Irreducible vector bundles on some elliptic non-Kahler threefolds

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Abstract

We study rank-2 vector bundles on non-Kähler threefolds $\pi : X \to B$, which are elliptic principal bundles with at least one non-zero Chern class over a complex surface B with no curves. In this case, we prove that every rank-2 irreducible vector bundle on X is a pull-back from B up to a twist by a line bundle. These 2-vector bundles are, via the Kobayashi-Hitchin correspondence, solutions of the Yang-Mills equations on the threefold X.

Keywords: Holomorphic vector bundles, principal elliptic bundles, surfaces without curves.

1. Introduction

The study of vector bundles over elliptic fibrations has been a very active area of research in both mathematics and physics over the last twenty-five years. There is by now a well-understood theory for projective elliptic fibrations (see [Don], [DonPa], [Fri], [FriMoWi], [BriMac],[RuiMu], [Brig], etc.) but not very much is known about the non-Kähler case: the study of rank-2 vector bundles on non-Kähler elliptic surfaces is done in [BrMo], [BrMo2], [BrMo3] and the study of relatively semi-stable vector bundles on non-Kähler principal elliptic bundles over complex manifolds of arbitrary dimensions is done in [BrHaTr].

One motivation for the study of vector bundles on non-Kähler elliptic n-folds comes from developments in superstring theory with non-vanishing background H-field (see [BeBeDas], [CaCuDal], [GoPr], [BeBeFuTsYa]).

The explanation for the interest in moduli spaces of vector bundles in both mathematics and physics comes from the Kobayashi-Hitchin correspondence,

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which relates the complex geometry concept of *stable holomorphic vector bundles* to the differential geometry concept of Hermite-Einstein connections (which are solutions of Yang-Mills equations). Here, the stability is defined with respect to a Gauduchon metric, which exists on any compact complex hermitian manifold (cf [Gau]).

Among the vector bundles on non-projective manifolds, the most striking are perhaps the *nonfiltrable* ones and even *irreducible* ones. Recall that a holomorphic vector bundle on a non-projective complex manifold is called *irreducible* if it has no coherent subsheaf of proper rank (more about nonfiltrable and irreducible vector bundles in the non-projective case see, for example: [ElFo], [BrFl], [BaLP], [BrLNM] chapter 4, [Vu2]).

In this paper we study irreducible (hence stable, with respect to *any* Gauduchon metric) rank-2 vector bundles on non-Kähler principal elliptic bundles $\pi : X \to B$ where the base B is a complex surface with no curves.

2. Basic facts about principal elliptic bundles

Through this section, we deal with elliptic principal bundles $\pi : X \to B$ with typical fiber an elliptic curve E, over an arbitrary complex manifold B.

Let $\mathcal{O}_B(E)$ be the sheaf of germs of holomorphic maps from B to E; then the set (of isomorphism classes) of holomorphic elliptic principal bundles over B is in natural bijection with $H^1(B, \mathcal{O}_B(E))$.

Let $\Lambda \simeq \mathbb{Z}^2$ be the lattice of periods of E; from the cohomology sequence of the exact sequence

$$0 \to \Lambda \to \mathcal{O}_B \to \mathcal{O}_B(E) \to 0$$

we see that any elliptic principal bundle π gives rise to an element $c(\pi) \in H^2(B, \Lambda)$. Fixing a basis in Λ gives an identification of $c(\pi)$ with a couple of elements $(c_1', c_1'') \in H^2(B, \mathbb{Z})$, called *the Chern classes* of π.

Assumption 1. *We assume that* at least one of the Chern classes is non-zero (modulo torsion)*; this ensures that the Poincaré dual of any fiber of* π *is zero in* $H^2(X, \mathbb{Z})$ *(modulo torsion) - see [Del] in the formulation of [Hö], Prop. 5.2, or [BrHaTr], Thm. 2.1.*

Eventually, we recall one more invariant of elliptic principal bundles. Such a bundle induces (cf [Hö], 1.5.c) a map

$$\varepsilon_\pi : H^0(B, \mathcal{R}^1\pi_*(\mathcal{O}_X)) \to H^2(B, \pi_*(\mathcal{O}_X))$$

which is the first possible nontrivial d_2 morphism in the Leray spectral sequence of $\mathcal{O}_X$. If the $H^2(B)$ has a Hodge decomposition (as it is always the case when B is a surface, Kähler or not), then this map is completely determined by the Chern classes.

3. Line bundles versus sections of the relative jacobian

Recall that if $\pi : X \to B$ is a principal elliptic bundle with fiber E, then the cokernel of the sheaves map (induced by the exponential sequence on X)

$$\mathcal{R}^1\pi_*(\mathbb{Z}_X) \to \mathcal{R}^1\pi_*(\mathcal{O}_X)$$

is the sheaf $\mathcal{O}_B(E^\vee)$ of germs of sections in the jacobian fibration of the dual of E, $J_{E^\vee} = B \times E^\vee$ - see e.g. [BaPeVV], pp 153 for the case of surfaces, and [BrHaTr], Thm. 2.4. for arbitrary dimension of the base.

The exact sequence

$$0 \to Pic(B) \to Pic(X) \stackrel{res}{\to} H^0(B, \mathcal{R}^1\pi_*(\mathcal{O}_X^*)) \tag{1}$$

induced by the Leray spectral sequence of $\mathcal{O}_X^*$ shows that any line bundle $\mathcal{L} \in Pic(X)$ gives rise to an element in $H^0(B, \mathcal{R}^1\pi_*(\mathcal{O}_X^*))$ given by restriction of $\mathcal{L}$ to the fibers of π.

Since we have also the exact sequence

$$0 \to H^0(B, \mathcal{O}_B(E^\vee)) \to H^0\left(B, \mathcal{R}^1\pi_*(\mathcal{O}_X^*)\right) \stackrel{degr}{\to} H^0(B, \mathcal{R}^2\pi_*(\mathbb{Z}_B))$$

we see that the group of global sections in the jacobian fibration identifies with a subgroup of $H^0\left(B, \mathcal{R}^1\pi_*(\mathcal{O}_X^*)\right)$.

We may ask for the following: given a global section in the jacobian, does there exists a line bundle inducing it? In the case B is a curve, this is true, (see e.g. [BrUe]) but for higher dimensions, this is no longer true in general.

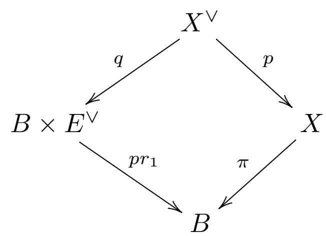
Under our assumption that the fibers of π are cohomologically trivial, we see that the restriction of any line bundle form X to the fibers has zero degree. Hence we get an exact sequence, which will be used later:

$$0 \to Pic(B) \stackrel{\pi^*}{\to} Pic(X) \stackrel{res}{\to} H^0(B, \mathcal{O}_B(E^\vee)) \tag{2}$$

4. The spectral space

We will further need the notion of spectral cover of a sheaf on X. In the case of elliptic principal bundles, this notion was defined in [BrHaTr]; since we will need only its support (*spectral space*), and not its structure sheaf, we briefly recall it below.

We fix a Poincaré bundle $\mathcal{P}_E \in Pic(E \times E^\vee)$ of E and let $\pi : X \to B$ be an elliptic principal bundle. We denote by $X^\vee = X \times E^\vee$, by $p = pr_1 : X^\vee \to X$ and by $q : X^\vee \to B \times E^\vee$ the map defined as $q(x, \mathcal{L}) = (\pi(x), \mathcal{L})$. In other words, we have a cartesian diagram



First we deal with elliptic principal bundles $\pi : X \to B$ that *do have* a "global Poincaré bundle", that is for which there exists $\mathcal{P} \in Pic(X^\vee)$ such that

$$\mathcal{P}_{|E_b \times E^\vee} \simeq \mathcal{P}_E$$

for all fibers $E_b = \pi^{-1}(b), b \in B$. In particular,

$$\mathcal{P}_{|E_b \times \{\mathcal{L}\}} \simeq \mathcal{L} \tag{3}$$

for all $\mathcal{L} \in E^\vee$.

Let V be any coherent sheaf on X; define the sheaf $\Phi(V)$ on $B \times E^\vee$ by:

$$\Phi(V) = R^1 q_*(p^*(V) \otimes \mathcal{P}).$$

Lemma 1. *Let V be any coherent sheaf on X. Then a point $(b, \mathcal{L})$ lives in $Supp(\Phi(V))$ iff $H^1(E_b, \mathcal{L} \otimes V_{|E_b}) \neq 0$.*

In particular, the support of $\Phi(V)$ does not depend on the choice of the Poincaré bundle $\mathcal{P}$.

Proof. We have

$$(b, \mathcal{L}) \in Supp(\Phi(V)) \Leftrightarrow H^1(q^{-1}(b, \mathcal{L}), p^*(V) \otimes \mathcal{P}_{|q^{-1}(b,\mathcal{L})}) \neq 0$$

since the dimension of the fibers of q is 1. But $q^{-1}(b, \mathcal{L})$ canonically identifies with $E_b \times \{\mathcal{L}\}$ so

$$(b, \mathcal{L}) \in Supp(\Phi(V)) \Leftrightarrow H^1(E_b, V_{|E_b} \otimes \mathcal{L}) \neq 0.$$

Q.E.D.

Remark. Notice that in the above Lemma there was no assumption on the base B, which in particular can be non-compact.

Next we consider the general case of an elliptic principal bundle $\pi : X \to B$. As X is locally trivial over B, we see we can cover B by open subsets U_i such that $\pi^{-1}(U_i)$ carries a global Poincaré bundle, since it is isomorphic to the product $U_i \times E$.

Now if $V \in Coh(X)$ is arbitrary, then for any two overlapping U_i, U_j the supports of $\Phi(V_{|U_i})$ and of $\Phi(V_{|U_j})$ agree on the intersection, hence defining a closed analytic subspace of $B \times E^\vee$. By definition, this space will be called *the spectral space* of V, denoted also Σ_V.

Remark. It is easy to see that if $\mathcal{L} \in Pic(X)$ is a line bundle, then $\Sigma_{\mathcal{L}}$ is its associated section $res(\mathcal{L})$.

Definition 1. *Let V be a rank-2 vector bundle on X. A fiber E_b of π will be called a* jump fiber *for V if $V_{|E_b} \simeq \mathcal{L}_1 \oplus \mathcal{L}_2$ with $deg(\mathcal{L}_1) > 0$.*

The following observation follows at once from Atiyah classification of rank-2 vector bundles on elliptic curves ([At]).

Lemma 2. *Suppose V is a rank-2 vector bundle and $b \in B$. Then:*
a) the fiber E_b is a jump fiber if and only if it is contained in Σ_V;
b) if E_b is not a jump fiber, then $\Sigma_V \cap E_b$ consists of at most two points. More precisely
1). $\Sigma_V \cap E_b = \{\mathcal{L}_1, \mathcal{L}_2\}$ where $\mathcal{L}_1, \mathcal{L}_2 \in E^\vee$ with distinct $\mathcal{L}_1 \neq \mathcal{L}_2$ iff $E_{|E_b} \simeq \mathcal{L}_1 \oplus \mathcal{L}_2$;
2). $\Sigma_V \cap E_b = \{\mathcal{L}\}$ if either $E_{|E_b} = \mathcal{L} \oplus \mathcal{L}$ or $E_{|E_b} \simeq F_2 \otimes \mathcal{L}$ where F_2 is Atyiah's bundle: the unique nonsplit extension of $\mathcal{O}_E$ by $\mathcal{O}_E$.

Notation. We will denote by $\Sigma^1_{\mathcal{F}}$ the codimension-1 part of $\Sigma_{\mathcal{F}}$.

5. Irreducible vector bundles

Through this section, we consider an elliptic principal bundle $\pi : X \to B$ with at least one non-zero Chern class (see our assumption 1) over a compact complex surface B with no curves. Notice that the assumption of the cohomological vanishing of the fibers of π and the absence of curves on B implies that the only proper closed analytic subspaces of X are the fibers of π.

Lemma 3. *Consider an elliptic principal bundle $\pi : X \to B$ with at least one non-zero Chern class and with $\epsilon_\pi = 0$. If B is a surface with no curves, then every section of the relative jacobian comes from a line bundle.*

Proof. Recall the exact sequence (2):

$$0 \to Pic(B) \stackrel{\pi^*}{\to} Pic(X) \stackrel{res}{\to} H^0(B, \mathcal{O}_B(E^\vee))$$

The main point is that the space of global sections in the relative jacobian $H^0(B, \mathcal{O}_B(E^\vee))$ consists of *constant* sections only. Indeed, assume $\Sigma \subset J_B = B \times E^\vee$ is a global section; then its projection onto $E^\vee$ must be constant, since otherwise its fibers would produce an infinity of curves on Σ, hence on B, absurd. So we can identify $H^0(B, \mathcal{O}_B(E^\vee))$ with $E^\vee$ itself.

On the other hand, the map res is a complex Lie group morphism. As the dimensions of $Pic(X)$ and $Pic(B)$ are equal to, respectively, $h^1(X, \mathcal{O}_X)$ and $h^1(B, \mathcal{O}_B)$ we see from the assumption $\epsilon_\pi = 0$ that $h^1(X, \mathcal{O}_X) = h^1(B, \mathcal{O}_B) + 1$. It follows that the dimension of the image of res is at least one. But since $H^0(B, \mathcal{O}_B(E^\vee)) \simeq E^\vee$ we see the map res is actually surjective.

Q.E.D.

Theorem 1. *Consider an elliptic principal bundle $\pi : X \to B$ with at least one non-zero Chern class and with $\epsilon_\pi = 0$. If B has no curves, then every rank-2 irreducible vector bundle V on X is a pull-back from B up to a twist by a line bundle.*

Proof. Let V be an irreducible rank-2 vector bundle on X.

First, we notice that by [Vu], the spectral space is of positive codimension, since otherwise by Lemma 2, all fibers of π would be jump fibers of V.

Next we claim the codimension 1 part of the spectral space Σ_V^1 must actually be irreducible.

Suppose this is not the case, so $\Sigma_V^1 = \Sigma_1 \cup \Sigma_2$ with $\Sigma_1 \neq \Sigma_2$. Using the previous Lemma 3, we can choose $\mathcal{L}_i \in Pic(X)$ such that $\Sigma_{\mathcal{L}_i} = \Sigma_i$ for $i = 1, 2$. As $\Sigma_1 \neq \Sigma_2$, it follows that the restriction to the general fiber of $V \otimes \mathcal{L}_1^\vee$ has exactly one section (up to constants). So $\pi_*(V \otimes \mathcal{L}_1^\vee)$ is a torsion-free sheaf of rank one. But then, as the canonical map

$$\pi^*(\pi_*(V \otimes \mathcal{L}_1^\vee)) \to V \otimes \mathcal{L}_1^\vee$$

is not identically zero, it follows that $V \otimes \mathcal{L}_1^\vee$ has a subsheaf of rank one, so it is reducible, absurd. We conclude that $\Sigma_1 = \Sigma_2 = \Sigma_V^1$. Hence, for a generic fiber E_b one has either $V_{|E_b} = \mathcal{L}_1 \oplus \mathcal{L}_1$ or $V_{|E_b} = F_2 \otimes \mathcal{L}_1$. The latter case is easily excluded by an argument similar to above; if this would be the case, then $\pi_*(V \otimes \mathcal{L}_1^\vee)$ would be a rank-one sheaf on B and its π^* would produce a rank-one subsheaf in $V \otimes \mathcal{L}_1$. So that $h^0(E_b, V \otimes \mathcal{L}_{1\,|E_b}^\vee) = 2$, for a generic fiber E_b.

Define $V_1 = V \otimes \mathcal{L}_1$ and let $\mathcal{V} = \pi_*(V_1)^{\vee\vee}$; notice that $\mathcal{V}$ is a reflexive sheaf on B, hence locally free as B is a surface. Since $\mathcal{V}$ is isomorphic to $\pi_*(V_1)$ outside an analytic subset of B, we see that we have a map between $\pi^*(\mathcal{V})$ and V_1 defined outside a proper, analytic subset Y of X. As any such Y is of codimension at least 2 since B has no curves, we see this map is actually defined on the whole X, and is obviously an isomorphism.

Q.E.D.

Example. We close by showing up some examples of elliptic principal bundles with non-zero Chern classes over a compact complex surface B without curves.

Let B be a torus with $a(B) = 0$ and with Picard number $\varrho(B) > 0$; such tori do exist - see for instance the Appendix in [ElFo]. Notice that since $a(B) = 0$ and as B is torus, it follows easily that B has no curves. Next choose some $\mathcal{L} \in Pic(B)$ with $c_1(L) \neq 0$ modulo torsion (which exists, as $\varrho(B) > 0$) and choose any elliptic curve $E = \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$. Then, using Proposition 7.1 in [Hö], one can associate to the element

$$(L \otimes \lambda_1) \oplus (\mathcal{O}_B \otimes \lambda_2) \in Pic(B) \otimes \Lambda$$

an elliptic principal bundle with Chern classes $(c_1(L), 0)$.

Remark. In the case when B is a projective manifold, a similar result was obtained by Verbitsky: see [Ve], Thm 1.4, pp 253.

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