A COHOMOLOGICAL FRAMEWORK FOR HOMOTOPY MOMENT MAPS

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Abstract. Given a Lie group acting on a manifold M preserving a closed $n+1$-form ω, the notion of homotopy moment map for this action was introduced in [6], in terms of L_∞-algebra morphisms. In this note we describe homotopy moment maps as coboundaries of a certain complex. This description simplifies greatly computations, and we use it to study various properties of homotopy moment maps: their relation to equivariant cohomology, their obstruction theory, how they induce new ones on mapping spaces, and their equivalences. The results we obtain extend some of the results of [6].

Contents

Introduction

Recall that a symplectic form is a closed, non-degenerate 2-form. It is natural to consider symmetries of a given symplectic manifold, that is, a Lie group acting on a manifold, preserving the symplectic form. Among such actions, a nice subclass is given by actions that admit a moment map; in that case the infinitesimal generators of the action are hamiltonian vector fields. Actions admitting a moment map enjoy remarkable geometric, algebraic and topological properties, that have been studied extensively in the literature (e.g. symplectic reduction, the relation to equivariant cohomology and localization, convexity theorems,...)

In this note we consider closed $n{+}1$-forms for some $n \geq 1$. When they are non-degenerate, they are called multisymplectic form, and are higher analogues of symplectic forms which appear naturally in classical field theory.

Recently Rogers [15] (see also [18]) showed that the algebraic structure underlying a manifold with a closed $n+1$-form ω is the one of an L_∞-algebra. This allowed [6] for a natural extension of the notion of moment map to closed forms of arbitrary degree, called *homotopy moment map*. The latter is phrased in terms of L_∞-algebra morphisms.

The first contribution of this note is to construct, out of the action of a Lie group G on a manifold M, a chain complex $\mathcal{C}$ with the following property:

- any invariant closed form ω gives rise to a cocycle $\tilde{\omega}$ in $\mathcal{C}$
- homotopy moment maps are given exactly by the primitives of $\tilde{\omega}$.

The chain complex $\mathcal{C}$ is simply the product of the Chevalley-Eilenberg complex of the Lie algebra of G, with the de Rham complex of M. The action is encoded by the cocycle $\tilde{\omega}$. Notice that by the above the set of homotopy moment maps (for a fixed ω) has the structure of an affine space, which is unexpected since L_∞-algebra morphisms are generally very non-linear objects.

This characterization of homotopy moment maps is very useful: L_∞-algebra morphisms are usually quite intricate and cumbersome to work with in an explicit way, while working with coboundaries in a complex is much simpler. In this note we use the above characterization to:

- show that certain extensions of ω in the Cartan model give rise to homotopy moment maps (see §4),
- give cohomological obstructions to the existence of homotopy moment maps (see §5),
- show that a homotopy moment map for a G-action on (M, ω) induces one on $Maps(\Sigma, M)$, the space of maps from any closed and oriented manifold Σ into M, endowed with the closed form obtained from ω by transgression (see §6),
- obtain a natural notion of equivalence of homotopy moment maps, both under the requirement that ω be kept fixed and allow ω to vary (see §7). We show that it is compatible with the geometric notion of equivalence induced by isotopies of the manifold M, and with the notion of equivalence of L_∞-morphisms (see Appendix A).

In §4 and §5 we obtain results similar to those of [6], but with much less computational effort. The results obtained in §7 are a significant extension of results obtained in [6], where only closed 3-forms and loop spaces were considered. The equivalences introduced in 7 and their properties extend and justify the work carried out for closed 3-forms in [6, §7.4].

One more application of the characterization of moment maps as coboundaries in $\mathcal{C}$ is the following. Given two manifolds endowed with closed forms, their cartesian product $(M_1 \times M_2, \omega_1 \wedge \omega_2)$ is again an object of the same kind. This construction restricts to the multisymplectic category, but not to the symplectic one. The above characterization of moment maps is used in [17] to construct homotopy moment maps for cartesian products.

Remark: Recall that if $\mathfrak{X}$ is a Lie algebra, a $\mathfrak{X}$-*differential algebra* [14, §3] is a graded commutative algebra $\Omega = \oplus_{i \geq 0} \Omega^i$ with graded derivations $\iota_v, \mathcal{L}_v$ of degrees $-1, 0$ (depending linearly on $v \in \mathfrak{X}$) and a derivation d of degree 1 such that the Cartan relations hold:

$$[d, d] = 0 \quad [\mathcal{L}_v, d] = 0, \quad [\iota_v, d] = \mathcal{L}_v$$
$$[\iota_v, \iota_w] = 0, \quad [\mathcal{L}_v, \mathcal{L}_w] = \mathcal{L}_{[v,w]_{\mathfrak{X}}}, \quad [\mathcal{L}_v, \iota_w] = \iota_{[v,w]_{\mathfrak{X}}}.$$

This note is written in terms of geometric objects, but most of it applies also to the algebraic setting obtained replacing the setting we assume in §2 with:

$\mathfrak{X}$ a Lie algebra, Ω a $\mathfrak{X}$-differential algebra, $\omega \in \Omega^{n+1}$ with $d\omega = 0$.
$\mathfrak{g}$ a Lie algebra and $\rho\colon \mathfrak{g} \to \mathfrak{X}$ a Lie algebra morphism, so that $\mathcal{L}_{\rho(x)}\omega = 0$ for all $x \in \mathfrak{g}$.

Remark: The existence and uniqueness of homotopy moment maps is also studied by Ryvkin and Wurzbacher in [16], where the authors obtain independently results similar to ours on this subject, putting an emphasis on the differential geometry of multisymplectic forms.

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1. Closed forms

We recall briefly how some notions from symplectic geometry apply to closed differential forms of arbitrary degree.

Definition 1.1. Let (M,ω) be a **pre-n-plectic** manifold, i.e., M is a manifold and ω a closed $n+1$-form. An $(n-1)$-form α is **Hamiltonian** iff there exists a vector field $v_\alpha \in \mathfrak{X}(M)$ such that

$$d\alpha = -\iota_{v_\alpha}\omega.$$

We say v_α is a **Hamiltonian vector field** for α. The set of Hamiltonian $(n-1)$-forms is denoted as $\Omega^{n-1}_{\mathrm{Ham}}(M)$.

In analogy to symplectic geometry, one can endow the set of Hamiltonian $(n-1)$-forms with a skew-symmetric bracket, which however is not a Lie bracket. If one passes from $\Omega^{n-1}_{\mathrm{Ham}}(M)$ to a larger space, one obtains an L_∞-algebra [12], which was constructed essentially in [15, Thm. 5.2], and generalized slightly in [18, Thm. 6.7].

Definition 1.2. Given a pre-n-plectic manifold (M,ω), the **observables** form an L_∞ algebra, denoted $L_\infty(M,\omega) := (L,\{l_k\})$. The underlying graded vector space is given by

$$L_i = \begin{cases} \Omega^{n-1}_{\mathrm{Ham}}(M) & i = 0, \\ \Omega^{n-1+i}(M) & -n+1 \le i < 0. \end{cases}$$

The maps $\left\{l_k\colon L^{\otimes k} \to L | 1 \le k < \infty\right\}$ are defined as

$$l_1(\alpha) = d\alpha,$$

if $\deg(\alpha) > 0$, and for all $k > 1$

$$l_k(\alpha_1,\dots,\alpha_k) = \begin{cases} 0 & \text{if } \deg(\alpha_1\otimes\cdots\otimes\alpha_k) < 0, \\ \varsigma(k)\iota(v_{\alpha_1}\wedge\cdots\wedge v_{\alpha_k})\omega & \text{if } \deg(\alpha_1\otimes\cdots\otimes\alpha_k) = 0, \end{cases}$$

where v_{α_i} is any Hamiltonian vector field associated to $\alpha_i \in \Omega^{n-1}_{\mathrm{Ham}}(M)$. Here[1] $\varsigma(k) = -(-1)^{k(k+1)/2}$. Notice that $\varsigma(k-1)\varsigma(k) = (-1)^k$ for all k.

Among the Lie group actions on M that preserve ω, it is natural to consider those whose infinitesimal generators are hamiltonian vector fields. This leads to the following notion [6, Def. 5.1].

[1] So $\varsigma(k) = 1, 1, -1, -1, 1, \dots$ for $k = 1, 2, 3, 4, 5, \dots$.

Definition 1.3. A **(homotopy) moment map** for the action of G on (M,ω) is a L_∞-morphism $f\colon \mathfrak{g} \to L_\infty(M,\omega)$ such that for all $x \in \mathfrak{g}$

$$d(f_1(x)) = -\iota(v_x)\omega. \tag{1}$$

Saying that f is a L_∞-morphism means that it consists of components $f_k\colon \wedge^k \mathfrak{g} \to \Omega^{n-k}(M)$ (for $k = 1,\dots,n$) satisfying

$$\begin{aligned}\sum_{1\le i<j\le k} &(-1)^{i+j+1} f_{k-1}([x_i,x_j],x_1,\dots,\widehat{x_i},\dots,\widehat{x_j},\dots,x_k)\\ &= df_k(x_1,\dots,x_k) + \varsigma(k)\iota(v_{x_1}\wedge\cdots\wedge v_{x_k})\omega\end{aligned} \tag{2}$$

for $2 \le k \le n$, as well as

$$\sum_{1\le i<j\le n+1} (-1)^{i+j+1} f_n([x_i,x_j],x_1,\dots,\widehat{x_i},\dots,\widehat{x_j},\dots,x_{n+1}) = \varsigma(n+1)\iota(v_{x_1}\wedge\cdots\wedge v_{x_{n+1}})\omega. \tag{3}$$

2. A double complex encoding moment maps

The set-up in the whole of this note is the following:

$$\boxed{\begin{array}{c}(M,\omega) \text{ is a pre-}n\text{-plectic manifold,}\\ G \text{ is a Lie group acting on } M \text{ preserving } \omega.\end{array}}$$

We denote the Lie algebra of G by $\mathfrak{g}$, elements of $\mathfrak{g}$ by x, and the corresponding infinitesimal generators of the action (which are vector fields on M) by v_x.

In this section we introduce a complex with the property that suitable coboundaries correspond bijectively to moment maps for the action of G on (M,ω).

The manifold M and the Lie algebra $\mathfrak{g}$ give rise to a double complex

$$(\wedge^{\ge 1}\mathfrak{g}^* \otimes \Omega(M), d_{\mathfrak{g}}, d), \tag{4}$$

where $d_{\mathfrak{g}}$ is the Chevalley-Eilenberg differential of $\mathfrak{g}$ and d is the De Rham differential of M. We consider the total complex, which we denote by $\mathcal{C}$, with differential

$$d_{tot} := d_{\mathfrak{g}} \otimes 1 + 1 \otimes d.$$

We use the Koszul sign convention, hence, on an element of $\wedge^k\mathfrak{g}^* \otimes \Omega(M)$, d_{tot} acts as $d_{\mathfrak{g}} + (-1)^k d$.

We first need a lemma, which appears (using a slightly different notation) as the Extended Cartan Formula in [13, Lemma 3.4], and which we present without[2] proof.

Lemma 2.1. *Let M be a manifold and let Ω be an N-form (not necessarily closed). For all $k \ge 2$ and all vector fields $v_1,\dots,v_k$ we have:*

$$\begin{aligned}(-1)^k d\iota(v_1\wedge\cdots\wedge v_k)\Omega = &\sum_{1\le i<j\le k} (-1)^{i+j}\iota([v_i,v_j]\wedge v_1\wedge\cdots\wedge\widehat{v_i}\wedge\cdots\wedge\widehat{v_j}\wedge\cdots\wedge v_k)\Omega\\ &+\sum_{1=1}^{k}(-1)^i\iota(v_1\wedge\cdots\wedge\widehat{v_i}\wedge\cdots\wedge v_k)\mathcal{L}_{v_i}\Omega\\ &+\iota(v_1\wedge\cdots\wedge v_k)d\Omega.\end{aligned}$$

[2] It can be proven by a direct computation, extending the proof of [6, Lemma 7.2].

Remark 2.2. The Lie derivative of a form Ω along an multivector field $V = v_1 \wedge \cdots \wedge v_k$ is defined by $\mathcal{L}_V\Omega := d\iota(V)\Omega - (-1)^k\iota(V)d\Omega$ [10, Def. A2]. From the above we deduce that $(-1)^k\mathcal{L}_V\Omega$ equals the first two terms on the right hand side of the identity in Lemma 2.1.

Lemma 2.3. *For any G-invariant $\sigma \in \Omega^N(M)$ define*

$$\sigma_k \colon \wedge^k \mathfrak{g} \to \Omega^{N-k}(M), \ \ (x_1, \ldots, x_k) \mapsto \iota(v_{x_1} \wedge \cdots \wedge v_{x_k})\sigma. \tag{5}$$

and $\widetilde{\sigma} := \sum_{k=1}^N (-1)^{k-1}\sigma_k$. The map

$$\widetilde{\ } : (\Omega(M)^G, d) \to (\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M), d_{tot}), \ \ \sigma \mapsto \widetilde{\sigma}$$

intertwines the differentials, that is: $d_{tot}\widetilde{\sigma} = \widetilde{d\sigma}$.

Proof. Lemma 2.1 implies that $(-1)^k d\sigma_k = d_{\mathfrak{g}}\sigma_{k-1} + (d\sigma)_k$ for all $k \geq 2$. Hence

$$\begin{aligned}
d_{tot}\widetilde{\sigma} &= \sum_{k=1}^{N} (-1)^{k-1}(d_{\mathfrak{g}}\sigma_k + (-1)^k d\sigma_k) \\
&= \sum_{k=2}^{N+1} (-1)^k d_{\mathfrak{g}}\sigma_{k-1} - \sum_{k=1}^{N} d\sigma_k \\
&= \sum_{k=2}^{N+1} ((-1)^k d_{\mathfrak{g}}\sigma_{k-1} - d\sigma_k) - d\sigma_1 = \widetilde{d\sigma},
\end{aligned}$$

where in the third equality we used $\sigma_{N+1} = 0$, and in the last one we used the above equation and $-d\sigma_1 = (d\sigma)_1$ (the latter follows from $d(\iota_{v_x}\sigma) + \iota_{v_x}d\sigma = \mathcal{L}_{v_x}\sigma = 0$). ☐

Since ω is a closed differential form, from Lemma 2.3 we obtain:

Corollary 2.4. *$\widetilde{\omega}$ is d_{tot}-closed.*

The next proposition states that moment maps for the action of G on (M, ω) correspond bijectively to primitives of $\widetilde{\omega}$ in $(\mathcal{C}, d_{tot})$. In particular, moment maps form an affine space, which is somewhat surprising since generally L_∞-morphisms are very non-linear objects.

Proposition 2.5. *Let $\varphi = \varphi_1 + \cdots + \varphi_n$, with $\varphi_k \in \wedge^k\mathfrak{g}^* \otimes \Omega^{n-k}(M)$. Then: $d_{tot}\varphi = \widetilde{\omega}$ iff*

$$f_k := \varsigma(k)\varphi_k \colon \wedge^k \mathfrak{g} \to \Omega^{n-k}(M),$$

for $k = 1, \ldots, n$, are the components of a homotopy moment map for the action of G on (M, ω).

Proof. $d_{tot}\varphi = \sum_{k=2}^{n+1} d_{\mathfrak{g}}\varphi_{k-1} + \sum_{k=1}^{n}(-1)^k d\varphi_k$ is equal to $\widetilde{\omega}$ iff we have

$$-d\varphi_1 = \omega_1 \tag{6}$$

$$d_{\mathfrak{g}}\varphi_{k-1} + (-1)^k d\varphi_k = (-1)^{k-1}\omega_k \quad \text{ for all } 2 \leq k \leq n \tag{7}$$

$$d_{\mathfrak{g}}\varphi_n = (-1)^n\omega_{n+1}. \tag{8}$$

Evaluating eq. (6) on $x \in \mathfrak{g}$ we obtain $d\varphi_1(x) = -\iota_{v_x}\omega$, which is equivalent to eq. (1).
Evaluating eq. (7) on $x_1, \ldots, x_k \in \mathfrak{g}$ we obtain

$$\begin{aligned}
\sum_{1 \leq i < j \leq k} &(-1)^{i+j}\varphi_{k-1}([v_{x_i}, v_{x_j}], v_{x_1}, \ldots, \widehat{v_{x_i}}, \ldots, \widehat{v_{x_j}}, \ldots, v_{x_k}) \\
&= -(-1)^k d\varphi_k(v_{x_1}, \ldots, v_{x_k}) + (-1)^{k-1}\iota(v_{x_1} \wedge \cdots \wedge v_{x_k})\omega.
\end{aligned}$$

Multiplying this equation by $-\varsigma(k-1) = -(-1)^k\varsigma(k)$ we obtain eq. (2). Similarly one sees that eq. (8) is equivalent to eq. (3). $\square$

Remark 2.6. The results of this section can be derived also from [9, §3]. See [6] for an explanation of how this derivation goes.

3. Closed forms and moment maps as cocycles

Recall that whenever $f : (A, d) \longrightarrow (A', d')$ is a map of complexes, $(A[1] \oplus A', d_f)$ is a complex with differential $d_f := \begin{pmatrix} d & 0 \\ f & -d' \end{pmatrix}$. This is known as the *cone construction.*

We apply this to the map of complexes $\widetilde{\ }$ of Lemma 2.3. We obtain:

Proposition 3.1. *Fix an action of a Lie group G on a manifold M. Then*

$$\mathcal{B} := \Omega(M)^G[1] \oplus (\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M)), \qquad D := \begin{pmatrix} d & 0 \\ \sim & -d_{tot} \end{pmatrix}$$

is a complex with the property: the D-closed elements in degree n are pairs $(\omega[1], \varphi)$ where ω is a pre-n-plectic form and φ corresponds (via Prop. 2.5) to a moment map for ω.

Proof. Compute $D(\omega[1], \varphi) = (d\omega[1], \widetilde{\omega} - d_{tot}\varphi)$ and apply Prop. 2.5. $\square$

4. Equivariant cohomology

In this section we recover in a quick way a result of [6], which states that suitable extensions of ω in the Cartan model give rise to moment maps (see Prop. 4.4).

The following is a variation of Lemma 2.3:

Lemma 4.1. *For any G-equivariant $F\colon \mathfrak{g} \to \Omega^N(M)$, such that $\iota_{v_x}F(x) = 0$ for all $x \in \mathfrak{g}$, define*

$$F_k\colon \wedge^k \mathfrak{g} \to \Omega^{N+1-k}(M), \quad F_k(x_1, \dots, x_k) = \iota(v_{x_1} \wedge \cdots \wedge v_{x_{k-1}})F(x_k)$$

and $\widetilde{F} := F_1 + \cdots + F_{N+1} \in \wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M)$. Then $d_{tot}\widetilde{F} = -\widetilde{dF}$.

Remark 4.2. 1) Notice that $F_1 = F$.

2) If $\alpha \in \Omega^{N+1}(M)$ is G-invariant, then $\alpha_1\colon \mathfrak{g} \to \Omega(M)^N, x \mapsto \iota_{v_x}\alpha$ is G-equivariant since $\mathcal{L}_{v_x}(\iota_{v_y}\alpha) = \iota_{[v_x, v_y]}\alpha$. We have $\widetilde{\alpha} = \widetilde{(\alpha_1)}$ (where the l.h.s. was defined in Lemma 2.3 and the r.h.s. in Lemma 4.1.)

3) $\widetilde{F}$ lies in the G-invariant part of $\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M)$, see [6, §6] for a proof.

Proof. Notice first that the condition $\iota_{v_x}F(x) = 0$ ensures that $\widetilde{F}$ is well-defined (as a totally skew-symmetric map). It also implies that $\iota_{v_x}d(F(x)) = \mathcal{L}_{v_x}(F(x)) = F([x, x]) = 0$, so that $\widetilde{dF}$ is well-defined.

We compute for all k:

$$\begin{aligned}
(9) \qquad (d_{\mathfrak{g}}F_{k-1})(x_1, \dots, x_k) &= \sum_{1 \leq i < j \leq k} (-1)^{i+j} F_{k-1}([x_i, x_j], \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_k) \\
&= \sum_{1 \leq i < j \leq k-1} (-1)^{i+j} \iota([v_{x_i}, v_{x_j}] \wedge \cdots \wedge \widehat{v_{x_i}} \wedge \cdots \wedge \widehat{v_{x_j}} \wedge \cdots \wedge v_{x_{k-1}})F(x_k) \\
&+ \sum_{i=1}^{k-1} (-1)^{i+k}(-1)^{k-2}\iota(v_{x_1} \wedge \cdots \wedge \widehat{v_{x_i}} \wedge \cdots \wedge v_{x_{k-1}})F([x_i, x_k])
\end{aligned}$$

and notice that $F([x_i,x_k]) = \mathcal{L}_{v_{x_i}} F(x_k)$ by the equivariance of F. Hence

$$\begin{aligned}(-1)^{k-1} d(F_k(x_1,\dots,x_k)) &= (-1)^{k-1} d(\iota(v_{x_1}\wedge\cdots\wedge v_{x_{k-1}})F(x_k))\\ &= (d_{\mathfrak{g}}F_{k-1})(x_1,\dots,x_k) + \iota(v_{x_1}\wedge\cdots\wedge v_{x_{k-1}})d(F(x_k)),\end{aligned}$$

where in the last equation we used Lemma 2.1 (applied to $\Omega := F(x_k)$) and eq. (9). In other words:

$$(-1)^{k-1} dF_k = d_{\mathfrak{g}}F_{k-1} + (dF)_k. \tag{10}$$

We conclude the proof computing

$$\begin{aligned} d_{tot}\widetilde{F} = \sum_{k=1}^{N+1}(d_{\mathfrak{g}}F_k + (-1)^k dF_k) &= \sum_{k=2}^{N+2} d_{\mathfrak{g}}F_{k-1} + \sum_{k=1}^{N+1}(-1)^k dF_k\\ &= \sum_{k=2}^{N+2}(d_{\mathfrak{g}}F_{k-1} + (-1)^k dF_k) - dF_1\\ &= -\widetilde{dF},\end{aligned}$$

using $F_{N+2}=0$ in the third equality and eq. (10) in the last one. $\square$

Given the action of G on M, recall that the Cartan model is the complex[3] $(\Omega(M)\otimes S\mathfrak{g}^*)^G$, where elements of $\mathfrak{g}^*$ are assigned degree two, together with the Cartan differential d_G (see for example [11]). If we choose a basis x_i of $\mathfrak{g}$ and denote by ξ^i the dual basis of $\mathfrak{g}^*$ (concentrated in degree two), we can write $d_G = d\otimes 1 - \sum_i \iota_{v_{x_i}}\otimes \xi^i$.

Remark 4.3. The invariant pre-n-plectic form ω (or, more precisely, $\omega\otimes 1$) is usually not closed in the Cartan model. Given an equivariant linear map $\mu\colon \mathfrak{g}\to\Omega^{n-1}(M)$, which we can regard as an element of $(\Omega^{n-1}(M)\otimes\mathfrak{g}^*)^G$, we have [6, §6.1]: $\omega-\mu$ is a closed element of the Cartan model iff for all $x,y\in\mathfrak{g}$

a) $d\mu(x) = -\iota_{v_x}\omega$ (i.e., v_x is the hamiltonian vector field of $\mu(x)$),
b) $\mathcal{L}_{v_x}\mu(y) = \mu([x,y])$ (i.e., $\mu\colon\mathfrak{g}\to\Omega^{n-1}_{ham}(M)$ is G-equivariant),
c) $\iota_{v_x}\mu(x) = 0$.

We recover the main statement[4] of [6, Thm. 6.3]:

Proposition 4.4. *Let $\mu\colon\mathfrak{g}\to\Omega^{n-1}(M)$ be an equivariant linear map so that $\omega-\mu$ is a cocycle in the Cartan model. Then $d_{tot}\widetilde{\mu}=\widetilde{\omega}$, and the maps ($1\le k\le n$)*

$$\begin{aligned} f_k\colon \wedge^k\mathfrak{g} &\to \Omega^{n-k}(M),\\ (x_1,\dots x_k) &\mapsto \varsigma(k)\iota(v_{x_1}\wedge\cdots\wedge v_{x_{k-1}})\mu(x_k)\end{aligned}$$

are the components of a homotopy moment map $\mathfrak{g}\to L_\infty(M,\omega)$.

Proof. By a) in Remark 4.3 we have the following equality of equivariant maps $\mathfrak{g}\to\Omega^n(M)$:

$$d\mu = -\omega_1,$$

where $\omega_1(x) = \iota_{v_x}\omega$. As $\iota_{v_x}\iota_{v_x}\omega = 0$ for all x, we can apply the map $\widetilde{\ }$ (see Lemma 4.1) to obtain $\widetilde{d\mu} = -\widetilde{\omega_1}$. We have $d_{tot}\widetilde{\mu} = -\widetilde{d\mu}$ by applying Lemma 4.1 to μ (we are allowed to do

[3]It calculates the equivariant cohomology when G is compact.
[4]In [6, Thm. 6.3] it is also shown that the moment map f is equivariant, using b) in Remark 4.3.

so because of c) in Remark 4.3), and we also have $\widetilde{\omega_1} = \widetilde{\omega}$ (see Rem. 4.2). Altogether we obtain

$$d_{tot}\widetilde{\mu} = \widetilde{\omega}.$$

We conclude applying Prop. 2.5 to $\varphi := \widetilde{\mu}$. $\square$

Remark 4.5. Prop. 4.4 can be extended [1] [6], as follows: every arbitrary extension of ω to a cocycle in the Cartan model gives rise to a moment map.

5. Obstruction theory

We consider the obstruction theory for the existence of moment maps, obtaining results similar to those contained in [6, §9.1, §9.2].

Fix a point $p \in M$. It is immediate to check that

$$r\colon (\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M), d_{tot}) \to (\wedge \mathfrak{g}^*, d_{\mathfrak{g}}), \quad \eta \otimes \alpha \mapsto \eta \cdot \alpha|_p$$

is a chain map. Here $\Omega|_p \in \mathbb{R}$ is declared to vanish if $\Omega \in \Omega^{\geq 1}(M)$. Since $\widetilde{\omega}$ is d_{tot}-closed by Cor. 2.4, it follows that $r(\widetilde{\omega}) = (-1)^n \omega_{n+1}|_p \in \wedge^{n+1}\mathfrak{g}^*$ is $d_{\mathfrak{g}}$-closed, hence it defines a class in the Chevalley-Eilenberg cohomology $H_{CE}(\mathfrak{g})$.

Corollary 5.1. *Let $p \in M$. If a homotopy moment map exists, then $[\omega_{n+1}|_p] = 0$.*

Proof. By Prop. 2.5, a homotopy moment map exists iff $[\widetilde{\omega}] = 0$. In this case $0 = H(r)([\widetilde{\omega}]) = (-1)^n[\omega_{n+1}|_p]$, where $H(r)$ denotes the map on cohomology induced by r. $\square$

Corollary 5.2. *If $[\omega_{n+1}|_p] = 0$ and*[5]

$$H^j_{CE}(\mathfrak{g}) \otimes H^{n+1-j}(M) = 0 \text{ for } j = 1, \dots, n$$

then there exists a moment map.

Proof. The algebraic Künneth formula ([3, Exerc. 14.23]) and the conditions on the vanishing of cohomology groups imply that $H(r)\colon H^{n+1}(\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M), d_{tot}) \to H^{n+1}(\wedge \mathfrak{g}^*, d_{\mathfrak{g}})$ is an isomorphism. Hence $[\widetilde{\omega}]$ vanishes iff $[\omega_{n+1}|_p]$ vanishes. The latter does vanish by assumption, so there exists φ with $d_{tot}\varphi = \widetilde{\omega}$, and by Prop. 2.5 the primitive φ gives rise to a homotopy moment map. $\square$

6. Actions on mapping spaces

In [6, §11] it is shown that a moment map for a pre-2-plectic manifold M gives rise to a moment map for the loop space LM and an induced presymplectic form. Recall that $LM = M^{S^1}$ consists of all differentiable maps from the circle S^1 to M. In this section we generalize this, allowing M to be any pre-n-plectic manifold and replacing the circle with any compact, orientable manifold.

6.1. **Loop spaces.** For the sake of exposition, consider first the case of the loop space LM (an infinite-dimensional Fréchet manifold). The action of G on M induces an action on LM, simply given by $(g\cdot\gamma)(t) := g\cdot\gamma(t)$ for all $\gamma \in LM$ and $t \in S^1$. Given an element x of the Lie algebra $\mathfrak{g}$, recall that we denote by v_x (a vector field on M) the corresponding infinitesimal generator of the action on M. The corresponding infinitesimal generator of the action on LM, which we denote by v^ℓ_x, is given as follows: $v^\ell_x|_\gamma = \gamma^* v_x \in \Gamma(\gamma^* TM) = T_\gamma LM$, for all $\gamma \in LM$.

[5] These cohomology classes vanish, for example, if $H^1(M) = \cdots = H^n(M) = 0$.

There is a degree preserving[6] map

$$\ell\colon \Omega(M) \to \Omega(LM)[-1]$$

called transgression, which commutes with the de Rham differential [4, §3.5]. Explicitly, it sends a form $\alpha \in \Omega^j(M)$ to $\alpha^\ell \in \Omega^{j-1}(LM)$ given by

$$\alpha^\ell|_\gamma(z_1,\dots,z_{j-1}) = \int_0^{2\pi} \alpha(z_1,\dots,z_{j-1},\dot\gamma)|_{\gamma(s)}\, ds \quad \forall \gamma \in LM,\ \forall z_1,\dots,z_{j-1} \in T_\gamma LM.$$

In particular, the closed form $\omega \in \Omega^{n+1}(M)$ transgresses to a closed form $\omega^\ell \in \Omega^n(LM)$.

Consider the complex $\mathcal{C} = (\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M), d_{tot})$ of eq. (4), as well as $\mathcal{C}' := (\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(LM)[-1], d_{tot})$. The transgression map extends trivially to a degree preserving map

$$Id \otimes \ell\colon \mathcal{C} \to \mathcal{C}',$$

which commutes with $d_{\mathfrak{g}}$ and the de Rham differential, and hence with d_{tot}. We use the superscript ℓ to denote this map too. In particular, given $\varphi = \varphi_1 + \dots + \varphi_n \in \mathcal{C}$ where $\varphi_k \in \wedge^k\mathfrak{g}^* \otimes \Omega^{n-k}(M)$, we obtain an element $\varphi^\ell \in \mathcal{C}'$ with components $\varphi^\ell = (\varphi^\ell)_1 + \dots + (\varphi^\ell)_{n-1}$ where $(\varphi^\ell)_k := (\varphi_k)^\ell \in \wedge^k\mathfrak{g}^* \otimes \Omega^{n-k-1}(LM)$.

Proposition 6.1. *If φ corresponds (in the sense of Prop. 2.5) to a homotopy moment map for (M,ω), then φ^ℓ corresponds to a homotopy moment map for (LM,ω^ℓ).*

Proof. If $d_{tot}\varphi = \widetilde{\omega}$ then

$$d_{tot}(\varphi^\ell) = (d_{tot}\varphi)^\ell = (\widetilde{\omega})^\ell = \widetilde{\omega^\ell}. \tag{11}$$

The last equality holds because for all k we have $(\omega_k)^\ell = (\omega^\ell)_k$, as a consequence of

$$\begin{aligned}(\omega_k)^\ell(x_1,\dots,x_k)|_\gamma = (\iota(v_{x_1}\wedge\dots\wedge v_{x_k})\omega)^\ell|_\gamma &= \int_0^{2\pi} \omega(v_{x_1},\dots,v_{x_k},\bullet,\dot\gamma)|_{\gamma(s)}\, ds\\ &= \iota(v_1^\ell\wedge\dots\wedge v_k^\ell)(\omega^\ell)|_\gamma = (\omega^\ell)_k(x_1,\dots,x_k)|_\gamma,\end{aligned}$$

where $x_1,\dots,x_k \in \mathfrak{g}$, $\gamma \in LM$, and $\bullet$ denotes $n-k$ slots for elements of $T_\gamma LM$. Recall that $(\omega^\ell)_k$ was defined in Lemma 2.3.

Thanks to eq. (11) we can now apply Prop. 2.5 (which holds in the setting of Fréchet manifolds too). □

6.2. **General mapping spaces.** We now generalize Prop. 6.1. Let Σ be a compact, oriented manifold of dimension s. The G action on M gives rise to a G action on M^Σ, the Fréchet manifold of smooth maps from Σ to M. (The formulae for the this action and the corresponding infinitesimal generators are exactly as in §6.1).

Transgression is the differential-preserving map

$$\ell := \int_\Sigma \circ\, ev^*\colon \Omega(M) \to \Omega(M^\Sigma)[-s]$$

where $ev\colon \Sigma \times M^\Sigma \to M$ is the evaluation map and $\int_\Sigma$ denotes integration along the fibers [2, Cap. VI.4] of the projection $\Sigma \times M^\Sigma \to M^\Sigma$, which lowers the degree of a differential form by s. Notice that the closed form $\omega \in \Omega^{n+1}(M)$ transgresses to a closed form $\omega^\ell \in \Omega^{n+1-s}(M^\Sigma)$.

The transgression map extends trivially to a map of complexes

$$Id \otimes \ell\colon \mathcal{C} \to \mathcal{C}',$$

[6] The notation $\Omega(LM)[-1]$ refers to the fact that here $\Omega^{k-1}(LM)$ is assigned degree k.

where now $\mathcal{C}' := (\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M^\Sigma)[-s], d_{tot})$. Let $\varphi = \varphi_1 + \cdots + \varphi_n$, where $\varphi_k \in \wedge^k\mathfrak{g}^* \otimes \Omega^{n-k}(M)$.

Proposition 6.2. *If φ corresponds (in the sense of Prop. 2.5) to a homotopy moment map for (M, ω), then φ^ℓ corresponds to a homotopy moment map for (M^Σ, ω^ℓ).*

Proof. The proof is the same as for Prop. 6.1. We just point out that the equation $(\omega_k)^\ell = (\omega^\ell)_k$ for all k is obtained applying the well-known relation $(\iota_{v_x}\omega)^\ell = \iota_{(v_x)^\ell}\omega^\ell$ for all $x \in \mathfrak{g}$. It can be proven also directly, exactly as in the proof of Prop.6.1, using the explicit description of the integration along the fiber given in [2, Cap. VI.4] and the fact that the derivative $d_{(p,\sigma)}ev\colon T_{(p,\sigma)}(\Sigma \times M^\Sigma) \to T_{\sigma(p)}M$ maps a tangent vector of the form $(Z, 0)$ to the vector $\sigma_*(Z)$ and, for all $x \in \mathfrak{g}$, the vector $(0, (v_x)^\ell)|_{(p,\sigma)}$ to $(v_x)|_{\sigma(p)}$. □

Spelling out Prop. 6.2 in terms of moment maps we obtain:

Corollary 6.3. *Let*

$$f\colon \mathfrak{g} \to L_\infty(M, \omega)$$

be a homotopy moment map for the pre-n-plectic manifold (M, ω) with components $f_k\colon \wedge^k \mathfrak{g} \to \Omega^{n-k}(M)$, where $k = 1, \ldots, n$.

Then

$$f^\ell\colon \mathfrak{g} \to L_\infty(M^\Sigma, \omega^\ell)$$

is a homotopy moment map for the pre-$(n-s)$-plectic manifold (M^Σ, ω^ℓ) with components $(f^\ell)_k := (f_k)^\ell\colon \wedge^k \mathfrak{g} \to \Omega^{n-s-k}(M^\Sigma)$, where $k = 1, \ldots, n-s$.

7. Equivalences

In this section we introduce notions of equivalence for: 1) certain cocycles in the Cartan model, and 2) pairs consisting of closed invariant forms and moment maps. Recall that Prop. 4.4 states that to the former Cartan cocycles one can canonically associate moment maps; we show that the above equivalences are compatible with this association. All along this section we fix an action of a Lie group G on a manifold M.

7.1. **Equivalences of Cartan cocycles.**

Definition 7.1. Two cocycles $C^0 = \omega^0 - \mu^0$ and $C^1 = \omega^1 - \mu^1$ in the Cartan model, with $\omega^0, \omega^1 \in \Omega^{n+1}(M)^G$ and $\mu^0, \mu^1 \in (\Omega^{n-1}(M) \otimes \mathfrak{g}^*)^G$, are **equivalent** iff they differ by a coboundary of the form[7] $d_G(\alpha + F)$, where

$$\alpha \in \Omega^n(M)^G \text{ and } F \in (\Omega^{n-2}(M) \otimes \mathfrak{g}^*)^G.$$

Explicitly, $C^1 - C^0 = d_G(\alpha + F)$ means that

a) $\omega^1 - \omega^0 = d\alpha$
b) $\mu^1 - \mu^0 = \iota_\bullet\alpha - dF$
c) $\iota(v_x)F(x) = 0$ for all $x \in \mathfrak{g}$,

where we use the short form $(\iota_\bullet\alpha)(x) := \iota_{v_x}\alpha$.

Remark 7.2. In the symplectic case (so $n = 1$), Def. 7.1 reduces to: $\omega^1 - \omega^0 = d\alpha$ and $\mu^1 - \mu^0 = \iota_\bullet\alpha$ for some $\alpha \in \Omega^1(M)^G$. In particular, if $\omega^1 = \omega^0$, each function $\iota_{v_x}\alpha$ is constant.

[7]If $C^0 - C^1$ is exact, in general we may not find a primitive of the form $\alpha + F$ as above. We justify our definition remarking that the choices of Cartan cocycles we allow are also not the most general ones.

The following proposition states that if two Cartan cocycles are related by a G-equivariant diffeomorphism of M isotopic to the identity, then they are equivalent.

Proposition 7.3. *Let the Lie group G act on M. Let $\omega^0, \omega^1 \in \Omega^{n+1}(M)^G$. Let $\mu^0, \mu^1 \in (\Omega^{n-1}(M) \otimes \mathfrak{g}^*)^G$ so that $C^i := \omega^i - \mu^i$ is a cocycle in the Cartan model. Suppose that there exists a G-equivariant diffeomorphism ψ, isotopic to Id_M by G-equivariant diffeomorphisms, with*

$$\psi^*\omega^1 = \omega^0, \quad \psi^*\mu^1 = \mu^0.$$

(Here μ^1 is viewed as a map $\mathfrak{g} \to \Omega^{n-1}(M)$ and $(\psi^*\mu^1)(x) := (\psi^*(\mu^1(x))$ for all $x \in \mathfrak{g}$). Then C^1 and C^0 are equivalent in the sense of Def. 7.1.

Proof. We construct explicitly equivariant Cartan cochains α, F such that $d_G(\alpha + F) = C^1 - C^0$.

Let $\{\psi_s\}_{s\in[0,1]}$ a isotopy from $\psi^0 = Id_M$ to $\psi = \psi^1$ by G-equivariant diffeomorphisms, and denote by $\{X_s\}_{s\in[0,1]}$ the time-dependent vector field generating $\{\psi_s\}_{s\in[0,1]}$. Define ω^s by $\psi_s^*(\omega^s) = \omega^0$ and $\mu^s \in (\Omega^{n-1}(M) \otimes \mathfrak{g}^*)^G$ by $\psi_s^*(\mu^s) = \mu^0$.

We claim that

$$\alpha := -\int_0^1 \iota_{X_s}\omega^s \in \Omega^n(M)$$

satisfies $\omega^1 - \omega^0 = d\alpha$ (condition a) in Def. 7.1). This follows integrating from $s = 0$ to $s = 1$ the equation

$$\frac{d}{ds}\omega^s = -\mathcal{L}_{X_s}\omega^s = -d\iota_{X_s}\omega^s$$

where in the first equality we use [7, Prop. 6.4]

$$0 = \frac{d}{ds}(\psi_s^*\omega^s) = \psi_s^*(\mathcal{L}_{X_s}\omega^s + \frac{d}{ds}\omega^s).$$

We claim that

$$F := \int_0^1 \iota_{X_s}\mu^s \in \Omega^{n-2}(M) \otimes \mathfrak{g}^*$$

satisfies $\mu^1 - \mu^0 = \iota_\bullet\alpha - dF$ (condition b) in Def. 7.1). Similarly to the above, this follows integrating from 0 to 1 the following expression, for all $x \in \mathfrak{g}$:

$$\begin{aligned}\frac{d}{ds}\mu^s(x) = -\mathcal{L}_{X_s}(\mu^s(x)) &= -\iota_{X_s}(d\mu^s(x)) - d\iota_{X_s}\mu^s(x)\\ &= \iota_{X_s}(\iota_{v_x}\omega^s) - d\iota_{X_s}\mu^s(x)\\ &= \iota_{v_x}(-\iota_{X_s}\omega^s) - d\iota_{X_s}\mu^s(x).\end{aligned}$$

Here in the first equality we used again [7, Prop. 6.4], and in the third one $\iota_{v_x}\omega^s = -d(\mu^s(x))$ for all $x \in \mathfrak{g}$ (see Remark 4.3 a)).

We are left with showing $\iota_{v_x}F(x) = 0$ for all $x \in \mathfrak{g}$ (condition c) in Def. 7.1). This holds since $\iota_{v_x}\mu^s(x) = 0$, a consequence of Remark 4.3 c) and the fact that $\omega^s - \mu^s$, being the pullback of a Cartan cocycle by a G-equivariant map, is itself a Cartan cocycle.

Notice that the X_s are G-invariant, as their flow $\{\psi_s\}$ commutes with the G-action. Further the ω^s are G-invariant, since ω is G-invariant. Hence α is G-invariant. By the same reasoning and the invariance of μ^0, we see that F is G-invariant. □

7.2. **Equivalences of moment maps.** Let $f\colon \mathfrak{g} \rightsquigarrow L_\infty(M,\omega)$ be a homotopy moment map for ω, and

$$\varphi_k := \varsigma(k) f_k \colon \wedge^k \mathfrak{g} \to \Omega^{n-k}(M) \qquad \text{for } k = 1, \dots, n. \tag{12}$$

We know that $\varphi = \varphi_1 + \cdots + \varphi_n$ satisfies $d_{tot}\varphi = \widetilde{\omega}$.

Indeed, by Prop. 2.5, this equation characterizes moment maps for ω. Therefore, if $\eta \in (\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M))_{n-1}$, then $\varphi + d_{tot}\eta$ naturally provides a new moment map for (M,ω). Further, notice that if $\alpha \in (\Omega^n)^G$, by Lemma 2.3 we have $d_{tot}(\varphi + \widetilde{\alpha}) = \widetilde{\omega + d\alpha}$, i.e. $\varphi + \widetilde{\alpha}$ provides a moment map for $\omega + d\alpha$. The following definition, which arises naturally considering the complex $\mathcal{B}$ introduced in §3, is made so that these two kinds of moment maps are equivalent to the original one.

Definition 7.4. Let ω be an invariant pre-n-plectic form on M and f a moment map for ω, for which we denote by φ the corresponding element of $(\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M))_n$ as in Prop. 2.5, and similarly for (ω', f'). The pairs (ω, f) and (ω', f') are **equivalent** if there exist $\eta \in (\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M))_{n-1}$ and $\alpha \in (\Omega^n)^G$ such that

$$\omega' - \omega = d\alpha \tag{13}$$

$$\varphi' - \varphi = d_{tot}\eta + \widetilde{\alpha}. \tag{14}$$

Remark 7.5. The equivalence introduced in Def. 7.4 can be phrased as a simple coboundary condition, thereby providing an algebraic justification for Def. 7.4. Indeed, in terms of the complex $\mathcal{B} = (\Omega(M)^G[1] \oplus (\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M)), D)$ introduced in §3, the conditions (13)-(14) are simply phrased as

$$(\omega'[1], \varphi') - (\omega[1], \varphi) = D(\alpha[1], -\eta).$$

A geometric justification for Def. 7.4 is given in Prop. 7.8.

In Appendix A we compare Def. 7.4 with the natural notion of equivalence for L_∞-morphisms. There we show that two homotopy moment maps are equivalent *with* $\alpha = 0$ (see Def. 7.4) iff they are equivalent as L_∞-morphisms.

Remark 7.6. Condition (14), explicitly, is that there exists $\eta = \eta_1 + \cdots + \eta_{n-1} \in (\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M))_{n-1}$ and $\alpha \in (\Omega^n)^G$ with

$$\begin{gathered} -d\eta_1 + \alpha_1 = (\varphi' - \varphi)_1, \\ d_{\mathfrak{g}}\eta_{k-1} + (-1)^k d\eta_k + (-1)^{k-1}\alpha_k = (\varphi' - \varphi)_k \quad \forall k = 2, \dots, n-1 \\ d_{\mathfrak{g}}\eta_{n-1} + (-1)^{n-1}\alpha_n = (\varphi' - \varphi)_n, \end{gathered}$$

where α_i is defined as in eq. (5).

Remark 7.7. Given a G-manifold, we can consider $\omega = 0 \in \Omega^{n+1}_{closed}(M)$ and the zero moment map $f = 0$. Given $\alpha \in \Omega^n(M)^G$, applying the operation described in Def. 7.4 provides a moment map for $(M, d\alpha)$, which agrees exactly with the one provided in [6, §8.1] for exact pre-n-plectic forms admitting an invariant primitive.

The following proposition provides a geometric justification for Def. 7.4. It states that if two moment maps are related by a G-equivariant diffeomorphism of M isotopic to the identity, then they are equivalent.

Proposition 7.8. *Let the Lie group G act on M. Let ω^0, ω^1 be closed $(n+1)$-forms preserved by the action. Suppose that there exists a G-equivariant diffeomorphism ψ, isotopic to Id_M by G-equivariant diffeomorphisms, such that $\psi^*\omega^1 = \omega^0$.*

Let $f^i\colon \mathfrak{g} \rightsquigarrow L_\infty(M,\omega_i)$ be homotopy moment maps ($i=0,1$) intertwined by ψ that is, for all their components ($k=1,\dots,n$) we have

$$\psi^* f^1_k = f^0_k.$$

Then f^0 and f^1 are equivalent in the sense of Def. 7.4.

Proof. Let $\{\psi_s\}_{s\in[0,1]}$ a isotopy from $\psi^0 = Id_M$ to $\psi=\psi^1$ by G-equivariant diffeomorphisms, and denote by $\{X_s\}_{s\in[0,1]}$ the time-dependent vector field generating $\{\psi_s\}_{s\in[0,1]}$. Define ω^s by $\psi_s^*(\omega^s)=\omega^0$ and f^s by $\psi_s^*(f^s)=f^0$.

The form

$$\alpha := -\int_0^1 \iota_{X_s}\omega^s \, ds \in \Omega^n(M)$$

satisfies $\omega^1-\omega^0 = d\alpha$, i.e. eq. (13), as we have already shown at the beginning of the proof of Prop. 7.3.

Now, for all $1\le k\le n$ (and defining $f^0=0$) we have

$$\begin{aligned}\frac{d}{ds} f^s_k = -\mathcal{L}_{X_s} f^s_k &= -d(\iota_{X_s} f^s_k) - \iota_{X_s} d f^s_k\\ &= -d(\iota_{X_s} f^s_k) + \iota_{X_s} d_{\mathfrak{g}} f^s_{k-1} + \varsigma(k)\iota_{X_s}\omega^s_k\\ &= -d(\iota_{X_s} f^s_k) + d_{\mathfrak{g}}\iota_{X_s} f^s_{k-1} + (-1)^k\varsigma(k)(\iota_{X_s}\omega^s)_k,\end{aligned}$$

where in the first equation we used [7, Prop. 6.4], and in the second one $df^s_k = -d_{\mathfrak{g}} f^s_{k-1} - \varsigma(k)\omega^s_k$ (which holds by eq. (7)).

Multiplying by $\varsigma(k)$ the equation $f^1_k - f^0_k = \int_0^1 \frac{d}{ds} f^s_k \, ds$ we hence obtain

$$\varphi^1_k - \varphi^0_k = (-1)^k d\eta_k + d_{\mathfrak{g}}\eta_{k-1} + (-1)^{k-1}\alpha_k,$$

where we define

$$\eta_k\colon \wedge^k \mathfrak{g} \to \Omega^{n-1-k}(M), \quad \eta_k(x^1,\dots,x_k) = (-1)^{k-1}\varsigma(k)\int_0^1 \iota(X_s) f^s_k \, ds.$$

As this holds for all $1\le k\le n$, we obtain $\varphi_1-\varphi_0 = d_{tot}\eta + \widetilde{\alpha}$, i.e. eq. (14). □

We finish this subsection discussing equivalences for which the pre-n-plectic form is fixed. Fix a pre-n-plectic form ω. Restricting Def. 7.4 to the space of moment maps for ω we obtain: two moment maps f, f' for ω are equivalent iff there exist $\eta \in (\wedge^{\ge 1}\mathfrak{g}^*\otimes\Omega(M))_{n-1}$ and a *closed* form $\alpha \in (\Omega^n)^G$ such that eq. (14) is satisfied[8].

The following proposition extends the results of [6, §7.5].

Proposition 7.9. *There exist a closed 3-form ω with the following property: There exist an* equivariant *moment map for ω which is not equivalent (in the sense of Def. 7.4) to any moment map for ω arising from a Cartan cocycle of the form $\omega-\mu$ as in Prop. 4.4.*

Proof. Consider an action of a Lie group G on a connected pre-2-plectic manifold (M,ω), and let f be an equivariant moment map. Let f' be another equivariant moment map (for the same action on (M,ω)) which is equivalent to f. This means exactly that there exist $\eta_1 \in \mathfrak{g}^*\otimes C^\infty(M)$ and a *closed* form $\alpha\in\Omega^2(M)^G$ satisfying eq. (14). In particular the equation

$$-d\eta_1 + \alpha_1 = (\varphi'-\varphi)_1 \tag{15}$$

[8]Loosely speaking, the action of α can be interpreted as induced by a gauge transformation on the higher Courant algebroid $TM\oplus\wedge^{n-1}T^*M$ endowed with the ω-twisted Courant bracket.

holds, where we denote $\varphi_1 = f_1$,$\varphi_2 = f_2$, etc.

For every $x \in \mathfrak{g}$, evaluating the l.h.s. of the eq. (15) on x and applying the interior product ι_{v_x}, we obtain a function. We claim that this function is a constant, which we denote by C_x. To see this, first notice that evaluating the l.h.s. of the eq. (15) on x and applying ι_{v_x} we obtain $\iota_{v_x}(-d\eta_1(x) + \alpha_1(x)) = -\mathcal{L}_{v_x}(\eta_1(x))$. Second, notice that since $\varphi_1, (\varphi')_1, \alpha_1$ are equivariant, it follows that $d\eta_1$ is also equivariant. In particular we have $d\mathcal{L}_{v_x}\eta_1(x) = \mathcal{L}_{v_x} d\eta_1(x) = d\eta_1([x,x]) = 0$, i.e., $\mathcal{L}_{v_x}(\eta_1(x)) := -C_x$ is a constant function.

From the above claim we conclude: if there exists $x \in \mathfrak{g}$ such that

i) $\iota_{v_x}\varphi_1(x) \neq 0$
ii) $C_x = 0$

then necessarily $\iota_{v_x}(\varphi')_1(x) \neq 0$, so f' can not arise from a Cartan cocycle (compare with Remark 4.3 c)).

Now, following [6, §7.5], we display an example of moment map f and $x \in \mathfrak{g}$ satisfying assumption i) and such that for *every* $\eta_1 \in \mathfrak{g}^* \otimes C^\infty(M)$ assumption ii) is satisfied. It follows that there exists no moment map which is equivalent to f and which arises from a Cartan cocycle.

Let G be the abelian group $S^1 \times S^1$, and $(M,\omega) = (S^1 \times S^1 \times \mathbb{R}, d\theta_1 \wedge d\theta_2 \wedge dz)$. We take the infinitesimal action of $\mathfrak{g}$ on M to be given by $(1,0) \in \mathfrak{g} \mapsto \partial_{\theta_1}$, $(0,1) \mapsto \partial_{\theta_2}$. It is easily checked that

$$\begin{aligned} f_1 \colon \mathfrak{g} \to \Omega^1_{Ham}(M), &\quad (1,0) \mapsto z d\theta_2 + d\theta_1, \quad (0,1) \mapsto -z d\theta_1 + d\theta_2, \\ f_2 \colon \wedge^2 \mathfrak{g} \to C^\infty(M), &\quad (1,0) \wedge (0,1) \mapsto -z \end{aligned}$$

are the components of an equivariant moment map. Let $x := (1,0) \in \mathfrak{g}$. Since $v_x = \partial_{\theta_1}$, we clearly have $\iota_{v_x} f_1(x) = 1 \neq 0$, hence assumption i) is satisfied. For *any* $h \in C^\infty(M)$ such that $\mathcal{L}_{v_x}(h) = \partial_{\theta_1}(h)$ is a constant, integrating $\mathcal{L}_{v_x}(h) d\theta_1$ along the circles $S^1 \times \{point\} \times \{point\}$ of M one sees by Stokes' theorem that the constant $\mathcal{L}_{v_x}(h)$ is necessarily zero. Hence, for *any* $\eta_1 \in \mathfrak{g}^* \otimes C^\infty(M)$ we have $\mathcal{L}_{v_x}(\eta_1(x)) = 0$, verifying that assumption ii) is satisfied. $\square$

Remark 7.10. The notion of equivalence on the space of moment maps for ω mentioned just before Prop. 7.9 should not be confused with the similar but more restrictive one in which $\alpha = 0$ is imposed. We refer to the latter notion of equivalence as **inner equivalence**. Explicitly: two moment maps f and f' for ω are inner equivalent if there exist $\eta \in (\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M))_{n-1}$ such that $\varphi' - \varphi = d_{tot}\eta$, where φ denotes the element of $(\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M))_n$ corresponding to f as in Prop. 2.5, and similarly for φ' and f'. The notion of inner equivalence arises naturally if one considers the complex $(\mathcal{C}, d_{tot})$ of §2 (as opposed to the complex $\mathcal{B}$ introduced in §3).

Notice that when $\alpha = 0$, the first equation in Rem. 7.6 says that, for all $x \in \mathfrak{g}$, the elements $(\varphi')_1(x)$ and $\varphi_1(x)$ of $\Omega^{n-1}_{ham}(M,\omega)$ – which have the same hamiltonian vector field v_x, and hence a priory differ by a closed form – actually differ by an exact form.

7.3. **Relation between the two notions of equivalence.** We end establishing the relation between the equivalences introduced in Def. 7.1 and Def. 7.4.

Proposition 7.11. *Let the Lie group G act on M. Take two Cartan cocycles $C^0 = \omega^0 - \mu^0$ and $C^1 = \omega^1 - \mu^1$, with $\omega^0, \omega^1 \in \Omega^{n+1}(M)^G$ and $\mu^0, \mu^1 \in (\Omega^{n-1}(M) \otimes \mathfrak{g}^*)^G$. Assume that C^0 and C^1 are equivalent in the sense of Def. 7.1.*

Then the homotopy moment maps f^0 and f^1, induced by the μ^i as in Prop. 4.4, are equivalent in the sense of Def. 7.4.

Proof. Since we assume that C^0 and C^1 are equivalent, there is $\alpha \in \Omega^n(M)^G$ and $F \in (\Omega^{n-2}(M) \otimes \mathfrak{g}^*)^G$ satisfying the equation appearing below Def. 7.1, that is

a) $\omega^1 - \omega^0 = d\alpha$
b) $\mu^1 - \mu^0 = \iota_\bullet \alpha - dF$
c) $\iota(v_x)F(x) = 0$ for all $x \in \mathfrak{g}$,

We now check that an equivalence between the homotopy moment maps is given by the form α and by $\eta := \widetilde{F}$ (notice that $\widetilde{F}$ is well-defined by c)). The relation (13) is just a).

Applying the map $\widetilde{\ }$ (see Lemma 4.1) to equation b) we obtain

$$\widetilde{\mu^1} - \widetilde{\mu^0} = \widetilde{\iota_\bullet \alpha} - \widetilde{dF}.$$

Denoting by $\varphi^i \in (\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M))_n$ the element corresponding to f^i as in Prop. 2.5, for $i = 0, 1$, we have $\varphi^i = \widetilde{\mu^i}$ (to see this, compare the formulae in Prop. 2.5 and Prop. 4.4). Using $\widetilde{\iota_\bullet \alpha} = \widetilde{\alpha}$ (by Rem. 4.2) and $d_{tot}\widetilde{F} = -\widetilde{dF}$ (by Lemma 4.1) we obtain exactly the relation (14). □

Appendix A. Equivalences of moment maps and L_∞-algebra morphisms

Let G be a Lie group acting on a pre-n-plectic manifold (M, ω). A moment map for this action (Def. 1.3) is in particular an L_∞-morphism $\mathfrak{g} \to L_\infty(M, \omega)$. There is a natural notion of equivalence of L_∞-morphisms, and the aim of this appendix is to show that it coincides with the inner equivalence introduced in Rem. 7.10 (that is, equivalence in the sense of Def. 7.4 imposing $\alpha = 0$.)

The notion of equivalence of L_∞-morphisms comes from homotopy theory, and coincides with the one given by equivalences of Maurer-Cartan elements [8]. We express it following [8, §5]: let $\widetilde{L}, L$ be L_∞-algebras. Consider $\Omega(\mathbb{R}) = \mathbb{R}[t] + \mathbb{R}[t]dt$, the differential graded algebra of polynomial forms on the real line, where t has degree 0 and dt degree 1. Then $L \otimes \Omega(\mathbb{R})$ is again an L_∞-algebra [5, §1].

Definition A.1. Let $\widetilde{L}, L$ be L_∞-algebras. Let $f, f' \colon \widetilde{L} \to L$ be L_∞-morphisms. f and f' are **equivalent** iff there exists an L_∞-morphism $H \colon \widetilde{L} \to L \otimes \Omega(\mathbb{R})$ such that

$$H|_{t=0,dt=0} = f, \quad H|_{t=1,dt=0} = f'. \tag{16}$$

Proposition A.2. *Two homotopy moment maps f and f' are inner equivalent (see Rem. 7.10) iff they are equivalent in the sense of Def. A.1.*

Proof. We first given a characterization of L_∞-morphisms from $\mathfrak{g}$ to $L_\infty(M, \omega) \otimes \Omega(\mathbb{R})$. $L_\infty(M, \omega) \otimes \Omega(\mathbb{R})$ is concentrated in degrees ≤ 1, and its multibrackets $\{l_k\}$ are as follows [5, §1]: for $k \geq 2$ they are given by the multibrackets of $L_\infty(M, \omega)$ extended by $\Omega(\mathbb{R})$-linearity (with no signs involved), and the differential is

$$l_1(\gamma \otimes \Gamma) = D\gamma \otimes \Gamma + (-1)^{deg(\gamma)} \gamma \otimes \frac{\partial}{\partial t}\Gamma dt,$$

where D denotes the differential in $L_\infty(M, \omega)$. All multibrackets, except for the differential, vanish unless all entries are in degree 0 or 1.

We observe that the truncation

$$T := (L_\infty(M, \omega) \otimes \Omega(\mathbb{R}))_{<0} \oplus \{c \in (L_\infty(M, \omega) \otimes \Omega(\mathbb{R}))_0 : l_1(c) = 0\}$$

is closed[9] under the multibrackets. Hence T is a L_∞-algebra for which the multibrackets (except for the differential) vanish unless all entries are in degree zero.

Let

$$H\colon \wedge^{\geq 1}\mathfrak{g} \to L_\infty(M,\omega)\otimes\Omega(\mathbb{R})$$

be a linear map such that $H|_{\wedge^k\mathfrak{g}}$ has degree $1-k$.

<u>Claim</u>: *H is an L_∞-morphism iff conditions* (21) *and* (22) *below are satisfied.*

We divide the proof of the claim in three steps.

A) H is a L_∞-morphism iff the image of the first component H_1 is annihilated by l_1 and for $2\leq m\leq n+1$, for all $x_i\in\mathfrak{g}$

$$\sum_{1\leq i<j\leq m}(-1)^{i+j+1}H_{m-1}([x_i,x_j],x_1,\dots,\widehat{x_i},\dots,\widehat{x_j},\dots,x_m) = l_1H_m(x_1,\dots,x_m)+l_m(H_1(x_1),\dots,H_1(x_m)) \tag{17}$$

where $H_{n+1}=0$. Indeed, if H is a L_∞-morphism, then H_1 is a chain map and takes values in l_1-closed elements, and therefore H takes values in T, so that we can apply [6, §3.2]. Conversely, if the image of H_1 is annihilated by l_1, we can apply [6, §3.2], and eq. (17) implies that H is a L_∞-morphism.

B) Write

$$H=h^0(t)+h^1(t)dt$$

where $h^0(t)$ and $h^1(t)$ are maps $\wedge^{\geq 1}\mathfrak{g}\to L_\infty(M,\omega)\otimes\mathbb{R}[t]$. Notice that the component $h^0(t)_k$ has degree $1-k$, while $h^1(t)_k$ has degree $-k$. The condition that the degree zero component of H_1 takes values in l_1-closed elements reads

$$\frac{\partial}{\partial t}h^0(t)_1+dh^1(t)_1=0. \tag{18}$$

Separating the terms without dt from those containing dt, eq. (17) is equivalent to ($2\leq m\leq n+1$)

$$\sum_{1\leq i<j\leq m}(-1)^{i+j+1}h^0(t)_{m-1}([x_i,x_j],x_1,\dots,\widehat{x_i},\dots,\widehat{x_j},\dots,x_m) = dh^0(t)_m(x_1,\dots,x_m)+l_m(h^0(t)_1(x_1),\dots,h^0(t)_1(x_m)) \tag{19}$$

and

$$\sum_{1\leq i<j\leq m}(-1)^{i+j+1}h^1(t)_{m-1}([x_i,x_j],x_1,\dots,\widehat{x_i},\dots,\widehat{x_j},\dots,x_m) = dh^1(t)_m(x_1,\dots,x_m)+(-1)^{1-m}\frac{\partial}{\partial t}h^0(t)_m(x_1,\dots,x_m), \tag{20}$$

where we used $m\geq 2$ and degree counting both to replace D by the de Rham differential d, and to conclude that l_m vanishes if one of its arguments is of the form $h^1(t)_1(x_i)$.

[9]Indeed, T is closed under l_1 since $(l_1)^2=0$. For the higher brackets, the only non-trivial case to consider is $l_2(\gamma\otimes\Gamma,\gamma'\otimes\Gamma')$ when $\gamma,\Gamma,\gamma',\Gamma'$ all have degree zero. This bracket lies in T since l_1 satisfies the Leibniz rule w.r.t. l_2.

C) Eq. (18), (19) and (20) are equivalent to the fact that the following two equations hold for all $t \in \mathbb{R}$:

$$h^0(t)\colon \mathfrak{g} \rightsquigarrow L_\infty(M,\omega) \text{ is an } L_\infty\text{- morphism} \tag{21}$$

$$d_{tot}\overline{h^1(t)} = \frac{\partial}{\partial t}\overline{h^0(t)}, \tag{22}$$

where the bar denotes the following: if $\xi = \sum_{k=1}^N \xi_k$ with $\xi_k \in \wedge^k \mathfrak{g}^* \otimes \Omega(M)$, then

$$\bar{\xi} := \sum_{k=1}^N \varsigma(k)\xi_k.$$

The equivalence between eq. (19) (for all $2 \le m \le n+1$) and eq. (21) is given again by [6, §3.2]. We show the equivalence between eq. (18) and eq. (20) (for all $2 \le m \le n+1$) on one side, and eq. (22) on the other. Notice that the L.H.S. of eq. (22) consists of three kinds of terms, exactly as it happens in eq. (6), (7), (8). The analogue of eq. (6) is equivalent to eq. (18). To take care of the analogues of eq. (7) and (8), write eq. (20) in the form $-d_{\mathfrak{g}}h^1(t)_{m-1} = dh^1(t)_m + (-1)^{1-m}\frac{\partial}{\partial t}h^0(t)_m$ for all $2 \le m \le n+1$, and use that $h^1(t)_n = 0$ by degree reasons. △

Now, given homotopy moment maps f, f', let $\varphi := \bar{f}, \varphi' := \bar{f}'$ (where the bar has been defined just above).

Assume first that f and f' are inner equivalent, so that there is $\eta \in (\wedge^{\ge 1}\mathfrak{g}^* \otimes \Omega(M))_{n-1}$ with $\varphi' - \varphi = d_{tot}\eta$. Define

$$H = h^0(t) + h^1(t)dt := (\overline{\varphi + t d_{tot}\eta}) + \overline{\eta}dt.$$

Notice that H satisfies eq. (16). Now we check the two conditions appearing in the above claim. Condition (21) is satisfied, because $d_{tot}\overline{h^0(t)} = d_{tot}\varphi = \widetilde{\omega}$ for all t and because of Prop. 2.5. Condition (22) is satisfied as both sides are equal to $d_{tot}\eta$. Therefore by the claim H is an L_∞-morphism. We conclude that f and f' are equivalent the sense of Def. A.1.

Conversely, assume we are given $H = h^0(t) + h^1(t)dt$ satisfying the conditions of Def. A.1, that is: $h^0(1) = f' = \overline{\varphi'}$ and $h^0(0) = f = \overline{\varphi}$, and H is an L_∞-morphism. Then integrating over t we define

$$\eta := \int_0^1 \overline{h^1(t)}.$$

It satisfies

$$\varphi' - \varphi = \overline{h^0(1)} - \overline{h^0(0)} = \int_0^1 \frac{\partial}{\partial t}\overline{h^0(t)} = d_{tot}\eta,$$

where in the last equation condition (22) is used. Hence f and f' are inner equivalent. □

References

[1] C. Arias Abad and B. Uribe. An A_∞-version of the Chern-Weil homomorphism. In progress.

[2] M. Audin. *Torus Actions On Symplectic Manifolds*. Progress in Mathematics. Birkhäuser Basel, 2004.

[3] R. Bott and L. W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[4] J. Brylinski. *Loop Spaces, Characteristic Classes and Geometric Quantization*. Modern Birkhäuser Classics. Birkhäuser Boston, 2007.

[5] U. Buijs and A. Murillo. Algebraic models of non-connected spaces and homotopy theory of L_∞ algebras. 04 2012, 1204.4999v1.

[6] M. Callies, Y. Fregier, C. L. Rogers, and M. Zambon. Homotopy moment maps. ArXiv:1304.2051.

[7] A. Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.
[8] V. A. Dolgushev. Erratum to: "A Proof of Tsygan's Formality Conjecture for an Arbitrary Smooth Manifold". math/0703113.
[9] D. Fiorenza, C. L. Rogers, and U. Schreiber. Higher geometric prequantum theory. *Homology Homotopy Appl.*, 16(2):107–142, 2014.
[10] M. Forger, C. Paufler, and H. Römer. The Poisson bracket for Poisson forms in multisymplectic field theory. *Rev. Math. Phys.*, 15(7):705–743, 2003.
[11] V. W. Guillemin and S. Sternberg. *Supersymmetry and equivariant de Rham theory*. Mathematics Past and Present. Springer-Verlag, Berlin, 1999. With an appendix containing two reprints by Henri Cartan [MR0042426 (13,107e); MR0042427 (13,107f)].
[12] T. Lada and J. Stasheff. Introduction to SH Lie algebras for physicists. *Internat. J. Theoret. Phys.*, 32(7):1087–1103, 1993.
[13] T. B. Madsen and A. Swann. Closed forms and multi-moment maps. *Geom. Dedicata*, 165:25–52, 2013.
[14] E. Meinrenken. Equivariant cohomology and the Cartan model. *Encyclopedia of Mathematical Physics, Elsevier. Availabe at* `http://www.math.toronto.edu/mein/research/pubnew.html`, 2006.
[15] C. L. Rogers. L_∞-algebras from multisymplectic geometry. *Lett. Math. Phys.*, 100(1):29–50, 2012.
[16] L. Ryvkin and T. Wurzbacher. Existence and unicity of co-moments in multisymplectic geometry. *Diff. Geom. and Appl.*, 41:1–11, 2015. Also available as ArXiv:1411.2287.
[17] C. S. Shahbazi and M. Zambon. Products of multisymplectic manifolds and homotopy moment maps. ArXiv:1504.08194.
[18] M. Zambon. L_∞-algebras and higher analogues of Dirac structures and Courant algebroids. *J. Symplectic Geom.*, 10(4):563–599, 2012.

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