{m} \otimes_A {}^*m)^* \rightarrow m \otimes_A {}^*\tilde{m}.$$

Since  $P_{\tilde{m},m} : \tilde{m} \otimes {}^*m \rightarrow \tilde{m} \otimes_A {}^*m$  is a projector onto  $\tilde{m} \otimes_A {}^*m$ , it follows that also its right dual  $P_{\tilde{m},m}^*$  is a projector, indeed it projects onto  $(\tilde{m} \otimes_A {}^*m)^*$ . The commutativity of the diagram (5.22) shows that the pivotal structure of  $\mathcal{C}$  descends to an isomorphism

$$1 \otimes_A {}^*a_{\tilde{m}} : (\tilde{m} \otimes_A {}^*m)^* \rightarrow m \otimes_A {}^*\tilde{m}.$$

It is clear that this isomorphism is natural in  $m$  and  $\tilde{m}$ , so that we are left with showing that it is a bimodule natural isomorphism. Since it induced from the identity on  $m \in \mathcal{M}$ , it is clearly a module natural isomorphism with respect to  $m$ . In the other argument it follows from the fact that the pivotal structure is a monoidal natural isomorphism, that  $1 \otimes_A {}^*a_{\tilde{m}}$  respects the module structure. This implies that  $\mathbf{Mod}(\mathcal{C})_A$  is an inner-product module category.  $\square$

Examples of special symmetric Frobenius algebras in finite tensor categories are obtained from certain coends in [31].

## 5.4 Inner-product bimodule categories in the semisimple case

We finally consider the case of semisimple bimodule categories over pivotal fusion categories. For a semisimple module category  ${}_e\mathcal{M}$  over a pivotal fusion category  $\mathcal{C}$ , a module trace is a  $\mathcal{C}$ -balanced natural isomorphism  $\mathbf{Hom}_{\mathcal{M}}(\tilde{m}, m) \simeq \mathbf{Hom}_{\mathcal{M}}(m, \tilde{m})^*$ . In [30] it is shown that there is a one to one correspondence between module categories with module traces and special symmetric Frobenius algebras in  $\mathcal{C}$ . In view of Theorem 5.20 a module trace thus provides an inner-product module category. In the sequel we clarify the relation of inner-product module categories and module categories with module traces more directly.

Let  $\mathcal{C}$  be a pivotal tensor category with pivotal structure  $a : \text{id} \rightarrow (-)^{**}$ . Then for every endomorphism  $f : c \rightarrow c$  of an object  $c \in \mathcal{C}$  there exists the left and right trace  $\text{tr}^L(f), \text{tr}^R(f) \in \text{End}(\mathbf{1}_c)$ , see Definition A.2 *ii*). To emphasis the dependence of the pivotal structure, we sometimes write  $\text{tr}^{L,a}$  and  $\text{tr}^{R,a}$ . We recall the existence of conjugate pivotal structure for pivotal fusion categories.

**Proposition 5.21** *Let  $\mathcal{C}$  be a pivotal fusion category with pivotal structure  $a : \text{id} \rightarrow (-)^{**}$ . There exists a pivotal structure  $\bar{a}$  for  $\mathcal{C}$  that is uniquely characterized by the property  $\text{tr}^{R,\bar{a}}(f) = \text{tr}^{L,a}(f)$  for all  $f : c \rightarrow c$  and all  $c \in \mathcal{C}$ .*

Proof. This follows from the existence of a canonical monoidal natural isomorphism  $\text{id} \rightarrow (-)^{****}$  for fusion categories, see [12]. Explicitly, the conjugate pivotal structure is constructed in [30, Sec. 4.3]. The property  $\text{tr}^{R,\bar{a}}(f) = \text{tr}^{L,a}(f)$  is shown in the proof of [30, Prop. 4.10].  $\square$

In particular, a pivotal fusion category is spherical if and only if the pivotal structure satisfies  $a = \bar{a}$ . If  $\mathcal{C}$  is a pivotal fusion category, we denote by  $\bar{\mathcal{C}}$  the fusion category  $\mathcal{C}$  equipped with the conjugate pivotal structure.

Let  $\mathcal{C}, \mathcal{D}$  be pivotal fusion categories,  ${}_{\mathcal{D}}\mathcal{M}_e$  a bimodule category. While the  $\mathbf{Hom}_{\mathcal{M}}$ -functor always defines a multi-balanced functor  $\mathbf{Hom} : {}^{\#}\mathcal{M}_{\mathcal{D}} \boxtimes {}_{\mathcal{D}}\mathcal{M}_e \rightarrow \mathbf{Vect}$  in the sense of Definition 4.9, see Example 4.10, a multi-balancing structure of the dual  $\mathbf{Hom}$ -functor depends on the choice of pivotal structures for  $\mathcal{C}$  and  $\mathcal{D}$ . We will consider the functor  $\mathcal{M}^{\text{op}} \boxtimes \mathcal{M} \ni \tilde{m} \boxtimes m \mapsto \mathbf{Hom}_{\mathcal{M}}(m, \tilde{m})^* \in \mathbf{Vect}$  as multi-balanced functor  ${}^{\#}\mathcal{M}_{\overline{\mathcal{D}}} \boxtimes {}_{\overline{\mathcal{D}}}\mathcal{M}_e \rightarrow \mathbf{Vect}$  and  ${}^{\#}\mathcal{M}_{\mathcal{D}} \boxtimes {}_{\mathcal{D}}\mathcal{M}_{\overline{\mathcal{C}}} \rightarrow \mathbf{Vect}$ .

For the unit bimodule category  ${}_e\mathcal{C}_e$  we obtain the following interpretation of the left and right traces in  $\mathcal{C}$ .

**Lemma 5.22** *Let  $\mathcal{C}$  be a pivotal fusion category.*

i) *The left trace  $\text{tr}^{L,a}$  defines a multi-balanced natural isomorphism*

$$\text{tr}^{L,a} : \mathbf{Hom}_e(\tilde{c}, c) \simeq \mathbf{Hom}_e(c, \tilde{c})^*, \quad (5.23)$$

*between multi-balanced functors  ${}^{\#}\mathcal{C}_e \boxtimes {}_e\mathcal{C}_{\overline{e}} \rightarrow \mathbf{Vect}$ .*

ii) *The right trace  $\text{tr}^{R,a}$  defines a multi-balanced natural isomorphism*

$$\text{tr}^{R,a} : \mathbf{Hom}_e(\tilde{c}, c) \simeq \mathbf{Hom}_e(c, \tilde{c})^*, \quad (5.24)$$

*between multi-balanced functors  ${}^{\#}\mathcal{C}_{\overline{e}} \boxtimes {}_{\overline{e}}\mathcal{C}_e \rightarrow \mathbf{Vect}$ .*

Proof. Due to the semisimplicity of  $\mathcal{C}$ ,  $\text{tr}^{L,a}$  is an isomorphism. To show that  $\text{tr}^{L,a}$  is balanced with respect to the action of  $\overline{\mathcal{C}}$  amounts to the commutativity of the following diagram for  $\tilde{c}, c, x \in \mathcal{C}$

$$\begin{array}{ccc} \mathbf{Hom}(\tilde{c}, c \otimes x) & \xrightarrow{\text{tr}^{L,a}} & \mathbf{Hom}(c \otimes x, \tilde{c})^* \\ \downarrow \simeq & & \downarrow \simeq \\ \mathbf{Hom}(\tilde{c}, c \otimes x) & & \mathbf{Hom}(c, \tilde{c} \otimes x^*) \\ & & \downarrow \mathbf{Hom}(1, 1 \otimes {}^*\overline{a}_x) \\ \mathbf{Hom}(\tilde{c} \otimes {}^*x, c) & \xrightarrow{\text{tr}^{L,a}} & \mathbf{Hom}(c, \tilde{c} \otimes {}^*x)^* \end{array} \quad (5.25)$$

This diagram commutes due to the characterization of the conjugate pivotal structure  $\overline{a}$  in Proposition 5.21, as can be seen most easily using the diagrammatic calculus for  $\mathcal{C}$ . The balancing with respect to  $\mathcal{C}$  and the second part are shown analogously.  $\square$

Note that in the case of non-semisimple pivotal finite tensor categories, the pairings induced by the pivotal structure are degenerate [7, Prop. 5.7].

Lemma 5.22 allows us to characterize inner-product bimodule categories in the semisimple case.

**Theorem 5.23** *Let  $\mathcal{C}, \mathcal{D}$  be pivotal fusion categories and  ${}_{\mathcal{D}}\mathcal{M}_e$  a semisimple bimodule category. The structure of a inner-product bimodule category on  ${}_{\mathcal{D}}\mathcal{M}_e$  is equivalent to the collection of the following structures:*

- *A multi-balanced natural isomorphism  $\eta^L : \mathbf{Hom}_{\mathcal{M}}(m, \tilde{m}) \simeq \mathbf{Hom}_{\mathcal{M}}(\tilde{m}, m)^*$  between multi-balanced functors  ${}^{\#}\mathcal{M}_{\overline{\mathcal{D}}} \boxtimes {}_{\overline{\mathcal{D}}}\mathcal{M}_e \rightarrow \mathbf{Vect}$ .*
- *A multi-balanced natural isomorphism  $\eta^R : \mathbf{Hom}_{\mathcal{M}}(m, \tilde{m}) \simeq \mathbf{Hom}_{\mathcal{M}}(\tilde{m}, m)^*$  between multi-balanced functors  ${}^{\#}\mathcal{M}_{\mathcal{D}} \boxtimes {}_{\mathcal{D}}\mathcal{M}_{\overline{\mathcal{C}}} \rightarrow \mathbf{Vect}$ .*

Proof. Assume that  ${}_{\mathcal{D}}\mathcal{M}_e$  has the structure of a  $\mathcal{D}$ -inner-product bimodule category with balanced bimodule natural isomorphism  $I^{\mathcal{D}} : {}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}} \simeq {}_{\mathcal{D}}\langle \tilde{m}, m \rangle^*$ . Composing with  $\mathbf{Hom} : {}^{\#}\mathcal{M}_{\mathcal{D}} \boxtimes {}_{\mathcal{D}}\mathcal{M}_e \rightarrow \mathbf{Vect}$ ,  $I^{\mathcal{D}}$  induces a multi-balanced natural isomorphism

$$\mathbf{Hom}_{\mathcal{D}}({}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}}, d) \simeq \mathbf{Hom}_{\mathcal{D}}({}_{\mathcal{D}}\langle \tilde{m}, m \rangle^*, d) : {}^{\#}\mathcal{M}_e \boxtimes {}^{\#}\mathcal{M}_{\mathcal{D}} \boxtimes {}_{\mathcal{D}}\mathcal{D} \rightarrow \mathbf{Vect}.$$

Here we used Lemma 4.7 to regard  $\tilde{m} \boxtimes m \boxtimes d$  as object in  ${}^{\#}\mathcal{M}_e \boxtimes {}^{\#}\mathcal{M}_{\mathcal{D}} \boxtimes {}_{\mathcal{D}}\mathcal{D}$ . Now consider the following chain of natural isomorphisms

$$\begin{aligned} \mathbf{Hom}_{\mathcal{M}}(m, d \triangleright \tilde{m}) &\simeq \mathbf{Hom}_{\mathcal{D}}({}_{\mathcal{D}}\langle m, \tilde{m} \rangle^{\mathcal{M}}, d) \stackrel{(I^{\mathcal{D}})^{-1}}{\simeq} \mathbf{Hom}_{\mathcal{D}}({}_{\mathcal{D}}\langle \tilde{m}, m \rangle^*, d) \\ &\simeq \mathbf{Hom}_{\mathcal{D}}({}^*d, {}_{\mathcal{D}}\langle \tilde{m}, m \rangle^{\mathcal{M}}) \stackrel{\text{tr}^{L,a}}{\simeq} \mathbf{Hom}_{\mathcal{D}}({}_{\mathcal{D}}\langle \tilde{m}, m \rangle^{\mathcal{M}}, {}^*d)^* \\ &\simeq \mathbf{Hom}_{\mathcal{M}}(\tilde{m}, {}^*d \triangleright m)^* \simeq \mathbf{Hom}_{\mathcal{M}}(d \triangleright \tilde{m}, m)^*. \end{aligned} \quad (5.26)$$

We claim that this chain consists of multi-balanced natural isomorphisms between multi-balanced functors  ${}^{\#}\mathcal{M}_{\mathcal{C}} \boxtimes {}^{\#}\mathcal{M}_{\mathcal{D}} \boxtimes {}_{\mathcal{D}}\mathcal{D}_{\overline{\mathcal{D}}} \rightarrow \mathbf{Vect}$ , where all functors with dual  $\mathbf{Hom}$ -spaces use the pivotal structures for the balancing isomorphisms in all arguments as indicated by the indices. The forth natural isomorphism is induced by the left trace in  $\mathcal{D}$ , which is a multi-balanced natural isomorphism between functors  ${}^{\#}\mathcal{D}_{\mathcal{D}} \boxtimes {}_{\mathcal{D}}\mathcal{D}_{\overline{\mathcal{D}}} \rightarrow \mathbf{Vect}$  by Lemma 5.22. Composing with the balanced bimodule functor  ${}_{\mathcal{D}}\langle -, - \rangle^{\mathcal{M}}$  thus gives the required multi-balanced natural isomorphism in step four. The remaining natural isomorphisms are multi-balanced for every choice of pivotal structure for the fusion category between neighboring arguments. Hence the composition (5.26) is multi-balanced and thus induces a multi-balanced natural isomorphism  $\mathbf{Hom}_{\mathcal{M}}(m, \tilde{m}) \simeq \mathbf{Hom}_{\mathcal{M}}(\tilde{m}, m)^*$  between multi-balanced functors  ${}^{\#}\mathcal{M}_{\overline{\mathcal{D}}} \boxtimes {}_{\overline{\mathcal{D}}}\mathcal{M}_{\mathcal{C}} \rightarrow \mathbf{Vect}$ .

Conversely, such a multi-balanced natural isomorphism  $\mathbf{Hom}_{\mathcal{M}}(m, \tilde{m}) \simeq \mathbf{Hom}_{\mathcal{M}}(\tilde{m}, m)^*$  induces the structure of a  $\mathcal{D}$ -inner-product bimodule category on  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$  by applying this argument in the other direction. The equivalence of a  $\mathcal{C}$ -inner-product bimodule category structure and a multi-balanced natural isomorphism  $\eta^R$  is shown analogously using the right trace in  $\mathcal{C}$ .  $\square$

The following example, that is taken from [30, Ex. 3.13], shows that not every module category has the structure of an inner-product module category.

**Example 5.24** Let  $G$  be a finite group and  $\mathcal{C} = \mathbf{Vect}[G]$  the corresponding fusion category of  $G$ -graded vector spaces. The pivotal structures on  $\mathcal{C}$  are in bijection with group homomorphisms  $\kappa : G \rightarrow \mathbb{k}^{\times}$ . A subgroup  $H \subset G$  yields a left  $\mathcal{C}$ -module category  ${}_{\mathcal{C}}\mathcal{M} = \mathbf{Vect}[H \setminus G]$  given by action of  $G$  on the left cosets in  $H \setminus G$ . The module category  ${}_{\mathcal{C}}\mathcal{M}$  has a module trace and thus the structure of an inner-product module category if and only if  $\kappa|_H = 1$ .

For the case of spherical fusion categories there is a simpler way to obtain inner-product bimodule categories. This is of particular importance since spherical fusion category define a prominent 3-dimensional oriented topological field theory [2, 32].

**Definition 5.25** ([30]) *A bimodule trace on an semisimple  $(\mathcal{D}, \mathcal{C})$ -bimodule category over pivotal fusion categories  $\mathcal{C}$  and  $\mathcal{D}$  is a multi-balanced natural isomorphism*

$$\eta_{\tilde{m}, m} : \mathbf{Hom}_{\mathcal{M}}(\tilde{m}, m) \rightarrow \mathbf{Hom}_{\mathcal{M}}(m, \tilde{m})^*. \quad (5.27)$$

It follows that this structure is sufficient to define an inner-product bimodule category.

**Corollary 5.26** *Let  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$  be an semisimple bimodule category over spherical fusion categories  $\mathcal{C}$  and  $\mathcal{D}$  with bimodule trace. Then  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$  is canonically an inner-product bimodule category.*

*Proof.* For a spherical structure the identity  $a = \bar{a}$  for the spherical structure holds. Thus Theorem 5.23 shows that a bimodule trace on  ${}_{\mathcal{D}}\mathcal{M}_{\mathcal{C}}$  defines the structure of an inner-product bimodule category on  $\mathcal{M}$ .  $\square$

Note however that conversely an inner-product bimodule category over spherical fusion categories might yield two different bimodule traces by the correspondence in Theorem 5.23.

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## A Duals in Bicategories

In this section we summarize our conventions regarding duals in monoidal categories and in bicategories and state some basic results about bicategories. In the next subsection we recall the definition of a finite tensor category.

### A.1 Duals in monoidal categories

In the following definition we assume for simplicity that all monoidal categories are strict.

**Definition A.1** *Let  $\mathcal{C}$  be a monoidal category.*

- i) *A right dual of an object  $x \in \mathcal{C}$  is an object  $x^* \in \mathcal{C}$ , together with morphisms  $\text{ev}_x : x^* \otimes x \rightarrow 1$  and  $\text{coev}_x : 1 \rightarrow x \otimes x^*$  satisfying the so-called snake identities*

$$(1_x \otimes \text{ev}_x) \cdot (\text{coev}_x \otimes 1_x) = 1_x, \quad (\text{A.1})$$

and

$$(\text{ev}_x \otimes 1_{x^*}) \cdot (1_{x^*} \otimes \text{coev}_x) = 1_{x^*}. \quad (\text{A.2})$$

*The morphisms  $\text{ev}_x$  and  $\text{coev}_x$  are called (right) duality morphisms of  $x$ .*

- ii) *A left dual of an object  $x \in \mathcal{C}$  is an object  ${}^*x \in \mathcal{C}$ , together with morphisms  $\text{ev}'_x : x \otimes {}^*x \rightarrow 1$  and  $\text{coev}'_x : 1 \rightarrow {}^*x \otimes x$  satisfying the identities*

$$(\text{ev}'_x \otimes 1_x) \cdot (1_x \otimes \text{coev}'_x) = 1_x, \quad (\text{A.3})$$

and

$$(1_{{}^*x} \otimes \text{ev}'_x) \cdot (\text{coev}'_x \otimes 1_{{}^*x}) = 1_{{}^*x}. \quad (\text{A.4})$$

*The morphisms  $\text{ev}'_x$  and  $\text{coev}'_x$  are called (left) duality morphisms of  $x$ .*

- iii) *A monoidal category  $\mathcal{C}$  is said to have right (left) duals if every object of  $\mathcal{C}$  has a right (left) dual object. In case every object of  $\mathcal{C}$  has both a right and a left dual object,  $\mathcal{C}$  is said to have duals and  $\mathcal{C}$  is called rigid.*

If  $\mathcal{C}$  has right duals, the duals are unique up to unique isomorphism and the double dual functor is canonically a monoidal functor  $(-)^{**} : \mathcal{C} \rightarrow \mathcal{C}$ .

**Definition A.2** *Let  $\mathcal{C}$  be a monoidal category with right duals.*

- i) *A pivotal structure  $a$  on  $\mathcal{C}$  is a monoidal natural isomorphism*

$$a : \text{id}_{\mathcal{C}} \rightarrow (-)^{**}. \quad (\text{A.5})$$

*A rigid monoidal category with pivotal structure is called a pivotal category.*

- ii) *Let  $f \in \text{Hom}_{\mathcal{C}}(c, c)$  be a morphism in a pivotal category. The right trace of  $f$  is defined as*

$$\text{tr}^R(f) = \text{ev}'_x(f \otimes {}^*a_x) \text{coev}_x \in \text{End}(1_{\mathcal{C}}) \quad (\text{A.6})$$

*and the left trace is defined as*

$$\text{tr}^L(f) = \text{ev}_x(a_{{}^*x} \otimes f) \text{coev}'_x \in \text{End}(1_{\mathcal{C}}). \quad (\text{A.7})$$

*The left dimension of an object  $x \in \mathcal{C}$  is defined as  $\dim^L = \text{tr}^L(1_x)$  and the right dimension as  $\dim^R = \text{tr}^R(1_x)$ .*

- iii) *A pivotal structure  $a$  on  $\mathcal{C}$  is called spherical if  $\text{tr}^L(f) = \text{tr}^R(f)$  for all  $f \in \text{Hom}_{\mathcal{C}}(c, c)$  and all  $c \in \mathcal{C}$ . In this case  $\mathcal{C}$  is called a spherical category.*

Recall that an abelian category is called finite if every object has finite length, it has enough projectives and there are only finitely many isomorphism classes of simple objects.

**Definition A.3** ([11]) *i) A tensor category  $\mathcal{C}$  over  $\mathbb{k}$  is a  $\mathbb{k}$ -linear abelian category with bilinear monoidal structure, finite dimensional  $\text{Hom}$ -spaces and right and left-duals for all objects, for which every object is of finite length and in which the tensor unit  $1_{\mathcal{C}}$  satisfies  $\text{End}(1_{\mathcal{C}}) = \mathbb{k}$ . A finite tensor category is a tensor category that is finite as abelian category.*

- ii) A tensor category  $\mathcal{C}$  is called a fusion category if it is finite semisimple as abelian category.
- iii) A pivotal (spherical) fusion category  $\mathcal{C}$  is a fusion category  $\mathcal{C}$  that is in addition a pivotal (spherical) category.

## A.2 Bicategories

We next recall the definitions of a bicategory, a 2-functor, a 2-natural transformation and a modification.

**Definition A.4** A bicategory  $\mathcal{B}$  consists of the following data:

- i) a collection of objects  $a, b \in \text{Obj}(\mathcal{B})$ ,
- ii) for any two objects  $a, b$  a category  $\mathcal{B}(a, b)$ , whose objects are called 1-morphisms and denoted  $F, G : a \rightarrow b$  and whose morphisms are called 2-morphisms and denoted  $\eta : F \Rightarrow G$ . The composition of 2-morphisms in  $\mathcal{B}(a, b)$  is called vertical composition,
- iii) for any three objects  $a, b, c$  a functor  $\circ : \mathcal{B}(b, c) \times \mathcal{B}(a, b) \rightarrow \mathcal{B}(a, c)$ , called horizontal composition, and for any object  $b$  a functor  $I_b : I \rightarrow \mathcal{B}(b, b)$ , where  $I$  is the unit category with one object and one morphism. The image of  $I_b$  on the object of  $I$  is called  $1_b : b \rightarrow b$  and the image on the morphism is called  $1_{1_b} : 1_b \Rightarrow 1_b$ ,
- iv) for any three 1-morphisms  $F : c \rightarrow d$ ,  $G : b \rightarrow c$  and  $H : a \rightarrow b$ , invertible 2-morphisms  $\omega_{F,G,H}^{\mathcal{B}} : (F \circ G) \circ H \Rightarrow F \circ (G \circ H)$ ,
- v) for any 1-morphism  $F : a \rightarrow b$  invertible 2-morphisms  $\lambda_F^{\mathcal{B}} : I_b \circ F \Rightarrow F$  and  $\rho_F^{\mathcal{B}} : F \circ I_a \Rightarrow F$ ,

such that the 2-morphisms  $\omega_{H,G,F}^{\mathcal{B}}, \lambda_F^{\mathcal{B}}$  and  $\rho_F^{\mathcal{B}}$  are natural in their arguments and the following diagrams commute for all 1-morphisms where these expressions are defined

$$\begin{array}{ccc}
 & ((F \circ G) \circ H) \circ K & \\
 \swarrow \omega_{F,G,H}^{\mathcal{B}} \circ 1_K & & \searrow \omega_{F \circ G, H, K}^{\mathcal{B}} \\
 (F \circ (G \circ H)) \circ K & & (F \circ G) \circ (H \circ K) \\
 \downarrow \omega_{F,G \circ H, K}^{\mathcal{B}} & & \downarrow \omega_{F,G,H \circ K}^{\mathcal{B}} \\
 F \circ ((G \circ H) \circ K) & \xrightarrow{1_F \circ \omega_{G,H,K}^{\mathcal{B}}} & F \circ (G \circ (H \circ K)),
 \end{array} \tag{A.8}$$

$$\begin{array}{ccc}
 (F \circ 1_a) \circ G & \xrightarrow{\omega_{F,1,G}^{\mathcal{B}}} & F \circ (1_a \circ G) \\
 \searrow \rho_F^{\mathcal{B}} \circ G & & \swarrow 1_F \circ \lambda_G \\
 & F \circ G. &
 \end{array} \tag{A.9}$$

A 2-category  $\mathcal{B}$  is a strict bicategory  $\mathcal{B}$ , i.e. a bicategory, in which all 2-morphisms  $\omega_{H,G,F}^{\mathcal{B}}, \lambda_F^{\mathcal{B}}$  and  $\rho_F^{\mathcal{B}}$  are identities.

The notion of equivalence of categories can be formulated in a general bicategory as follows.

**Definition A.5** Let  $\mathcal{B}$  be a bicategory.

- i) Two objects  $b, c$  in  $\mathcal{B}$  are called equivalent, if there exist 1-morphism  $F : b \rightarrow c$  and  $G : c \rightarrow b$  together with invertible 2-morphisms  $\eta : F \circ G \Rightarrow 1_c$  and  $\rho : G \circ F \Rightarrow 1_b$ .
- ii) An adjoint equivalence  $(f, g, \alpha, \beta)$  between objects  $b$  and  $c$  in  $\mathcal{B}$  consists of 1-morphisms  $f : b \rightarrow c$  and  $g : c \rightarrow b$ , together with isomorphisms  $\alpha : fg \rightarrow 1_c$  and  $\beta : 1_b \rightarrow gf$ , such that the snake identities (A.1) and (A.2) hold in the monoidal categories  $\mathcal{B}(b, b)$  and  $\mathcal{B}(c, c)$ .

**Definition A.6** A 2-functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between bicategories  $\mathcal{C}, \mathcal{D}$  is given by the following data



- i) A function  $F_0 : \text{Obj}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{D})$ .
- ii) For all objects  $a, b$  of  $\mathcal{C}$ , a functor  $F_{a,b} : \mathcal{C}_{a,b} \rightarrow \mathcal{D}_{F_0(a), F_0(b)}$ .
- iii) For all objects  $a, b, c$  of  $\mathcal{C}$  a natural isomorphism  $\Phi_{abc} : \circ (F_{b,c} \times F_{a,b}) \rightarrow F_{a,c} \circ$ . These determine, for all 1-morphisms  $H : a \rightarrow b$ ,  $G : b \rightarrow c$ , a invertible 2-morphism  $\Phi_{G,H} : F_{b,c}(G) \circ F_{a,b}(H) \rightarrow F_{a,c}(G \circ H)$ .
- iv) For all objects  $a$ , an invertible 2-morphism  $\Phi_a : 1_{F_0(a)} \rightarrow F_{a,a}(1_a)$ .

The function  $F_0$ , the functors  $F_{a,b}$  and the 2-morphisms  $\Phi_{G,H}$  and  $\Phi_a$  are required to satisfy the following consistency conditions

- v) For all 1-morphisms  $H : a \rightarrow b$ :

$$\begin{array}{ccc}
F_{a,b}(H) = F_{a,b}(H) \circ 1_{F_0(a)} & \xrightarrow{1_{F_{a,b}(H)} \circ \Phi_a} & F_{a,b}(H) \circ F_{a,a}(1_a) \\
= 1_{F_0(b)} \circ F_{a,b}(H) & \searrow \text{id} & \downarrow \Phi_{H,1_a} \\
\downarrow \Phi_b \circ 1_{F_{a,b}(H)} & & \\
F_{b,b}(1_b) \circ F_{a,b}(H) & \xrightarrow{\Phi_{1_b, H}} & F_{a,b}(H \circ 1_a) = F_{a,b}(1_b \circ H) = F_{a,b}(H)
\end{array} \quad (\text{A.10})$$

- vi) For all 1-morphisms  $H : a \rightarrow b$ ,  $G : b \rightarrow c$ ,  $K : c \rightarrow d$ , the following diagram commutes

$$\begin{array}{ccc}
F_{c,d}(K) \circ F_{b,c}(G) \circ F_{a,b}(H) & \xrightarrow{\Phi_{K,G} \circ 1} & F_{b,d}(K \circ G) \circ F_{a,b}(H) \\
\downarrow 1 \circ \Phi_{G,H} & & \downarrow \Phi_{K \circ G, H} \\
F_{c,d}(K) \circ F_{a,c}(G \circ H) & \xrightarrow{\Phi_{K,G \circ H}} & F_{a,d}(K \circ G \circ H).
\end{array} \quad (\text{A.11})$$

A 2-functor is said to have strict units if the 2-morphisms  $\Phi_a$  are all identities, and it is called strict if the 2-morphisms  $\Phi_{G,H}$  and  $\Phi_a$  are all identities. In this case, one has

$$F_{a,c}(G \circ H) = F_{b,c}(G) \circ F_{a,b}(H) \quad 1_{F_0(a)} = F_{a,a}(1_a).$$

The following notion of natural 2-transformation of 2-functors adopts the convention of [18, 20] and is sometimes also referred to as ‘oplax 2-transformation’.

**Definition A.7** i) A natural 2-transformation  $\rho : F \rightarrow G$  between 2-functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  is given by the following data:

- (a) For all objects  $a$  of  $\mathcal{C}$ , a 1-morphism  $\rho_a : F_0(a) \rightarrow G_0(a)$ .
- (b) For all objects  $a, b$  of  $\mathcal{C}$  a natural transformation

$$\rho_{a,b} : (\rho_b \circ -)F_{a,b} \rightarrow (- \circ \rho_a)G_{a,b},$$

where  $- \circ \rho_a : \mathcal{D}_{G_0(a), G_0(b)} \rightarrow \mathcal{D}_{F_0(a), G_0(b)}$  and  $\rho_b \circ - : \mathcal{D}_{F_0(a), F_0(b)} \rightarrow \mathcal{D}_{F_0(a), G_0(b)}$  denote the functors given by pre- and post-composition with  $\rho_a$  and  $\rho_b$ . These natural transformations determine for all 1-morphisms  $H : a \rightarrow b$  a 2-morphism  $\rho_H : \rho_b \circ F_{a,b}(H) \rightarrow G_{a,b}(H) \circ \rho_a$ .

The 1-morphisms  $\rho_a$  and 2-morphisms  $\rho_H$  are required to satisfy the following consistency conditions:

- (a) For all 1-morphisms  $H : a \rightarrow b$  and  $K : b \rightarrow c$  the following diagram commutes

$$\begin{array}{ccc}
\rho_c \circ F_{b,c}(K) \circ F_{a,b}(H) & \xrightarrow{\rho_{K \circ 1}} & G_{b,c}(K) \circ \rho_b \circ F_{a,b}(H) \\
\downarrow 1 \circ \Phi_{K,H} & & \downarrow 1 \circ \rho_H \\
\rho_c \circ F_{a,c}(K \circ H) & & G_{b,c}(K) \circ G_{a,b}(H) \circ \rho_a \\
\downarrow \rho_{K \circ H} & \swarrow \Psi_{K,H} \circ 1 & \\
G_{a,c}(K \circ H) \circ \rho_a & & 
\end{array}$$

(b) For all objects  $a$  of  $\mathcal{C}$  the following diagram commutes

$$\begin{array}{ccc}
1_{G_0(a)} \circ \rho_a = \rho_a = \rho_a \circ 1_{F_0(a)} & & \\
\downarrow 1 \circ \Phi_a & \searrow \Psi_a \circ 1 & \\
\rho_a \circ F_{a,a}(1_a) & \xrightarrow{\rho_{1_a}} & G_{a,a}(1_a) \circ \rho_a.
\end{array}$$

- ii) A pseudo-natural transformation  $\rho : F \rightarrow G$  of 2-functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  is a natural 2-transformation of 2-functors in which all 2-morphisms  $\rho_H : \rho_a \circ F_{a,b}(H) \rightarrow G_{a,b}(H) \circ \rho_a$  are isomorphisms.
- iii) A pseudo-natural transformation  $\rho$  is called an equivalence if all the 1-morphisms  $\rho_a$  are equivalences in the bicategory  $\mathcal{D}$ , see Definition A.5 i).
- iv) A 1-identity natural 2-transformation  $\rho : F \rightarrow G$  between 2-functors  $F$  and  $G$  such that  $F_0(a) = G_0(a)$  for all objects  $a$  of  $\mathcal{C}$  is a natural  $F$  2-transformation  $\rho$  such that all 1-morphisms  $\rho_a$  are the identities for all objects  $a$  of  $\mathcal{C}$ .
- v) A natural 2-isomorphism is a pseudo-natural transformation which is a 1-identity natural 2-transformation.

**Definition A.8** Let  $\rho = (\rho_a, \rho_{a,b}) : F \rightarrow G$  and  $\tau = (\tau_a, \tau_{a,b}) : F \rightarrow G$  be natural 2-transformations between 2-functors  $F = (F_0, F_{a,b}, \Phi_{H,K}, \Phi_a), G = (G_0, G_{a,b}, \Psi_{H,K}, \Psi_a) : \mathcal{C} \rightarrow \mathcal{D}$ . A modification  $\Psi : \rho \Rightarrow \tau$  is a collection of 2-morphisms  $\Psi_a : \rho_a \Rightarrow \tau_a$  for every object  $a$  of  $G$  such that for all 1-morphisms  $H : a \rightarrow b$

$$\tau_H \cdot (\Psi_a \circ 1_{F_{a,b}(H)}) = (1_{G_{a,b}(H)} \circ \Psi_b) \cdot \rho_H$$

A modification is called invertible if all 2-morphisms  $\Psi_a$  are invertible.

**Lemma A.9** Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a 2-functor between bicategories  $\mathcal{A}$  and  $\mathcal{B}$ . If  $(f, g, \alpha, \beta)$  is an adjoint equivalence between two objects  $x$  and  $y$  in  $\mathcal{A}$ , then  $(F(f), F(g), F(\alpha), F(\beta))$  is an adjoint equivalence between  $F_0(x)$  and  $F_0(y)$  in  $\mathcal{B}$ .

Proof. The proof of this statement is a combination of the proof that monoidal functors respect duality and the fact that functors respect isomorphisms.  $\square$

### A.3 Duals in Bicategories

In this subsection we discuss duals and pivotal structures in bicategories.

**Definition A.10** Let  $\mathcal{X}$  be a bicategory.

- i) A right dual of a 1-morphism  $F : c \rightarrow d$  in  $\mathcal{X}$  is a 1-morphism  $F^* : d \rightarrow c$  such that there exist 2-morphisms  $\text{ev}_F : F^* \circ F \rightarrow 1_c$  and  $\text{coev}_F : 1_d \rightarrow F \circ F^*$  that satisfy the snake identities (A.1) and (A.2) with the monoidal product replaced by the horizontal composition. The 2-morphisms  $\text{ev}_F$  and  $\text{coev}_F$  are called right duality morphisms. If every 1-morphism in  $\mathcal{X}$  has a right dual then the bicategory  $\mathcal{X}$  is said to have right duals.
- ii) A left dual of a 1-morphism  $F : c \rightarrow d$  is a 1-morphism  ${}^*F : d \rightarrow c$  such that there exist 2-morphisms  $\text{ev}'_F : F \circ {}^*F \rightarrow 1_d$  and  $\text{coev}'_F : 1_c \rightarrow {}^*F \circ F$  that satisfy the snake identities (A.3) and (A.4). The 2-morphisms  $\text{ev}'_F$  and  $\text{coev}'_F$  are called left duality 2-morphisms. If every 1-morphism in  $\mathcal{X}$  has a left dual then the bicategory  $\mathcal{X}$  is said to have left duals.

As for a monoidal category, duals in a bicategory are unique up to unique isomorphism.

**Lemma A.11** Let  $F : \mathcal{X} \rightarrow \mathcal{Y}$  be a 2-functor between bicategories.

- i) For every right dual  $G^* : c \rightarrow b$  of a 1-morphism  $G : b \rightarrow c$  in  $\mathcal{X}$ ,  $F(G^*)$  is a right dual of  $F(G)$ .

- ii) Let  $\mathcal{X}$  and  $\mathcal{Y}$  be bicategories with right duals. There exists a natural 2-isomorphism  $\xi^F : (-)^* \circ F \rightarrow F \circ (-)^*$ , that is uniquely determined by

$$(1_{F(G)} \circ \xi_G^F) \cdot \text{coev}_{F(G)} = F(\text{coev}_G) \quad (\text{A.12})$$

for all 1-morphisms  $G$  in  $\mathcal{X}$ .

- iii) Let  $F : \mathcal{X} \rightarrow \mathcal{Y}$  be a biequivalence of bicategories  $\mathcal{X}$  and  $\mathcal{Y}$ . If  $\mathcal{X}$  has (right) duals, then  $\mathcal{Y}$  has (right) duals as well.

**Definition A.12** Let  $\mathcal{X}$  be a bicategory with right duals. A pivotal structure  $a$  on  $\mathcal{X}$  is a natural 2-isomorphism

$$a : \text{id}_{\mathcal{X}} \rightarrow (-)^{**}. \quad (\text{A.13})$$

**Definition A.13** A 2-functor  $F : \mathcal{X} \rightarrow \mathcal{Y}$  between pivotal bicategories is called pivotal, if the diagram

$$\begin{array}{ccc} F(H)^{**} & \xleftarrow{a_{F(H)}^{\mathcal{Y}}} & F(H) \\ (\xi_H^F)^* \uparrow & & \downarrow F(a_H^{\mathcal{X}}) \\ F(H^*)^* & \xrightarrow{\xi_{H^*}^F} & F(H^{**}) \end{array} \quad (\text{A.14})$$

commutes for all 1-morphisms  $H : a \rightarrow b$ .

## B Tricategories

The following definition is a slight modification from [20, Def. 3.1.2].

**Definition B.1** A tricategory  $\mathcal{T}$  consists of the following data

- i) A set of objects  $a, b \in \text{Obj}(\mathcal{T})$ .
- ii) For any two objects  $a, b$  a bicategory  $\mathcal{T}(a, b)$  of 1- and 2-morphisms with horizontal composition  $\circ$  and vertical composition  $\cdot$ .
- iii) For any three objects  $a, b, c$ , 2-functors

$$\square : \mathcal{T}(b, c) \times \mathcal{T}(a, b) \rightarrow \mathcal{T}(a, c), \quad (\text{B.1})$$

called  $\square$ -product of 1-morphisms.

- iv) For any object  $a$  a 2-functor  $I_a : I \rightarrow \mathcal{T}(a, a)$ , where  $I$  denotes the unit 2-category with one object  $1$ , one 1-morphism  $1_1$  and one 2-morphism  $1_{1_1}$ . The image of the functor  $I_a$  on the object of  $I$  is the 1-morphism also denoted  $I_a : a \rightarrow a$ .
- v) For any four objects  $a, b, c, d$ , an adjoint equivalence  $a : \square(\square \times 1) \Rightarrow \square(1 \times \square)$ , called associator. More precisely,  $a$  consists of a pseudo-natural transformation

$$\begin{array}{ccc} \mathcal{T}(c, d) \times \mathcal{T}(b, c) \times \mathcal{T}(a, b) & \xrightarrow{\square \times 1} & \mathcal{T}(b, d) \times \mathcal{T}(a, b) \\ \downarrow 1 \times \square & & \downarrow \square \\ \mathcal{T}(c, d) \times \mathcal{T}(a, c) & \xrightarrow{\square} & \mathcal{T}(a, d), \end{array} \quad (\text{B.2})$$

and, a pseudo-natural transformation  $a^- : \square(1 \times \square) \rightarrow \square(\square \times 1)$ , such that  $a$  and  $a^-$  form an adjoint equivalence, see Definition A.5.

- vi) For any two objects  $a, b$ , there are adjoint equivalences  $l : \square(I_b \times 1) \Rightarrow 1$  and  $r : \square(1 \times I_a) \Rightarrow 1$ , called the unit 2-morphisms,

$$\begin{array}{ccc} & \mathcal{T}(b, b) \times \mathcal{T}(a, b) & \\ I_b \times 1 \nearrow & \Downarrow \eta & \searrow \square \\ \mathcal{T}(a, b) & \xrightarrow{1} & \mathcal{T}(a, b) \end{array} \quad (\text{B.3})$$

and

$$\begin{array}{ccc}
 & \mathcal{T}(a,b) \times \mathcal{T}(a,a) & \\
 1 \times I_a \nearrow & \Downarrow r & \square \searrow \\
 \mathcal{T}(a,b) & \xrightarrow{1} & \mathcal{T}(a,b)
 \end{array} \quad (\text{B.4})$$

By definition of an adjoint equivalence,  $l$  and  $r$  are pseudo-natural transformations. Furthermore there are corresponding pseudo-natural transformations  $l^- : 1 \Rightarrow \square(I_b \times 1)$  and  $r^- : 1 \Rightarrow \square(1 \times I_a)$ .

vii) For all objects  $a, b, c$ , an invertible modification  $\mu$

$$\begin{array}{ccc}
 \mathcal{T}^2 \xrightarrow{1 \times I \times 1} \mathcal{T}^3 \xrightarrow{\square \times 1} \mathcal{T}^2 & & \mathcal{T}^2 \xrightarrow{1 \times I \times 1} \mathcal{T}^3 \xrightarrow{\square \times 1} \mathcal{T}^2 \\
 \downarrow 1 \swarrow \llcorner 1 \times \square \searrow a & \Downarrow \mu & \downarrow \square \\
 \mathcal{T}^2 & \xrightarrow{\square} & \mathcal{T}
 \end{array} \quad (\text{B.5})$$

where we used for example the abbreviation  $\mathcal{T}^3 = \mathcal{T}(b, c) \times \mathcal{T}(b, b) \times \mathcal{T}(a, b)$ .

viii) For all objects  $a, b, c$ , an invertible modification  $\lambda$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \mathcal{T}^3 & \\
 I \times 1 \times 1 \nearrow & \Downarrow l \times 1 & \square \times 1 \searrow \\
 \mathcal{T}^2 & \xrightarrow{1} & \mathcal{T}^2 \\
 \downarrow \square & & \downarrow \square \\
 \mathcal{T} & \xrightarrow{1} & \mathcal{T}
 \end{array} & \xrightarrow{\lambda} & \begin{array}{ccc}
 & \mathcal{T}^3 & \\
 I \times 1 \times 1 \nearrow & \downarrow 1 \times \square & \square \times 1 \searrow \\
 \mathcal{T}^2 & \xrightarrow{1} & \mathcal{T}^2 \\
 \downarrow I \times 1 & & \downarrow l \\
 \mathcal{T} & \xrightarrow{1} & \mathcal{T}
 \end{array}
 \end{array} \quad (\text{B.6})$$

ix) For all objects  $a, b, c$ , an invertible modification  $\rho$

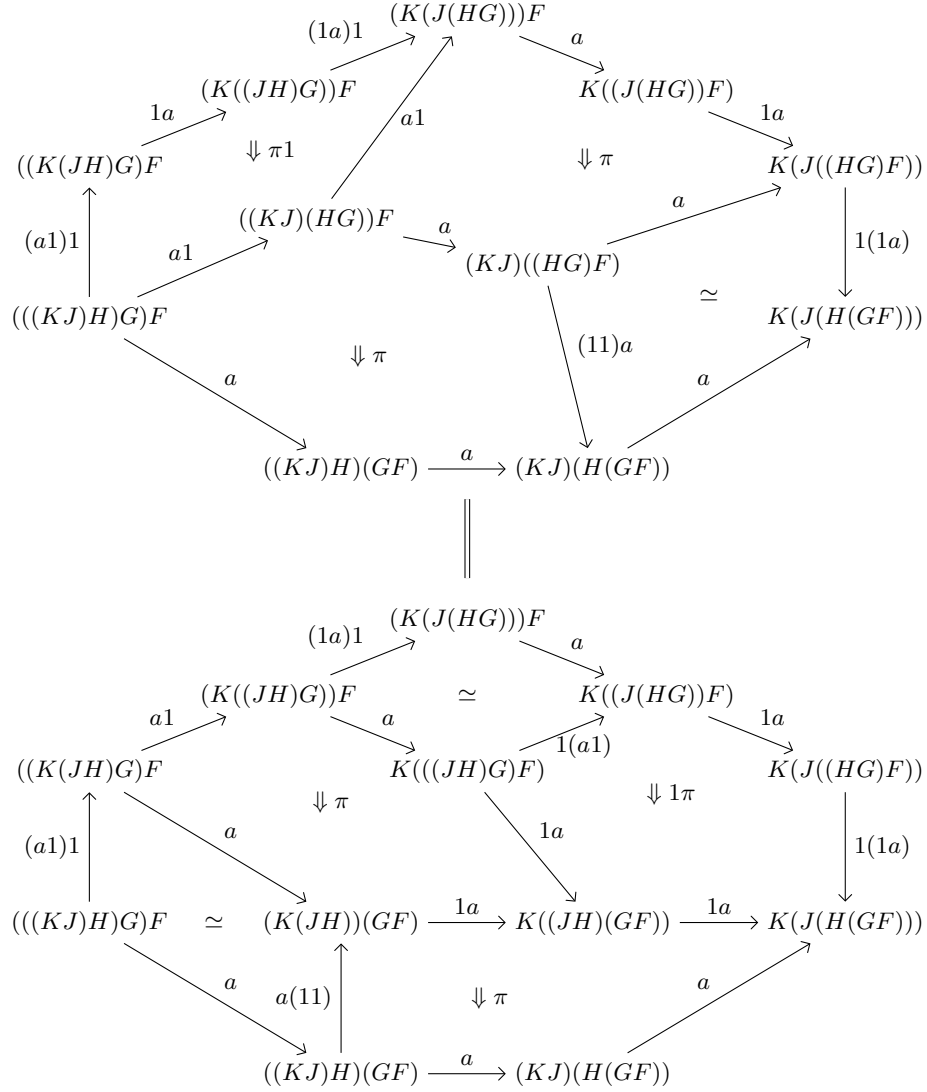
$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \mathcal{T}^3 & \xrightarrow{\square \times 1} \mathcal{T}^2 \\
 1 \times 1 \times I \nearrow & \Downarrow 1 \times r & \square \times 1 \searrow \\
 \mathcal{T}^2 & \xrightarrow{1} & \mathcal{T}^2 \\
 \downarrow \square & & \downarrow \square \\
 \mathcal{T} & \xrightarrow{1} & \mathcal{T}
 \end{array} & \xrightarrow{\rho} & \begin{array}{ccc}
 & \mathcal{T}^3 & \xrightarrow{\square \times 1} \mathcal{T}^2 \\
 1 \times 1 \times I \nearrow & \downarrow 1 \times I & \square \times 1 \searrow \\
 \mathcal{T}^2 & \xrightarrow{\square} & \mathcal{T} \\
 \downarrow \square & & \downarrow \square \\
 \mathcal{T} & \xrightarrow{1} & \mathcal{T}
 \end{array}
 \end{array} \quad (\text{B.7})$$

x) For all objects  $a, b, c, d, e$ , an invertible modification  $\pi$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \mathcal{T}^4 & \xrightarrow{\square \times 1 \times 1} \mathcal{T}^3 \\
 1 \times 1 \times \square \nearrow & \downarrow 1 \times \square \times 1 & \square \times 1 \searrow \\
 \mathcal{T}^3 & \xrightarrow{1 \times \square \times 1} \mathcal{T}^3 & \xrightarrow{\square \times 1} \mathcal{T}^2 \\
 \downarrow 1 \times \square & \swarrow \llcorner 1 \times \square \searrow a & \downarrow \square \\
 \mathcal{T}^2 & \xrightarrow{1 \times \square} \mathcal{T} & \xrightarrow{\square} \mathcal{T}
 \end{array} & \Downarrow \pi & \begin{array}{ccc}
 & \mathcal{T}^4 & \xrightarrow{\square \times 1 \times 1} \mathcal{T}^3 \\
 1 \times 1 \times \square \nearrow & \swarrow \llcorner \square \times 1 & \square \times 1 \searrow \\
 \mathcal{T}^3 & \xrightarrow{\square \times 1} \mathcal{T}^2 & \xrightarrow{1 \times \square} \mathcal{T}^2 \\
 \downarrow 1 \times \square & \swarrow \llcorner \square \searrow a & \downarrow \square \\
 \mathcal{T}^2 & \xrightarrow{\square} \mathcal{T} & \xrightarrow{\square} \mathcal{T}
 \end{array}
 \end{array} \quad (\text{B.8})$$

This data is required to satisfy the following three axioms. In the first axiom, the unmarked isomorphisms are isomorphisms induced by the naturality of the associator  $a$ .

i)



(B.9)



**Remark B.2** Our definition of a tricategory differs from [20] in that we replaced the arrow of the right unit  $r^-$  in the definition of the pseudo natural transformation  $\mu$  in [20] with its adjoint  $r$ . Consequently the axioms (B.10) and (B.11) have a different shape. It is straightforward to see that the two definitions are equivalent.

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