

Welschinger invariants of real del Pezzo surfaces of degree ≥ 2

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Abstract

We compute the purely real Welschinger invariants, both original and modified, for all real del Pezzo surfaces of degree ≥ 2 . We show that under some conditions, for such a surface X and a real nef and big divisor class $D \in \text{Pic}(X)$, through any generic collection of $-DK_X - 1$ real points lying on a connected component of the real part $\mathbb{R}X$ of X one can trace a real rational curve $C \in |D|$. This is derived from the positivity of appropriate Welschinger invariants. We furthermore show that these invariants are asymptotically equivalent, in the logarithmic scale, to the corresponding genus zero Gromov-Witten invariants. Our approach consists in a conversion of Shoal-Shustin recursive formulas counting complex curves on the plane blown up at seven points and of Vakil's extension of the Abramovich-Bertram formula for Gromov-Witten invariants into formulas computing real enumerative invariants.

MSC2010: Primary 14N10. Secondary 14P05, 14N35.

Keywords: real rational curves, enumerative geometry, Welschinger invariants, Caporaso-Harris formula, Abramovich-Bertram-Vakil formula.

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The cashier gave us a sad smile, took a small hammer out of her mouth, and moving her nose slightly back and forth, she said:

- In my opinion, a seven comes after an eight, only if an eight comes after a seven.

(Daniil Kharms "A sonnet")

1 Introduction

Welschinger invariants can be regarded as real analogues of genus zero Gromov-Witten invariants. They were introduced in [25], [26] and count, with appropriate signs, the real rational pseudo-holomorphic curves which pass through given real collections of points in a given real rational symplectic four-fold. In the case of real del Pezzo surfaces, the Welschinger count is equivalent to the enumeration of real rational algebraic curves. In the present paper, we continue the study of purely real Welschinger invariants (that is, Welschinger invariants in the situation when all the point constraints are real) of del Pezzo surfaces. These invariants, as well as their modifications introduced in [16], can be used to prove the existence of interpolating real rational curves.

As we proved in [11, 12, 14, 15, 16], if X is either the plane blown up at a real points and b pairs of complex conjugate points, where $a + 2b \leq 6$, $b \leq 1$, or a minimal two-component real conic bundle over \mathbb{P}^1 , or a two-component real cubic surface, then the (modified) Welschinger invariants of X are positive and are asymptotically equivalent in the logarithmic scale to the corresponding Gromov-Witten invariants. These results not only prove the existence of interpolating real rational curves, but also show their abundance.

In the present paper, we extend these results to almost all del Pezzo surfaces of degree ≥ 2 (see Theorems 2 and 3) and, in particular, cover all the missing cases

in degree ≥ 3 . The main novelty is the use of nodal del Pezzo surfaces in a way which is similar to Vakil's approach to computation of Gromov-Witten invariants of the plane blown up at six points [23]. We derive new real Caporaso-Harris type formulas (see Theorems 4 and 5) and real analogues of Abramovich-Bertram-Vakil formula [1, 23] (see Theorems 6 and 7). These formulas combined together allow one to compute the purely real Welschinger invariants of all real del Pezzo surfaces of degree ≥ 2 from finitely many explicitly determined initial values (see Propositions 11 and 16).

As a technical tool, we introduce certain numbers (called ordinary w -numbers and sided w -numbers) that count with signs some specifically constrained real rational curves on real nodal del Pezzo surfaces, and exhibit a case when sided w -numbers are independent of the choice of point constraints (see Corollary 28).

A new phenomenon for del Pezzo surfaces of degree 2 is the absence of real rational curves in some cases (see Section 2.3). In this regard, note that in the case of multicomponent real surfaces, the original Welschinger invariants often happen to vanish (see [5, Proposition 3.3]). However, by (2) in Theorem 2, in many situations such a vanishing is not related to the non-existence of real rational curves, but only states that the real rational curves under consideration cancel each other when supplied with the original Welschinger signs.

Several results related to that of the present paper should be mentioned here. When we were working on Theorems 6 and 7, E. Brugallé and N. Puignau communicated to us similar real versions of Abramovich-Bertram-Vakil formula in the case of del Pezzo surfaces of degree ≥ 3 ; afterwards, they extended these formulas to the symplectic setting and arbitrary real rational symplectic 4-manifolds, see [5].²

J. Solomon [22, 10] suggested a completely different and very powerful recursive tool for computing Welschinger invariants of real blown ups of the projective plane. His recursion is based on analogues of Kontsevich-Manin axioms and WDVV equation, and involves the Gromov-Witten invariants and a finite number of initial values. However, the presence of plenty of terms of opposite signs (contrary to our formulas which, in most of cases, contain only non-negative terms) makes not evident the use of these recursive formulas for getting general statements on positivity and asymptotic behavior.

In an unpublished joint work with R. Rasdeaconu, J. Solomon has considered a kind of w -numbers which count curves subject to point constraints and odd tangency conditions to a fixed divisor, and showed that some combinations of such numbers are independent of point constraints. Let us underline that our sided w -numbers are defined via even tangency conditions and, in some cases, are individually invariant with respect to point constraints.

²When our text was finished, there appeared a new preprint by E. Brugallé [4], where he uses a slightly different approach and besides introduces floor diagrams with respect to a conic; this technique allows him to advance in the calculation of Gromov-Witten and Welschinger invariants of del Pezzo surfaces and to treat some del Pezzo surfaces of degrees 3, 2, and 1; in particular, he proves the non-negativity of Welschinger invariants in one of the two cases not covered by our Theorem 2.

The paper is organized as follows. Section 2 describes the main results of the paper; these results concern positivity and asymptotics of Welschinger invariants. In Section 3, we define ordinary and sided w -numbers and prove Caporaso-Harris type recursive formulas for these numbers in the case of real rational surfaces Y with a given real smooth rational curve E such that the classes $-K_Y$ and $-K_Y - E$ are nef (we call (Y, E) a monic log-del Pezzo pair). In Section 4, we consider nodal degenerations of del Pezzo surfaces and derive Abramovich-Bertram-Vakil type formulas relating the ordinary and sided w -numbers of the degeneration with Welschinger invariants. Section 5 contains the proof of the main results of the paper. The further two sections are devoted to other applications of the results of Sections 3 and 4: monotonicity behavior of Welschinger invariants is studied in Section 6, and Mikhalkin type congruences in Section 7. The index of notations can be found at the end of the paper.

Acknowledgments. A considerable part of the work on this text was done during authors' *Research in Pairs* stay at the *Mathematisches Forschungsinstitut Oberwolfach*. The final version of the paper was prepared during author's stay at the *Max-Planck-Institut für Mathematik* in Bonn. We thank these institutions for hospitality and excellent working conditions. The first two authors were partially funded by the ANR-09-BLAN-0039-01 grant of *Agence Nationale de la Recherche*. The third author enjoyed a support from the Israeli Science Foundation grant no. 448/09, from the German-Israeli Foundation research grant no. 1174-197.6/2011, and from the Hermann-Minkowski-Minerva Center for Geometry at the Tel Aviv University.

We are deeply grateful to E. Brugallé for the help in uncovering a mistake in the first draft of this paper. We are thankful to the referee for the helpful comments and suggestions on the manuscript that allowed us to improve the presentation.

2 Positivity and asymptotics of Welschinger invariants: statement of results

2.1 Purely real (modified) Welschinger invariants of real del Pezzo surfaces

Let X be a real del Pezzo surface (*i.e.*, a real smooth rational surface having an ample anticanonical class $-K_X$) with a non-empty real point set $\mathbb{R}X$. Let $D \in \text{Pic}(X)$ be a real effective divisor class with $D^2 \geq -1$, assumed to be primitive in $\text{Pic}(X)$ if $D^2 = 0$.

The set $R(X, D)$ of reduced irreducible rational curves in $|D|$ is a non-empty quasi-projective variety of pure dimension $-DK_X - 1$ with nodal curves as generic elements (see, for instance, [17, Lemma 4]). Denote by $\mathbb{R}R(X, D)$ the set of real rational curves in $R(X, D)$.

We intend to count curves in $\mathbb{R}R(X, D)$ that match a suitable number of real point constraints. If $-DK_X > 1$, we pick a generic collection \mathbf{w} of $-DK_X - 1$ points in $\mathbb{R}X$. Since a curve in $\mathbb{R}R(X, D)$ passing through \mathbf{w} must contain all these points in its (unique) real one-dimensional component, we have to suppose that \mathbf{w} lies in one connected component of $\mathbb{R}X$. Notice also that if $-DK_X = 1$, each curve in the (finite) set $\mathbb{R}R(X, D)$ has a one-dimensional real branch. Indeed, a real curve with a finite real part must have an even self-intersection, whereas $D^2 = -DK_X \pmod{2}$ by the adjunction formula.

To introduce (*modified*) *purely real Welschinger numbers*, let us fix a connected component F of the real part $\mathbb{R}X$ of X and, in addition, a conjugation invariant class $\varphi \in H_2(X \setminus F, \mathbb{Z}/2)$. If $-DK_X = 1$, we set $\mathbb{R}R(X, D, F) = \{C \in \mathbb{R}R(X, D) : |C \cap F| = \infty\}$ and put

$$W(X, D, F, \varphi) = \sum_{C \in \mathbb{R}R(X, D, F)} (-1)^{s(C) + C_{1/2} \circ \varphi},$$

where $C_{1/2}$ is the image of one of the halves of $\mathbb{P}^1 \setminus \mathbb{R}P^1$ by the normalization map $\mathbb{P}^1 \rightarrow C$, and $s(C)$ is the number of real solitary nodes of C . If $-DK_X > 1$, we pick a generic collection \mathbf{w} of $-DK_X - 1$ points of F , set $\mathbb{R}R(X, D, \mathbf{w}) = \{C \in \mathbb{R}R(X, D) : C \supset \mathbf{w}\}$, and put

$$W(X, D, F, \varphi, \mathbf{w}) = \sum_{C \in \mathbb{R}R(X, D, \mathbf{w})} (-1)^{s(C) + C_{1/2} \circ \varphi}. \quad (1)$$

The following statement is a version of the Welschinger theorem [25] (*cf.* also [17, Theorem 7]).

Theorem 1 (1) *If $-DK_X > 1$, the number $W(X, D, F, \varphi, \mathbf{w})$ does not depend on the choice of a generic collection \mathbf{w} of $-DK_X - 1$ points in F .*

(2) *With the given data X, D, F, φ as above, let X_t , $t \in [0, 1]$, $X_0 = X$, be a smooth family of smooth real rational surfaces with non-empty real part such that for all but finitely many $t \in [0, 1]$, X_t is a real del Pezzo surface. Let $\Theta_t : X_0 \rightarrow X_t$, $t \in [0, 1]$, $\Theta_0 = \text{Id}$, be a smooth family of conjugation invariant C^∞ -diffeomorphisms that trivializes our family of surfaces. Then*

$$W(X, D, F, \varphi, \mathbf{w}) = W(X_1, (\Theta_1)_*(D), \Theta_1(F), (\Theta_1)_*(\varphi), \Theta_1(\mathbf{w})).$$

In the sequel we write $W(X, D, F, \varphi)$ omitting the notation of point constraints.

2.2 Real del Pezzo surfaces of degree 2

The classification of real del Pezzo surfaces up to equivariant deformations is well known and goes back to A. Comessatti; the details can be found, for example, in [7, Section 17.3].

Here, we recall this classification in the case of del Pezzo surfaces of degree 2. According to this classification, a real del Pezzo surface of degree 2 is determined up to equivariant deformation by the topology of its real part.

The anticanonical linear system on a real del Pezzo surface X of degree 2 defines a double covering $X \rightarrow \mathbb{P}^2$ branched in a nonsingular real quartic curve $Q_X \subset \mathbb{P}^2$, and thus identifies X with a hypersurface defined in the weighted projective space $P^3(1, 1, 1, 2)$ by an equation $u^2 = \varepsilon f_X(x, y, z)$, where f_X is a real defining polynomial of Q_X and $\varepsilon = \pm 1$. Therefore, as a topological space, $\mathbb{R}X$ is the result of gluing two copies of $\mathbb{R}f_{X,\varepsilon} = \{p \in \mathbb{R}\mathbb{P}^2 : \varepsilon f_X(p) \geq 0\}$ along their common boundary, if this boundary is non-empty, and the disjoint union of two copies otherwise. Below we always choose the sign for f_X so that $\mathbb{R}f_{X,-}$ is non-orientable.

As is known, the real part of a real non-singular quartic is isotopic in $\mathbb{R}P^2$ either to the union of $0 \leq q \leq 4$ null-homologous circles placed outside each other (denote this isotopy type by $\langle q \rangle$), or to a pair of null-homologous circles placed one inside the other (denote this isotopy type by $1\langle 1 \rangle$). In accordance with this notation and the above sign-convention, the deformation types of real del Pezzo surfaces X of degree 2 with $\mathbb{R}X \neq \emptyset$ are denoted below by $\langle 0 \rangle^-$, $\langle q \rangle^\varepsilon$, $1 \leq q \leq 4$, and $1\langle 1 \rangle^\varepsilon$. For example, the deformation type of the plane blown up at a real points and b pairs of complex conjugate points, $a + 2b = 7$, which we denote by $\mathbb{P}_{a,b}^2$, coincides with $\langle 4 - b \rangle^-$.

For surfaces X of type $\langle q \rangle^\varepsilon$, $1 \leq q \leq 4$, $\langle 0 \rangle^-$, and $1\langle 1 \rangle^+$, the choice of a connected component F of $\mathbb{R}X$ does not affect the computation of Welschinger invariants; indeed, for X of type $\langle 0 \rangle^-$ the two connected components of $\mathbb{R}X$ are interchanged by the deck transformation of the above double covering, while for other types of X with disconnected $\mathbb{R}X$ such an independence follows from Theorem 1(2).

As to surfaces X of type $1\langle 1 \rangle^-$, they have two connected components: one, which we denote F^o , is orientable, and the other one, F^{no} , is not.

Notice also that the 28 bitangents of Q_X lift into the 56 curves in X with self-intersection -1 , and that the curves of the linear system $|-K_X|$ are the pull-backs of the straight lines in \mathbb{P}^2 .

2.3 Main results

Let X be a real del Pezzo surface. Denote by $\text{Pic}^{\mathbb{R}}(X)$ the subgroup of $\text{Pic}(X)$ formed by real divisor classes of X , and denote by $\text{bh} : \text{Pic}^{\mathbb{R}}(X) \rightarrow H_1(\mathbb{R}X; \mathbb{Z}/2)$ the natural homomorphism which sends each real effective divisor class D that is represented by a real reduced curve, say C , to $[\mathbb{R}C \cap \mathbb{R}X] \in H_1(\mathbb{R}X, \mathbb{Z}/2)$ (cf. [3, 24]). If \mathcal{F} is a union of some connected components of $\mathbb{R}X$, then denote by $\text{bh}_{\mathcal{F}}$ the composition of bh with the projection $H_1(\mathbb{R}X; \mathbb{Z}/2) \rightarrow H_1(\mathcal{F}; \mathbb{Z}/2)$.

Let F be a connected component of $\mathbb{R}X$. We say that a real effective divisor class D on X is F -compatible, if $\text{bh}_{\mathbb{R}X \setminus F}(D) = 0$. It is clear that if a real effective

divisor class D is not F -compatible, then $W(X, D, F, \varphi)$ vanishes for any conjugation invariant class $\varphi \in H_2(X \setminus F, \mathbb{Z}/2)$.

Theorem 2 *Let X be a real del Pezzo surface of degree ≥ 2 with a non-empty real point set, and let F be a connected component of $\mathbb{R}X$. Assume that $F \neq S^2$ if X is of degree 2 and $\mathbb{R}X$ is S^2 or $S^2 \sqcup \mathbb{R}P^2 \# \mathbb{R}P^2$. Then, for any F -compatible nef and big divisor class D on X , one has*

$$W(X, D, F, [\mathbb{R}X \setminus F]) > 0 . \quad (2)$$

In particular, through any collection of $-K_X D - 1$ points of F , one can trace a real rational curve $C \in |D|$. Furthermore,

$$\log W(X, nD, F, [\mathbb{R}X \setminus F]) = -K_X D \cdot n \log n + O(n), \quad n \rightarrow +\infty. \quad (3)$$

The proof of Theorem 2 is presented in Sections 5.5.1 and 5.5.2.

Remark 1 Theorem 2 covers all the cases studied in [12, 14, 15, 16, 21] (notice that the proof of [16, Theorem 2] in the journal version contains an inaccuracy in the case of the plane blown up at two real points and two pairs of complex conjugate points; this inaccuracy is corrected in the ArXiv version of [16]).

Theorem 3 *Let X be a real del Pezzo surface of degree 2 with $\mathbb{R}X = S^2$. Then,*

- (i) *for any real effective divisor class D on X , we have $W(X, D, \mathbb{R}X, 0) \geq 0$;*
- (ii) *the big and nef real effective divisor classes D on X such that $W(X, D, \mathbb{R}X, 0) > 0$ form a subsemigroup in $\text{Pic}(X)$; this subsemigroup contains $-mK_X$ with $m \geq 2$ and all the divisor classes D' and $D' - K_X$, where D' is big, nef, and disjoint from a pair of complex conjugate (-1) -curves;*
- (iii) *if a big and nef real effective divisor class D on X satisfies $W(X, D, \mathbb{R}X, 0) > 0$, then*

$$\log W(X, nD, \mathbb{R}X, 0) = -K_X D \cdot n \log n + O(n), \quad n \rightarrow +\infty; \quad (4)$$

- (iv) *if a big and nef real effective divisor class D on X satisfies $D^2 \leq 2$, then $W(X, D, \mathbb{R}X, 0) = 0$ as long as $-K_X D \neq 4$.*

The proof is given in Section 5.6.

Notice that Theorem 3 (iv) implies the following statement: for a real Del Pezzo surface X of degree 2 with $\mathbb{R}X = S^2$ there are infinitely many nef and big real divisors D such that $W(X, D, \mathbb{R}X, 0) = 0$. Indeed, represent X as an ellipsoid (that is, a real quadric with spherical real part) blown up at 3 pairs of complex conjugate points and choose the basis $L_1, L_2, E_1, \dots, E_6$ of $\text{Pic}(X)$ where L_1, L_2 are generators of the quadric and E_1, \dots, E_6 are the exceptional divisors of the blow up; then, each

divisor $D = m(L_1 + L_2) - n(E_1 + \dots + E_6)$, where $m^2 - 3n^2 = 1$ and $m \neq 7$, is real and nef, and it satisfies $D^2 = 2$ and $-K_X D = 4m - 6n \neq 4$.

Such a vanishing is sometimes "sharp": if the only oval of a real plane quartic of type $\langle 1 \rangle$ is convex, then there is no real tangent through a point inside the oval, and hence there are no real rational curves $C \in | -K_X |$ at all.

The case $\mathbb{R}X = S^2 \sqcup \mathbb{R}P^2 \# \mathbb{R}P^2$ and $F = S^2$, not covered by the above theorems, is discussed in Section 5.7.

The following table contains the values of Welschinger invariants $W(X, D, F, \varphi)$ for $D = -K_X$ or $-2K_X$ and $\varphi = 0$ or $\varphi = \varphi_F = [\mathbb{R}X \setminus F]$ (for surfaces of types $\langle 2 \rangle^+$, $\langle 3 \rangle^+$ or $\langle 4 \rangle^+$, the invariants do not depend on the choice of F among the components of $\mathbb{R}X$).

D	φ	$\langle 4 \rangle^-$	$\langle 3 \rangle^-$	$\langle 2 \rangle^-$	$\langle 1 \rangle^-$	$\langle 0 \rangle^-$	$1\langle 1 \rangle^+$	$1\langle 1 \rangle^-, F^{no}$	$1\langle 1 \rangle^-, F^o$	$\langle 1 \rangle^+$	$\langle 2 \rangle^+$	$\langle 3 \rangle^+$	$\langle 4 \rangle^+$
$-K_X$	0	8	6	4	2	0	2	0	0	0	-2	-4	-6
$-K_X$	φ_F	8	6	4	2	0	2	4	0	0	2	4	6
$-2K_X$	0	224	128	64	24	0	32	0	0	8	0	0	0
$-2K_X$	φ_F	224	128	64	24	32	32	48	16	8	16	32	64

Notice that the original Welschinger invariants ($\varphi = 0$) may take negative values or vanish for the multi-component del Pezzo surfaces. This reflects the following general phenomenon.

Proposition 2 *Let X be a real del Pezzo surface of degree ≥ 2 with disconnected real point set, let F and F' be two distinct connected components of $\mathbb{R}X$, and let $\varphi \in H_2(X \setminus (F \cup F'); \mathbb{Z}/2)$ be a conjugation invariant class. Then, $W(X, D, F, \varphi) = 0$ for any big and nef real effective divisor class D on X such that $-K_X D \geq 3$.*

This proposition immediately follows from the formulas (27) and (49) proved, respectively, in Sections 3.10 and 4.3.4. In a more general setting, the vanishing statement given by Proposition 2 is found in [5, Proposition 3.3].

3 Recursive formulas for w -numbers of real monic log-del Pezzo pairs

3.1 Surfaces under consideration

Let Y be a smooth rational surface which is a blow-up of \mathbb{P}^2 , and let $E \subset Y$ be a smooth rational curve. Suppose that $-K_Y$ is positive on all curves different from E and $K_Y E \geq 0$, and that the log-anticanonical class $-(K_Y + E)$ is nef, effective, and satisfies $(K_Y + E)^2 = 0$. We call such a pair (Y, E) a *monic log-del Pezzo pair*. Throughout Section 3, we assume that (Y, E) is a monic log-del Pezzo pair.

Observe that $-(K_Y + E)E = 2$, $E^2 \leq -2$, and $K_Y(K_Y + E) = 2$, so that the latter implies, once more by adjunction, that $| -(K_Y + E) |$ is a one-dimensional

linear system, whose generic element is a smooth rational curve. This linear system contains precisely two smooth curves L', L'' (quadratically) tangent to E , and $4 - E^2$ reducible curves, all of type $L_1 + L_2$ where $L_1^2 = L_2^2 = -1$, $L_1 L_2 = 1$, $L_1 E = L_2 E = 1$. In particular, it provides a conic bundle structure on Y and shows that Y can be regarded as the plane blown up at ≥ 6 points on a smooth conic (E is the strict transform of the conic) and at one more point outside the conic. We will assume that the blown up points are in general position subject to the above allocation with respect to the conic. The curves L' and L'' are called *supporting curves*.

Introduce the sets

$$\mathcal{E}(E) = \{E' \in \text{Pic}(Y) : (E')^2 = -1, E'K_Y = -1, E'E > 0\} .$$

$$\mathcal{E}(E)^{\perp D} = \{E' \in \mathcal{E}(E) : E'D = 0\}, \quad D \in \text{Pic}(Y) .$$

Suppose that (Y, E) is equipped with a real structure such that $\mathbb{R}Y \supset \mathbb{R}E \neq \emptyset$. Denote by F the connected component of $\mathbb{R}Y$ containing $\mathbb{R}E$. We also choose a conjugation invariant class $\varphi \in H_2(Y \setminus F, \mathbb{Z}/2)$.

Quadruples (Y, E, F, φ) as above are called *basic quadruples*.

3.2 Some notations

Let \mathbb{Z}_+^∞ be the direct sum of countably many additive semigroups $\mathbb{Z}_+ = \{k \in \mathbb{Z} \mid k \geq 1\}$, labeled by the positive integer numbers, with the basis formed by the summand generators e_i , $i = 1, 2, \dots$. For $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{Z}_+^\infty$, put

$$\|\alpha\| = \sum_{i=1}^{\infty} \alpha_i, \quad I\alpha = \sum_{i=1}^{\infty} i\alpha_i, \quad I^\alpha = \prod_{i=1}^{\infty} i^{\alpha_i}, \quad \alpha! = \prod_{i=1}^{\infty} \alpha_i! .$$

For $\alpha, \beta \in \mathbb{Z}_+^\infty$, we write $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for any positive integer number i . For $\alpha^{(0)}, \dots, \alpha^{(m)}, \alpha \in \mathbb{Z}_+^\infty$ such that $\alpha^{(0)} + \dots + \alpha^{(m)} \leq \alpha$, put

$$\binom{\alpha}{\alpha^{(0)}, \dots, \alpha^{(m)}} = \frac{\alpha!}{\alpha^{(0)}! \dots \alpha^{(m)}! (\alpha - \alpha^{(0)} - \dots - \alpha^{(m)})!} .$$

Introduce also the semigroups

$$\begin{aligned} \mathbb{Z}_+^{\infty, \text{ odd}} &= \text{Span}\{e_{2i+1} : i \geq 0\} , \\ \mathbb{Z}_+^{\infty, \text{ even}} &= \text{Span}\{e_{2i} : i \geq 1\} , \\ \mathbb{Z}_+^{\infty, \text{ odd} \cdot \text{ even}} &= \text{Span}\{e_{4i+2} : i \geq 0\} . \end{aligned}$$

3.3 Divisor classes

Let Σ be a smooth real surface. We denote by $\text{Pic}^{\mathbb{R}}(\Sigma)$ the subgroup of $\text{Pic}(\Sigma)$ formed by real divisor classes of Σ and denote by $\text{Pic}_+^{\mathbb{R}}(\Sigma)$ the subsemigroup of $\text{Pic}^{\mathbb{R}}(\Sigma)$ generated by effective real divisor classes. Let $E \subset \Sigma$ be a smooth real curve. Put

$\text{Pic}_{++}(\Sigma, E)$ to be the subsemigroup of $\text{Pic}(\Sigma)$ generated by complex irreducible curves C such that $CE \geq 0$. The involution of complex conjugation $\text{Conj} : \Sigma \rightarrow \Sigma$ naturally acts on $\text{Pic}(\Sigma)$ and preserves $\text{Pic}_{++}(\Sigma, E)$. Denote by $\text{Pic}_{++}^{\mathbb{R}}(\Sigma, E)$ the disjoint union of the sets

$$\{D \in \text{Pic}_{++}(\Sigma, E) : \text{Conj } D = D\}$$

and

$$\{\{D_1, D_2\} \in \text{Sym}^2(\text{Pic}_{++}(\Sigma, E)) : \text{Conj } D_1 = D_2\} .$$

For an element $\mathcal{D} \in \text{Pic}_{++}^{\mathbb{R}}(\Sigma, E)$, define $[\mathcal{D}] \in \text{Pic}_{++}(\Sigma, E)$ by

$$[\mathcal{D}] = \begin{cases} D, & \mathcal{D} = D, \text{ a divisor class,} \\ D_1 + D_2, & \mathcal{D} = \{D_1, D_2\}, \text{ a pair of divisor classes.} \end{cases}$$

For a element $\mathcal{D} \in \text{Pic}_{++}^{\mathbb{R}}(\Sigma, E)$ and a vector $\beta \in \mathbb{Z}_+^{\infty}$, put

$$R_{\Sigma}(\mathcal{D}, \beta) = -[\mathcal{D}](K_{\Sigma} + E) + \|\beta\| - \begin{cases} 1, & \mathcal{D} = D, \text{ a divisor class,} \\ 2, & \mathcal{D} = \{D_1, D_2\}, \text{ a pair of divisor classes.} \end{cases}$$

3.4 Families of real curves

Let (Y, E) be a real monic log-del Pezzo pair. An *admissible tuple* $(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat})$ consists of an element \mathcal{D} in $\text{Pic}_{++}^{\mathbb{R}}(Y, E)$, vectors $\alpha, \beta^{\text{re}}, \beta^{\text{im}} \in \mathbb{Z}_+^{\infty}$ satisfying $I(\alpha + \beta^{\text{re}} + 2\beta^{\text{im}}) = [\mathcal{D}]E$, and a sequence $\mathbf{p}^{\flat} = \{p_{i,j} : i \geq 1, 1 \leq j \leq \alpha_i\}$ of $\|\alpha\|$ distinct real generic points on E . Denote by $V_Y^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat})$ the closure in the linear system $[[\mathcal{D}]]$ of the family of real reduced curves C such that

- (i) if $\mathcal{D} = D$, a divisor class, then $C \in |D|$ is an irreducible over \mathbb{C} rational curve; if $\mathcal{D} = \{D_1, D_2\}$, a pair of divisor classes, then $C = C_1 \cup C_2$, where $C_1 \in |D_1|$, $C_2 \in |D_2|$ are distinct, irreducible, rational, complex conjugate curves;
- (ii) $C \cap E$ consists of \mathbf{p}^{\flat} and of $\|\beta^{\text{re}} + 2\beta^{\text{im}}\|$ other points: $\|\beta^{\text{re}}\|$ of them real, and $2\|\beta^{\text{im}}\|$ form pairs of complex conjugate points;
- (iii) at each point of $C \cap E$, the curve C has one local branch, and the intersection multiplicities of C and E are described as follows:
 - $(C \cdot E)(p_{i,j}) = i$ for all $i \geq 1, 1 \leq j \leq \alpha_i$,
 - for each $i \geq 1$, there are β_i^{re} real points $q \in (C \cap E) \setminus \mathbf{p}^{\flat}$ such that $(C \cdot E)(q) = i$;
 - for each $i \geq 1$ there are β_i^{im} pairs q, q' of complex conjugate points of $C \cap E$ such that $(C \cdot E)(q) = (C \cdot E)(q') = i$.

If $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ a divisor class, introduce also the variety $V_Y(D, \alpha, \beta, \mathbf{p}^b)$ which is the closure in $|D|$ of the family of complex reduced irreducible rational curves C such that $C \cap E$ consists of \mathbf{p}^b and of $\|\beta\|$ other points, at each point of $C \cap E$, the curve C has one local branch, and the intersection multiplicities of C and E are as follows:

- $(C \cdot E)(p_{i,j}) = i$ for all $i \geq 1, 1 \leq j \leq \alpha_i$,
- for each $i \geq 1$, there are β_i points $q \in (C \cap E) \setminus \mathbf{p}^b$ such that $(C \cdot E)(q) = i$.

Lemma 3 *If $\mathcal{D} = D$ is a divisor class and $V_Y^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b)$ is nonempty, then $R_Y(\mathcal{D}, \beta^{\text{re}} + 2\beta^{\text{im}}) \geq 0$, and each component of $V_Y^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b)$ has dimension $\leq R_Y(\mathcal{D}, \beta^{\text{re}} + 2\beta^{\text{im}})$. Moreover, a generic element C of any component of $V_Y^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b)$ of dimension $R_Y(\mathcal{D}, \beta^{\text{re}} + 2\beta^{\text{im}})$ is an immersed curve, nonsingular along E . If, in addition, $E^2 \geq -3$, then C is nodal.*

Proof. If D is a multiple of a divisor class orthogonal to $K_Y + E$, then $V_Y^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b)$ cannot be nonempty, since such a linear system contains only non-reduced curves. In the other case, the statement follows from [19, Proposition 2.1]. \square

Suppose that $R_Y(\mathcal{D}, \beta^{\text{re}} + 2\beta^{\text{im}}) \geq 0$. Pick a set $\mathbf{p}^{\#}$ of $R_Y(\mathcal{D}, \beta^{\text{re}} + 2\beta^{\text{im}})$ generic points of $F \setminus E$ and denote by $V_Y^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b, \mathbf{p}^{\#})$ the set of curves $C \in V_Y^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b)$ passing through $\mathbf{p}^{\#}$.

Lemma 4 *Assume that $V_Y^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b)$ is nonempty.*

(1) *If $\mathcal{D} = D$, a divisor class, then $V_Y^{\mathbb{R}}(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b, \mathbf{p}^{\#})$ is a finite set of real immersed irreducible rational curves which are nonsingular along E .*

(2) *If $\mathcal{D} = \{D_1, D_2\}$, a pair of divisor classes, then $V_Y^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b, \mathbf{p}^{\#})$ is finite, and it is nonempty only if $\alpha = \beta^{\text{re}} = 0$, $R_Y(\mathcal{D}, 2\beta^{\text{im}}) = 0$, and $\mathbf{p}^b = \mathbf{p}^{\#} = \emptyset$.*

Proof. By Lemma 3 we have to show only that $R_Y(\mathcal{D}, 2\beta^{\text{im}}) = 0$ is necessary for the nonemptiness of $V_Y^{\mathbb{R}}(\mathcal{D}, 0, 0, \beta^{\text{im}}, \emptyset, \mathbf{p}^{\#})$ with $\mathcal{D} = \{D_1, D_2\}$, and the proof of this fact literally coincides with the proof of [16, Lemma 3(2)]. \square

Lemma 5 *The only nonempty sets $V_Y^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b)$ for admissible tuples $(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b)$ such that $R_Y(\mathcal{D}, \beta^{\text{re}} + 2\beta^{\text{im}}) = 0$ are the following ones:*

(1) *if $\mathcal{D} = D$ is a divisor class and $I(\alpha + \beta^{\text{re}} + 2\beta^{\text{im}}) = DE > 0$,*

(1i) *$V_Y^{\mathbb{R}}(E', 0, e_1, 0, \emptyset)$ consists of one element, where E' is a real (-1) -curve crossing E ;*

(1ii) *$V_Y^{\mathbb{R}}(-(K_Y + E), 0, e_2, 0, \emptyset)$ consists of two elements L', L'' , if the supporting curves L', L'' are both real;*

- (1iii) $V_Y^{\mathbb{R}}(-(K_Y + E), e_1, e_1, 0, \mathbf{p}^b)$ consists of one element;
- (1iv) $V_Y^{\mathbb{R}}(D, \alpha, 0, 0, \mathbf{p})$ consists of one element, if $(K_Y + E)D = -1$, $I\alpha = DE$;
- (2) if $\mathcal{D} = \{D_1, D_2\}$ is a pair of divisor classes and $I(\alpha + \beta^{\text{re}} + 2\beta^{\text{im}}) = [\mathcal{D}]E > 0$,
 - (2i) $V_Y^{\mathbb{R}}(\{E'_1, E'_2\}, 0, 0, e_1, \emptyset)$ consists of one element, where E'_1, E'_2 are complex conjugate (-1) -curves crossing E ;
 - (2ii) $V_Y^{\mathbb{R}}(\{-(K_Y + E), -(K_Y + E)\}, 0, 0, e_2, \emptyset)$ consists of one element $\{L', L''\}$, if L', L'' are complex conjugate;
- (3) if $I(\alpha + \beta^{\text{re}} + 2\beta^{\text{im}}) = [\mathcal{D}]E = 0$,
 - (3i) $V_Y^{\mathbb{R}}(E', 0, 0, 0, \emptyset)$ consists of one element, where E' of a real (-1) -curve disjoint from E ;
 - (3ii) $V_Y^{\mathbb{R}}(\{E'_1, E'_2\}, 0, 0, 0, \emptyset)$ consists of one element, where E'_1, E'_2 are complex conjugate (-1) -curves disjoint from E .

Proof. Straightforward. □

3.5 Deformation diagrams and CH position

3.5.1 Deformation diagrams

Let (Y, E) be a monic log-del Pezzo pair such that Y and E are real and $\mathbb{R}E \neq \emptyset$. Denote by F the connected component of $\mathbb{R}Y$ containing $\mathbb{R}E$ and pick a conjugation invariant class $\varphi \in H_2(\mathbb{R}Y \setminus F, \mathbb{Z}/2)$. Let $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b)$ be an admissible tuple, where $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ is a divisor class and $R_Y(D, \beta^{\text{re}} + 2\beta^{\text{im}}) > 0$. Pick a set $\tilde{\mathbf{p}}^{\sharp}$ of $R_Y(D, \beta^{\text{re}} + 2\beta^{\text{im}}) - 1$ generic real points of $F \setminus E$, a generic real point $p \in E \setminus \mathbf{p}^b$, and a smooth real algebraic curve germ Λ , crossing E transversally at p . Denote by $\Lambda^+ = \{p(t) : t \in (0, \varepsilon)\}$ a parameterized connected component of $\Lambda \setminus \{p\}$ with $\lim_{t \rightarrow 0} p(t) = p$. There exists $\varepsilon_0 > 0$ such that, for all $t \in (0, \varepsilon_0]$, the sets $V_Y(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b, \tilde{\mathbf{p}}^{\sharp} \cup \{p(t)\})$ are finite, their elements remain immersed, nonsingular along E as t runs over the interval $(0, \varepsilon_0]$, and the closure in $V_Y(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b)$ of the family

$$V = \bigcup_{t \in (0, \varepsilon_0]} V_Y(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b, \tilde{\mathbf{p}}^{\sharp} \cup \{p(t)\}) \quad (5)$$

is a union of real algebraic arcs, disjoint for $t > 0$. This closure is called a *deformation diagram* of $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b, \tilde{\mathbf{p}}^{\sharp}, p)$, cf. [16, Section 3.3], and the real algebraic arcs under consideration are called *branches* of the deformation diagram. The elements of $V_Y(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b, \tilde{\mathbf{p}}^{\sharp} \cup \{p(1)\})$ are called *leaves* of the deformation diagram, and the elements of $\overline{V} \setminus V$ are called *roots* of the deformation diagram.

Lemma 6 *Each connected component of a deformation diagram of $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \tilde{\mathbf{p}}^{\sharp}, p)$ contains exactly one root. Each root is either a generic member of an $(R_Y(D, \beta^{\text{re}} + 2\beta^{\text{im}}) - 1)$ -dimensional component of one of the families*

$$V_Y(D, \alpha + e_j, \beta^{\text{re}} - e_j, \beta^{\text{im}}, \mathbf{p}^{\flat} \cup \{p\}, \tilde{\mathbf{p}}^{\sharp}) ,$$

where j is a natural number such that $\beta_j^{\text{re}} > 0$, or a reducible curve having E as a component.

Proof. The statement follows from [19, Proposition 2.6]. \square

For any root ρ of a deformation diagram, the leaves belonging to the connected component of ρ is said to be *generated* by ρ .

3.5.2 CH position

Pick a divisor class $D_0 \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ and put $N = \dim |D_0|$. Note that the set

$$\text{Prec}(D_0) = \{D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E) : D \text{ is a divisor class and } D_0 \geq D\}$$

is finite, and we have $\dim |D| \leq N$ for each $D \in \text{Prec}(D_0)$. Furthermore, for each nonempty variety $V_Y^{\mathbb{R}}(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat})$ with $D \in \text{Prec}(D_0)$, we have

$$\|\alpha\| + R_Y(D, \beta^{\text{re}} + 2\beta^{\text{im}}) \leq N .$$

Lemma 7 (cf. [16, Lemma 10]) *Let $D_0 \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ be a divisor class with $N = \dim |D_0| > 0$. Then, there exists a sequence $\Lambda(D_0) = (\Lambda_i)_{i=1, \dots, N}$ of N disjoint smooth real algebraic arcs in Y , which are parameterized by $t \in [-1, 1] \mapsto p_i(t) \in \Lambda_i$, such that $p_i(0) \in E$, the arcs Λ_i are transverse to E at $p_i(0)$, $i = 1, \dots, N$, and the following condition holds:*

for any admissible tuple $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat})$, any disjoint subsets $J^{\flat}, J^{\sharp} \subset \{1, \dots, N\}$, any positive integer $k \leq N$, and any sequence $\sigma = (\sigma_i)_{i=1, \dots, N}$ such that

- (i) $D \in \text{Prec}(D_0)$,
- (ii) $R_Y(D, \beta^{\text{re}} + 2\beta^{\text{im}}) > 0$,
- (iii) $i < k < j$ for all $i \in J^{\flat}$, $j \in J^{\sharp}$,
- (iv) the number of elements in J^{\sharp} is equal to $R_Y(D, \beta^{\text{re}} + 2\beta^{\text{im}}) - 1$,
- (v) $\mathbf{p}^{\flat} = \{p_i(0) : i \in J^{\flat}\}$,
- (vi) $\sigma_i = \pm 1$ for any integer $1 \leq i \leq N$,

the closure of the family

$$\bigcup_{t \in (0, 1]} V_Y(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \tilde{\mathbf{p}}^{\sharp} \cup \{p_k(\sigma_k t)\}) ,$$

where $\tilde{\mathbf{p}}^{\sharp} = \{p_j(\sigma_j)\}_{j \in J^{\sharp}}$, is a deformation diagram of $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \tilde{\mathbf{p}}^{\sharp}, p_k(0))$.

Proof. Take a sequence $\widehat{\Lambda}_i$, $i = 1, \dots, N$, of disjoint smooth real algebraic arcs in Y , which are parameterized by $t \in [-1, 1] \mapsto p_i(t) \in \Lambda_i$, such that $(p_i(0))_{i=1, \dots, N}$ is a generic sequence of points in E , and the arcs $\widehat{\Lambda}_i$ are transverse to E at $p_i(0)$, $i = 1, \dots, N$. We will inductively shorten these arcs in order to satisfy the condition required in Lemma.

Take an integer $1 \leq k \leq N$, and suppose that we have already constructed arcs $\Lambda_1, \dots, \Lambda_{k-1}$ parameterized respectively by intervals $[0, \varepsilon_i]$, $1 \leq i < k$. There are finitely many admissible tuples $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat})$, subsets $J^{\flat}, J^{\sharp} \subset \{1, \dots, N\}$, and sequences σ satisfying the restrictions (i)-(vi) above. Given such a datum $D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, J^{\flat}, J^{\sharp}, \sigma$, we take a small positive number ε_k such that the closure of the family

$$\bigcup_{t \in (0, \varepsilon_k]} V_Y(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \widetilde{\mathbf{p}}^{\sharp} \cup \{p_k(\sigma_k t)\}) ,$$

where $\widetilde{\mathbf{p}}^{\sharp} = \{p_i(\varepsilon_i)\}_{1 \leq i < k}$, is a deformation diagram of $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \widetilde{\mathbf{p}}^{\sharp}, p_k(0))$, and put

$$\Lambda_k(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, J^{\flat}, J^{\sharp}, \sigma) = \bigcup_{t \in [-\varepsilon_k, \varepsilon_k]} p_k(t).$$

Then, we define

$$\Lambda_k = \bigcap_{(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, J^{\flat}, J^{\sharp}, \sigma)} \Lambda_k(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, J^{\flat}, J^{\sharp}, \sigma).$$

It remains now to reparameterize by the interval $[-1, 1]$ the arcs $\Lambda_1, \dots, \Lambda_N$ obtained. \square

Take a divisor class $D_0 \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ such that $N = \dim |D_0| > 0$ and a sequence of arcs $(\Lambda_i)_{i=1, \dots, N}$ as in Lemma 7. Given a sequence $\sigma = (\sigma_i)_{i=1, \dots, N}$ of ± 1 and two subsets $J^{\flat}, J^{\sharp} \subset \{1, \dots, N\}$ such that $i < j$ for all $i \in J^{\flat}, j \in J^{\sharp}$, we say that the pair of point sequences

$$\mathbf{p}^{\flat} = \{p_i(0) : i \in J^{\flat}\}, \quad \mathbf{p}^{\sharp} = \{p_j(\sigma_j) : j \in J^{\sharp}\}$$

is in D_0 -CH position. A pair of point sequences

$$(\mathbf{p}^{\flat})' = \{p_i(0) : i \in (J^{\flat})'\}, \quad (\mathbf{p}^{\sharp})' = \{p_j(\sigma_j) : j \in (J^{\sharp})'\}$$

in D_0 -CH position is said to be a *predecessor* of a pair of point sequences

$$\mathbf{p}^{\flat} = \{p_i(0) : i \in J^{\flat}\}, \quad \mathbf{p}^{\sharp} = \{p_j(\sigma_j) : j \in J^{\sharp}\}$$

in D_0 -CH position if $(J^{\sharp})' = \{j \in J^{\sharp} : j > k\}$ for a certain integer k .

Let $(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat})$ be an admissible tuple such that $\mathcal{D} = D$ is a divisor class. Choose a sequence \mathbf{p}^{\sharp} of $R_Y(\mathcal{D}, \beta^{\text{re}} + 2\beta^{\text{im}})$ points in F , and assume that the pair of point sequences $\mathbf{p}^{\flat}, \mathbf{p}^{\sharp}$ is in a D_0 -CH position. Then, $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp})$ is

called a D_0 -proper tuple. The elements of $V_Y^{\mathbb{R}}(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp})$ are called *interpolating curves* constrained by the D_0 -proper tuple $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp})$. We say that a D_0 -proper tuple $(D', \alpha', (\beta^{\text{re}})', (\beta^{\text{im}})', (\mathbf{p}^{\flat})', (\mathbf{p}^{\sharp})')$ *precedes* a D_0 -proper tuple $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp})$ if $R_Y(D', (\beta^{\text{re}})' + 2(\beta^{\text{im}})') < R_Y(D, \beta^{\text{re}} + 2\beta^{\text{im}})$ and the pair $(\mathbf{p}^{\flat})', (\mathbf{p}^{\sharp})'$ is a predecessor of $\mathbf{p}^{\flat}, \mathbf{p}^{\sharp}$.

Lemma 8 *Let $D_0 \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ be a divisor class, and let $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp})$ be a D_0 -proper tuple such that $R_Y(D, \beta^{\text{re}} + 2\beta^{\text{im}}) > 0$ and $\beta^{\text{im}} \neq 0$. Then, $V_Y^{\mathbb{R}}(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp}) = \emptyset$.*

Proof. Assume that $V_Y^{\mathbb{R}}(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp}) \neq \emptyset$, and put $k = \min J^{\sharp}$, where $\mathbf{p}^{\sharp} = \{p_j(\sigma_j) : j \in J^{\sharp}\}$ (see Lemma 7). We obtain inductively a contradiction showing that $V_Y^{\mathbb{R}}(D', \alpha', (\beta^{\text{re}})', (\beta^{\text{im}})', (\mathbf{p}^{\flat})', (\mathbf{p}^{\sharp})') \neq \emptyset$ for a certain D_0 -proper tuple $(D', \alpha', (\beta^{\text{re}})', (\beta^{\text{im}})', (\mathbf{p}^{\flat})', (\mathbf{p}^{\sharp})')$ that precedes $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp})$ and such that $R_Y(D', (\beta^{\text{re}})' + 2(\beta^{\text{im}})') > 0$ and $(\beta^{\text{im}})' \neq 0$.

Consider the degeneration of $C \in V_Y^{\mathbb{R}}(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp})$ when $p_k \in \mathbf{p}^{\sharp}$ tends to E along the arc Λ_k . By [19, Proposition 2.6], the degenerate curve is either an irreducible interpolating curve constrained by a D_0 -proper tuple that precedes $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp})$, or of the form $E \cup C'$. In the latter case, the curve C' has a real component belonging to $V_Y^{\mathbb{R}}(D', \alpha', (\beta^{\text{re}})', (\beta^{\text{im}})', (\mathbf{p}^{\flat})', (\mathbf{p}^{\sharp})')$, where $(D', \alpha', (\beta^{\text{re}})', (\beta^{\text{im}})', (\mathbf{p}^{\flat})', (\mathbf{p}^{\sharp})')$ is a D_0 -proper tuple which precedes $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp})$ and $R_Y(D', (\beta^{\text{re}})' + 2(\beta^{\text{im}})') > 0$, $(\beta^{\text{im}})' \neq 0$. This statement follows from [19, Lemma 2.9] and the fact that any imaginary component of C' avoids \mathbf{p}^{\sharp} , and thus has a unique intersection point with E (see Lemma 5). The former lemma states that the intersection points of C with $E \setminus \mathbf{p}^{\flat}$ all come from the intersection points of C' with $E \setminus \mathbf{p}^{\flat}$, and that, in the deformation of $E \cup C'$ into C , each component of C' glues up with E via smoothing out one of its intersection points with $E \setminus \mathbf{p}^{\flat}$. \square

3.6 Ordinary w -numbers

Let C be a real curve on a real smooth surface Σ , and let z be a real singular point of C such that all local branches of C at z are smooth. Denote by $s(C, z)$ the number of pairs of imaginary complex conjugate local branches of C at z , each pair being counted with the weight equal to the intersection number of the branches.

Lemma 9 *Let $C(t)$, $-\varepsilon < t < \varepsilon$, be a continuous family of real curves in Σ , and let z_0 be a real singular point of $C(0)$ such that all local branches of $C(0)$ at z_0 are smooth. Assume that for a certain neighborhood $U(z_0) \subset \Sigma$ of z_0 and a sufficiently small number $\varepsilon' > 0$, the curves $C(t)$, $-\varepsilon' < t < \varepsilon'$, are transversal to the boundary of $U(z_0)$, and the curves $C(t) \cap U(z_0)$, $-\varepsilon' < t < \varepsilon'$, admit simultaneous parametrizations by a continuous family of immersions $\Delta_i(t) \rightarrow U(z_0)$, $i = 1, \dots, b(z_0)$, where $b(z_0)$ is the number of local branches of $C(0)$ at $z(0)$, and $\Delta_i(t)$, $-\varepsilon' <$*

$t < \varepsilon'$, is a continuous family of discs in \mathbb{C} . Then, $\sum_{z \in \text{Sing}(C(t)) \cap U(z_0)} s(C(t), z)$ does not depend on t .

Proof. Straightforward. \square

For an immersed real curve $C \subset \Sigma$, put $s(C) = \sum_{z \in \text{Sing}(C)} s(C, z)$.

Let $(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat})$ be an admissible tuple such that $\mathcal{D} = D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ is a divisor class, and let \mathbf{p}^{\sharp} be a generic set of $R_Y(\mathcal{D}, \beta^{\text{re}} + 2\beta^{\text{im}})$ points in $F \setminus E$. The set $V_Y^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp})$ is finite and consists of immersed curves (see Lemma 4). We put

$$W_{Y,E,\varphi}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp}) = \sum_{C \in V_Y^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp})} \mu_{\varphi}(C), \quad (6)$$

where

$$\mu_{\varphi}(C) = (-1)^{s(C) + C_{1/2} \circ \varphi}. \quad (7)$$

and $C_{1/2}$ is the image of one of the halves of $\mathbb{P}^1 \setminus \mathbb{R}\mathbb{P}^1$ by the normalization map $\mathbb{P}^1 \rightarrow C$ if C is an irreducible real curve, and one of the irreducible components of C if C is a pair of complex conjugate irreducible curves. The number $\mu_{\varphi}(C)$ is called (*modified*) *Welschinger sign*.

The proof of the following proposition literally coincides with the proof of [16, Proposition 11].

Proposition 10 *Let (Y, E, F, φ) be a basic quadruple. Fix a tuple $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}})$, where $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ is a divisor class, $\alpha, \beta^{\text{re}} \in \mathbb{Z}_+^{\infty, \text{odd}}$, and $\beta^{\text{im}} \in \mathbb{Z}_+^{\infty}$ are such that $R_Y(D, \beta^{\text{re}} + 2\beta^{\text{im}}) > 0$. Choose two point sequences \mathbf{p}^{\flat} and \mathbf{p}^{\sharp} satisfying the following restrictions:*

- (r1) *the tuple $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat})$ is admissible,*
- (r2) *the number of points in \mathbf{p}^{\sharp} is equal to $R_Y(D, \beta^{\text{re}} + 2\beta^{\text{im}})$,*
- (r3) *the pair $(\mathbf{p}^{\flat}, \mathbf{p}^{\sharp})$ is in D_0 -CH position for some divisor class $D_0 \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$, $D_0 \geq D$.*

Then, the number $W_{Y,E,\varphi}(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp})$ does not depend on the choice of sequences \mathbf{p}^{\flat} and \mathbf{p}^{\sharp} subject to (r1)-(r3). \square

Proposition 11 *The only non-zero numbers $W_{Y,E,\varphi}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \emptyset)$ for admissible tuples $(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat})$ such that $\mathcal{D} \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$, $\alpha, \beta^{\text{re}} \in \mathbb{Z}_+^{\infty, \text{odd}}$, $\beta^{\text{im}} \in \mathbb{Z}_+^{\infty}$, and*

$$I(\alpha + \beta^{\text{re}} + 2\beta^{\text{im}}) = [\mathcal{D}]E > 0, \quad R_Y(\mathcal{D}, \beta^{\text{re}} + 2\beta^{\text{im}}) = 0,$$

are the following ones:

(1) if $\mathcal{D} = D$ is a divisor class,

(1i) $W_{Y,E,\varphi}(E', 0, e_1, 0, \emptyset, \emptyset) = (-1)^{E'_{1/2} \circ \varphi}$, where $E' \in \mathcal{E}(E)$ is real;

(1ii) $W_{Y,E,\varphi}(-(K_Y + E), e_1, e_1, 0, \mathbf{p}^\flat, \emptyset) = (-1)^{L_{1/2} \circ \varphi}$, where $L \in |-(K_Y + E)|$ is real, $\mathbb{R}L \subset F$;

(1iii) $W_{Y,E,\varphi}(D, \alpha, 0, 0, \mathbf{p}^\flat, \emptyset) = (-1)^{C_{1/2} \circ \varphi}$, where $-(K_Y + E)D = 1$, $I\alpha = DE$, $C \in V_Y^{\mathbb{R}}(D, \alpha, 0, 0, \mathbf{p}^\flat, \emptyset)$;

(2) if \mathcal{D} is a pair of divisor classes,

(2i) if $E'_1, E'_2 \in \mathcal{E}(E)$ are complex conjugate, $E'_1 E'_2 = 1$, then

$$W_{Y,E,\varphi}(\{E'_1, E'_2\}, 0, 0, e_1, \emptyset, \emptyset) = -(-1)^{E'_1 \circ \varphi} ,$$

(2ii) if $E'_1, E'_2 \in \mathcal{E}(E)$ are disjoint complex conjugate, then

$$W_{Y,E,\varphi}(\{E'_1, E'_2\}, 0, 0, e_1, \emptyset, \emptyset) = (-1)^{E'_1 \circ \varphi} ,$$

(2iii) $W_{Y,E,\varphi}(\{-(K_Y + E), -(K_Y + E)\}, 0, 0, e_2, \emptyset, \emptyset) = 1$, if L', L'' are complex conjugate.

Proof. Proposition 11 can easily be derived from Lemma 5. Notice only that, in case (2iii), $L' \circ \varphi = 0 \pmod{2}$ since the linear system $|-(K_Y + E)|$, which contains L' , contains also a real rational curve whose complex locus is divided into two halves by its real locus located in F . \square

The numbers $W_{Y,E,\varphi}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^\flat, \emptyset)$ in Proposition 11 do not depend on the choice of \mathbf{p}^\flat .

We skip \mathbf{p}^\flat and \mathbf{p}^\sharp in the notation of the numbers appearing in Propositions 10 and 11, and write $W_{Y,E,\varphi}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}})$ for these numbers calling them *ordinary w-numbers*.

3.7 Formula for ordinary w -numbers

Theorem 4 Let (Y, E, F, φ) be a basic quadruple.

(1) For any divisor class $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ and vectors $\alpha, \beta^{\text{re}} \in \mathbb{Z}_+^{\infty, \text{odd}}$, $\beta^{\text{im}} \in \mathbb{Z}_+^{\infty}$ such that $I(\alpha + \beta^{\text{re}} + 2\beta^{\text{im}}) = DE$, $R_Y(D, \beta^{\text{re}} + 2\beta^{\text{im}}) \geq 0$, and $\beta^{\text{im}} \neq 0$, one has

$$W_{Y,E,\varphi}(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}) = 0 . \tag{8}$$

(2) For any divisor class $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ and vectors $\alpha, \beta \in \mathbb{Z}_+^{\infty, \text{odd}}$ such that $I(\alpha + \beta) = DE$ and $R_Y(D, \beta) > 0$, one has

$$W_{Y,E,\varphi}(D, \alpha, \beta, 0) = \sum_{j \geq 1, \beta_j > 0} W_{Y,E,\varphi}(D, \alpha + e_j, \beta - e_j, 0)$$

$$\begin{aligned}
& + (-1)^{E_{1/2} \circ \varphi} \sum (-1)^{(I\beta^{(0)} + I\alpha^{(0)})(L_{1/2} \circ \varphi)} \cdot \frac{2^{\|\beta^{(0)}\|}}{\beta^{(0)}!} (l+1) \binom{\alpha}{\alpha^{(0)} \alpha^{(1)} \dots \alpha^{(m)}} \frac{(n-1)!}{n_1! \dots n_m!} \\
& \quad \times \prod_{i=1}^m \left(\binom{(\beta^{\text{re}})^{(i)}}{\gamma^{(i)}} W_{Y,E,\varphi}(\mathcal{D}^{(i)}, \alpha^{(i)}, (\beta^{\text{re}})^{(i)}, (\beta^{\text{im}})^{(i)}) \right), \tag{9}
\end{aligned}$$

where L is any real curve in $|-(K_Y + E)|$ with $\mathbb{R}L \subset F$,

$$n = R_Y(D, \beta), \quad n_i = R_Y(\mathcal{D}^{(i)}, (\beta^{\text{re}})^{(i)} + 2(\beta^{\text{im}})^{(i)}), \quad i = 1, \dots, m,$$

and the second sum in (9) is taken

- over all integers $l \geq 0$ and vectors $\alpha^{(0)} \leq \alpha$, $\beta^{(0)} \leq \beta^{\text{re}}$;
- over all sequences

$$(\mathcal{D}^{(i)}, \alpha^{(i)}, (\beta^{\text{re}})^{(i)}, (\beta^{\text{im}})^{(i)}), \quad 1 \leq i \leq m, \tag{10}$$

such that, for all $i = 1, \dots, m$,

(1a) $\mathcal{D}^{(i)} \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$, and $\mathcal{D}^{(i)}$ is neither the divisor class $-(K_Y + E)$, nor the pair $\{-(K_Y + E), -(K_Y + E)\}$,

(1b) $I(\alpha^{(i)} + (\beta^{\text{re}})^{(i)} + 2(\beta^{\text{im}})^{(i)}) = [\mathcal{D}^{(i)}]E$, and $R_Y(\mathcal{D}^{(i)}, (\beta^{\text{re}})^{(i)} + 2(\beta^{\text{im}})^{(i)}) \geq 0$,

(1c) $\mathcal{D}^{(i)}$ is a pair of divisor classes if and only if $(\beta^{\text{im}})^{(i)} \neq 0$,

(1d) if $\mathcal{D}^{(i)}$ is a pair of divisor classes, then $n_i = 0$ and $\alpha^{(i)} = (\beta^{\text{re}})^{(i)} = 0$,

and

(1e) $D - E = \sum_{i=1}^m [\mathcal{D}^{(i)}] - (2l + I\alpha^{(0)} + I\beta^{(0)})(K_Y + E)$,

(1f) $\sum_{i=0}^m \alpha^{(i)} \leq \alpha$, $\sum_{i=0}^m (\beta^{\text{re}})^{(i)} \geq \beta$,

(1g) each tuple $(\mathcal{D}^{(i)}, 0, (\beta^{\text{re}})^{(i)}, (\beta^{\text{im}})^{(i)})$ with $n_i = 0$ appears in (10) at most once,

- over all sequences

$$\gamma^{(i)} \in \mathbb{Z}_+^{\infty, \text{odd}}, \quad \|\gamma^{(i)}\| = \begin{cases} 1, & \mathcal{D}^{(i)} \text{ is a divisor class,} \\ 0, & \mathcal{D}^{(i)} \text{ is a pair of divisor classes,} \end{cases} \quad i = 1, \dots, m, \tag{11}$$

satisfying

$$(2a) \quad (\beta^{\text{re}})^{(i)} \geq \gamma^{(i)}, \quad i = 1, \dots, m, \quad \text{and} \quad \sum_{i=1}^m ((\beta^{\text{re}})^{(i)} - \gamma^{(i)}) = \beta^{\text{re}} - \beta^{(0)},$$

and the second sum in (9) is factorized by simultaneous permutations in the sequences (10) and (11).

(3) All ordinary w -numbers $W_{Y,E,\varphi}(D, \alpha, \beta, 0)$, where $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ is a divisor class and $R_Y(D, \beta) > 0$, are recursively determined by the formula (9) and the initial conditions given by Proposition 11.

Remark 12 *It is easy to verify that $n - 1 = \sum_i n_i + \|\beta^{(0)}\|$ (in the notation of Theorem 4).*

The proof of Theorem 4 literally coincides with the proof of [16, Theorem 1 and Corollary 14] (notice that in the journal version of [16, Section 3.3] one deformation label of type (DL1) is missing; the complete list of the deformation labels can be found in the ArXiv version of [16]).

We present here an immediate consequence that will be used below.

Corollary 13 *Under the hypotheses of Theorem 4(2), assume in addition that $F \setminus \mathbb{R}E$ is disconnected, $DE = 0$, and $R_Y(D, 0) \geq 2$. Then*

$$W_{Y,E,\varphi}(D, 0, 0, 0) = 0 .$$

Proof. This follows from Proposition 10: indeed, we may choose two of the points of \mathbf{p}^\sharp in different components of $F \setminus \mathbb{R}E$ making the set $V_Y^{\mathbb{R}}(\mathcal{D}, 0, 0, 0, \emptyset, \mathbf{p}^\sharp)$ empty, since a real rational curve cannot have two one-dimensional real components. \square

3.8 Sided w -numbers

Let (Y, E, F, φ) be a basic quadruple. Suppose in addition that $F \setminus \mathbb{R}E$ splits into two connected components F_+ and F_- . In this case, (Y, E, F, φ) is called *dividing basic quadruple*.

Let $(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat})$ be an admissible tuple. Choose a sequence \mathbf{p}^\sharp of $R_Y(\mathcal{D}, \beta^{\text{re}} + 2\beta^{\text{im}})$ points in F_+ . Suppose that the pair of point sequences $\mathbf{p}^{\flat}, \mathbf{p}^\sharp$ is in a D_0 -CH position with respect to some divisor class $D_0 \in \text{Pic}_{++}(Y, E)$, $D_0 \geq [\mathcal{D}]$. Put

$$V_{Y,F_+}^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^\sharp) = \{C \in V_Y^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^\sharp) : \text{card}(C \cap F_-) < \infty\} .$$

Clearly, if \mathcal{D} is a pair of divisor classes, then

$$V_{Y,F_+}^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^\sharp) = V_Y^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^\sharp) .$$

Set

$$W_{Y,F_+,\varphi}^\varepsilon(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^\sharp) = \sum_{C \in V_{Y,F_+}^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^\sharp)} \mu_\varphi^\varepsilon(C), \quad \varepsilon = \pm, \quad (12)$$

where $\mu_\varphi^+(C) = \mu_\varphi(C)$ is defined by (7) and

$$\mu_\varphi^-(C) = (-1)^{s(C) + C_{1/2} \circ \varphi + \text{card}(C_{1/2} \cap F_-)} . \quad (13)$$

Remark 14 By Lemma 4(2), if $\mathcal{D} \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ is a pair of divisor classes, then

$$W_{Y, F_+, \varphi}^+(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b, \mathbf{p}^\sharp) = 0$$

as long as $\alpha + \beta^{\text{re}} > 0$ or $R_Y(\mathcal{D}, \beta^{\text{re}} + 2\beta^{\text{im}}) > 0$.

Proposition 15 Let (Y, E, F, φ) be a dividing basic quadruple. Fix an admissible tuple $(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b)$, where $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ is a divisor class, $\alpha, \beta^{\text{re}} \in \mathbb{Z}_+^{\infty, \text{even}}$, $\beta^{\text{im}} \in \mathbb{Z}_+^{\infty}$, and $R_Y(D, \beta^{\text{re}} + 2\beta^{\text{im}}) > 0$. Choose two point sequences $\mathbf{p}^b \subset \mathbb{R}E$ and $\mathbf{p}^\sharp \subset F_+$ satisfying the restrictions (r1)-(r3) of Proposition 10. Then, the numbers $W_{Y, F_+, \varphi}^\pm(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b, \mathbf{p}^\sharp)$ do not depend on the choice of sequences \mathbf{p}^b and \mathbf{p}^\sharp subject to (r1)-(r3).

The proof of Proposition 15 is given in Section 3.10.

Proposition 16 Let (Y, E, F, φ) be a dividing basic quadruple, and let $(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b)$ be an admissible tuple such that $\alpha, \beta^{\text{re}} \in \mathbb{Z}_+^{\infty, \text{even}}$ and

$$I(\alpha + \beta^{\text{re}} + 2\beta^{\text{im}}) = [\mathcal{D}]E > 0, \quad R_Y(\mathcal{D}, \beta^{\text{re}} + 2\beta^{\text{im}}) = 0.$$

(1) Assume that $W_{Y, F_+, \varphi}^\pm(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b, \emptyset) \neq 0$ and $\mathcal{D} = D$ is a divisor class. Then,

- (1i) either $D = -(K_Y + E)$, $\alpha = 0$, $\beta^{\text{re}} = e_2$, $\beta^{\text{im}} = 0$, $\mathbf{p}^b = \emptyset$, and the supporting curves L', L'' are both real,
- (1ii) or $-(K_Y + E)D = 1$, $I\alpha = DE$, and D is represented by a curve $C \in V_{Y, F_+}^{\mathbb{R}}(\mathcal{D}, \alpha, 0, 0, \mathbf{p}^b, \emptyset)$ with $\mathbb{R}C \subset \overline{F}_+$.

In the first case, $W_{Y, F_+, \varphi}^\pm(-(K_Y + E), 0, e_2, 0, \emptyset, \emptyset) = \lambda(-1)^{L_{1/2} \circ \varphi}$, where λ is the number of supporting curves L', L'' whose real part is contained in \overline{F}_+ . In the second case, $W_{Y, F_+, \varphi}^\pm(D, \alpha, 0, 0, \mathbf{p}^b, \emptyset) = (-1)^{C_{1/2} \circ \varphi}$.

(2) Assume that $W_{Y, F_+, \varphi}^\pm(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b, \emptyset) \neq 0$ and \mathcal{D} is a pair of divisor classes. Then,

- (2i) either $\mathcal{D} = \{E'_1, E'_2\}$, where $E'_1, E'_2 \in \mathcal{E}(E)$ are complex conjugate,
- (2ii) or $\mathcal{D} = \{L', L''\}$ and the supporting curves L', L'' are complex conjugate.

In the first case,

$$\begin{aligned} W_{Y, F_+, \varphi}^+(\{E'_1, E'_2\}, 0, 0, e_1, \emptyset, \emptyset) &= (-1)^{E'_1 \circ \varphi + E'_1 \circ E'_2}, \\ W_{Y, F_+, \varphi}^-(\{E'_1, E'_2\}, 0, 0, e_1, \emptyset, \emptyset) &= (-1)^{E'_1 \circ \varphi + \text{card}(E'_1 \cap F_-) + E'_1 \circ E'_2}. \end{aligned}$$

In the second case,

$$W_{Y, F_+, \varphi}^\pm(\{-(K_Y + E), -(K_Y + E)\}, 0, 0, e_2, \emptyset, \emptyset) = 1.$$

Proof. The statement can be easily derived from Lemma 5, taking into account that $L' \circ \varphi = 0 \pmod{2}$ in (2ii) (*cf.*, the proof of Proposition 11). \square

We skip \mathbf{p}^b and \mathbf{p}^\sharp in the notation of the numbers appearing in Propositions 15 and 16, and write $W_{Y, F_+, \varphi}^\pm(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}})$ for these numbers calling them *sided w -numbers*.

3.9 Sided w -numbers in deformation diagrams

Let (Y, E, F, φ) be a dividing basic quadruple, and let $(\mathcal{D}, \alpha, \beta, 0, \mathbf{p}^b)$ be an admissible tuple such that $\mathcal{D} = D$ is a divisor class and $R_Y(\mathcal{D}, \beta) > 0$. Choose a sequence \mathbf{p}^\sharp of $R_Y(\mathcal{D}, \beta)$ points in F_+ , and assume that the pair of point sequences $\mathbf{p}^b, \mathbf{p}^\sharp$ is in a D_0 -CH position with respect to some divisor class $D_0 \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$.

Put $k = \min J^\sharp$, where $\mathbf{p}^\sharp = \{p_j(\sigma_j) : j \in J^\sharp\}$

(see Lemma 7), and denote by \bar{V} a deformation diagram of $(D, \alpha, \beta, 0, \mathbf{p}^b, \mathbf{p}^\sharp \setminus \{p_k\}, p_k)$.

Lemma 17 *Let $\beta_j > 0$, and $C \in V_{Y, F_+}^{\mathbb{R}}(D, \alpha + e_j, \beta - e_j, 0, \mathbf{p}^b \cup \{p_k(0)\}, \mathbf{p}^\sharp \setminus \{p_k\})$ intersects E at $p_k(0)$ with multiplicity j . Then, the real leaves of \bar{V} that are generated by the root C consist of two curves $C_1, C_2 \in V_{Y, F_+}^{\mathbb{R}}(D, \alpha, \beta, 0, \mathbf{p}^b, \mathbf{p}^\sharp)$, and $\mu_\varphi^\pm(C_i) = \mu_\varphi^\pm(C)$, $i = 1, 2$.*

Proof. Choose local coordinates x, y in a neighborhood of $p_k(0)$ so that $E = \{y = 0\}$, $F_+ = \{y > 0\}$, $\Lambda_k = \{(0, t) : t \geq 0\}$, and

$$C = \left\{ ay + bx^j + \sum_{m+n>j} c_{mn}x^m y^n = 0 \right\}, \quad a, b \in \mathbb{R}^* .$$

Since $\text{card}(\mathbb{R}C \cap F_-) < \infty$, the multiplicity j is even, and $ab < 0$. Hence, the root C has two real branches given in a neighborhood of $p_k(0)$ by (*cf.* [19, Formulas (22) and (23)])

$$C_i(t) = \left\{ ay + b(x + \tau)^j + \sum_{m+n>j} O(\tau) \cdot (x + \tau)^m y^n = 0 \right\},$$

where $\tau = (-\frac{a}{b})^{1/j} t^{1/j} + o(t^{1/j})$ for $i = 1$, and $\tau = -(-\frac{a}{b})^{1/j} t^{1/j} + o(t^{1/j})$ for $i = 2$. For each curve $C_i(t)$, $i = 1, 2$, one has $\mu_\varphi^\pm(C_i) = \mu_\varphi^\pm(C)$. Indeed, the above local formula insures that the topology of the curves is preserved in a neighborhood of $p_k(0)$; outside of a neighborhood of $p_k(0)$, the equality required follows from Lemma 9. \square

Lemma 18 *Let $C = E \cup \check{C}$ be a root of \bar{V} such that C generates at least one leaf belonging to $V_{Y, F_+}^{\mathbb{R}}(D, \alpha, \beta, 0, \mathbf{p}^b, \mathbf{p}^\sharp)$. Then, \check{C} splits in primary components from the following list:*

- (i) *pairs of reduced complex conjugate components as described in Lemma 5(2);*
- (ii) *real reduced components, whose all intersection points with E are real and have even multiplicity; each of these components belongs to $V_{Y, F_+}^{\mathbb{R}}(D', \alpha', \beta', 0, (\mathbf{p}^b)', (\mathbf{p}^\sharp)')$ for a certain D_0 -proper tuple $(D', \alpha', \beta', 0, (\mathbf{p}^b)', (\mathbf{p}^\sharp)')$;*
- (iii) *non-reduced components $s'L', s''L''$, where $L', L'' \in |-(K_Y + E)|$ are the supporting curves, and, in addition, $s' = s''$ if L', L'' are complex conjugate, and s' (respectively, s'') is even if L' (respectively, L'') is real with $\mathbb{R}L' \subset \overline{F_-}$ (respectively, $\mathbb{R}L'' \subset \overline{F_-}$);*
- (iv) *non-reduced components $sL(z)$, where s is even, $z \in \mathbf{p}^\sharp \setminus \{p_k\}$, and L is the curve belonging to $|-(K_Y + E)|$ and passing through z ;*
- (v) *non-reduced components $iL(p_{ij})$, where i is even, $p_{ij} \in \mathbf{p}^b$, and $L(p_{ij})$ is the curve belonging to $|-(K_Y + E)|$ and passing through p_{ij} .*

Moreover,

- (1) *pairwise intersections of distinct primary components are either empty, or transversal contained in $Y \setminus E$, and all reduced primary components are immersed and are nonsingular along E ,*
- (2) *the intersection multiplicities of \check{C} and E at the points $\check{C} \cap \mathbf{p}^b$ are encoded by a vector $\check{\alpha} \leq \alpha$,*
- (3) *there exists a conjugation invariant set $\mathbf{z} \subset \check{C} \cap E \setminus \mathbf{p}^b$ containing exactly one point of each irreducible component of \check{C} and such that the intersection multiplicities of \check{C} and E at the points of $\check{C} \cap E \setminus (\mathbf{p}^b \cup \mathbf{z})$ are encoded by the vector β .*

Proof. The statement follows from [19, Proposition 2.6]. The fact that each real reduced primary component of \check{C} intersects E only at real points is guaranteed by Lemmas 5 and 8; these intersection points have even multiplicity due to the assumption that the C generates a leaf belonging to $V_{Y, F_+}^{\mathbb{R}}(D, \alpha, \beta, 0, \mathbf{p}^b, \mathbf{p}^\sharp)$. \square

To describe the leaves of deformation diagrams with reducible roots, we use certain deformation labels from the list (DL1)-(DL9) introduced in [16, Section 3.3]. Each label is seen as a curve on the toric surface that is determined by the Newton polygon of the polynomial defining the label. We recall these defining polynomials (some of them are slightly modified by a conjugation-invariant coordinate change), where $\text{cheb}_k(t) = \cos \arccos kt$ is the k -th Chebyshev polynomial and y_k is the only simple positive root of $\text{cheb}_k(t) - 1$:

(DL2) $_j$ for an even $j > 0$, two deformation labels defined by the equations

$$\psi_1(x, y) = y^2 + 1 - y \cdot \text{cheb}_j(x), \quad \psi_2(x, y) = \psi_1(x\sqrt{-1}, y),$$

(DL3)_i for an even $i > 0$, a deformation label

$$(x - 1)(y^i - x) = 0 ,$$

(DL5)_s for an even $s > 0$, two deformation labels

$$(x - 1)(x((y \pm 1)^s + 1) - 1) = 0 ,$$

(DL6)_{s,ε₀} for an integer $s > 0$ and $\varepsilon_0 = \pm 1$, a deformation label

$$\frac{y + x^2}{2y} \left(\text{cheb}_{s+1} \left(y_{s+1} - \frac{y\varepsilon_0}{2^{(s-1)/(s+1)}} \right) - 1 \right) + 1 = 0 ,$$

(DL7)_s for $s > 0$, $s + 1$ pairs of complex conjugate deformation labels

$$\frac{y + \sqrt{-1}x^2}{2y} \left(\text{cheb}_{s+1} \left(\frac{y\varepsilon}{2^{(s-1)/(s+1)}} + y_{s+1} \right) - 1 \right) + 1 = 0 ,$$

$$\frac{y - \sqrt{-1}x^2}{2y} \left(\text{cheb}_{s+1} \left(\frac{y\bar{\varepsilon}}{2^{(s-1)/(s+1)}} + y_{s+1} \right) - 1 \right) + 1 = 0 ,$$

where $\varepsilon^{s+1} = 1$.

Lemma 19 *Let a curve $C = E \cup \check{C} \in |D|$ be such that the primary components of \check{C} belong to the list (i)-(v) and satisfy conditions (1)-(3) in the statement of Lemma 18. Then, the curve C is a root of \bar{V} . It generates at least one leaf that belongs to $V_{Y,F_+}^{\mathbb{R}}(D, \alpha, \beta, 0, \mathbf{p}^b, \mathbf{p}^\sharp)$ if and only if the following condition holds: \check{C} has no any real primary component $s'L'$ or $s''L''$ whose multiplicity (s' or s'') is odd and the real point set is contained in \bar{F}_- . If the set of leaves generated by C and belonging to $V_{Y,F_+}^{\mathbb{R}}(D, \alpha, \beta, 0, \mathbf{p}^b, \mathbf{p}^\sharp)$ is non-empty, then it is in one-to-one correspondence with the following data:*

- a set $\mathbf{z} \subset \check{C} \cap E \setminus \mathbf{p}^b$ satisfying the condition (3) of Lemma 18,
- a collection \mathcal{DL} of deformation labels chosen as follows:
 - a deformation label of type (DL2)_j for each point $q \in \mathbf{z}'$ satisfying $(\check{C} \cdot E)_q = j$, where $\mathbf{z}' \subset \mathbf{z}$ consists of the points which do not lie on the primary components $s'L'$, $s''L''$, $iL(p_{ij})$, $sL(z)$ of \check{C} ,
 - a deformation label of type (DL3)_i for each primary component $iL(p_{ij})$ of \check{C} ,
 - a deformation label of type (DL5)_s for each primary component $sL(z)$ of \check{C} ,
 - a deformation label of type (DL6)_{s',+1} (respectively, (DL6)_{s'',+1}) for a real primary component $s'L'$ (respectively, $s''L''$) of \check{C} with $\mathbb{R}L' \subset \bar{F}_+$ (respectively, $\mathbb{R}L'' \subset \bar{F}_+$), if s' (respectively, s'') is even,
 - a deformation label of type (DL6)_{s',-1} (respectively, (DL6)_{s'',-1}) for a real primary component $s'L'$ (respectively, $s''L''$) of \check{C} with $\mathbb{R}L' \subset \bar{F}_-$ (respectively, $\mathbb{R}L'' \subset \bar{F}_-$), if s' (respectively, s'') is even,

- a deformation label of type $(DL6)_{s',+1}$ or $(DL6)_{s',-1}$ (respectively, $(DL6)_{s'',+1}$ or $(DL6)_{s'',-1}$) for a real primary component $s'L'$ (respectively, $s''L''$) of \check{C} with $\mathbb{R}L' \subset \overline{F}_+$ (respectively, $\mathbb{R}L'' \subset \overline{F}_+$), if s' (respectively, s'') is odd,
- a pair of complex conjugate deformation labels of type $(DL7)_s$ for a pair of complex conjugate primary components sL', sL'' of \check{C} .

Proof. Since the primary components of \check{C} belong to the list (i)-(v) and satisfy conditions (1)-(3) in the statement of Lemma 18, we obtain (see [19, Lemma 2.20]) that C is a root of the deformation diagram \overline{V} , and the leaves of \overline{V} are in one-to-one correspondence with the set of pairs $(\mathbf{z}, \mathcal{DP})$, where $\mathbf{z} \subset \check{C} \cap E \setminus \mathbf{p}^\flat$ is a set of points satisfying the condition (3) of Lemma 18, and \mathcal{DP} is a conjugation-invariant collection of deformation patterns

$$\{\Psi_q : q \in \mathbf{z}'\}, \quad \{\Psi_{iL(p_{ij})} : iL(p_{ij}) \subset \check{C}\}, \quad \{\Psi_{sL(z)} : sL(z) \subset \check{C}\},$$

$$\{\Psi_{s'L'} : s'L' \subset \check{C}\}, \quad \{\Psi_{s''L''} : s''L'' \subset \check{C}\},$$

where $iL(p_{ij})$, $sL(z)$, $s'L'$, or $s''L''$ run over the corresponding primary components of \check{C} , and Ψ_* denote specific curves on toric surfaces introduced in [19, Section 2.5.3]. We consider these pairs $(\mathbf{z}, \mathcal{DP})$ in detail and prove that they induce leaves belonging to $V_{Y,F_+}^{\mathbb{R}}(D, \alpha, \beta, 0, \mathbf{p}^\flat, \mathbf{p}^\sharp)$ if and only if \check{C} does not have a primary component $s'L'$ or $s''L''$ such that this component is real, its real point set is contained in \overline{F}_- , and its multiplicity (s' or s'') is odd.

A deformation of C into any leaf-curve $\check{C} \in V_{Y,F_+}(D, \alpha, \beta, 0, \mathbf{p}^\flat, \mathbf{p}^\sharp(t))$, where $\mathbf{p}^\sharp(t) = (\mathbf{p}^\sharp \setminus \{p_k\}) \cup \{p_k(t)\}$, $t > 0$, can be described by a family of sections of $H^0(Y, \mathcal{O}_Y(D))$:

$$S\check{T}_\tau + \tau^\kappa T_\tau, \quad \tau \in (\mathbb{C}, 0), \quad \tau^\kappa = t, \quad (14)$$

where S, \check{T}_0, T_0 are real, $S^{-1}(0) = E$, $\check{T}_0^{-1}(0) = \check{C}$, and κ is the least common multiple of all the parameters $j, i, s, s'+1, s''+1$ in the assertion of Lemma 19 (cf. [19, Formula (57)]). In addition, S and \check{T}_0 are negative in F_- (except possibly for a finite set of points and lines), and

$$\begin{cases} S(p_k(t)) > 0, \check{T}_0(p_k(t)) < 0, T_0(p_k(t)) > 0 & \text{as } t > 0, \\ T_0(z) > 0 & \text{for all } z \in \mathbb{R}E \setminus \mathbf{p}^\sharp. \end{cases} \quad (15)$$

Indeed, formula (14) follows from [19, Lemma 2.10]. Observe that S does not divide T_0 in view of $t = \tau^\kappa$ (see (14)). Hence $T_0^{-1}(0)$ intersects E at finitely many points and with even multiplicities, since the same holds for each curve $C_\tau = \{S\check{T}_\tau + \tau^\kappa T_\tau = 0\}$, $\tau \neq 0$. This claim combined with the facts that $\text{card}(\mathbb{R}\check{C} \cap F_-) < \infty$, $\text{card}(\mathbb{R}\check{C} \cap F_-) < \infty$, $(S\check{T}_\tau + \tau^\kappa T_\tau)(p_k(t)) = 0$ for all $t > 0$, and with the assumption $\alpha + \beta \in \mathbb{Z}_+^{\infty, \text{even}}$ yields all sign conditions (15).

Given a point $q \in \mathbf{z}'$, the intersection multiplicity $j = (\check{C} \cdot E)_q$ is even by Lemma 18. Choose local real coordinates x, y in a neighborhood of q in Y so that $E = \{y = 0\}$, $q = (0, 0)$, $F_- = \{y < 0\}$. Then in formula (14), we get $\check{T}_0 = y - 2x^j + \text{h.o.t.}$,

$T_0 = a + \text{h.o.t.}$, where $a > 0$ due to (15). Thus, by [19, Lemma 2.11], there are two real deformation patterns Ψ_q given by $y^2 - 2yP(x) + a = 0$ with $P(x) = x^j + \text{l.o.t.}$, and they can be brought to the form $(\text{DL2})_j$ by a transformation

$$\psi(x, y) \mapsto \lambda\psi(\mu x, \nu y), \quad \lambda, \mu, \nu > 0. \quad (16)$$

Given a primary component $iL(p_{ij})$ of \check{C} (with i even by Lemma 18), it has a unique deformation pattern $\Psi_{iL(p_{ij})}$ (see [19, Lemma 2.15]) which is real and can be brought to the form $(\text{DL3})_i$ by transformation (16).

Given a primary component $sL(z)$ of \check{C} (with s even by Lemma 18) and an intersection point $q \in L(z) \cap E$ belonging to \mathbf{z} , we are interested in deformation patterns that describe a deformation of C in a neighborhood of $L(z)$ such that the intersection point q of $sL(z)$ and E smoothes out, and the other intersection point of $sL(z)$ and E turns into an intersection point of multiplicity s of the deformed curve with E . Choose real coordinates x, y in a neighborhood of $L(z)$ so that $L(z) = \{y = 0\}$, $q = (0, 0)$, $z = (x_0, 0)$, $E = \{x^2 - x + xy = 0\}$, where $L(z) \cap F_+ = \{(x, 0) : 0 < x < 1\}$. In particular, $0 < x_0 < 1$, since $z \in \mathbf{p}^\# \subset F_+$, and in (14) we have $S(x, y) = x - x^2 - xy$ and $\check{T}_0(x, y) = y^s(\varphi(x) - y\psi(x, y))$ with $\varphi(0) = -1$, and hence $S(x, y)\check{T}_0(x, y) = y^s(x^2 - x + xy)(y\psi(x, y) - \varphi(x))$. By [19, Section 2.5.3(8) and Lemma 2.16(1)], deformation of the curve $E \cup \check{C}$ in a neighborhood of $L(z)$ can be viewed as a patchworking of the curve $E \cup \check{C}$, given by $S(x, y)\check{T}_0(x, y) = 0$, and a curve, given by a polynomial $-(x-1)h(x, y)\varphi(x)$, where the factor $\Psi_{sL(z)} = (x-1)h(x, y)$ (the deformation pattern for the pair $(sL(z), (0, 0))$) satisfies the relations

$$h(x, y) = xf(y) + a, \quad f(y) + a = (y + \xi)^s, \quad \xi \in \mathbb{C}, \quad x_0f(0) + a = 0.$$

Notice that the coefficients of the common monomials $x^i y^s$, $i \geq 0$, for $S(x, y)\check{T}_0(x, y)$ and $-(x-1)h(x, y)\varphi(x)$ respectively coincide. Furthermore, in this presentation, $T_0 = -(x-1)h(x, 0)\varphi(x)$, and, in particular, $-a = T_0(q) > 0$ by (15). From this we easily derive that $\xi^s = -a \frac{1-x_0}{x_0} > 0$, obtaining two real deformation patterns that can be brought to the form $(\text{DL5})_s$ via transformation (16).

If $\check{C} \supset s(L' \cup L'')$, where L', L'' are complex conjugate, then there are s pairs of complex conjugate deformation patterns for these primary components (see [19, Lemma 2.13]), which can be brought to the form $(\text{DL7})_s$.

Let L' be real, $\mathbb{R}L' \subset \overline{F}_+$, and $\check{C} \supset sL'$. By (14) and (15), we can choose real coordinates x, y in a neighborhood of L' so that

$$\begin{cases} L' = \{y = 0\}, & S = y + x^2 + xy, & q = (0, 0) = E \cap L', \\ F_+ = \{S > 0\}, & \check{T}_0 = y^s((-1)^{s+1} + \text{h.o.t.}), & c = T_0(q) > 0. \end{cases} \quad (17)$$

Substituting this data to the formulas of [19, Lemma 2.13], we obtain that the (complex) deformation patterns of sL' are given by the formula

$$\Psi_{sL'} = \{(y + x^2)f(y) + c(-1)^{s+1} = 0\}, \quad (18)$$

where

$$yf(y) + c(-1)^{s+1} = \frac{c(-1)^{s+1}}{2} \left(\text{cheb}_{s+1} \left(\frac{\xi y}{(2^{s-1}c(-1)^{s+1})^{1/(s+1)}} + y_{s+1} \right) + 1 \right), \quad (19)$$

$\xi^{s+1} = 1$. If s is even, then there exists a unique real deformation pattern, and via transformation (16) preserving the terms $y + x^2$ in the above equation $S(x, y) = 0$ for E we can bring it to the form (DL6) $_{s,+1}$. If s is odd, then there exist two real deformation patterns. Indeed, the equation for $\Psi_{sL'}$ can be rewritten as

$$x^2 = -y \cdot \frac{\text{cheb}_{s+1} \left(\frac{\xi_0 y}{(2^{s-1}c)^{1/(s+1)}} + y_{s+1} \right) + 1}{\text{cheb}_{s+1} \left(\frac{\xi_0 y}{(2^{s-1}c)^{1/(s+1)}} + y_{s+1} \right) - 1} \quad (20)$$

with $(2^{s-1}c)^{1/(s+1)} > 0$, $\xi_0 = \pm 1$. It is easy to bring them to the form (DL6) $_{s,-\xi_0}$ via transformation of type (16).

Suppose that L' is real, $\mathbb{R}L' \subset \overline{F_-}$, and \check{C} contains a primary component sL' , $s > 0$. Then s must be even, since otherwise the leaf-curves would contain a one-dimensional branch in the domain F_- , which is not possible by the definition of $V_{Y,F_+}^{\mathbb{R}}(D, \alpha, \beta, 0, \mathbf{p}^\flat, \mathbf{p}^\sharp)$. In view of sign conditions (15), formulas (17) of the preceding paragraph turn into

$$\begin{cases} L' = \{y = 0\}, & S = -(y + x^2 + xy), & q = (0, 0) = E \cap L', \\ F_+ = \{S > 0\}, & \check{T}_0 = y^s(-1 + \text{h.o.t.}), & c = T_0(q) > 0. \end{cases}$$

Substituting these data to the formulas of [19, Lemma 2.13], we obtain the (complex) deformation patterns of sL' in the form

$$\Psi_{sL'} = \{(y + x^2)f(y) + c = 0\},$$

where

$$yf(y) + c = \frac{c}{2} \left(\text{cheb}_{s+1} \left(\frac{\xi y}{(2^{s-1}c)^{1/(s+1)}} + y_{s+1} \right) + 1 \right), \quad \xi^{s+1} = 1;$$

hence there exists a unique real deformation pattern, corresponding to $\xi = 1$, and it can easily be transformed to the type (DL6) $_{s,-1}$. \square

Introduce the following numbers:

- for a deformation label Ψ of type (DL2) $_j$, put $\mu^+(\Psi) = (-1)^{s(\Psi)}$ and $\mu^-(\Psi) = (-1)^{s^-(\Psi)}$, where $s(\Psi)$ is the number of solitary nodes of Ψ , and $s^-(\Psi)$ is the number of solitary nodes lying in the domain $y > 0$;
- for a deformation label Ψ of type (DL3) $_i$ or (DL5) $_s$, put $\mu^+(\Psi) = \mu^-(\Psi) = 1$;
- for a deformation label Ψ of type (DL6) $_{i,\varepsilon_0}$ with even i , put $\mu^+(\Psi) = (-1)^{s(\Psi)}$ and $\mu^-(\Psi) = (-1)^{s^-(\Psi)}$, where $s(\Psi)$ is the number of solitary nodes of Ψ , and $s^-(\Psi)$ is the number of solitary nodes lying in the domain $\varepsilon_0(y + x^2) > 0$;

- for a deformation label Ψ of type $(\text{DL6})_{i,\varepsilon_0}$ with odd i , put $\mu^+(\Psi) = (-1)^{s(\Psi)}$ and $\mu^-(\Psi) = (-1)^{s^-(\Psi)}$, where $s(\Psi)$ is the number of solitary nodes of Ψ , and $s^-(\Psi)$ is the number of solitary nodes lying in the domain $y + x^2 > 0$.

Let

- $\mu_{2,j}^\pm, \mu_{3,i}^\pm, \mu_{5,s}^\pm$ be the sums of the numbers $\mu^\pm(\Psi)$ over all deformation labels of type $(\text{DL2})_j, (\text{DL3})_i, (\text{DL5})_s$, respectively,
- $\mu_{6,s,\varepsilon_0}^\pm$, where s is even, be the value of $\mu^\pm(\Psi)$ for the deformation label Ψ of type $(\text{DL6})_{s,\varepsilon_0}$, $\varepsilon_0 = \pm 1$,
- $\mu_{6,s}^\pm$, where s is odd, be the sum of the numbers $\mu^\pm(\Psi)$ over the two deformation labels of type $(\text{DL6})_{s,+1}$ and $(\text{DL6})_{s,-1}$.

Lemma 20 *We have*

$$\mu_{2,j}^+ = 0, \quad \mu_{2,j}^- = \begin{cases} 0, & j = 0 \pmod{4}, \\ 2, & j = 2 \pmod{4}, \end{cases} \quad (21)$$

$$\mu_{3,i}^\pm = 1, \quad \mu_{5,s}^\pm = 2, \quad (22)$$

$$\mu_{6,s,\varepsilon_0}^\sigma = \begin{cases} 1, & s = 0 \pmod{2}, \varepsilon_0 = 1, \sigma = \pm \\ 1, & s = 0 \pmod{2}, \varepsilon_0 = -1, \sigma = +, \\ (-1)^{s/2}, & s = 0 \pmod{2}, \varepsilon_0 = -1, \sigma = - \end{cases} \quad (23)$$

$$\mu_{6,s}^\sigma = \begin{cases} 0, & s = 1 \pmod{2}, \sigma = +, \\ 2, & s = 1 \pmod{4}, \sigma = -, \\ 0, & s = 3 \pmod{4}, \sigma = -. \end{cases} \quad (24)$$

Proof. All the relations follow from a direct computation. \square

Let $C = E \cup \check{C}$ be a root of \bar{V} such that C generates a leaf $\check{C} \in V_{Y,F_+}^{\mathbb{R}}(D, \alpha, \beta, 0, \mathbf{p}^\flat, \mathbf{p}^\sharp)$ corresponding to a pair $(\mathbf{z}, \mathcal{DL})$ (see Lemma 19). Introduce the following numbers:

- for each point $q \in \mathbf{z}'$ (where $\mathbf{z}' \subset \mathbf{z}$ consists of the points which do not lie on the primary components $s'L', s''L'', iL(p_{ij}), sL(z)$ of \check{C}), put $\mu^\pm(C, q) = \mu_{2j}^\pm$, where $j = (\check{C} \cdot E)_q$;
- for each primary component $sL(z)$ of \check{C} , put $\mu^\pm(C, sL(z)) = \mu_{5,s}^\pm$;
- for each real primary component sL' (respectively, sL'') of \check{C} such that $\mathbb{R}L' \subset \bar{F}_+$ (respectively, $\mathbb{R}L'' \subset \bar{F}_+$), put $\mu^\pm(C, sL')$ (respectively, $\mu^\pm(C, sL'')$) equal to $\mu_{6,s,+1}^\pm$ if s is even, and equal to $\mu_{6,s}^\pm$ if s is odd,

- for each real primary component sL' (respectively, sL'') of \check{C} such that $\mathbb{R}L' \subset \overline{F}_-$ (respectively, $\mathbb{R}L'' \subset \overline{F}_-$) and s is even, put $\mu^\pm(C, sL')$ (respectively, $\mu^\pm(C, sL'')$) equal to $\mu_{6,s,-1}^\pm$.

Lemma 21 *Let $C = E \cup \check{C}$ be a root of \overline{V} such that C generates at least one leaf belonging to $V_{Y,F_+}^{\mathbb{R}}(D, \alpha, \beta, 0, \mathbf{p}^\flat, \mathbf{p}^\sharp)$, and let $\text{Lf}(C)$ be the set of all such leaves. Then,*

$$\sum_{\tilde{C} \in \text{Lf}(C)} \mu_\varphi^\pm(\tilde{C}) = (-1)^{E_{1/2} \circ \varphi} \cdot \mu_\varphi^\pm(\check{C}^{red}) \cdot 2^m \cdot M^\pm(C) \cdot \sum_{\mathbf{z}} \prod_{q \in \mathbf{z}'} \mu^\pm(C, q), \quad (25)$$

where \check{C}^{red} is the union of all reduced primary components of \check{C} different from L', L'' , the exponent m is the number of primary components $sL(z)$ of \check{C} , the factor $M^\pm(C)$ equals $(-1)^{(s'+s'')(L'_{1/2} \circ \varphi)} \mu^\pm(C, s'L') \mu^\pm(s''L'')$ if $s'L'$ and $s''L''$ are real primary components of \check{C} , and equals $s+1$ if \check{C} contains a pair of complex conjugate primary components sL', sL'' , and finally, \mathbf{z} runs over all subsets of $\check{C} \cap E \setminus \mathbf{p}^\flat$ satisfying condition (3) of Lemma 18, and $\mathbf{z}' \subset \mathbf{z}$ consists of the points which do not lie on the primary components $s'L', s''L'', iL(p_{ij}), sL(z)$ of \check{C} .

Proof. Let $\tilde{C} \in \text{Lf}(C)$. By [19, Lemma 2.9], singular points of \tilde{C} (regarded as a small deformation of C) appear in a neighborhood of $\text{Sing}(C)$. Furthermore, the local branches do not glue up in local deformation of singular points in $\text{Sing}(\check{C}^{red})$, of intersection points $q \in \mathbf{p}^\flat \cup (\check{C} \cap E) \setminus \mathbf{z}$, and of the intersection points of \check{C}^{red} with the other primary components of \check{C} . In particular, first, local deformations of the intersection points of \check{C}^{red} with other primary components of \check{C} and of the points $q \in \mathbf{p}^\flat \cup (\check{C} \cap E) \setminus \mathbf{z}$ do not bear solitary nodes, and, second, due to Lemma 9, the multiplicative contribution of $\text{Sing}(\check{C}^{red})$ to $\mu_\varphi^\pm(\tilde{C})$ is $\mu_\varphi^\pm(\check{C}^{red})$. Local deformations of the primary components $iL(p_{ij}), sL(z), s'L', s''L''$ of \check{C} and of the points $q \in \mathbf{z}'$ are determined by the corresponding deformation labels so that the solitary nodes of \tilde{C} , which appear in these deformations are in one-to-one correspondence with the solitary nodes of all deformation labels involved. Deformation labels of type (DL3)_i, (DL5)_s, and (DL7)_s do not have solitary nodes. The solitary nodes of the other deformation labels, which belong to the domains indicated in the definition of numbers $\mu^-(\Psi)$, correspond precisely to the nodes of \tilde{C} belonging to F_+ . It follows from the fact that, in the coordinates x, y in the equations of a deformation label Ψ , this domain defines an intersection of F_+ with a neighborhood of a point $q \in \mathbf{z}'$ or with a neighborhood of real primary components $s'L', s''L''$ of \check{C} .

Then, formula (25) immediately follows from Lemmas 18 and 19. \square

3.10 Formula for sided w -numbers

Theorem 5 *Let (Y, E, F, φ) be a dividing basic quadruple.*

(1) For a divisor class $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ and vectors $\alpha, \beta^{\text{re}}, \beta^{\text{im}} \in \mathbb{Z}_+^{\infty}$ such that $I(\alpha + \beta^{\text{re}} + 2\beta^{\text{im}}) = DE$ and $R_Y(D, \beta^{\text{re}} + 2\beta^{\text{im}}) > 0$, one has

$$W_{Y, F_+, \varphi}^{\pm}(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}) = 0, \quad (26)$$

provided that either $\alpha \notin \mathbb{Z}_+^{\infty, \text{even}}$, or $\beta^{\text{re}} \notin \mathbb{Z}_+^{\infty, \text{even}}$, or $\beta^{\text{im}} \neq 0$.

(2) For a divisor class $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ and vectors $\alpha, \beta \in \mathbb{Z}_+^{\infty, \text{even}}$ such that $I(\alpha + \beta) = DE$ and $R_Y(D, \beta) > 0$, one has:

(2i) If $(K_Y + E)D = 0$ or $(K_Y + E)D < -2$, then

$$W_{Y, F_+, \varphi}^+(D, \alpha, \beta, 0) = 0. \quad (27)$$

(2ii) If $(K_Y + E)D = -1$, then

$$W_{Y, F_+, \varphi}^+(D, \alpha, \beta, 0) = 2^{\|\beta\|} W_{Y, F_+, \varphi}^+(D, \alpha + \beta, 0, 0). \quad (28)$$

(2iii) If $(K_Y + E)D = -2$, then

$$\begin{aligned} W_{Y, F_+, \varphi}^+(D, \alpha, \beta, 0) &= 2 \sum_{j \geq 2, \beta_j > 0} W_{Y, F_+, \varphi}^+(D, \alpha + e_j, \beta - e_j, 0) \\ &+ 4^{n-1} (-1)^{E_{1/2} \circ \varphi} \cdot \sum (-1)^{I\alpha^{(0)} \cdot (L_{1/2} \circ \varphi)} (l/2 + 1) \binom{\alpha}{\alpha^{(0)}} \prod_{i=1}^m W_{Y, F_+, \varphi}^+(\mathcal{D}^{(i)}, 0, 0, e_1), \end{aligned} \quad (29)$$

where $L \in |-(K_Y + E)|$ is real with $\mathbb{R}L \subset F$, $n = R_Y(D, \beta)$, and the second sum in (29) is taken over all even integers $l \geq 0$, vectors $\alpha^{(0)} \leq \alpha$, and sequences of distinct tuples

$$(\mathcal{D}^{(i)}, 0, 0, e_1), \quad 1 \leq i \leq m, \quad (30)$$

such that

- each $\mathcal{D}^{(i)} \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ is a pair of divisor classes that is different from $-(K_Y + E)$, $-(K_Y + E)$ and satisfies $[\mathcal{D}^{(i)}]E = 2$, $R_Y(\mathcal{D}^{(i)}, 2e_1) = 0$,
- $D - E = \sum_{i=1}^m [\mathcal{D}^{(i)}] - (l + I\alpha^{(0)} + I\beta)(K_Y + E)$;

the second sum in (29) is factorized by permutations of sequences (30).

(3) For any divisor class $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ and vectors $\alpha, \beta \in \mathbb{Z}_+^{\infty, \text{even}}$ such that $I(\alpha + \beta) = DE$ and $R_Y(D, \beta) > 0$, one has

$$\begin{aligned} W_{Y, F_+, \varphi}^-(D, \alpha, \beta, 0) &= 2 \sum_{j \geq 2, \beta_j > 0} W_{Y, F_+, \varphi}^-(D, \alpha + e_j, \beta - e_j, 0) \\ &+ (-1)^{E_{1/2} \circ \varphi} \cdot \sum (-1)^{(I\alpha^{(0)} + \beta^{(0)})(L_{1/2} \circ \varphi)} \cdot \frac{4^{\|\beta^{(0)}\|}}{\beta^{(0)}!} \eta(l) \binom{\alpha}{\alpha^{(0)} \alpha^{(1)} \dots \alpha^{(m)}} \frac{(n-1)!}{n_1! \dots n_m!} \end{aligned}$$

$$\times \prod_{i=1}^m \left(\left(\begin{matrix} (\beta^{\text{re}})^{(i)} \\ \gamma^{(i)} \end{matrix} \right) W_{Y, F_+, \varphi}^-(\mathcal{D}^{(i)}, \alpha^{(i)}, (\beta^{\text{re}})^{(i)}, (\beta^{\text{im}})^{(i)}) \right), \quad (31)$$

where $L \in |-(K_Y + E)|$ is real with $\mathbb{R}L \subset F$,

$$n = R_Y(D, \beta), \quad n_i = R_Y(\mathcal{D}^{(i)}, (\beta^{\text{re}})^{(i)} + 2(\beta^{\text{im}})^{(i)}), \quad i = 1, \dots, m,$$

$$\eta(l) = \begin{cases} 1, & \text{if } l = 0, \\ l/2 + 1, & \text{if } l \text{ is even, } L', L'' \text{ are imaginary,} \\ 0, & \text{if } l \text{ is odd, } L', L'' \text{ are imaginary} \\ & \text{or are real with } \mathbb{R}L', \mathbb{R}L'' \subset \overline{F}_-, \\ (l/2 + 1)(2 - (-1)^{l/2}), & \text{if } l \text{ is even, } L', L'' \text{ are real,} \\ & \mathbb{R}L', \mathbb{R}L'' \subset \overline{F}_+, \\ 4([l/4] + 1)(-1)^{L'_{1/2} \circ \varphi}, & \text{if } l \text{ is odd, } L', L'' \text{ are real,} \\ & \mathbb{R}L', \mathbb{R}L'' \subset \overline{F}_+, \\ (1 + (-1)^{l/2})/2, & \text{if } l \text{ is even, } L', L'' \text{ are real,} \\ & \mathbb{R}L' \subset \overline{F}_\pm, \mathbb{R}L'' \subset \overline{F}_\mp, \\ 2([l/4] + 1)(-1)^{(l-1)/2 + L'_{1/2} \circ \varphi}, & \text{if } l \text{ is odd, } L', L'' \text{ are real,} \\ & \mathbb{R}L' \subset \overline{F}_\pm, \mathbb{R}L'' \subset \overline{F}_\mp, \\ (-1)^{l/2}(l/2 + 1), & \text{if } l \text{ is even, } L', L'' \text{ are real,} \\ & \mathbb{R}L', \mathbb{R}L'' \subset \overline{F}_-, \end{cases} \quad (32)$$

and the second sum in (31) is taken

- over all integers $l \geq 0$ and vectors $\alpha^{(0)} \leq \alpha$, $\beta^{(0)} \leq \beta^{\text{re}}$;
- over all sequences

$$(\mathcal{D}^{(i)}, \alpha^{(i)}, (\beta^{\text{re}})^{(i)}, (\beta^{\text{im}})^{(i)}), \quad 1 \leq i \leq m, \quad (33)$$

such that, for all $i = 1, \dots, m$,

(3a) $\mathcal{D}^{(i)} \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$, and $\mathcal{D}^{(i)}$ is neither the divisor class $-(K_Y + E)$, nor the pair $\{-(K_Y + E), -(K_Y + E)\}$,

(3b) $I(\alpha^{(i)} + (\beta^{\text{re}})^{(i)} + 2(\beta^{\text{im}})^{(i)}) = [\mathcal{D}^{(i)}]E$, and $R_Y(\mathcal{D}^{(i)}, (\beta^{\text{re}})^{(i)} + 2(\beta^{\text{im}})^{(i)}) \geq 0$,

(3c) $\mathcal{D}^{(i)}$ is a pair of divisor classes if and only if $(\beta^{\text{im}})^{(i)} \neq 0$,

(3d) if $\mathcal{D}^{(i)}$ is a pair of divisor classes, then $n_i = 0$, $\alpha^{(i)} = (\beta^{\text{re}})^{(i)} = 0$, and $(\beta^{\text{im}})^{(i)} = e_1$,

and

(3e) $D - E = \sum_{i=1}^m [\mathcal{D}^{(i)}] - (l + I\alpha^{(0)} + I\beta^{(0)})(K_Y + E)$,

(3f) $\sum_{i=0}^m \alpha^{(i)} \leq \alpha$, $\sum_{i=0}^m (\beta^{\text{re}})^{(i)} \geq \beta$,

(3g) each tuple $(\mathcal{D}^{(i)}, 0, (\beta^{\text{re}})^{(i)}, (\beta^{\text{im}})^{(i)})$ with $n_i = 0$ appears in (30) at most once,

- over all sequences

$$\gamma^{(i)} \in \mathbb{Z}_+^{\infty, \text{odd} \cdot \text{even}}, \quad i = 1, \dots, m, \quad (34)$$

satisfying

$$(3h) \quad \|\gamma^{(i)}\| = \begin{cases} 1, & \mathcal{D}^{(i)} \text{ is a divisor class,} \\ 0, & \mathcal{D}^{(i)} \text{ is a pair of divisor classes,} \end{cases} \quad i = 1, \dots, m,$$

$$(3i) \quad (\beta^{\text{re}})^{(i)} \geq \gamma^{(i)}, \quad i = 1, \dots, m, \quad \text{and} \quad \sum_{i=1}^m ((\beta^{\text{re}})^{(i)} - \gamma^{(i)}) = \beta^{\text{re}} - \beta^{(0)};$$

the second sum in (31) is factorized by simultaneous permutations in the sequences (33) and (34).

(4) All sided w -numbers $W_{Y, F_+, \varphi}^{\pm}(D, \alpha, \beta, 0)$, where $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ is a divisor class and $R_Y(D, \beta) > 0$, are recursively determined by the formulas (27), (28), (29), (31) and the initial conditions given by Proposition 16.

Proof. We follow the main lines of the proof of the recursive formula in [16, Section 3].

Proof of (1). The statement is clear for $\alpha \notin \mathbb{Z}_+^{\infty, \text{even}}$ or $\beta^{\text{re}} \notin \mathbb{Z}_+^{\infty, \text{even}}$, since the curves in count must have a non-empty one-dimensional part in F_- contrary to the definition of $V_{Y, F_+}^{\mathbb{R}}(D, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp})$. In the case $\beta^{\text{im}} \neq 0$, the statement follows from Lemma 8.

Proof of (2i). If $(K_Y + E)D = 0$, then either $D^2 = -1$, $DE = 1$, or $D = -(K_Y + E)$, $DE = 2$, and in both situations, $V_{Y, F_+}^{\mathbb{R}}(D, \alpha, \beta, 0, \mathbf{p}^{\flat}) = \emptyset$. Indeed, in the former case, we have $DE = -DK_Y = 1$; in the latter case, the condition $R_Y(-(K_Y + E), \beta) > 0$ yields $\beta = 2e_1$, and both conclusions contradict the assumption $\beta \in \mathbb{Z}_+^{\infty, \text{even}}$.

Let $(K_Y + E)D < -2$. The leaf-curves from $V_{Y, F_+}^{\mathbb{R}}(D, \alpha, \beta, 0, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp})$ generated by any reducible root-curve $C = E \cup \check{C}$ do not contribute to $W_{Y, F_+, \varphi}^+(D, \alpha, \beta, 0, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp})$. Indeed, \check{C} must contain a reduced real primary component, since $(K_Y + E)(D - E) < 0$ and $K_Y + E$ vanishes on all imaginary or non-reduced primary components of \check{C} (cf. Lemma 18). Hence, the total contribution of the leaves $\tilde{C} \in \text{Lf}(C)$ is zero in view of the factor $\mu^+(C, q) = 0$ (see formula (21)) in (25). Thus, by Lemma 17,

$$W_{Y, F_+, \varphi}^+(D, \alpha, \beta, 0, \mathbf{p}^{\flat}, \mathbf{p}^{\sharp}) = 2^{|\beta|} W_{Y, F_+, \varphi}^+(D, \alpha + \beta, 0, 0, (\mathbf{p}^{\flat})', (\mathbf{p}^{\sharp})'). \quad (35)$$

However, $R_Y(D, 0) = -(K_Y + E)D - 1 > 0$, that is $(\mathbf{p}^{\sharp})' \neq \emptyset$, and, as explained above, the right-hand side of (35) must vanish.

Proof of (2ii). Notice that $D - E$ is not effective, since $-(K_Y + E)(D - E) = -1$ and $-(K_Y + E)$ is nef. Hence, there are only irreducible root-curves, and the formula follows from Lemma 17.

Proof of Proposition 15 and assertions (2iii) and (3). All these statements follow by induction on $n = R_Y(D, \beta)$ from Lemmas 17 and 21. Proposition 16 serves as the base of induction.

In the right-hand side of formulas (29) and (31), the first sum runs over irreducible root-curves and the second sum runs over root-curves containing E . We only explain notations and coefficients occurring in the second sum:

- the vector $\alpha^{(0)}$ encodes the multiplicities of the primary components of type $iL(p_{ij})$,
- the vector $\beta^{(0)}$ encodes the multiplicities of the primary components of type $sL(z)$, and their multiplicative contribution amounts to $4^{\|\beta^{(0)}\|}$,
- the factors $(l/2 + 1)$ and $\eta(l)$ are the sums of the contributions of primary components sL' , $s''L''$ (computed using (23) and (24)) over the range $s' + s'' = l$,
- the vectors $\gamma^{(i)}$ encode the intersection multiplicities $j = (\check{C} \cdot E)_q$ of the points $q \in \mathbf{z}'$ (see Lemma 19) for the reducible root-curves $C = E \cup \check{C}$.

The statement (4) is straightforward. □

4 ABV formula over the reals

Let Y be a smooth rational surface, $E \subset Y$ a smooth rational curve. If the anti-canonical class $-K_Y$ is effective, positive on all curves different from E , and $K_Y E = 0$, we call the pair (Y, E) a *nodal del Pezzo pair*. (It follows from the adjunction formula that $E^2 = -2$.) Notice that a nodal del Pezzo pair may be not monic log-del Pezzo, and vice versa. Throughout this section we assume that (Y, E) is a nodal del Pezzo pair.

An example of a nodal del Pezzo pair is provided by the plane blown up at a generic collection of ≤ 8 points subject to the condition that six of them belong to a conic.

A nodal del Pezzo pair (Y, E) is an almost Fano surface in the sense of [23, Section 4.1], and thus by [23, Theorem 4.2] we have the following Abramovich-Bertram-Vakil formula (briefly *ABV formula*):

$$GW_0(Y, D) = \sum_{m \geq 0} \binom{DE + 2m}{m} N_Y(D - mE, 0, (DE + 2m)e_1), \quad (36)$$

where $D \in \text{Pic}(Y)$ is a divisor class, $GW_0(Y, D)$ states for the genus zero Gromov-Witten invariant, and $N_Y(D', 0, (D'E)e_1)$ is the number of rational curves $C \in |D'|$ passing through a generic collection of $-D'K_Y - 1$ points in $Y \setminus E$.

4.1 Deformation representation of ABV formula

ABV formula (36) can be represented geometrically. Let $\pi : \mathfrak{X} \rightarrow (\mathbb{C}, 0)$ be a proper holomorphic submersion of a smooth three-dimensional variety \mathfrak{X} (with $(\mathbb{C}, 0)$ being understood as a disc germ), where each fiber $\mathfrak{X}_t, t \neq 0$, is a del Pezzo surface and the central fiber $Y = \mathfrak{X}_0$ contains a smooth rational curve E such that (Y, E) is a nodal del Pezzo pair.

Remark 22 *There is a natural isomorphism*

$$\mathrm{Pic}(\mathfrak{X}_t) \xrightarrow{\simeq} \mathrm{Pic}(Y), \quad t \neq 0, \quad (37)$$

preserving the intersection form; for the sake of brevity we use the same symbol for corresponding divisor classes in $\mathfrak{X}_t, t \in (\mathbb{C}, 0)$. To distinguish linear systems themselves we use the notation $|D|_{\mathfrak{X}_t}$.

Let $D \in \mathrm{Pic}(\mathfrak{X}_t)$ be effective for $t \neq 0$ and satisfy $-K_{\mathfrak{X}_t}D - 1 \geq 0$. Pick $r = -K_{\mathfrak{X}_t}D - 1$ disjoint sections $z_i : (\mathbb{C}, 0) \rightarrow \mathfrak{X}, 1 \leq i \leq r$, so that $\mathbf{p}^\sharp(0) = \{z_i(0), 1 \leq i \leq r\}$ is a generic point collection in $Y \setminus E$, and $\mathbf{p}^\sharp(t) = \{z_i(t), 1 \leq i \leq r\}$ is a generic point collection in $\mathfrak{X}_t, t \neq 0$. For each $t \in (\mathbb{C}, 0)$, denote by $V_t(D, \mathbf{p}^\sharp(t))$ the set of reduced irreducible rational curves $C \in |D|_{\mathfrak{X}_t}$ that pass through $\mathbf{p}^\sharp(t)$. It is well-known (see, for instance, [8]) that $V_t(D, \mathbf{p}^\sharp(t)), t \neq 0$, is finite, contains $GW_0(Y, D)$ elements, and each element is a nodal curve. By [19, Proposition 2.1], for each $m \geq 0$, the set $V_0(D - mE, \mathbf{p}^\sharp(0))$ is finite, and its elements are immersed curves crossing E transversally at $DE + 2m$ distinct points. Thus, we have a diagram

$$\begin{array}{ccccc} \tilde{\mathcal{C}}' & \xrightarrow{\nu'} & \mathcal{C}' & \hookrightarrow & \mathfrak{X} \\ \downarrow \tilde{\pi}' & & \downarrow & & \downarrow \pi \\ (\mathbb{C}, 0) \setminus \{0\} & = & (\mathbb{C}, 0) \setminus \{0\} & \hookrightarrow & (\mathbb{C}, 0) \end{array} \quad (38)$$

where \mathcal{C}' is the union of $GW_0(Y, D)$ families of curves $C \in V_t(D, \mathbf{p}^\sharp(t)), t \in (\mathbb{C}, 0) \setminus \{0\}$, and $\tilde{\mathcal{C}}'$ is its normalization. The following statement follows from [23, Theorem 4.2].

Proposition 23 *There exists a diagram*

$$\begin{array}{ccccc} \tilde{\mathcal{C}} & \xrightarrow{\nu} & \mathcal{C} & \hookrightarrow & \mathfrak{X} \\ \downarrow \tilde{\pi} & & \downarrow & & \downarrow \pi \\ (\mathbb{C}, 0) & = & (\mathbb{C}, 0) & = & (\mathbb{C}, 0) \end{array} \quad (39)$$

which extends the diagram (38) so that

- (1)
 - \mathcal{C} is the closure of \mathcal{C}' in \mathfrak{X} ;
 - the fiber over 0 of each component of \mathcal{C} is $C_0 \cup mE$ for some $m \geq 0$, where $C_0 \in V_0(D, \mathbf{p}^\sharp(0))$;
 - each curve $C_0 \cup mE$ with $m \geq 0, C_0 \in V_0(D - mE, \mathbf{p}^\sharp(0))$ appears as the fiber over 0 for exactly $\binom{DE+2m}{m}$ components of \mathcal{C} ;

- (2) • $\tilde{\mathcal{C}}$ is the union of $GW_0(Y, D)$ disjoint nonsingular surfaces;
- the fiber over 0 of each component of $\tilde{\mathcal{C}}$ is either isomorphic to \mathbb{P}^1 with $\nu : \mathbb{P}^1 \rightarrow C_0 \in V_0(D, \mathbf{p}^\sharp(0))$ birational, or is a nodal reducible rational curve $\bigcup_{i=0}^m \mathbb{P}_{(i)}^1$ with some $m \geq 1$, $\mathbb{P}_{(i)}^1 \simeq \mathbb{P}^1$ for all $i = 0, \dots, m$, $\mathbb{P}_{(1)}^1, \dots, \mathbb{P}_{(m)}^1$ disjoint from each other, $\mathbb{P}_{(0)}^1$ intersecting each $\mathbb{P}_{(1)}^1, \dots, \mathbb{P}_{(m)}^1$ at one point, and such that $\nu : \mathbb{P}_{(0)}^1 \rightarrow C_0 \in V_0(D - mE, \mathbf{p}^\sharp(0))$ is birational, $\nu : \mathbb{P}_{(i)}^1 \rightarrow E$ is an isomorphism for all $i = 1, \dots, m$;
 - for each $C_0 \in V_0(D - mE, \mathbf{p}^\sharp(0))$, $m \geq 0$, there are exactly $\binom{DE+2m}{m}$ components of $\tilde{\mathcal{C}}$ whose fiber $\bigcup_{i=0}^m \mathbb{P}_{(i)}^1$ over 0 covers C_0 , and they differ from each other by the image of the m -tuple $\mathbb{P}_{(0)}^1 \cap (\bigcup_{i=1}^m \mathbb{P}_{(i)}^1)$ in the $(DE + 2m)$ -tuple $C_0 \cap E$.

If the families \mathfrak{X} , \mathcal{C}' , and $\tilde{\mathcal{C}}'$ are defined over the reals, then so are the families \mathcal{C} and $\tilde{\mathcal{C}}$. \square

4.2 Nodal degenerations

Let $\pi' : \mathfrak{X}' \rightarrow (\mathbb{C}, 0)$ be a holomorphic map of a smooth three-dimensional variety \mathfrak{X}' such that the fibers \mathfrak{X}'_τ , $\tau \in (\mathbb{C}, 0) \setminus \{0\}$, are del Pezzo surfaces, the central fiber \mathfrak{X}'_0 is a surface with one singular point z of type A_1 (node), at the point z the map π' is given, with respect to appropriate local coordinates x_1, x_2, x_3 , by

$$\pi'(x_1, x_2, x_3) = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2, \quad a_1 a_2 a_3 \neq 0, \quad (40)$$

and π' is a submersion at each point of $\mathfrak{X}' \setminus \{z\}$. Such a family is called *nodal degeneration*.

Make the base change $\tau = t^2$, perform the blow up $\tilde{\mathfrak{X}}' \rightarrow \mathfrak{X}'$ at the node of the new family, $\tilde{\mathfrak{X}}'' = (\mathbb{C}, 0) \times_{t^2=\pi'} \tilde{\mathfrak{X}}'$, and obtain a family $\tilde{\pi}' : \tilde{\mathfrak{X}}' \rightarrow (\mathbb{C}, 0)$, whose fibers $\tilde{\mathfrak{X}}'_t$, $t \neq 0$, are del Pezzo surfaces, and $\tilde{\mathfrak{X}}'_0 = Y \cup Z$, where $Z \simeq (\mathbb{P}^1)^2$, $E = Y \cap Z$ is a smooth rational (-2) -curve in Y , and (Y, E) is a nodal del Pezzo pair. Here, E represents the class $C_1 + C_2$ in $\text{Pic}(Z)$, C_1, C_2 being the generators of the two rulings of Z . We call the family $\tilde{\pi}' : \tilde{\mathfrak{X}}' \rightarrow (\mathbb{C}, 0)$ the *unscrew* of the nodal degeneration $\pi' : \mathfrak{X}' \rightarrow (\mathbb{C}, 0)$.

Contracting Z to E along the lines of one of the rulings (see [2]), say, generated by C_1 , we get a family

$$\pi : \mathfrak{X} \rightarrow (\mathbb{C}, 0) \quad (41)$$

of smooth surfaces as in Section 4.1. Let $D \in \text{Pic}(\mathfrak{X}_t)$ be effective for $t \neq 0$ and satisfy $-K_{\mathfrak{X}_t} D - 1 \geq 0$. Pick $r = -K_{\mathfrak{X}_t} D - 1$ disjoint sections $z_i : (\mathbb{C}, 0) \rightarrow \mathfrak{X}$, $1 \leq i \leq r$, so that $\mathbf{p}^\sharp(0) = \{z_i(0), 1 \leq i \leq r\}$ is a generic point collection in $Y \setminus E$, and $\mathbf{p}^\sharp(t) = \{z_i(t), 1 \leq i \leq r\}$ is a generic point collection in \mathfrak{X}_t , $t \neq 0$. The following lemma is straightforward.

Lemma 24 *Let \mathcal{C}' be a family of rational curves $\mathcal{C}'_t \in |D|_{\mathfrak{X}_t}$ which pass through $\mathbf{p}^\sharp(t)$, $t \in (\mathbb{C}, 0) \setminus \{0\}$. The family \mathcal{C}' lifts to a family of curves in $\tilde{\pi}' : \tilde{\mathfrak{X}}' \rightarrow (\mathbb{C}, 0)$ as follows: if \mathcal{C}' closes up in \mathfrak{X} with the central fiber $C'_m \cup mE$, $C'_m \in V_0(D - mE, \mathbf{p}^\sharp(0))$, then \mathcal{C}' closes up in $\tilde{\mathfrak{X}}'$ with a central fiber $C'_m \cup C_1^{(DE+m)} \cup C_2^{(m)}$, where $C_2^{(m)}$ is the union of m lines in $|C_2|_Z$ attached to m intersection points of C'_m and E , and $C_1^{(DE+m)}$ is the union of $DE + m$ lines from $|C_1|_Z$ attached to the remaining points of $C'_m \cap E$. \square*

Remark 25 *A family of plane quartics with the central fiber Q having one node z induces a family of del Pezzo surfaces of degree 2 degenerating into a nodal del Pezzo pair. In this setting, E is the exceptional divisor of the blow up of the node of the double cover of the plane ramified along the nodal quartic, the six pairs of intersecting (-1) -curves crossing E respectively cover the six lines in the plane passing through z and tangent to Q outside z , and, finally, the supporting curves L', L'' doubly cover the tangent lines to Q at the node z .*

Assume that a nodal degeneration $\pi' : \mathfrak{X}' \rightarrow (\mathbb{C}, 0)$ possesses a real structure Conj which lifts the standard complex conjugation $\text{conj} : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$. Then, the point z is real and with respect to appropriate real local coordinates at z the map π' is given by

$$\pi'(x_1, x_2, x_3) = a_1x_1^2 + a_2x_2^2 + a_3x_3^2, \quad a_1, a_2, a_3 \in \mathbb{R}, \quad a_1a_2a_3 \neq 0. \quad (42)$$

The real structure Conj gives rise to two real structures on the unscrew $\tilde{\pi}' : \tilde{\mathfrak{X}}' \rightarrow (\mathbb{C}, 0)$ of $\pi' : \mathfrak{X}' \rightarrow (\mathbb{C}, 0)$. One real structure covers the complex conjugation $t \mapsto \bar{t}$, and we call the resulting real unscrew a θ -unscrew, where θ is the signature of the quadratic form (42). The quadric $Z \subset \tilde{\mathfrak{X}}'_0$ is real, and

$$\mathbb{R}Z \simeq \begin{cases} S^2, & \text{if } \theta = 3 \text{ or } -1, \\ (S^1)^2, & \text{if } \theta = 1, \\ \emptyset, & \text{if } \theta = -3. \end{cases}$$

The other real structure on $\tilde{\pi}' : \tilde{\mathfrak{X}}' \rightarrow (\mathbb{C}, 0)$ covers the conjugation $t \mapsto -\bar{t}$ and defines a *mirror* $(-\theta)$ -unscrew.

Proposition 26 *Let $\tilde{\pi}' : \tilde{\mathfrak{X}}' \rightarrow (\mathbb{C}, 0)$ be a θ -unscrew. Then, the isomorphism $\text{Pic}(\mathfrak{X}_t) \xrightarrow{\sim} \text{Pic}(Y)$ is conjugation invariant. If θ is equal to 1 or -3 , this isomorphism induces an isomorphism $\text{Pic}^{\mathbb{R}}(\mathfrak{X}_t) \xrightarrow{\sim} \text{Pic}^{\mathbb{R}}(Y)$, $t \in (\mathbb{R}, 0)$. If θ is equal to -1 or 3 , the isomorphism $\text{Pic}(\mathfrak{X}_t) \xrightarrow{\sim} \text{Pic}(Y)$, $t \neq 0$, induces a monomorphism $\text{Pic}^{\mathbb{R}}(\mathfrak{X}_t) \rightarrow \text{Pic}^{\mathbb{R}}(Y)$, $t \in (\mathbb{R}, 0) \setminus \{0\}$, and the image of the latter is orthogonal to $[E] \in \text{Pic}^{\mathbb{R}}(Y)$.*

Proof. If θ is equal to 1 or -3 , the action of Conj in $\text{Pic}(\tilde{\mathfrak{X}}'_t) = H^2(\tilde{\mathfrak{X}}'_t; \mathbb{Z})$ and $\text{Pic}(Y) = H^2(Y; \mathbb{Z})$ commutes with the natural (as in Remark 22) isomorphism $H^2(\tilde{\mathfrak{X}}'_t; \mathbb{Z}) \rightarrow H^2(Y; \mathbb{Z})$. If θ is equal to -1 or 3 , the action of Conj in $\text{Pic}(\tilde{\mathfrak{X}}'_t)$ and

$\text{Pic}(Y)$ does not commute with the isomorphism $H^2(\tilde{\mathfrak{X}}'_t; \mathbb{Z}) \rightarrow H^2(Y; \mathbb{Z})$, the defect being the composition with the reflection in $[E] \in H^2(Y; \mathbb{Z})$. \square

4.3 Real versions of ABV formula

4.3.1 Ordinary and sided u -numbers

Let (Y, E) be a nodal del Pezzo pair such that Y and E are real, and $\mathbb{R}E \neq \emptyset$. Denote by F the connected component of $\mathbb{R}Y$ containing $\mathbb{R}E$ and pick a conjugation invariant class $\varphi \in H_2(Y \setminus F, \mathbb{Z}/2)$. Let $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$. Choose a generic collection \mathbf{p}^\sharp of $-K_Y D - 1$ points in $F \setminus E$.

By [23, Proposition 4.1(b)], the set $V_Y(D, \mathbf{p}^\sharp)$ of rational curves in the linear system $|D|$ which pass through the points of \mathbf{p}^\sharp is finite and consists of immersed curves crossing E transversally at DE distinct points.

For any nonnegative integers k and l such that $k + 2l = DE$, define an *ordinary u -number* $U_{Y,F,\varphi}(D, ke_1, le_1, \mathbf{p}^\sharp)$ putting (cf. the definition of ordinary w -numbers in Section 3.6)

$$U_{Y,E,\varphi}(D, ke_1, le_1, \mathbf{p}^\sharp) = \sum_{C \in V_Y^{\mathbb{R}}(D, ke_1, le_1, \mathbf{p}^\sharp)} \mu_\varphi(C), \quad (43)$$

where $V_Y^{\mathbb{R}}(D, ke_1, le_1, \mathbf{p}^\sharp) \subset V_Y(D, \mathbf{p}^\sharp)$ is formed by the curves intersecting $\mathbb{R}E$ in k points (and intersecting $E \setminus \mathbb{R}E$ in l pairs of complex conjugate points) and $\mu_\varphi(C)$ is defined by (7). If $F \setminus E$ splits into two components F_+ and F_- , the configuration \mathbf{p}^\sharp lies in F_+ , and DE is even, then define a *sided u -number* $U_{Y,F_+,\varphi}^\pm(D, 0, (DE/2)e_1, \mathbf{p}^\sharp)$, putting

$$U_{Y,F_+,\varphi}^+(D, 0, (DE/2)e_1, \mathbf{p}^\sharp) = \sum_{C \in V_{Y,F_+}^{\mathbb{R}}(D, 0, (DE/2)e_1, \mathbf{p}^\sharp)} \mu_\varphi(C), \quad (44)$$

$$U_{Y,F_+,\varphi}^-(D, 0, (DE/2)e_1, \mathbf{p}^\sharp) = \sum_{C \in V_{Y,F_+}^{\mathbb{R}}(D, 0, (DE/2)e_1, \mathbf{p}^\sharp)} \mu_\varphi^-(C), \quad (45)$$

where

$$V_{Y,F_+}^{\mathbb{R}}(D, 0, (DE/2)e_1, \mathbf{p}^\sharp) = \{C \in V_Y^{\mathbb{R}}(D, 0, (DE/2)e_1, \mathbf{p}^\sharp) : \text{card}(C \cap F_-) < \infty\}$$

and $\mu_\varphi^-(C)$ is defined by (13).

We say that a quadruple (Y, E, F, φ) has *property (R)* if for any divisor class $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ and for any connected component F_+ of $F \setminus E$, there exists a generic collection \mathbf{p}^\sharp of $-K_Y D - 1$ points in F_+ (referred to as R_{D,F_+} -collection or R_D -collection) such that, for any $m \geq 0$ with $D - mE \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$, the following holds:

$$(R1) \quad U_{Y,F,\varphi}(D - mE, ke_1, le_1, \mathbf{p}^\sharp) = 0 \text{ whenever } l > 0,$$

(R2) if $F \setminus E$ splits into two components and the intersection DE is even, then $U_{Y,F_+,\varphi}^+(D - mE, 0, (DE/2)e_1, \mathbf{p}^\sharp) = 0$.

Proposition 27 *Let (Y, E) be a nodal del Pezzo pair such that Y and E are real, and $\mathbb{R}E \neq \emptyset$. Denote by F the connected component of $\mathbb{R}Y$ containing $\mathbb{R}E$ and pick a conjugation invariant class $\varphi \in H_2(Y \setminus F, \mathbb{Z}/2)$. Assume in addition that (Y, E) is monic log-del Pezzo.*

(1) *Pick a divisor class $D_0 \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ such that $\dim |D_0| > 0$. Let $D \in \text{Prec}(D_0)$ be a divisor class, and let \mathbf{p}^\sharp be a collection of $-K_Y D - 1$ points in D_0 -CH position in $F \setminus E$. Then, for any nonnegative integers k and l such that $k + 2l = DE$, one has*

$$U_{Y,E,\varphi}(D, ke_1, le_1, \mathbf{p}^\sharp) = W_{Y,E,\varphi}(D, 0, ke_1, le_1, \emptyset, \mathbf{p}^\sharp).$$

If $F \setminus E$ splits into two components F_+ and F_- , the collection \mathbf{p}^\sharp is contained in F_+ , and DE is even, then

$$U_{Y,F_+,\varphi}^\pm(D, 0, (DE/2)e_1, \mathbf{p}^\sharp) = W_{Y,F_+,\varphi}^\pm(D, 0, 0, (DE/2)e_1, \emptyset, \mathbf{p}^\sharp).$$

The numbers $U_{Y,E,\varphi}(D, ke_1, le_1, \mathbf{p}^\sharp)$ and $U_{Y,F_+,\varphi}^\pm(D, 0, (DE/2)e_1, \mathbf{p}^\sharp)$ do not depend on the choice of a collection \mathbf{p}^\sharp in D_0 -CH position.

(2) *The quadruple (Y, E, F, φ) has property (R).*

Proof. The equality of ordinary (respectively, sided) u - and w -numbers is tautological. The invariance of the u -numbers considered follows from the invariance of w -numbers; for the latter invariance see Propositions 10 and 15. \square

4.3.2 External u -numbers

Let (Y, E) be a nodal del Pezzo pair such that Y and E are real, and $\mathbb{R}Y \neq \emptyset$. Let $F \subset \mathbb{R}Y$ be a connected component such that $F \cap \mathbb{R}E = \emptyset$. Let $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$. Choose a generic collection \mathbf{p}^\sharp of $-K_Y D - 1$ points in F . Notice that $DE \geq 0$ is even, and each real curve in $V_Y(D, \mathbf{p}^\sharp)$ intersects E at $DE/2$ distinct pairs of complex conjugate points.

If $\mathbb{R}E \neq \emptyset$, we denote by F' the connected component of $\mathbb{R}Y$ containing $\mathbb{R}E$. If $\mathbb{R}E = \emptyset$, we put $F' = \emptyset$. Choose a conjugation invariant class $\varphi \in H_2(Y \setminus (F \cup F'), \mathbb{Z}/2)$ and define *external u -numbers*

$$U_{Y,E,\varphi'}(D, \mathbf{p}^\sharp) = \sum_{C \in V_Y^{\mathbb{R}}(D, 0, (DE/2)e_1, \mathbf{p}^\sharp)} \mu_{\varphi'}(C), \quad \text{for } \varphi' = \varphi \text{ or } \varphi + [F'] .$$

4.3.3 ABV formulas for Welschinger invariants, I

As in Section 4.1, let $\pi : \mathfrak{X} \rightarrow (\mathbb{C}, 0)$ be holomorphic submersion of a smooth three-dimensional variety \mathfrak{X} , where each fiber $\mathfrak{X}_t, t \neq 0$, is a del Pezzo surface and the

central fiber $Y = \mathfrak{X}_0$ contains a smooth rational curve E such that (Y, E) is a nodal del Pezzo pair.

Suppose that \mathfrak{X} possesses a real structure $\text{Conj} : \mathfrak{X} \rightarrow \mathfrak{X}$ such that

$$\pi \circ \text{Conj} = \text{conj} \circ \pi \quad (46)$$

(where conj is the standard real structure on $(\mathbb{C}, 0)$). We get a family $\pi : \mathbb{R}\mathfrak{X} = \pi^{-1}(\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0) \hookrightarrow (\mathbb{C}, 0)$ of real surfaces, the fibers \mathfrak{X}_t , $t \in (\mathbb{R}, 0) \setminus \{0\}$, being real del Pezzo surfaces, and (Y, E) being a real nodal del Pezzo pair. Assume that $\mathbb{R}E \neq \emptyset$. Denote by F_0 the connected component of $\mathbb{R}Y$ containing $\mathbb{R}E$ and pick a conjugation invariant class $\varphi_0 \in H_2(Y \setminus F_0, \mathbb{Z}/2)$. The family $\mathbb{R}\mathfrak{X} \rightarrow (\mathbb{R}, 0)$ is topologically trivial. We extend F_0 to a continuous family F_t of connected components of $\mathbb{R}\mathfrak{X}_t$, and extend φ_0 to a continuous family of conjugation invariant classes $\varphi_t \in H_2(\mathbb{R}\mathfrak{X}_t \setminus F_t, \mathbb{Z}/2)$, $t \in (\mathbb{R}, 0)$.

Theorem 6 (1) *For any real effective divisor class D on \mathfrak{X}_t , $t \neq 0$, one has*

$$W(\mathfrak{X}_t, D, F_t, \varphi_t) = W(\mathfrak{X}_t, D + (DE)E, F_t, \varphi_t) .$$

(2) *Assume that the quadruple (Y, E, F_0, φ_0) has property (R). Then, for any $t \in (\mathbb{R}, 0)$, $t \neq 0$, any divisor class $D \in \text{Pic}^{\mathbb{R}}(\mathfrak{X}_t)$, and any R_D -collection $\mathbf{p}^\sharp \subset F_0 \setminus E$, the following equality holds:*

$$W(\mathfrak{X}_t, D, F_t, \varphi_t) = \sum_{m \geq 0} (-1)^{m(E_{1/2} \circ \varphi_0)} \binom{DE + 2m}{m} U_{Y, E, \varphi_0}(D - mE, (DE + 2m)e_1, 0, \mathbf{p}^\sharp) . \quad (47)$$

Proof. For the first statement, without loss of generality, assume that $DE = -d < 0$ and choose a continuous family of collections $\mathbf{p}_t^\sharp \subset F_t$ of $-K_X D - 1$ distinct points so that \mathbf{p}_0^\sharp is generic in $F_0 \setminus E$. We establish a one-to-one correspondence between the sets M_1 and M_2 of real rational curves in $|D|_{\mathfrak{X}_t}$ and $|D - dE|_{\mathfrak{X}_t}$, respectively, passing through \mathbf{p}_t^\sharp , such that the correspondence preserves the Welschinger signs. Indeed, by Proposition 23, degenerations of curves from M_1 are of type $C \cup (d + s)E$, $s \geq 0$, where $C \in |D - (d + s)E|_Y$ is a real rational curve passing through \mathbf{p}_0^\sharp ; furthermore, to each such a curve C and a conjugation invariant subset $w \subset C \cap E$ of $d + s$ points there corresponds a unique curve in M_1 , and its Welschinger sign coincides with that of C . Similarly, degenerations of curves from M_2 are of type $C \cup sE$, $C \in |D - (d + s)E|_Y$ as above, and to each subset $(C \cap E) \setminus w \subset C \cap E$ of s points there corresponds a unique curve in M_2 , and its Welschinger sign coincides with that of C .

To prove the second statement, notice that, since the quadruple (Y, E, F_0, φ_0) has property (R), the degenerate real curves to consider are of type $C \cup mE$, $m \geq 0$, where $C \in V_Y^{\mathbb{R}}(D - mE, (DE + 2m)e_1, 0, \mathbf{p}^\sharp)$, and the Welschinger sign of each real rational curve in \mathfrak{X}_t , $t \neq 0$, appearing in a deformation of $C \cup mE$ is $(-1)^{m(E_{1/2} \circ \varphi_0)} \mu_\varphi(C)$. Thus, formula (47) follows from Proposition 23. \square

4.3.4 ABV formulas for Welschinger invariants, II

Assume that $\tilde{\pi}' : \tilde{\mathfrak{X}}' \rightarrow (\mathbb{C}, 0)$ is a real unscrew of a real nodal degeneration $\pi' : \mathfrak{X}' \rightarrow (\mathbb{C}, 0)$, and $\mathbb{R}E \neq \emptyset$ (see Section 4.2). In this case, the signature θ of the unscrew is equal either to 1, or to -1 . If $\theta = 1$ (respectively, $\theta = -1$), one has $\mathbb{R}Z \simeq (S^1)^2$ (respectively, $\mathbb{R}Z \simeq S^2$). Let $F \subset \mathbb{R}Y$ be the connected component containing $\mathbb{R}E$. Pick a conjugation invariant class $\varphi \in H_2(Y \setminus F, \mathbb{Z}/2)$.

In the case of a 1-unscrew, both rulings of Z are real. The contraction of Z along one of the rulings leads to a family of smooth real surfaces $\pi : \mathfrak{X} \rightarrow (\mathbb{C}, 0)$ (cf. Section 4.2). Thus, Theorem 6 applies and gives Welschinger invariants of the real del Pezzo surfaces \mathfrak{X}_t , $t \in (\mathbb{R}, 0) \setminus \{0\}$, via w -numbers of the real nodal del Pezzo pair (Y, E) by formula (47).

Theorem 7 *Assume that the unscrew $\tilde{\pi}' : \tilde{\mathfrak{X}}' \rightarrow (\mathbb{C}, 0)$ of a real nodal degeneration $\pi' : \mathfrak{X}' \rightarrow (\mathbb{C}, 0)$ is of signature -1 . Then, the following holds.*

(1) *Suppose that $F \setminus E$ is connected. Let F_t be a component of $\mathbb{R}\tilde{\mathfrak{X}}'_t$ merging to F as $t \rightarrow 0$. Choose a class $\varphi \in H_2(\mathbb{R}Y \setminus F, \mathbb{Z}/2)$, and denote by φ_t the class in $H_2(\mathbb{R}\tilde{\mathfrak{X}}'_t \setminus F_t, \mathbb{Z}/2)$, $t \in (\mathbb{R}, 0) \setminus \{0\}$, which converges to φ as $t \rightarrow 0$. If the quadruple (Y, E, F, φ) has property (R), then for any $t \in (\mathbb{R}, 0)$, $t \neq 0$, any divisor class $D \in \text{Pic}_+^{\mathbb{R}}(\tilde{\mathfrak{X}}'_t)$ and any R_D -collection $\mathbf{p}^\sharp \subset F \setminus E$, one has*

$$W(\tilde{\mathfrak{X}}'_t, D, F_t, \varphi_t) = U_{Y, E, \varphi}(D, 0, 0, \mathbf{p}^\sharp). \quad (48)$$

(2) *Suppose that $F \setminus E$ splits into two components F_+, F_- , and $\mathbb{R}\tilde{\mathfrak{X}}'_t$ contains two connected components $F_{+,t}$ and $F_{-,t}$ which merge to F_+ and F_- , respectively. Choose a class $\varphi \in H_2(\mathbb{R}Y \setminus F, \mathbb{Z}/2)$, and denote by φ_t be the class in $H_2(\mathbb{R}\tilde{\mathfrak{X}}'_t \setminus F_t, \mathbb{Z}/2)$, $t \in (\mathbb{R}, 0) \setminus \{0\}$, which converges to φ as $t \rightarrow 0$.*

(i) *If the quadruple (Y, E, F, φ) has property (R), then, for any $t \in (\mathbb{R}, 0)$, $t \neq 0$, any divisor class $D \in \text{Pic}_+^{\mathbb{R}}(\tilde{\mathfrak{X}}'_t)$, and any R_{D, F_+} -collection $\mathbf{p}^\sharp \subset F_+$, one has*

$$W(\tilde{\mathfrak{X}}'_t, D, F_{+,t}, \varphi_t) = U_{Y, F_+, \varphi}^+(D, 0, 0, \mathbf{p}^\sharp). \quad (49)$$

(ii) *For any $t \in (\mathbb{R}, 0)$, $t \neq 0$, any divisor class $D \in \text{Pic}_+^{\mathbb{R}}(\tilde{\mathfrak{X}}'_t)$, and any generic configuration $\mathbf{p}^\sharp \subset F_+$ of $-K_Y D - 1$ points, one has*

$$W(\tilde{\mathfrak{X}}'_t, D, F_{+,t}, F_{-,t} \cup \varphi_t) = U_{Y, F_+, \varphi}^-(D, 0, 0, \mathbf{p}^\sharp). \quad (50)$$

Proof. Pick a divisor class $D \in \text{Pic}_+^{\mathbb{R}}(\tilde{\mathfrak{X}}'_t)$, $t \neq 0$, take disjoint sections $z_i : (\mathbb{R}, 0) \rightarrow \mathbb{R}\tilde{\mathfrak{X}}'$, $1 \leq i \leq -K_Y D - 1$, and consider the limits that the real rational curves in $|D|_{\tilde{\mathfrak{X}}'_t}$ passing through $\{z_i(t)\}_{1 \leq i \leq -K_Y D - 1}$, $t \neq 0$, have in the central fiber $Y \cup Z$. These limits are of type $C'_m \cup C_1^{(DE+m)} \cup C_2^{(m)}$ (see Lemma 24).

To prove statements (1) and (2i), assume that the quadruple (Y, E, F, φ) has property (R), and that $\mathbf{p}^\sharp = \{z_i(0)\}_{1 \leq i \leq -K_Y D - 1} \subset F \setminus E$ is an R_D -collection. Due

to property (R), the components $C'_m \in |D - mE|_Y$, $m > 0$, of the limits of real rational curves in $|D|_{\tilde{\mathfrak{X}}'_t}$ must have real intersection points with E . However, such curves C'_m cannot be completed to real curves in $|D|_Y$. Taking this into account, we prove statements (2) and (3i) in the same manner as Theorem 6(2).

To prove statement (2ii), put $\mathbf{p}^\sharp = \{z_i(0)\}_{1 \leq i \leq -K_Y D - 1}$, and notice that if the limit curve contains as a component a real rational curve $C \in |D - mE|_Y$, $m > 0$, then $C \cap \mathbb{R}E = \emptyset$. In the family $\{\tilde{\mathfrak{X}}'_t\}_{t \in [0,1]}$, the surface $\tilde{\mathfrak{X}}'_0 = Y \cup Z$ deforms so that F_+ glues up with a component Z_+ of $\mathbb{R}Z \setminus E$, whereas F_- glues up with the other component Z_- of $\mathbb{R}Z \setminus E$. In turn, each real rational curve $C \in |D - mE|_Y$ with $C \cap \mathbb{R}E = \emptyset$ can be completed up to a real curve on $Y \cup Z$ in 2^m ways, when attaching to each pair $z', z'' \in C \cap E$ of complex conjugate points either the pair $C'_1 \supset \{z'\}$, $C''_2 \supset \{z''\}$, or the pair $C'_1 \supset \{z''\}$, $C''_2 \supset \{z'\}$, where C'_1, C''_1 belong to one ruling of Z , and C'_2, C''_2 to the other. Observe that one of the pairs (C'_1, C''_2) , (C''_1, C'_2) has a solitary node in Z_+ , which contributes the factor (-1) to the Welschinger sign μ_{φ^-} of the corresponding deformed curve in $|D|_{\tilde{\mathfrak{X}}'_t}$, $t > 0$, whereas the other pair has a solitary node in Z_- that does not affect μ_{φ^-} . Hence, the total contribution $W(\tilde{\mathfrak{X}}'_t, D, F_{+,t}, F_{-,t} \cup \varphi_t)$ of the curves coming from C is zero, which proves formula (50). \square

Corollary 28 *Let (Y, E) be a real nodal del Pezzo pair such that $\mathbb{R}E$ divides a connected component F of $\mathbb{R}Y$ into two parts, F_+ and F_- . Let $D \in \text{Pic}^{\mathbb{R}}(Y)$ be an effective divisor class such that $DE = 0$. Let $\varphi \in H_2(Y \setminus F, \mathbb{Z}/2)$ be a conjugation invariant class. Then, the sided u-number $U_{Y, F_+, \varphi}^-(D, 0, 0, \mathbf{p}^\sharp)$ does not depend on the choice of a generic configuration $\mathbf{p}^\sharp \subset F_+$ of $-K_Y D - 1$ points.* \square

Let $\pi' : \mathfrak{X}' \rightarrow (\mathbb{C}, 0)$ be a real nodal degeneration. Assume that its unscrew is of signature 1, denote this unscrew by $\pi^h : \mathfrak{X}^h \rightarrow (\mathbb{C}, 0)$. Its mirror unscrew $\pi^e : \mathfrak{X}^e \rightarrow (\mathbb{C}, 0)$ is of signature -1 . Let (Y, E) be the real nodal del Pezzo pair that appears as a component of both \mathfrak{X}_0^h and \mathfrak{X}_0^e . Let F be the connected component of $\mathbb{R}Y$ containing $\mathbb{R}E$, and let F_t^h , respectively, F_t^e , be the component (or the union of components) of $\mathbb{R}\mathfrak{X}_t^h$, respectively, $\mathbb{R}\mathfrak{X}_t^e$, $t \neq 0$, which merges to F as $t \rightarrow 0$. Let $\varphi_t^h \in H_2(\mathfrak{X}_t^h \setminus F_t^h, \mathbb{Z}/2)$, respectively, $\varphi_t^e \in H_2(\mathfrak{X}_t^e \setminus F_t^e, \mathbb{Z}/2)$, $t \neq 0$, be families of conjugation invariant classes converging to the same class $\varphi \in H_2(Y \setminus F, \mathbb{Z}/2)$ as $t \rightarrow 0$.

Corollary 29 *Assume that the quadruple (Y, E, F, φ) has property (R). Let D be a real effective divisor class on Y such that $DE = 0$. Then,*

$$W(\mathfrak{X}_t^e, D, F_{+,t}^e, \varphi_t^e) = \sum_{m \in \mathbb{Z}} (-1)^m W(\mathfrak{X}_t^h, D - mE, F_t^h, \varphi_t^h), \quad t \neq 0,$$

where $F_{+,t}^e$ is any component of F_t^e .

Proof. This is an immediate consequence of Theorems 6(1,2) and 7(1,2i), and of the relation $\sum_{m \in \mathbb{Z}} (-1)^m \binom{2k}{m-k} = 0$ which holds for any integer $k \neq 0$. \square

4.3.5 ABV formulas for Welschinger invariants, III

Let $\tilde{\pi}' : \tilde{\mathfrak{X}}' \rightarrow (\mathbb{C}, 0)$ be a real unscrew of a real nodal degeneration $\pi' : \mathfrak{X}' \rightarrow (\mathbb{C}, 0)$ such that $\mathbb{R}Y \neq \emptyset$ (see Section 4.2). Assume that there is a connected component $F \subset \mathbb{R}Y$ disjoint from $\mathbb{R}E$. If $\mathbb{R}E \neq \emptyset$, denote by F' the connected component of $\mathbb{R}Y$ containing $\mathbb{R}E$; if $\mathbb{R}E = \emptyset$, put $F' = \emptyset$. Choose a conjugation invariant class $\varphi \in H_2(Y \setminus (F \cup F'), \mathbb{Z}/2)$. Extend F to a family F_t of connected components of $\mathbb{R}\tilde{\mathfrak{X}}'_t$, $t \in (\mathbb{R}, 0)$, and denote by F'_t the part of $\mathbb{R}\tilde{\mathfrak{X}}'_t$, $t \in (\mathbb{R}, 0) \setminus \{0\}$, which converges to $F' \cup \mathbb{R}Z$ as $t \rightarrow 0$. Extend φ to a family of conjugation invariant classes $\varphi_t \in H_2(\mathbb{R}\tilde{\mathfrak{X}}'_t \setminus (F_t \cup F'_t), \mathbb{Z}/2)$.

Theorem 8 (1) *Let the signature of the unscrew $\tilde{\pi}' : \tilde{\mathfrak{X}}' \rightarrow (\mathbb{C}, 0)$ be 1 or -3 . Then, the following holds.*

(1i) *For any real effective divisor class D on $\tilde{\mathfrak{X}}'_t$, $t \neq 0$, one has*

$$W(\tilde{\mathfrak{X}}'_t, D, F_t, \varphi_t) = W(\tilde{\mathfrak{X}}'_t, D + (DE)E, F_t, \varphi_t) ,$$

$$W(\tilde{\mathfrak{X}}'_t, D, F_t, \varphi_t + [F'_t]) = W(\tilde{\mathfrak{X}}'_t, D + (DE)E, F_t, \varphi_t + [F'_t]) .$$

(1ii) *For any real effective divisor class D on $\tilde{\mathfrak{X}}'_t$, $t \neq 0$, and any generic collection \mathbf{p}^\sharp of $-DK_Y - 1$ distinct points of F , one has*

$$W(\tilde{\mathfrak{X}}'_t, D, F_t, \varphi_t) = \sum_{m \geq 0} \binom{DE/2 + 2m}{m} U_{Y,E,\varphi}(D - 2mE, \mathbf{p}^\sharp) ,$$

$$W(\tilde{\mathfrak{X}}'_t, D, F_t, \varphi_t + [F'_t]) = \sum_{m \geq 0} \binom{DE/2 + 2m}{m} U_{Y,E,\varphi+[F'_t]}(D - 2mE, \mathbf{p}^\sharp) .$$

(2) *Let the signature of the unscrew $\tilde{\pi}' : \tilde{\mathfrak{X}}' \rightarrow (\mathbb{C}, 0)$ be 3 or -1 . Then, for any real effective divisor class D on $\tilde{\mathfrak{X}}'_t$, $t \neq 0$, one has*

$$W(\tilde{\mathfrak{X}}'_t, D, F_t, \varphi_t) = \sum_{m \geq 0} (-2)^m U_{Y,E,\varphi}(D - mE, \mathbf{p}^\sharp) ,$$

$$W(\tilde{\mathfrak{X}}'_t, D, F_t, \varphi_t + [F'_t]) = \sum_{m \geq 0} 2^m U_{Y,E,\varphi+[F'_t]}(D - mE, \mathbf{p}^\sharp) .$$

Proof. Statement (1i) can be proved as Theorem 6(1), if we notice that both the rulings of the quadric Z are conjugation-invariant in the considered situation. Formulas of (1iii) follow from Proposition 23 and Lemma 24: a curve $C \in |D - 2mE|_Y$ intersects E at $DE/2 + 2m$ pairs of complex conjugate points, to each pair we attach two lines belonging to the same ruling of Z , and, finally, one has to choose m pairs of conjugate intersection points to attach the lines of a marked ruling (denoted $|C_2|_Y$ in Lemma 24).

In the situation of assertion (2), the complex conjugation interchanges the rulings of Z . Thus, the formulas required follow from Proposition 23 and Lemma 24: to

each of the m pairs of complex conjugate intersection points we attach a line from $|C_1|_Y$ and a line from $|C_2|_Y$, and we have two ways to do so. For the sign relations, notice that $E \circ \varphi = 0 \in \mathbb{Z}/2$, and that a pair of complex conjugate lines on Z has a solitary intersection point in $\mathbb{R}Z$. \square

Proposition 30 *External u -numbers do not depend on the choice of \mathbf{p}^\sharp .*

Proof. From formulas of Theorem 8(1ii), we can express $U_{Y,E,\varphi}(D, \mathbf{p}^\sharp)$ (resp. $U_{Y,E,\varphi+[F']}(D, \mathbf{p}^\sharp)$) as a linear combination of $W(\tilde{\mathfrak{X}}'_t, D - 2m, F_t, \varphi_t)$, $m \geq 0$, (resp. $W(\tilde{\mathfrak{X}}'_t, D - 2m, F_t, \varphi_t + [F'_t])$, $m \geq 0$). \square

Corollary 31 *Let $\pi' : \mathfrak{X}' \rightarrow (\mathbb{C}, 0)$ be a real nodal degeneration, $\pi^\theta : \mathfrak{X}^\theta \rightarrow (\mathbb{C}, 0)$ and $\pi^{-\theta} : \mathfrak{X}^{-\theta} \rightarrow (\mathbb{C}, 0)$ the mirror unscrews obtained from π' (of signature θ and $-\theta$, respectively). Let the (common for \mathfrak{X}_0^θ and $\mathfrak{X}_0^{-\theta}$) real nodal del Pezzo pair (Y, E) have a connected component F of $\mathbb{R}Y \neq \emptyset$ disjoint from E . Choose a conjugation invariant class $\varphi \in H_2(Y \setminus (F \cup F'), \mathbb{Z}/2)$.*

If $\theta = 3$ or -1 , then, for any real effective divisor class $D \in \text{Pic}(Y)$ such that $DE = 0$ and $t \in (\mathbb{R}, 0) \setminus \{0\}$, one has

$$W(\mathfrak{X}_t^\theta, D, F_t, \varphi_t) + W(\mathfrak{X}_t^\theta, D, F_t, \varphi_t + [F'_t]) = 2 \sum_{m \in \mathbb{Z}} W(\mathfrak{X}_t^{-\theta}, D + 2mE, F_t, \varphi_t + [F'_t]).$$

Proof. This is an immediate consequence of Theorem 8. \square

Remark 32 *By [5, Proposition 3.3] (see also Proposition 2 below for the case of del Pezzo surfaces of degree ≥ 2), if $-DK_Y \geq 3$, then $W(\mathfrak{X}_t^\theta, D, F_t, \varphi_t) = 0$, and hence the left-hand side of the formula in Corollary 31 reduces to $W(\mathfrak{X}_t^\theta, D, F_t, \varphi_t + [F'_t])$.*

5 Proof of positivity and asymptotics statements

5.1 ABV families

Any two real del Pezzo surfaces of degree 2 that have homeomorphic real parts are deformation equivalent in the class of such surfaces (see, for example, [7], Theorem 17.3), and, as a result, they have the same system of Welschinger invariants. Therefore, in the proof of Theorems 2 and 3, for each topological type, it is sufficient to pick a particular real del Pezzo surface, X , that we include into an appropriate family with a special fiber containing a real nodal del Pezzo pair (Y, E) ; the family depends on the choice of a connected component F of $\mathbb{R}X$.

If X is of type $\mathbb{P}_{a,b}^2$, $a + 2b = 7$, (or, equivalently, of type $\langle q \rangle^-$, $1 \leq q \leq 4$), we include X into a family $\mathfrak{X} \rightarrow (\mathbb{C}, 0)$, which is a holomorphic submersion possessing

a real structure subject to (46) and whose central fiber is a real nodal del Pezzo pair (Y, E) (cf. Section 4.3.3). Namely, we specialize a conjugation-invariant set of 6 blown up points on a real conic C_2 , E being the strict transform of C_2 . We call $\mathfrak{X} \rightarrow (\mathbb{C}, 0)$ a *regular ABV family* of X .

The real del Pezzo surfaces X of degree 2 of other deformation types can be included into the following unscrews of real nodal degenerations corresponding to nodal degenerations of quartics Q_X :

(a) *hyperbolic ABV families:*

- if X is of type $1\langle 1 \rangle^+$, we degenerate Q_X into a nodal quartic shown in Figure 1(a) and choose a 1-unscrew assuming that the component $F \subset \mathbb{R}X$ doubly covers the annulus merging to the domain F_1 ;
- if X is of type $\langle 2 \rangle^+$, we degenerate Q_X into a real nodal quartic as shown in Figure 1(b) and choose a 1-unscrew, assuming that F doubly covers the disc merging to the domain $F_1 \cup F_2$;

(b) *elliptic ABV families:*

- if X is of type $1\langle 1 \rangle^-$, $F = F^{no}$, we degenerate Q_X into a nodal quartic shown in Figure 1(a) and choose a (-1) -unscrew, assuming that F doubly covers the Möbius band merging to the domain F_2 ;
- if X is of type $\langle 1 \rangle^+$, we degenerate Q_X into a nodal quartic shown in Figure 1(a) and choose a (-1) -unscrew, assuming that F doubly covers the disc merging to the domain F_1 ;
- if X is of type $\langle 0 \rangle^-$, we degenerate the quartic Q_X (having an empty real part) into a nodal quartic with a one-point real part, and choose a (-1) -unscrew;

(c) *external ABV families:*

- if X is of type $1\langle 1 \rangle^-$, $F = F^o$, we degenerate Q_X into a nodal quartic shown in Figure 1(c) and choose a $(2, 1)$ -unscrew, assuming that F doubly covers the disc merging to F_1 , and F' doubly covers a Möbius band merging to the domain F_2 ;
- if X is of type $\langle q \rangle^+$, $q = 3, 4$, we degenerate Q_X into a nodal quartic so that one of the ovals collapses to a point and choose a $(3, 0)$ -unscrew, assuming that F' doubly covers the disc merging to a point.

5.2 F -compatible divisor classes

The F -compatibility condition (see Section 2.3) holds for all real divisor classes if $H_1(\mathbb{R}X \setminus F, \mathbb{Z}/2) = 0$. Hence, the only cases with a non-trivial condition for del Pezzo surfaces of degree 2 are as follows: either X is of type $\langle 0 \rangle^-$, or X is of type

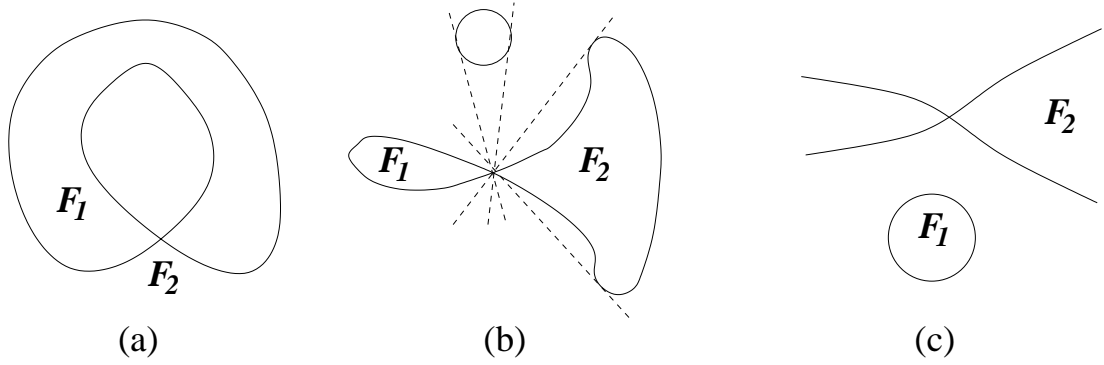


Figure 1: Nodal quartics

$1\langle 1 \rangle^-$ and $F = F^\circ$. For example, $-K_X$ is not F -compatible in either of these two cases.

We use below the following characterization of the F -compatibility condition for the two cases mentioned above.

The elliptic ABV family for a surface of type $\langle 0 \rangle^-$ (see Section 5.2) has the central fiber $Y \cup Z$, where Z is a quadric and Y is the plane blown up at three pairs of complex conjugate points on a real conic C_2 such that $\mathbb{R}C_2 \neq \emptyset$, and at one more real point belonging to the orientable component of $\mathbb{R}\mathbb{P}^2 \setminus \mathbb{R}C_2$. Consider the basis of $\text{Pic}(Y)$ consisting of the pull-back L of a generic line and the exceptional divisors E_1, \dots, E_7 , where E_{2i}, E_{2i+1} are complex conjugate, $i = 1, 2, 3$, and E_1 is real. Since the components of $\mathbb{R}X$ for X of type $\langle 0 \rangle^-$ are interchanged by an automorphism, we can choose any of them, and we assume that F is disjoint from $\mathbb{R}E_1$. Following Proposition 26, let us identify $\text{Pic}^{\mathbb{R}}(X)$ with a subgroup of $\text{Pic}^{\mathbb{R}}(Y)$.

Proposition 33 *For a surface X of type $\langle 0 \rangle^-$, a divisor class $D \in \text{Pic}(X)$, represented as $D = dL - d_1E_1 - \dots - d_7E_7$ in $\text{Pic}(Y)$,*

- *is real if and only if $d_{2i} = d_{2i+1}$, $i = 1, 2, 3$, and $2d = d_2 + \dots + d_7$,*
- *is F -compatible if and only if the number $d_1 = DE_1$ is even.*

Proof. Straightforward. □

We say that a divisor class $D \in \text{Pic}^{\mathbb{R}}(Y)$ is F_+ -compatible if it satisfies conditions of Proposition 33.

Let X be a surface of type $1\langle 1 \rangle^-$. Consider a degeneration of Q_X into $C_1 \cup C_3$, where C_3 is a real two-component cubic, and C_1 a line crossing the one-sided component of C_3 in three real points. This degeneration induces a conjugation invariant family of surfaces, in which the component $F = F^\circ \subset \mathbb{R}X$ merges to the sphere doubly covering the disc bounded by the oval of C_3 . Making the base change $\tau = t^2$ and blowing up the three nodes of the central fiber, we can realize

a triple $(2, 1)$ -unscrew with the central fiber $Y \cup Z^{(1)} \cup Z^{(2)} \cup Z^{(3)}$, where Y is a smooth real surface with $\mathbb{R}Y$ diffeomorphic to $\mathbb{R}X$, $Z^{(i)}$, $i = 1, 2, 3$, are disjoint real quadrics with $\mathbb{R}Z^{(i)} \simeq (S^1)^2$, intersecting Y along real (-2) -curves $E^{(i)}$, $i = 1, 2, 3$, respectively. Contracting each quadric along one of its rulings, we obtain a real family $\pi : \mathfrak{X} \rightarrow (\mathbb{C}, 0)$, where \mathfrak{X} is smooth, π is a submersion, and $\mathfrak{X}_0 = Y$.

Proposition 34 *The subgroup $\text{Pic}^{\mathbb{R}}(X) \simeq \text{Pic}^{\mathbb{R}}(Y) \subset \text{Pic}(Y)$ is generated by $K_X = K_Y$ and $E^{(i)}$, $i = 1, 2, 3$. A divisor class $D = -dK_X + d_1E^{(1)} + d_2E^{(2)} + d_3E^{(3)}$ is F -compatible if and only if $d + d_1 + d_2 + d_3$ is even.*

Proof. Straightforward. □

5.3 Auxiliary statements

Let $\pi : \mathfrak{X} \rightarrow (\mathbb{C}, 0)$ be a proper holomorphic submersion of a smooth three-dimensional variety \mathfrak{X} (with $(\mathbb{C}, 0)$ being understood as a disc germ), where each fiber \mathfrak{X}_t , $t \neq 0$, is a del Pezzo surface of degree 2 and the central fiber $Y = \mathfrak{X}_0$ contains a smooth rational curve E such that (Y, E) is a monic log-del Pezzo pair. In what follows we identify the Picard groups of the fibers as in Remark 22.

Lemma 35 *Let $X = \mathfrak{X}_t$ for some $t \neq 0$, and let $D \in \text{Pic}(X)$.*

- (i) *If D is big and X -nef, then $-K_X D > 1$, and the linear system $|D|_X$ contains an irreducible rational curve.*
- (ii) *The divisor class D is X -nef if and only if its intersection with any (-1) -curve on X is non-negative. In this case $D^2 \geq 0$. The divisor class D is Y -nef if and only if its intersection with E and any (-1) -curve on Y is non-negative. If D is Y -nef then it is X -nef.*
- (iii) *If D is nonzero and X -nef, and satisfies $D^2 = 0$, then $D = kD'$, where D' is primitive (i.e., not multiple of another divisor). Furthermore, $-K_X D' = 2$, $\dim |D'|_X = 1$, and a generic element of $|D'|_X$ is a smooth connected rational curve. If $D'E \geq 0$, then $|D'|_Y$ is one-dimensional with a smooth, connected, rational curve as a generic element. Furthermore, if $D'E > 1$, then $D' = -(K_Y + E)$.*
- (iv) *If $D \in \text{Pic}(Y, E)$ a Y -nef and big divisor, satisfying $R_Y(D, 0) > 0$. Then the divisor class $D' = D - E - \sum_{E' \in \mathcal{E}(E)^{\perp D}} E'$ is Y -nef and satisfies $D'E' = 0$ for all $E' \in \mathcal{E}(E)^{\perp D}$; furthermore, if $D' \neq 0$, then D' is presented by the union of curves different from E and crossing E positively.*
- (v) *If D is big and X -nef such that $\mathcal{E}(E)^{\perp D} \neq \emptyset$, then $D - mE$ with $m > 0$ cannot be represented by an irreducible curve in Y .*

Proof. It is known that big and nef divisors on del Pezzo surfaces are effective and can be represented by irreducible rational curves (see, for instance, [9, Theorems 3, 4, and Remark 3.1.4]). Hence $-K_X D > 0$. In the case when $D = -K_X$ or when $-K_X - D$ is effective, the inequality $-K_X D > 1$ can easily be verified. If $-K_X - D$ is not effective, then $-K_X D > 1$ due to $\dim | -K_X | = 2$.

Statement (ii) on the X -nefness (respectively, Y -nefness) follows from the fact that the effective cone in $\text{Pic}(X)$ (respectively, $\text{Pic}(Y)$) is generated by (-1) -curves (respectively, by (-1) -curves and E).

If D is Y -nef, then $DE' \geq 0$ for all (-1) -curves E' on Y , and $DE \geq 0$. Any (-1) -curve E'' in X degenerates either into a (-1) -curve of Y , or into a curve $E + E'$ with a (-1) -curve E' on Y , and hence in both the cases $DE'' \geq 0$.

The nonnegativity of D^2 in statement (ii) and the part of statement (iii), concerning divisors and linear systems on X , follow, for instance, from [9, Theorems 3, 4, and Remark 3.1.4]. In particular, if D' is primitive and satisfies $(D')^2 = 0$, then a general curve in $|D'|_X$ is non-singular, rational.

Let $D'E \geq 0$ in statement (iii). Suppose that $D'E' = 0$ for some (-1) -curve $E' \in \mathcal{E}(E)$. Then we can blow down E' and reduce the degenerating family to a family of del Pezzo surfaces, which immediately yields that $\dim |D'|_Y = 1$ as well as the fact that a generic element of $|D'|_Y$ can be chosen to be a smooth rational curve. Suppose that $D'E' > 0$ for all $E' \in \mathcal{E}(E)$. By Proposition 23, a general curve $C_t \in |D'|_{x_t}$, $t \neq 0$, degenerates into a curve $C_0 + mE \in |D'|_Y$ with some $m \geq 0$ and $C_0 \not\supset E$. If m were positive, we would have $(C_0)^2 = (D')^2 - 2D'E - 2m^2 \leq -2$, and, in view of $-K_Y D' = 2$ (comes from genus formula) and $-K_Y C \geq -1$ for all irreducible curves $C \neq E$, we would get C_0 consisting of components with negative self-intersection, a contradiction to $\dim |D'|_Y \geq 1$.

Let $D'E > 1$ in statement (iii), then $-(K_Y + E)D' = -K_Y D' - D'E = 2 - D'E \leq 0$, which in view of the nefness of $-(K_Y + E)$, yields $D' = -(K_Y + E)$.

In view statement (iii), to prove (iv) it is enough to check that $D - E$ non-negatively crosses each (-1) -curve of Y . If $\mathcal{E}(E)^{\perp D} = \emptyset$, then this immediately follows from the fact that $EE' = 1$ for all $E' \in \mathcal{E}(E)$.

In the case of $\mathcal{E}(E)^{\perp D} \neq \emptyset$, we have $D'E' < 0$ for all $E' \in \mathcal{E}(E)^{\perp D}$, and hence $D - mE$ with $m > 0$ cannot be represented by an irreducible curve in Y : any curve in $|D - mE|_Y$ must contain all $E' \in \mathcal{E}(E)^{\perp D}$ as components, and such a component cannot be unique, otherwise D would not be big. This proves (v). Furthermore, formula (36) and statement (i) yield

$$N_Y(D, 0, (DE)e_1) = GW_0(X, D) > 0 . \quad (51)$$

Since $R_Y(D, 0) > 0$, computing $N_Y(D, 0, (DE)e_1)$ via a sequence of formulas (66) from [19] written in the form

$$N_Y(D, 0, je_1, (DE - j)e_1) = N_Y(D, 0, (j + 1)e_1, (DE - j - 1)e_1) + S_j^C , \quad (52)$$

$$j = 0, \dots, DE ,$$

where $S_j^{\mathbb{C}}$ stands for the second sum in the right-hand side of the cited formula, and $N_Y(D, 0, (DE+1)e_1, -e_1)$ is zero by definition, we get $S_0^{\mathbb{C}} + \dots + S_{D'E}^{\mathbb{C}} = GW_0(X, D) > 0$. That means the divisor $D - E$ is effective, and the divisor class $D' = D - E - \sum_{E' \in \mathcal{E}(E)^{\perp D}} E'$ is represented by a curve C' , whose all components are disjoint from the (-1) -curves $E' \in \mathcal{E}(E)^{\perp D}$ and intersect with E . Notice, first, that C' does not contain (-1) -curves disjoint from E , and hence $D'E'' \geq 0$ for all (-1) -curves E'' with $E''E = 0$, and, second, $(D')^2 \geq 0$, since otherwise, C' would contain a (-1) -curve crossing E and disjoint from the other components of C' and from $E' \in \mathcal{E}(E)^{\perp D}$, contrary to the definition of $\mathcal{E}(E)^{\perp D}$. Altogether this yields the required statement. \square

Remark 36 Notice that Lemma 35 can be applied to all ABV families introduced in Section 5.1. Namely, over \mathbb{C} we can contract the quadric surface in the central fiber of the family along one of the rulings and thus obtain a family exactly as in Lemma 35.

The following two claims will be used in the proof of the asymptotic statements in Theorems 2 and 3.

Lemma 37 Let $\{a_n\}_{n \geq 0}$ be a sequence of positive numbers, $a_0 = 1$, and let $0 \leq f(n) \leq n$ an integral-valued function. If

- either

$$a_{n+1} \geq \lambda a_{f(n)} a_{n-f(n)}, \quad \text{for all } n \geq n_0 \text{ and some } \lambda > 0, \quad (53)$$

- or

$$a_n \geq \lambda a_{f(n)} a_{n-f(n)}, \quad \text{for all } n \geq n_0 \text{ and some } \lambda > 0, \quad (54)$$

then there exist $\xi_1, \xi_2 > 0$ such that $a_n \geq \xi_1 \xi_2^n$ for all $n \geq n_0$.

Proof. Straightforward induction on n with ξ, η found from the equations

$$\lambda \xi_1 = \xi_2, \quad \xi_1 \xi_2^{n_0} = a_0$$

in the first case, and the equations

$$\lambda \xi_1 = 1, \quad \xi_1 \xi_2^{n_0} = a_{n_0}$$

in the second case. \square

Lemma 38 Asymptotic relations (3) and (4) follow from

$$\log W(X, nD, \mathbb{R}X, [\mathbb{R}X \setminus F]) \geq -K_X D \cdot n \log n + O(n), \quad n \rightarrow +\infty.$$

Proof. Straightforward from

$$\begin{aligned} \log |W(X, nD, \mathbb{R}X, [\mathbb{R}X \setminus F])| &\leq \log GW_0(X, nD) \\ &= -K_X D \cdot n \log n + O(n), \quad n \rightarrow +\infty, \end{aligned}$$

(see [13, Theorem 1]). \square

5.4 Non-negativity of w -numbers

In each of the regular, elliptic, or hyperbolic ABV families introduced in Section 5.1, the central fiber coincides with or contains a real del Pezzo surface Y of degree 2 with a smooth real rational curve $E \subset Y$ whose real part $\mathbb{R}E$ lies in some connected component F of $\mathbb{R}Y$ (recall that $\mathbb{R}E \neq \emptyset$).

Lemma 39 *If Y appears in a regular or hyperbolic ABV family introduced in Section 5.1, then, for any $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ such that $-K_Y D \geq 1$ and, for any $\alpha, \beta \in \mathbb{Z}_+^{\infty, \text{odd}}$ such that $I(\alpha + \beta) = DE$, we have*

$$W_{Y, E, [\mathbb{R}Y \setminus F]}(D, \alpha, \beta, 0) \geq 0. \quad (55)$$

If Y appears in an elliptic ABV family introduced in Section 5.1, then, for any component F_+ of $F \setminus \mathbb{R}E$, for any $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ such that $-K_Y D \geq 2$ and DE is even, and for any $\alpha, \beta \in \mathbb{Z}_+^{\infty, \text{even}}$ such that $I(\alpha + \beta) = DE$, we have

$$W_{Y, F_+, [\mathbb{R}Y \setminus F]}^-(D, \alpha, \beta, 0) \geq 0. \quad (56)$$

Proof. Suppose, first, that Y appears in a regular, hyperbolic, or elliptic ABV family of a del Pezzo surface X whose type is different from $\langle 2 \rangle^+$. We use induction on $R_Y(D, \beta)$ to prove (55) via formula (9) and to prove (56) via formula (31). Notice that the coefficients in both these formulas are non-negative. Indeed, the values of $\eta(l)$ given in (32) and related to the considered cases are all non-negative, since, if L', L'' are real, then always $\mathbb{R}L', \mathbb{R}L'' \subset \overline{F}_+$. Thus, it remains to verify the non-negativity of the initial values given in Propositions 11 and 16. In the case of X of type $\mathbb{P}_{a,b}^2$, $a + 2b = 7$, this is so, since $\mathcal{E}(E)$ consists of $2b$ pairs of disjoint complex conjugate lines and of $6 - 2b$ pairs of intersecting real lines. In the case of X of type $1\langle 1 \rangle^+$, $\langle 0 \rangle^-$, $1\langle 1 \rangle^-$, or $\langle 1 \rangle^+$ this is so, since the corresponding real nodal plane quartic curve Q_Y has no real lines passing through the node z of Q_Y and tangent to Q_Y at a point $z' \neq z$ (*cf.* Remark 25).

Suppose now that Y appears in a hyperbolic ABV family of a del Pezzo surface X of type $\langle 2 \rangle^+$. We again use induction on $R_Y(D, \beta)$ and prove (55) via formula (9). The base of induction is provided by Proposition 11(1), where all values equal 1. However, some terms in the second sum of the left-hand side of (9) can be negative, and to proceed by induction, we will modify formula (9) in order to cancel out the negative summands. Consider the nodal quartic Q_Y (see Figure 1(b)). It has 6 tangents passing through the node z : four real (shown by dashed lines), L_1, L_2 (tangent to the domain F_2), L_3, L_4 (tangent to the oval), and two complex conjugate L_5, L_6 . Each tangent line L_i lifts to a pair of (-1) -curves $E'_i, E''_i \subset Y$ intersecting at one point, and we have

$$E''_i = \text{Conj}(E'_i), \quad i = 1, 2, 3, 4, \quad E'_5 = \text{Conj}(E'_5), \quad E''_5 = \text{Conj}(E''_5), \quad (57)$$

and $E'_i + E''_i \in | -K_Y - E |$, $i = 1, \dots, 6$. By Proposition 11(2i,ii),

$$W_{Y, E, [\mathbb{R}Y \setminus F]}(\{E'_i, E''_i\}, 0, 0, e_1) = \begin{cases} -1, & i = 1, 2, \\ 1, & i = 3, 4, \end{cases} \quad (58)$$

$$W_{Y,E,[\mathbb{R}Y \setminus F]}(\{E'_5, E'_6\}, 0, 0, e_1) = W_{Y,E,[\mathbb{R}Y \setminus F]}(\{E''_5, E''_6\}, 0, 0, e_1) = 1 \quad (59)$$

(here we denote by F the connected component of $\mathbb{R}Y$, which deforms into the considered component $\mathbb{R}X$; no confusion will arise). Now notice that the coefficients in the second sum of LHS of (9) are positive, since $E_{1/2} \circ [\mathbb{R}Y \setminus F] = L_{1/2} \circ [\mathbb{R}Y \setminus F] = 0$. Each summand of that sum can be written either as $(l+1)AB_m$, or $(l+1)A'B'_m$, or $(l+1)A''B''_m$, where all the factors of type (58) and (59) are separated to A , and the sum of the divisor classes in the factors, participating in B_m , B'_m , or B''_m equals either $D - E + m(K_Y + E)$, $D - E + m(K_Y + E) - (E'_5 + E'_6)$, or $D - E + m(K_Y + E) - (E''_5 + E''_6)$, respectively. Observe that, by Theorem 4(1g) the factors (58) and (59) appear in A at most once, and that A' (resp. A'') necessarily contains the factor $W_{Y,E,[\mathbb{R}Y \setminus F]}(\{E'_5, E'_6\}, 0, 0, e_1)$ (resp. $W_{Y,E,[\mathbb{R}Y \setminus F]}(\{E''_5, E''_6\}, 0, 0, e_1)$), and, for a given $m \geq 0$, all combinations of l and A (resp., A' or A'') subject to the above restrictions are allowed. Thus, an easy computation gives that, combining together all summands with the same B_m , B'_m or B''_m , $m \geq 0$, we finally reduce formula (9) to the form

$$\begin{aligned} W_{Y,E,[\mathbb{R}Y \setminus F]}(D, \alpha, \beta, 0) &= \sum_{j \geq 1, \beta_j > 0} W_{Y,E,[\mathbb{R}Y \setminus F]}(D, \alpha + e_j, \beta - e_j, 0) \\ &\quad + B_0 + B_2 + B'_0 + B''_0, \end{aligned} \quad (60)$$

which completes the proof in view of $B_0, B_2, B'_0, B''_0 \geq 0$ (by the induction assumption).

□

5.5 Proof of Theorem 2

It is clear that the case of any real del Pezzo surface of degree ≥ 3 can be reduced to that of degree 2 by blowing up suitable real points. To treat the degree 2 case, we include X into a suitable ABV family, defined in Section 5.1 and apply Theorems 6 and 7 expressing Welschinger invariants in terms of w -numbers. The nodal del Pezzo pairs in the central fibers of the considered ABV families are monic log-del Pezzo, which allows us to compute and estimate the above w -numbers using Theorems 4 and 5. The key observation is the non-negativity of the considered w -numbers (see Lemma 39).

5.5.1 Positivity and asymptotics statements for surfaces of types $\mathbb{P}_{a,b}^2$, $a + 2b = 7$, $1\langle 1 \rangle^+$, and $\langle q \rangle^+$, $q = 2, 3, 4$

Consider first surfaces X of types $\mathbb{P}_{a,b}^2$, $a + 2b = 7$, $1\langle 1 \rangle^+$, and $\langle 2 \rangle^+$ and the regular or hyperbolic ABV families for them introduced in Section 5.1. Following Proposition 26, we identify $\text{Pic}^{\mathbb{R}}(X)$ and $\text{Pic}^{\mathbb{R}}(Y)$, where Y is the central fiber of the corresponding ABV family. Furthermore, we restrict ourselves to the case

$DE \geq 0$ (cf. Theorem 6(1)). Fix a connected component $F \subset \mathbb{R}X$ and put $\varphi = [\mathbb{R}X \setminus F] \in H_2(Y \setminus F, \mathbb{Z}/2)$.

Positivity for surfaces of types $\mathbb{P}_{a,b}^2$, $a+2b=7$, $1\langle 1 \rangle^+$, and $\langle 2 \rangle^+$. From formula (47), Proposition 27, and inequality (55) it follows that

$$W(X, D, F, \varphi) \geq W(X, D - mE, F, \varphi) \quad \text{for all } m \geq 0. \quad (61)$$

Indeed, the both terms are sums of non-negative w -numbers, and all the w -numbers occurring in the development of the right-hand side appear in the development of the left-hand side with non-smaller coefficients:

$$\binom{DE + 2m + 2k}{m + k} \geq \binom{DE + 2m + 2k}{k} \quad \text{for all } k \geq 0.$$

We will prove inequality (2) for all real big and X -nef divisors $D \in \text{Pic}(X)$ such that $DE \geq 0$, using induction on $\rho(D) = -(K_X + E)D$.

Observe that $\rho(D) > 0$, since D is big and $|-K_X - E|$ defines a conic bundle. Suppose that $\rho(D) = 1$, or, equivalently, $R_Y(D, 0) = 0$. From formula (47), Proposition 27, and inequality (55) we get

$$W(X, D, F, \varphi) \geq W_{Y,E,\varphi}(D, 0, (DE)e_1, 0), \quad (62)$$

and then, applying formula (9) (resp. (60) for X of type $\langle 2 \rangle^+$) DE times, we end up with

$$W_{Y,E,\varphi}(D, 0, (DE)e_1, 0) \geq W_{Y,E,\varphi}(D, (DE)e_1, 0, 0) = 1,$$

the latter equality coming from Proposition 11(1iii) and inequality

$$DE = -K_Y D - \rho(D) = -K_X D - 1 \stackrel{\text{Lemma 35(i)}}{>} 0.$$

Assume that $\rho(D) > 1$, or, equivalently,

$$R_Y(D, 0) > 0. \quad (63)$$

Suppose that $\mathcal{E}(E)^{\perp D} = \emptyset$. Then, $D - E$ is Y -nef and satisfies $(D - E)^2 \geq 0$ (see Lemma 35(ii)).

If $(D - E)^2 > 0$, then $D - E$ is big and X -nef, and $\rho(D - E) = \rho(D) - 2 < \rho(D)$. Thus, by the induction assumption and (61)

$$W(X, D, F, \varphi) \geq W(X, D - E, F, \varphi) > 0.$$

If $(D - E)^2 = 0$, then by Lemma 35(iii), $D - E = kD''$ with $k \geq 1$ and a primitive $D'' \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ such that $D''E > 0$, $(D'')^2 = 0$, $\dim |D''|_Y = -K_Y D'' - 1 = 1$, and the linear system $|D''|_Y$ contains a real, rational, smooth curve. If $k = 1$, then

$W(X, D - E, F, \varphi) = 1$, and again (2) follows. If $k \geq 2$, then $R_Y(D, (DE)e_1) = -K_Y D - 1 = -kK_Y D'' - 1 = 2k - 1$, and we get

$$\begin{aligned} W(X, D, F, \varphi) &\stackrel{(47) \ \& \ (55)}{\geq} W_{Y,E,\varphi}(D, 0, (DE)e_1, 0) \\ &\stackrel{(9),(60) \ \& \ (55)}{\geq} W_{Y,E,\varphi}(D, (k-2)e_1, k((D''E) - 1)e_1, 0) \\ &\stackrel{(9),(60) \ \& \ (55)}{\geq} (W_{Y,E,\varphi}(D'', 0, (D''E)e_1, 0))^k \stackrel{\text{Lemma 35(iii)}}{=} 1. \end{aligned} \quad (64)$$

Suppose now that $\mathcal{E}(E)^{\perp D} \neq \emptyset$. By Lemma 35(v) and Theorem 6(2), we have

$$W(X, D, F, \varphi) = W_{Y,E,\varphi}(D, 0, (DE)e_1, 0). \quad (65)$$

By Lemma 35(iv), the divisor class $D' = D - E - \sum_{E' \in \mathcal{E}(E)^{\perp D}} E'$ is Y -nef, and hence, by Lemma 35(ii), is also X -nef.

Assume that $(D')^2 > 0$. Since $\rho(D') = \rho(D) - 2$, we have $W(X, D', F, \varphi) > 0$, which due to $\mathcal{E}(E)^{\perp D'} \supset \mathcal{E}(E)^{\perp D} \neq \emptyset$ yields (cf. (65))

$$W_{Y,E,\varphi}(D', 0, (D'E)e_1, 0) = W(X, D', F, \varphi) > 0.$$

Appropriately applying formula (9) (resp. (60) for X of type $\langle 2 \rangle^+$) and using (55), we obtain

$$\begin{aligned} W(X, D, F, \varphi) &= W_{Y,E,\varphi}(D, 0, (DE)e_1, 0) \\ &\geq W_{Y,E,\varphi}(D, (s-1)e_1, (DE - s + 1)e_1, 0), \end{aligned} \quad (66)$$

where $s = \text{card}(\mathcal{E}(E)^{\perp D})$. In addition,

$$W_{Y,E,\varphi}(D, (s-1)e_1, (DE - s + 1)e_1, 0) \geq W_{Y,E,\varphi}(D', 0, (D'E)e_1, 0) > 0. \quad (67)$$

Indeed,

- $D'E > 0$ (cf. Lemma 35(1iv)), $s = -K_Y D + K_Y D' \leq -K_Y D - 1$, $s - 1 = (D - E - D')E - 1 = DE + 1 - D'E \leq DE$, and $DE - s + 1 = D'E - 1$;
- if X is of type $\langle 2 \rangle^+$, then the non-empty set $\mathcal{E}(E)^{\perp D}$ must be either $\{E'_5, E'_6\}$, or $\{E''_5, E''_6\}$ (cf. (57)), and hence $W_{Y,E,\varphi}(D', 0, (D'E)e_1, 0)$ appears in the summand B'_0 , resp. B''_0 in (60).

Assume that $(D')^2 = 0$. If $D' = 0$, then the relations (66) and (67) transform to

$$W(X, D, F, \varphi) \geq \prod_{\mathcal{D}_1} W_{Y,E,\varphi}(\mathcal{D}_1, 0, e_1, 0) \prod_{\mathcal{D}_2} W_{Y,E,\varphi}(\mathcal{D}_2, 0, 0, e_1) > 0,$$

where \mathcal{D}_1 runs over the real elements of $\mathcal{E}(E)^{\perp D}$, and \mathcal{D}_2 runs over the pairs of complex conjugate elements in $\mathcal{E}(E)^{\perp D}$. If $D' \neq 0$, by Lemma 35(iii), $D' = kD''$

with one-dimensional linear system $|D''|_Y$ represented by a smooth real rational curve C'' , and the relations (66) and (66) transform to

$$\begin{aligned} W(X, D, F, \varphi) &\geq W_{Y,E,\varphi}(D, le_1, (DE - l)e_1, 0) \\ &\geq (W_{Y,E,\varphi}(D'', 0, (D''E)e_1, 0))^k = 1, \end{aligned} \quad (68)$$

where $l = -K_Y D - 2 - k$. Note that

$$k = (-K_Y D'')^{-1}(-K_Y D - s) \leq (-K_Y D - 1)/2 \leq -K_Y D - 2$$

and $l \leq DE$, the latter inequality coming from the relations

$$\begin{aligned} DE = D'E - 2 + E \cdot \sum_{E' \in \mathcal{E}(E)^{\perp D}} E' &= D'E - 2 + s \geq k - 2 + s, \\ l = -K_Y D - 2 - k = -K_Y D' - K_Y E - K_Y \cdot \sum_{E' \in \mathcal{E}(E)^{\perp D}} E' - 2 - k &= k - 2 + s. \end{aligned}$$

Asymptotics for surfaces of types $\mathbb{P}_{a,b}^2$, $a + 2b = 7$, $1\langle 1 \rangle^+$, and $\langle 2 \rangle^+$. Let $D \in \text{Pic}(X)$ be a real, big and X -nef divisor. By Theorem 6(1) we can suppose that D is Y -nef. We prove the asymptotic relation (3) by induction on $\tau(D) = \min\{DE' : E' \in \mathcal{E}(E)\}$.

Let $\tau(D) = 0$, or, equivalently $\mathcal{E}(E)^{\perp D} \neq \emptyset$. Put $s = \text{card}(\mathcal{E}(E)^{\perp D})$. By Lemma 35(iv), there exists an integer $m_0 \geq 1$ such that the divisors $D'_m = mD - E - \sum_{E' \in \mathcal{E}(E)^{\perp D}} E'$ are big and Y -nef and satisfy $D'_m E > 0$ for all integers $m \geq m_0$. Note also that by (2) and (65), one has

$$W_{Y,E,\varphi}(D'_m, 0, (D'_m E)e_1, 0) > 0, \quad m \geq m_0.$$

Put $\tilde{D} = D'_{m_0}$ and $\tilde{s} = \text{card}(\mathcal{E}(E)^{\perp \tilde{D}})$. Again there exists an integer $m_1 \geq 1$ such that \tilde{D}'_{m_1} is big and Y -nef and satisfy

$$\tilde{D}'_m E > 0, \quad W_{Y,E,\varphi}(\tilde{D}'_m, 0, (\tilde{D}'_m E)e_1, 0) > 0 \quad \text{for all integers } m \geq m_1.$$

For any integer $n \geq 2$, we have decompositions

$$\begin{cases} \tilde{D}'_{nm_1} - E = \tilde{D}'_{m_1 i(n)} + \tilde{D}'_{(n-i(n))m_1} + \sum_{E' \in \mathcal{E}(E)^{\perp D}} E', \\ -K_Y \tilde{D}'_{nm_1} - 2 = (-K_Y \tilde{D}'_{m_1 i(n)} - 1) + (-K_Y \tilde{D}'_{(n-i(n))m_1} - 1) + s, \end{cases} \quad i(n) = \left\lfloor \frac{n}{2} \right\rfloor,$$

and inequality $\tilde{D}'_{nm_1} E \geq \tilde{s}$ coming from the first relation. (Note that $\tilde{s} \geq s$ and, moreover, if we choose $m_0 \geq 3$ then $\mathcal{E}(E)^{\perp \tilde{D}} = \mathcal{E}(E)^{\perp D}$ and $\tilde{s} = s$.) By Proposition 11, the product of all the terms $W_{Y,E,0}(\mathcal{D}, 0, \beta^{\text{re}}, \beta^{\text{im}})$ with \mathcal{D} combined from $E' \in \mathcal{E}(E)^{\perp D}$ equals 1, and hence by formula (9) (resp. (60) for X of type $\langle 2 \rangle^+$) and inequality (55), one has

$$W_{Y,E,\varphi}(\tilde{D}'_{nm_1}, 0, (\tilde{D}'_{nm_1} E)e_1, 0) \geq W_{Y,E,\varphi}(\tilde{D}'_{nm_1}, \tilde{s}e_1, (\tilde{D}'_{nm_1} E - \tilde{s})e_1, 0)$$

$$\begin{aligned} &\geq \frac{1}{2}(-K_Y \tilde{D}'_{nm_1} - 2)! (\tilde{D}'_{m_1 i(n)} E) (\tilde{D}'_{(n-i(n))m_1} E) \\ &\quad \times \frac{W_{Y,E,\varphi}(\tilde{D}'_{m_1 i(n)}, 0, (\tilde{D}'_{m_1 i(n)} E) e_1, 0)}{(-K_Y \tilde{D}'_{m_1 i(n)} - 1)!} \cdot \frac{W_{Y,E,\varphi}(\tilde{D}'_{(n-i(n))m_1}, 0, (\tilde{D}'_{(n-i(n))m_1} E) e_1, 0)}{(-K_Y \tilde{D}'_{(n-i(n))m_1} - 1)!}. \end{aligned}$$

There exists $\lambda > 0$ such that, for all integers $n \geq 2$, one has

$$\frac{(\tilde{D}'_{m_1 i(n)} E) (\tilde{D}'_{(n-i(n))m_1} E)}{2(-K_Y \tilde{D}'_{nm_1} - 1)} = \frac{(m_1 i(n) \tilde{D} E + 2 - \tilde{s}) ((n - i(n)) m_1 \tilde{D} E + 2 - \tilde{s})}{2(-nm_1 K_Y \tilde{D} - 1 - \tilde{s})} > \lambda.$$

Hence, the sequence

$$a_n = \frac{W_{Y,E,\varphi}(\tilde{D}'_{nm_1}, 0, (\tilde{D}'_{nm_1} E) e_1, 0)}{(-K_Y \tilde{D}'_{nm_1} - 1)!}, \quad n \geq 1,$$

satisfies the relation (53). Thus, by Lemma 37, one has

$$\begin{aligned} &\log W_{Y,E,\varphi}(\tilde{D}'_{nm_1}, 0, (\tilde{D}'_{nm_1} E) e_1, 0) \geq \\ &\quad - K_Y \tilde{D}'_{m_1} \cdot n \log n + O(n) = -K_X D \cdot m_0 m_1 n \log n + O(n), \quad n \rightarrow +\infty. \end{aligned} \quad (69)$$

Observe that

$$\begin{aligned} (n+1)m_0 m_1 D - E &= D'_{(n+1)m_0 m_1} + \sum_{E' \in \mathcal{E}(E)^{\perp D}} E', \\ D'_{(n+1)m_0 m_1 + j} - E &= D'_{m_0 m_1 + j} + D'_{nm_0 m_1} + \sum_{E' \in \mathcal{E}(E)^{\perp D}} E', \quad 0 \leq j < m_0 m_1, \\ D'_{nm_0 m_1} &= \tilde{D}'_{nm_1} + nm_1 (E + \sum_{E' \in \mathcal{E}(E)^{\perp D}} E'), \end{aligned}$$

and hence, applying formula (9) and inequality (55) as above and omitting positive integer coefficients, we obtain

$$\begin{aligned} &W_{Y,E,\varphi}(((n+1)m_0 m_1 + j)D, 0, ((n+1)m_0 m_1 + j)(DE) e_1, 0) \\ &\quad \geq W_{Y,E,\varphi}(D'_{(n+1)m_0 m_1 + j}, 0, (D'_{(n+1)m_0 m_1 + j} E) e_1, 0) \\ &\quad \geq W_{Y,E,\varphi}(D'_{nm_0 m_1}, 0, (D'_{nm_0 m_1} E) e_1, 0) \\ &\quad \geq W_{Y,E,\varphi}(D'_{nm_0 m_1} - E - \sum_{E' \in \mathcal{E}(E)^{\perp D}} E', 0, (D'_{nm_0 m_1} E + 2 - s) e_1, 0) \geq \dots \\ &\quad \geq W_{Y,E,\varphi}(D'_{nm_0 m_1} - nm_1 (E + \sum_{E' \in \mathcal{E}(E)^{\perp D}} E'), 0, (D'_{nm_0 m_1} E + nm_1 (2 - s)) e_1, 0) \\ &\quad = W_{Y,E,\varphi}(\tilde{D}'_{nm_1}, 0, (\tilde{D}'_{nm_1} E) e_1, 0) \quad \text{for all integers } n \geq 2, 0 \leq j < m_0 m_1. \end{aligned}$$

These inequalities, together with (65) and (69), imply

$$\begin{aligned} \log W(X, nD, F, \varphi) &= \log W_{Y,E,\varphi}(nD, 0, n(DE)e_1, 0) \\ &\geq -K_X D \cdot n \log n + O(n), \quad n \rightarrow +\infty. \end{aligned}$$

and hence (3) by Lemma 38.

Now suppose that $\tau(D) > 0$. By Lemma 35(ii), $D - E$ is Y -nef, $(D - E)^2 \geq 0$ and $\tau(D - E) = \tau(D) - 1$.

If $(D - E)^2 > 0$, then by (61) and the induction assumption

$$\begin{aligned} \log W(X, nD, F, \varphi) &\geq \log W(X, n(D - E), F, \varphi) = \\ &= -K_X(D - E) \cdot n \log n + O(n) = -K_X D \cdot n \log n + O(n), \end{aligned}$$

which as above implies (3).

If $(D - E)^2 = 0$, then by Lemma 35(iii), $D - E = kD''$, where $k \geq 1$, D'' is a primitive Y -nef divisor represented by a real smooth rational curve crossing E , and $\dim |D''| = 1$. Consider the divisor $D'_2 = 2D - E$. It is Y -nef and satisfies $D'_2 E = 2DE + 2 \geq 2$. It follows from formula (9) (resp. (60) for X of type $\langle 2 \rangle^+$), inequality (55), decompositions

$$D'_2 - E = 2kD'', \quad -K_Y D'_2 - 2 = 2k(-K_Y D'' - 1) + 2(k - 1)$$

and inequality $D'_2 E = 2kD'' E - 2 \geq 2k - 2$ that

$$\begin{aligned} W_{Y,E,\varphi}(D'_2, 0, (D'_2 E)e_1, 0) &\geq W_{Y,E,\varphi}(D'_2, 2(k - 1)e_1, (D'_2 E - 2(k - 1))e_1, 0) \\ &\geq W_{Y,E,\varphi}(D'', 0, (D'' E)e_1, 0)^{2k} = 1. \end{aligned}$$

In the same way from decompositions

$$\begin{cases} D + nD'_2 - E = kD'' + nD'_2, \\ -K_Y(D + nD'_2) - 2 = k(-K_Y D'' - 1) + (-nK_Y D'_2 - 1) + (k - 1) \end{cases} \quad (70)$$

and

$$\begin{cases} (n + 1)D'_2 - E = 2kD'' + [n/2]D'_2 + [(n + 1)/2]D'_2, \\ -(n + 1)K_Y D'_2 - 2 = 2k(-K_Y D'' - 1) + (-[n/2]K_Y D'_2 - 1) \\ \quad + (-[(n + 1)/2]K_Y D'_2 - 1) + 2k \end{cases} \quad (71)$$

for all $n \geq 2$, we obtain

$$W_{Y,E,\varphi}(D + nD'_2, 0, (D + nD'_2 E)e_1, 0) \geq W_{Y,E,\varphi}(nD'_2, 0, n(D'_2 E)e_1, 0)$$

and

$$\begin{aligned} W_{Y,E,\varphi}((n + 1)D'_2, 0, (n + 1)(D'_2 E)e_1, 0) &\geq \frac{1}{2}(-(n + 1)K_Y D'_2 - 2)! \cdot (D'_2 E)^2 \\ &\times \left[\frac{n}{2} \right] \cdot \left[\frac{n + 1}{2} \right] \cdot \frac{W_{Y,E,\varphi}(\left[\frac{n}{2} \right] D'_2, 0, \left[\frac{n}{2} \right] (D'_2 E)e_1, 0)}{(-\left[\frac{n}{2} \right] K_Y D'_2 - 1)!} \cdot \frac{W_{Y,E,\varphi}(\left[\frac{n+1}{2} \right] D'_2, 0, \left[\frac{n+1}{2} \right] (D'_2 E)e_1, 0)}{(-\left[\frac{n+1}{2} \right] K_Y D'_2 - 1)!}. \end{aligned}$$

Since there exists $\lambda > 0$ such that

$$\frac{[n/2] \cdot [(n+1)/2](D'_2 E)^2}{2(-(n+1)K_Y D'_2 - 1)} \geq \lambda > 0 \quad \text{for all } n \geq 2,$$

by Lemma 37 we get

$$\begin{aligned} & \log W_{Y,E,\varphi}((2n+j+1)D, 0, (2n+j+1)(DE)e_1, 0) \geq \\ & \log W_{Y,E,\varphi}(D + nD'_2, 0, ((D + nD'_2)E)e_1, 0) \geq \log W_{Y,E,\varphi}(nD'_2, 0, n(D'_2 E)e_1, 0) \geq \\ & -K_Y D'_2 \cdot n \log n + O(n) = -K_X D \cdot 2n \log n + O(n) \end{aligned}$$

which in view of (61), (62), and hence (3) by Lemma 38.

The case of surfaces of type $\langle q \rangle^+$, $q = 3, 4$. Consider the $(3, 0)$ -ABV family for a surface X of type $\langle 3 \rangle^+$ introduced in Section 5.1. The general fiber of its mirror $(0, 3)$ -ABV family is a surface of type $\langle 2 \rangle^+$. Hence (2) and (3) follow from the same relations for surfaces of type $\langle 2 \rangle^+$, Corollary 31, Proposition 2, and relation $W(X, -K_X, F, [\mathbb{R}X \setminus F]) = 4$ (*cf.* the table in Section 2.3). In the same manner, (2) and (3) for surfaces of type $\langle 4 \rangle^+$ follow from these relations for surfaces of type $\langle 3 \rangle^+$.

5.5.2 Positivity and asymptotics statements for surfaces of types $\langle 0 \rangle^-$ and $1\langle 1 \rangle^-$

Let X be a real del Pezzo surface as in the title. Let F be a non-orientable connected component of $\mathbb{R}X$ (in the case of X of type $\langle 0 \rangle^-$ assume that F is chosen as specified in Section 5.2). Consider an elliptic ABV family of X (see Section 5.1).

The central fiber of that ABV family contains a real nodal del Pezzo pair (Y, E) . Observe that, for all divisors $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$, the intersection number DE is even. Denote by \hat{F} the connected component of $\mathbb{R}Y$ containing $\mathbb{R}E$, and by F^+ the component of $\hat{F} \setminus \mathbb{R}E$ to which merges the component F of $\mathbb{R}X$.

Following Proposition 26, we identify $\text{Pic}^{\mathbb{R}}(X)$ with a subgroup of $\text{Pic}^{\mathbb{R}}(Y)$.

Positivity. Let $D \in \text{Pic}^{\mathbb{R}}(X)$ be X -nef, big, and F -compatible. Then $DE = 0$, and by formula (50), one has

$$W(X, D, F, [\mathbb{R}X \setminus F]) = W_{Y, F^+, [\mathbb{R}Y \setminus \hat{F}]}^-(D, 0, 0, 0). \quad (72)$$

Thus, to prove (2) it is sufficient to show that for any big, Y -nef, and F_+ -compatible divisor class $D' \in \text{Pic}^{\mathbb{R}}(Y)$, one has

$$W_{Y, F^+, [\mathbb{R}Y \setminus \hat{F}]}^-(D', 0, (D'E/2)e_2, 0) > 0. \quad (73)$$

In what follows, we prove this inequality by induction on $\rho(D') = -(K_Y + E)D'$.

As we know, $\rho(D') = -(K_Y + E)D' > 0$, since D' is a big X -nef divisor and $\dim | -K_X - E| = 1$. If $\rho(D') = 1$, then due to (31) and inequality (56) one has

$$W_{Y, F^+, [\mathbb{R}Y \setminus \hat{F}]}(D', 0, (D'E/2)e_2, 0) \geq 2^{D'E/2} W_{Y, F^+, [\mathbb{R}Y \setminus \hat{F}]}(D', (D'E/2)e_2, 0, 0) > 0,$$

where the latter (strict) inequality follows from the fact that the real part of the unique rational curve $C \in |D'|_Y$ quadratically tangent to E at $D'E/2$ generic real points lies in \overline{F}_+ .

Let $\rho(D') \geq 2$, which is equivalent to $R_Y(D', 0) > 0$. By Lemma 35(iv), the divisor $D'' = D' - E - \sum_{E' \in \mathcal{E}(E)^{\perp D'}} E'$ is Y -nef, and it is easy to see that D'' is F_+ -compatible. Clearly, $\rho(D'') \leq \rho(D') - 2$. Since $D''E$ is even, we have $\text{card}(\mathcal{E}(E)^{\perp D'}) = 2s$, $1 \leq s \leq 3$.

So, if $(D'')^2 > 0$, then formula (31), inequality (56), the induction assumption, and the relations

$$2s = (D' - E - D'')E \leq (D' - E)E - 2 = D'E, \quad D''E/2 = D'E/2 - s + 1$$

result in

$$\begin{aligned} & W_{Y, F_+, [\mathbb{R}Y \setminus \hat{F}]}^-(D', 0, (D'E/2)e_2, 0) \geq 2^s W_{Y, F_+, [\mathbb{R}Y \setminus \hat{F}]}^-(D', se_2, (D'E/2 - s)e_2, 0) \\ & \geq 2^s \prod_{\mathcal{D}} W_{Y, F_+, [\mathbb{R}Y \setminus \hat{Y}]}^-(\mathcal{D}, 0, 0, e_1) \cdot (D''E) \cdot W_{Y, F_+, [\mathbb{R}Y \setminus \hat{F}]}^-(D'', 0, (D''E/2)e_2, 0) > 0, \end{aligned}$$

where \mathcal{D} runs over all elements $\mathcal{D} \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ combined from $E' \in \mathcal{E}(E)^{\perp D'}$.

If $D'' = 0$ (which is relevant only if $\mathcal{E}(E)^{\perp D'} \neq \emptyset$), we get by the same arguments the same expression but without $(D''E) \cdot W_{Y, F_+, [\mathbb{R}Y \setminus \hat{F}]}^-(D'', 0, (D''E)/2, 0)$ in the very end, which again leads to the required positivity.

Thus, it remains to treat the case $D'' \neq 0$ and $(D'')^2 = 0$. Then, since $D''E$ is positive and even, Lemma 35(iii) implies that $D'' = -k(K_Y + E)$, $k \geq 1$.

If X is of type $1\langle 1 \rangle^-$ and $F = F^{no}$, then the both $L', L'' \in |-K_Y - E|_Y$ (see Section 3.1) are real, $\mathbb{R}L' \cup \mathbb{R}L'' \subset \overline{F}_+$. Using

$$R_Y(D', (D'E/2)e_2) = -(K_Y + E)D' + D'E/2 - 1 = 2 + (k + s - 1) - 1 = k + s, \quad (74)$$

and applying formula (31) and inequality (56), we derive that

$$\begin{aligned} W_{Y, F_+, [\mathbb{R}Y \setminus \hat{F}]}^-(D', 0, (D'E/2)e_2, 0) &= W_{Y, F_+, [\mathbb{R}Y \setminus \hat{F}]}^-(D', 0, (k + s - 1)e_2, 0, 0) \\ &\geq 2^{k+s-1} W_{Y, F_+, [\mathbb{R}Y \setminus \hat{F}]}^-(D', (k + s - 1)e_2, 0, 0) \\ &\geq 2^{k+s-1} \eta(k) \prod_{\mathcal{D}} W_{Y, F_+, [\mathbb{R}Y \setminus \hat{Y}]}^-(\mathcal{D}, 0, 0, e_1) > 0, \end{aligned}$$

where the latter expression corresponds to the summand in the second sum of the right-hand side of (31), in which the product runs over pairs of divisors $\mathcal{D} \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$, combined out of the lines $E' \in \mathcal{E}(E)^{\perp D'}$, the parameters in the condition (3e) of Theorem 5 are chosen to be $\alpha^{(0)} = \beta^{(0)} = 0$, $l = k = (D - E)E/2$, and the value of $\eta(k)$ is given by formula (32):

$$\eta(k) = \begin{cases} (k/2 + 1)(2 - (-1)^{k/2}), & k \text{ is even,} \\ 2k + 2(-1)^{(k-1)/2}, & k \text{ is odd} \end{cases}$$

If X is of type $\langle 0 \rangle^-$, then k is even by the F_+ -compatibility condition (cf. Proposition 33). Now from (74), formula (31), and inequality (56), we derive

$$\begin{aligned} W_{Y,F_+,[\mathbb{R}Y \setminus \hat{F}]}^-(D', 0, (D'E/2)e_2, 0) &= W_{Y,F_+,[\mathbb{R}Y \setminus \hat{F}]}^-(D', 0, (k+s-1)e_2, 0, 0) \\ &\geq 2^{k/2+s-1} W_{Y,F_+,[\mathbb{R}Y \setminus \hat{F}]}^-(D', (k/2+s-1)e_2, (k/2)e_2, 0, 0) \\ &\geq 2^{k/2+s-1} 4^{k/2} \prod_{\mathcal{D}} W_{Y,F_+,[\mathbb{R}Y \setminus \hat{Y}]}^-(\mathcal{D}, 0, 0, e_1) > 0, \end{aligned}$$

where the latter expression corresponds to the summand in the second sum in the right-hand side of (31), matching the parameter values $\alpha^{(0)} = 0$, $\beta^{(0)} = (k/2)e_2$, and $l = 0$ in the condition (3e) in Theorem 5.

Asymptotics. Let $D \in \text{Pic}^{\mathbb{R}}(X)$ be X -nef, big, and F -compatible. In particular, $DE = 0$, which by Lemma 35(ii) yields that D is Y -nef, and hence $W_{Y,F_+,[\mathbb{R}Y \setminus \hat{Y}]}^-(D, 0, 0, 0) > 0$ (see (73)). Since $-K_Y D > 1$ (see Lemma 35(i)), formula (31) for $W_{Y,F_+,[\mathbb{R}Y \setminus \hat{F}]}^-(D, 0, 0, 0)$ reads

$$W_{Y,F_+,[\mathbb{R}Y \setminus \hat{F}]}^-(D, 0, 0, 0) = 2W_{Y,F_+,[\mathbb{R}Y \setminus \hat{F}]}^-(D - E, 0, e_2, 0), \quad (75)$$

thus,

$$W_{Y,F_+,[\mathbb{R}Y \setminus \hat{F}]}^-(D - E, 0, e_2, 0) > 0.$$

Again, using formula (31) and non-negativity statement (56), we obtain

$$\begin{aligned} W_{Y,F_+,[\mathbb{R}Y \setminus \hat{F}]}^-(nD - E, 0, e_2, 0) &\geq 2(-nK_Y D - 2)! \\ &\times \frac{W_{Y,F_+,[\mathbb{R}Y \setminus \hat{F}]}^-(\lfloor \frac{n}{2} \rfloor D - E, 0, e_2, 0)}{(-\lfloor \frac{n}{2} \rfloor K_Y D - 1)!} \cdot \frac{W_{Y,F_+,[\mathbb{R}Y \setminus \hat{F}]}^-(\lceil \frac{n+1}{2} \rceil D - E, 0, e_2, 0)}{(-\lceil \frac{n+1}{2} \rceil K_Y D - 1)!} \end{aligned}$$

for all $n \geq 2$, and hence the sequence

$$a_n = \frac{W_{Y,F_+,[\mathbb{R}Y \setminus \hat{F}]}^-(nD - E, 0, e_2, 0)}{(-nK_Y D)!}, \quad n \geq 1,$$

satisfies (54) with

$$\lambda = \inf_{n > 3} \frac{\lfloor n/2 \rfloor \cdot \lceil (n+1)/2 \rceil K_Y D}{n(-nK_Y D - 1)} > 0.$$

So, from Lemma 37 we derive

$$\liminf_{n \rightarrow \infty} \frac{\log W_{Y,F_+,[\mathbb{R}Y \setminus \hat{Y}]}^-(nD - E, 0, e_2, 0)}{n \log n} \geq -K_Y D = -K_X D,$$

and hence in view of (61), (75) applied to nD , and of Lemma 38, we obtain the desired relation (3).

5.6 Proof of Theorem 3

Take a real quadric surface in \mathbb{P}^3 with a spherical real point set and blow up this surface at three pairs of complex conjugate points. The resulting del Pezzo surface X is of degree 2 and of type $\langle 1 \rangle^+$. We have a natural basis $L_1, L_2, E_1, \dots, E_6$ in $\text{Pic}(X)$, where:

- L_1 and L_2 are complex conjugate, $L_1^2 = L_2^2 = 0$, and $L_1L_2 = 1$,
- $E_i^2 = -1$, and $L_1E_i = L_2E_i = 0$ for $1 \leq i \leq 6$,
- $E_iE_j = 0$ for $1 \leq i < j \leq 6$,
- E_{2i-1}, E_{2i} are complex conjugate for $i = 1, 2, 3$.

Any real big effective divisors D in X can be represented as

$$D = d(L_1 + L_2) - d_1(E_1 + E_2) - d_2(E_3 + E_4) - d_3(E_5 + E_6), \quad d > 0, \quad d_1, d_2, d_3 \geq 0,$$

in particular, D^2 is even.

Consider an elliptic ABV family of X (see Section 5.1), and denote by (Y, E) the real nodal del Pezzo pair in the central fiber of this family. From Proposition 26 and relation $E^2 = -2$, we immediately derive that the divisor class E is either $\pm(L_1 - L_2)$, or $\pm(E_{2i-1} - E_{2i})$, $i = 1, 2, 3$. Since the corresponding nodal quartic Q_Y does not have real tangents passing through the node (except for tangent lines at the node), there is no real (-1) -curve on Y crossing E , and we are left with the only option $E = \pm(L_1 - L_2)$ (we can assume that $E = L_1 - L_2$).

Let $D \in \text{Pic}(X)$ be an X -nef and big real divisor. By Theorem 7(1) and non-negativity statement (55), one obtains

$$W(X, D, \mathbb{R}X, 0) = W_{Y,E,0}(D, 0, 0, 0) \geq 0,$$

proving statement (i). If $W(X, D, \mathbb{R}X, 0) = W_{Y,E,0}(D, 0, 0, 0) > 0$, then formula (9) applied to $W_{Y,E,0}(D, 0, 0, 0)$ must contain in the right-hand side a summand

$$c \cdot W_{Y,E,0}(D^{(1)}, 0, e_1, 0) \cdot W_{Y,E,0}(D^{(2)}, 0, e_1, 0)$$

with a positive integer c , and

$$D^{(1)}, D^{(2)} \in \text{Pic}_{++}^{\mathbb{R}}(Y, E), \quad D^{(1)} + D^{(2)} = D - E, \quad W_{Y,E,0}(D^{(i)}, 0, e_1, 0) > 0, \quad i = 1, 2. \quad (76)$$

Here $D^{(1)}, D^{(2)}$ cannot be represented by (-1) -curves, since $(D - E)^2 = D^2 - 2 \geq 0$, and complex conjugate (-1) -curves, crossing E , are disjoint.

Assume now that $D^2 \leq 2$. Then $(D - E)^2 = D^2 - 2 \leq 0$, which in the case of $W(X, D, \mathbb{R}X, 0) > 0$ leaves the only option $D^{(1)} = D^{(2)} = D'$, where $(D')^2 = 0$ and $\dim |D'|_Y = 1$; hence $-K_Y D' = 2$ and $-K_X D = -2K_Y D' = 4$ as asserted in statement (iv).

Assume now that $D_1, D_2 \in \text{Pic}^{\mathbb{R}}(X)$ are X -nef and big, and satisfy $W(X, D_i, \mathbb{R}X, 0) > 0$, $i = 1, 2$. Show that $W(X, D_1 + D_2, \mathbb{R}X, 0) > 0$ in agreement with statement (ii). As we have seen above, there are $D_i^{(j)} \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$, $i, j = 1, 2$, such that

$$D_i^{(j)} E = 1, \quad W_{Y,E,0}(D_i^{(j)}, 0, e_1, 0) > 0, \quad i, j = 1, 2,$$

$$D_i^{(1)} + D_i^{(2)} = D_i - E, \quad i = 1, 2.$$

Appropriately applying formula (9), we obtain the required inequality from

$$W(X, D_1 + D_2, \mathbb{R}X, 0) \geq c_1 W_{Y,E,0}(D_1^{(1)} + D_2, 0, e_1, 0) W_{Y,E,0}(D_1^{(2)}, 0, e_1, 0),$$

and from

$$\begin{aligned} W_{Y,E,0}(D_1^{(1)} + D_2, 0, e_1, 0) &\geq W_{Y,E,0}(D_1^{(1)} + D_2, e_1, 0, 0) \\ &\geq c_{2j} W_{Y,E,0}(D_1^{(1)}, 0, e_1, 0) W_{Y,E,0}(D_2^{(1)}, 0, e_1, 0) W_{Y,E,0}(D_2^{(2)}, 0, e_1, 0) > 0, \end{aligned} \quad (77)$$

where c_1, c_{21}, c_{22} are some positive integers. If $D \in \text{Pic}^{\mathbb{R}}(X)$ is X -nef and big, and is disjoint from a pair of complex conjugate (-1) -curves on X , then we can regard D as a real divisor on a surface of type $\mathbb{P}_{1,3}^2$, whose Welschinger invariants are positive by Theorem 2. If $D \in \text{Pic}^{\mathbb{R}}(X)$ is X -nef and big with $W(X, D, \mathbb{R}X, 0) > 0$, then

$$\begin{aligned} W(X, D - K_X, \mathbb{R}X, 0) &= W_{Y,E,0}(D - K_X, 0, 0, 0) \\ &\geq c_0 \cdot W_{Y,E,0}(D^{(1)}, 0, e_1, 0) \cdot W_{Y,E,0}(D^{(2)} - K_X, 0, e_1, 0) \\ &\geq 2c_0 \cdot W_{Y,E,0}(D^{(1)}, 0, e_1, 0) \cdot W_{Y,E,0}(D^{(2)}, 0, e_1, 0) \cdot W_{Y,E,0}(-K_X - E, 0, 2e_1, 0) > 0. \end{aligned}$$

To complete the proof of statement (ii), we have to show the positivity of $W(X, -2K_X, \mathbb{R}X, 0)$: a direct application of Theorem 7(1) and formula (9) gives $W(X, -2K_X, \mathbb{R}X, 0) = 8$.

Let $D \in \text{Pic}^{\mathbb{R}}(X)$ be an X -nef and big divisor such that $W(X, D, \mathbb{R}X, 0) > 0$, and let $D^{(1)}, D^{(2)} \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ be as in (76). By (77), $W_{Y,E,0}(D^{(1)} + mD, 0, e_1, 0) > 0$ for all $m \geq 0$. Hence, first, for any $n \geq 1$,

$$\begin{aligned} W(X, (n+2)D, \mathbb{R}X, 0) &= W_{Y,E,0}((n+2)D, 0, 0, 0) \\ &\geq c_0 W_{Y,E,0}(D^{(2)}, 0, e_1, 0) W_{Y,E,0}(D^{(1)} + (n+1)D, 0, e_1, 0) \\ &\geq W_{Y,E,0}(D^{(1)} + (n+1)D, 0, e_1, 0), \end{aligned}$$

and further on,

$$\begin{aligned} W_{Y,E,0}(D^{(1)} + (n+1)D, 0, e_1, 0) &\geq W_{Y,E,0}(D^{(1)} + (n+1)D, e-1, 0, 0) \\ &\quad \frac{(-K_Y(D^{(1)} + (n+1)D) - 2)!}{2} \cdot W_{Y,E,0}(D^{(2)}, 0, e_1, 0) \\ &\quad \times \frac{W_{Y,E,0}(D^{(1)} + [\frac{n}{2}]D, 0, e_1, 0)}{(-K_Y(D^{(1)} + [\frac{n}{2}]D) - 1)!} \cdot \frac{W_{Y,E,0}(D^{(1)} + [\frac{n+1}{2}]D, 0, e_1, 0)}{(-K_Y(D^{(1)} + [\frac{n+1}{2}]D) - 1)!}. \end{aligned}$$

Then, the sequence

$$a_n = \frac{W_{Y,E,0}(D^{(1)} + nD, 0, e_1, 0)}{(-K_Y(D^{(1)} + nD))!}, \quad n \geq 0,$$

satisfies (53) in Lemma 37 with

$$\lambda = \inf_{n \geq 3} \frac{W_{Y,E,0}(D^{(2)}, 0, e_1, 0) \cdot (-K_Y(D^{(1)} + [\frac{n}{2}]D)) \cdot (-K_Y(D^{(1)} + [\frac{n+1}{2}]D))}{2(-K_Y(D^{(1)} + (n+1)D)) \cdot (-K_Y(D^{(1)} + (n+1)D) - 1)} > 0.$$

Hence (4) follows. \square

5.7 Case $\mathbb{R}X = S^2 \sqcup \mathbb{R}P^2 \# \mathbb{R}P^2$ and $F = S^2$

Let X be a real del Pezzo surface of degree 2 and of type $1\langle 1 \rangle^-$, and let $F = F^\circ$. Including such a surface in an ABV-family does not lead to non-negative w -numbers, so the approach used in the proof of Theorems 2 and 3 cannot be performed in this case without any modification.

Proposition 40 *Let X be a real del Pezzo surface of degree 2 and of type $1\langle 1 \rangle^-$. Then,*

- (i) *for any real F° -compatible big and nef divisor class $D \in \text{Pic}(X)$, we have for each $i = 1, 2, 3$,*

$$\sum_{m \in \mathbb{Z}} W(X, D + 2mE^{(i)}, F^\circ, [F^{no}]) > 0,$$

$$\log \sum_{m \in \mathbb{Z}} W(X, nD + 2mE^{(i)}, F^\circ, [F^{no}]) = -K_X D \cdot n \log n + O(n), \quad n \rightarrow \infty,$$

where $E^{(1)}, E^{(2)}, E^{(3)}$ are the real divisors classes introduced in Section 5.2;

- (ii) *if, in addition, $DE' = 0$ for some real (-1) -curve $E' \subset X$, then the relations (2) and (3) with $F = F^\circ$ hold true.*

Proof. For statement (i), consider an external ABV family of X (see Section 5.1). The mirror unscrew has a surface X' of type $\langle 2 \rangle^+$ as a general fiber, and hence both formulas follow from relations (2) and (3) for X' , claimed in Theorem 2, and from Corollary 31 and Remark 32 as far as $-DK_X \geq 3$, which by Proposition 34 holds for all big and nef F° -compatible divisor classes $D \in \text{Pic}(X)$.

For statement (ii), we blow down the curve E' and reduce the problem to the case of a real del Pezzo surface of degree 3, covered by Theorem 2. \square

Using a slightly different approach, E. Brugallé [4, Corollary 6.10] proved the non-negativity of the invariants $W(X, D, F^\circ, [F^{no}])$ for all real effective divisor classes $D \in \text{Pic}(X)$.

6 Monotonicity

Lemma 41 (cf. [15, Lemma 7.6]) *Let D_1, D_2 be X -nef and big real divisor classes on a del Pezzo surface X of type $\mathbb{P}_{a,b}^2$, $a + 2b = 7$. If $D_2 - D_1$ is effective, then $D_2 - D_1$ can be decomposed into a sum $E^{(1)} + \dots + E^{(k)}$, where $E^{(i)}$ is either a real (-1) -curve, or a pair of disjoint complex conjugate (-1) -curves, $i = 1, \dots, k$, and, moreover, each real divisor $D^{(i)} = D_1 + \sum_{j \leq i} E^{(j)}$ is X -nef and big, and satisfies $D^{(i)} E^{(i+1)} > 0$, $i = 0, \dots, k - 1$.*

Proof. It is well known that the effective cone in $\text{Pic}(X)$ is generated by (-1) -curves. It is easy to verify that two complex conjugate (-1) -curves in X intersect in at most one point, and if they intersect, then their sum is linearly equivalent to a pair of real (-1) -curves, thus, $D_2 - D_1$ can be decomposed into a sum $E^{(1)} + \dots + E^{(k)}$, where $E^{(i)}$ is either a real (-1) -curve, or a pair of disjoint complex conjugate (-1) -curves, $i = 1, \dots, k$. We show that a suitable reordering of $E^{(1)}, \dots, E^{(k)}$ ensures the X -nefness and bigness of $D^{(i)}$ together with $D^{(i)} E^{(i+1)} > 0$, $i = 0, \dots, k - 1$. The divisor $D^{(0)} = D_1$ is X -nef and big. Suppose now that $D^{(i)}$ is X -nef and big for some $0 \leq i < k$. If $i = k - 1$, then $D^{(k)} = D_2$ is X -nef and big, and furthermore

$$D^{(k-1)} E^{(k)} = (D_2 - E^{(k)}) E^{(k)} = D_2 E^{(k)} - (E^{(k)})^2 > 0 .$$

If $i \leq k - 2$, then there exists $i < j \leq k$ such that $D^{(i)} E^{(j)} > 0$. Indeed, otherwise, we would have

- either all $E^{(i+1)}, \dots, E^{(k)}$ orthogonal to each other and to $D^{(i)}$, and thus, $D_2 E^{(j)} < 0$, $i < j \leq k$ contrary to the X -nefness of D_2 ,
- or we would have some $i < j < j' \leq k$ such that $E^{(j)} E^{(j')} > 0$, but then $\dim |E^{(j)} + E^{(j')}| > 0$, contradicting to the bigness of $D^{(i)}$.

So, we may assume that $D^{(i)} E^{(i+1)} > 0$. Then $D^{(i+1)} = D^{(i)} + E^{(i+1)}$ is X -nef and big:

$$\begin{aligned} D^{(i+1)} E^{(i+1)} &= D^{(i)} E^{(i+1)} + (E^{(i+1)})^2 \geq 0 , \\ (D^{(i+1)})^2 &= (D^{(i)})^2 + 2D^{(i)} E^{(i+1)} + (E^{(i+1)})^2 \geq (D^{(i)})^2 > 0 . \quad \square \end{aligned}$$

Theorem 9 *Let D_1, D_2 be X -nef and big divisor classes on a real del Pezzo surface X of type $\mathbb{P}_{a,b}^2$, $a + 2b = 7$, $0 \leq b \leq 2$, such that $D_2 - D_1$ is effective. Then*

$$W(X, D_2, \mathbb{R}X, 0) \geq W(X, D_1, \mathbb{R}X, 0) . \tag{78}$$

Proof. By Lemma 41, we should only consider the case of $E^* = D_2 - D_1$ either a real (-1) -curve, or a pair of disjoint complex conjugate (-1) -curves. Let λ be the number of irreducible components of E^* . We can assume that E^* consists of λ exceptional divisors of the blow up $X \rightarrow \mathbb{P}^2$. Specializing $6 - \lambda$ other exceptional divisors so that their blow-downs and the blow-downs of the components of E^*

appear on a real conic, we degenerate X in a regular ABV family into a real nodal del Pezzo pair (Y, E) where E is the strict transform of the above plane conic. Since $D_2E = D_1E + \lambda > D_1E$, we have $\binom{D_2E+2m}{m} \geq \binom{D_1E+2m}{m}$ for all $m \geq 0$, and hence using the non-negativity statement (55) and formula (47) for the both sides of (78), we reduce the problem to establishing inequality

$$W_{Y,E,0}(D, \alpha, \beta + \lambda e_1, 0) \geq W_{Y,E,0}(D - E^*, \alpha, \beta, 0) \quad (79)$$

for all divisors $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ such that $DE \geq \lambda$, and for all vectors $\alpha, \beta \in \mathbb{Z}_+^{\infty, \text{odd}}$ such that $I(\alpha + \beta) = DE - \lambda$. We prove (79) by induction on $R_Y(D, \beta + \lambda e_1)$. The case $R_Y(D, \beta + \lambda e_1) < \lambda$ is trivial, since then $R_Y(D - E^*, \beta) = R_Y(D, \beta + \lambda e_1) - \lambda < 0$. If $R_Y(D, \beta + \lambda e_1) = \lambda$ and, respectively, $R_Y(D - E^*, \beta) = 0$, the only relevant case is that of Proposition 11(1iii) with $D - E^*$ playing the role of D and $\beta = 0$, in which case by (55), formula (9), and Proposition 11(1iii) we have

$$W_{Y,E,0}(D, \alpha, \lambda e_1, 0) \geq W_{Y,E,0}(D, \alpha + \lambda e_1, 0, 0) = 1 = W_{Y,E,0}(D - E^*, \alpha, 0, 0).$$

If $R_Y(D, \beta + \lambda e_1) = R_Y(D - E^*, \beta) + \lambda > \lambda$, we compute both sides of (79) by formula (9) and compare them using the induction assumption (in the sequel we shortly write $\text{RHS}(9)_l$ and $\text{RHS}(9)_r$ for the right-hand side of (9) expressing the left and the right terms of (79) respectively). So, for the summands in the first sum in $\text{RHS}(9)_l$ and $\text{RHS}(9)_r$ the induction assumption yields

$$W_{Y,E,0}(D, \alpha + e_j, \beta - e_j + \lambda e_1, 0) \geq W_{Y,E,0}(D - E^*, \alpha + e_j, \beta - e_j, 0).$$

For the second sum in $\text{RHS}(9)_l$ and $\text{RHS}(9)_r$ we perform the following comparison. Let

$$S_r = c \cdot \frac{2^{\|\beta^{(0)}\|}}{\beta^{(0)}!} \cdot \frac{(n-1-\lambda)!}{n_1! \dots n_m!} \cdot \prod_{i=1}^m \binom{(\beta^{\text{re}})^{(i)}}{\gamma^{(i)}} W_{Y,E,0}(\mathcal{D}^{(i)}, \alpha^{(i)}, (\beta^{\text{re}})^{(i)}, (\beta^{\text{im}})^{(i)})$$

be a summand in the second sum of $\text{RHS}(9)_r$, where $n = R_Y(D, \beta + \lambda e_1)$ and $n_i = R_Y(\mathcal{D}^{(i)}, (\beta^{\text{re}})^{(i)} + 2(\beta^{\text{im}})^{(i)})$, $1 \leq i \leq m$. Notice that $m \geq 1$, since $DE^* > 0$, and hence there is $\mathcal{D}^{(i)}$ such that $[\mathcal{D}^{(i)}]E^* > 0$. Pick $\mathcal{D}^{(j)}$ with $[\mathcal{D}^{(j)}]E^* > 0$ and associate with S_r the following summand S_l in the second sum of $\text{RHS}(9)_l$:

- if $[\mathcal{D}^{(j)}] \neq -\lambda(K_Y + E) - E^*$ (in which case $\mathcal{D}^{(j)}$ is a real divisor), then

$$S_l = c \cdot \frac{s_j^r}{s_j^l} \cdot \frac{2^{\|\hat{\beta}^{(0)}\|}}{\hat{\beta}^{(0)}!} \cdot \frac{(n-1)!}{\hat{n}_1! \dots \hat{n}_m!} \cdot \prod_{i=1}^m \binom{(\hat{\beta}^{\text{re}})^{(i)}}{\gamma^{(i)}} W_{Y,E,0}(\hat{\mathcal{D}}^{(i)}, \alpha^{(i)}, (\hat{\beta}^{\text{re}})^{(i)}, (\beta^{\text{im}})^{(i)}), \quad (80)$$

where $\hat{\beta}^{(0)} = \beta^{(0)}$, $\hat{n}_i = R_Y(\hat{\mathcal{D}}^{(i)}, (\hat{\beta}^{\text{re}})^{(i)} + 2(\beta^{\text{im}})^{(i)})$,

$$\hat{\mathcal{D}}^{(i)} = \begin{cases} \mathcal{D}^{(i)}, & i \neq j, \\ \mathcal{D}^{(j)} + E^*, & i = j, \end{cases} \quad \text{and} \quad (\hat{\beta}^{\text{re}})^{(i)} = \begin{cases} (\beta^{\text{re}})^{(i)}, & i \neq j, \\ (\beta^{\text{re}})^{(j)} + \lambda e_1, & i = j, \end{cases}$$

s_j^r counts how many times the tuple $(\mathcal{D}^j, \alpha^{(j)}, (\beta^{\text{re}})^{(j)}, \gamma^{(j)})$ occurs in the list $\{(\mathcal{D}^{(i)}, \alpha^{(i)}, (\beta^{\text{re}})^{(i)}, \gamma^{(i)})\}_{i=1}^m$, and s_j^l counts how many times the tuple $(\hat{\mathcal{D}}^j, \alpha^{(j)}, (\hat{\beta}^{\text{re}})^{(j)}, \gamma^{(j)})$ occurs in the list $\{(\hat{\mathcal{D}}^{(i)}, \alpha^{(i)}, (\hat{\beta}^{\text{re}})^{(i)}, \gamma^{(i)})\}_{i=1}^m$;

- if $D^{(j)} = -\lambda(K_Y + E) - E^*$, then

$$S_l = c \cdot \frac{2^{\|\hat{\beta}^{(0)}\|}}{\hat{\beta}^{(0)}!} \cdot \frac{(n-1)!}{\hat{n}_1! \dots \hat{n}_m!} \cdot \prod_{\substack{1 \leq i \leq m \\ i \neq j}} \binom{(\hat{\beta}^{\text{re}})^{(i)}}{\gamma^{(i)}} W_{Y,E,0}(\hat{\mathcal{D}}^{(i)}, \alpha^{(i)}, (\hat{\beta}^{\text{re}})^{(i)}, (\beta^{\text{im}})^{(i)}), \quad (81)$$

where $\hat{\beta}^{(0)} = \beta^{(0)} + \lambda e_1$, $(\hat{\beta}^{\text{re}})^{(i)} = (\beta^{\text{re}})^{(i)}$, $\hat{n}_i = n_i$, and $\hat{\mathcal{D}}^{(i)} = \mathcal{D}^{(i)}$ for $1 \leq i \leq m$, $i \neq j$, and $\hat{n}_j = 0$.

It is easy to verify (again using the induction assumption) that

$$S_r \leq \begin{cases} S_l \cdot \frac{s_j^l \hat{n}_j}{s_j^{r(n-1)}}, & \text{if } \lambda = 1, \\ S_l \cdot \frac{s_j^l \hat{n}_j (\hat{n}_j - 1)}{s_j^{r(n-1)(n-2)}}, & \text{if } \lambda = 2, \end{cases} \quad \text{in (80) ,}$$

and

$$S_r \leq \begin{cases} S_r \cdot \frac{\hat{\beta}_1^{(0)}}{2(n-1)}, & \text{if } \lambda = 1, \\ S_r \cdot \frac{\hat{\beta}_1^{(0)}(\hat{\beta}_1^{(0)} - 1)}{4(n-1)(n-2)}, & \text{if } \lambda = 2, \end{cases} \quad \text{in (81) .}$$

Since $n - 1 = \sum_j s_j^l \hat{n}_j + \|\hat{\beta}^{(0)}\|$ (cf. Remark 12), we conclude that the total value of the terms in the second sum in $\text{RHS}(9)_r$ associated with a given summand S_l in the second sum in $\text{RHS}(9)_l$ does not exceed S_l , which completes the proof. \square

7 Mikhalkin's congruence

Theorem 10 *For any X -nef and big divisor class D on a surface X of type $\mathbb{P}_{7,0}^2$, one has*

$$W(X, D, \mathbb{R}X, 0) = GW_0(X, D) \pmod{4} .$$

Proof. Using a regular ABV family of X and formulas (36), (47), we reduce the question to the congruence

$$W_{Y,E,0}(D, 0, (DE)e_1, 0) = N_Y(D, 0, (DE)e_1) \pmod{4}$$

for all divisors $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$.

In fact, a more general statement holds: for any divisor class $D \in \text{Pic}_{++}^{\mathbb{R}}(Y, E)$ and any vectors $\alpha, \beta \in \mathbb{Z}_+^{\infty}$ such that $I(\alpha + \beta) = DE$, one has

$$W_{Y,E,0}(D, \alpha, \beta, 0) = I^{\beta} N_Y(D, \alpha, \beta) \pmod{4} \quad \text{if } \beta \in \mathbb{Z}_+^{\infty, \text{odd}}, \quad (82)$$

and

$$I^{\beta} \cdot N_Y(D, \alpha, \beta) = 0 \pmod{4} \quad \text{if } \beta \notin \mathbb{Z}_+^{\infty, \text{odd}}, \quad (83)$$

where the numbers $N_Y(D, \alpha, \beta)$ are the degrees of varieties $V_Y(D, \alpha, \beta, \mathbf{p}^b)$ (here $\mathbf{p}^b = \{p_{i,j} : i \geq 1, 1 \leq j \leq \alpha_i\}$ are sequences of $\|\alpha\|$ distinct generic points on E) introduced in Section 3.4. The proof literally coincides with that of [16, Theorem 5] and uses induction on $R_Y(D, \beta)$ and the recursive formulas [19, Formula (66)] and (9). \square

Index of notations

$\ \alpha\ $	Section 3.2	$\text{Pic}^{\mathbb{R}}(\Sigma)$	Section 2.3
$I\alpha$	Section 3.2	$\text{Pic}_+^{\mathbb{R}}(\Sigma)$	Section 3.3
I^α	Section 3.2	$\text{Pic}_{++}(\Sigma, E)$	Section 3.3
e_i	Section 3.2	$\text{Pic}_{++}^{\mathbb{R}}(\Sigma)$	Section 3.3
\mathbb{Z}_+^∞	Section 3.2	$[\mathcal{D}]$	Section 3.3
$\mathbb{Z}_+^\infty, \text{ odd}$	Section 3.2	$R_\Sigma(\mathcal{D}, \beta)$	Section 3.3
$\mathbb{Z}_+^\infty, \text{ even}$	Section 3.2	$\mathbb{P}_{a,b}^2$	Section 2.2
$\mathbb{Z}_+^\infty, \text{ odd}\cdot\text{even}$	Section 3.2	$\langle q \rangle^\pm$	Section 2.2
$W(X, D, F, \varphi)$	Section 2.1	$\langle 1 \langle 1 \rangle \rangle^\pm$	Section 2.2
$W_{Y,E,\varphi}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}})$	Section 3.6	L', L''	Section 3.1
$W_{Y,F_+,\varphi}^\pm(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}})$	Section 3.8	$\mathcal{E}(E)$	Section 3.1
$U_{Y,F,\varphi}(D, ke_1, le_1, \mathbf{p}^\sharp)$	Section 4.3.1	$\mathcal{E}(E)^{\perp D}$	Section 3.1
$U_{Y,F_+,\varphi}^\pm(D, 0, (DE/2)e_1, \mathbf{p}^\sharp)$	Section 4.3.1	Q_X	Section 2.2
$U_{Y,E,\varphi'}(D, \mathbf{p}^\sharp)$	Section 4.3.2	F^o	Section 2.2
$V_Y^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b)$	Section 3.4	F^{no}	Section 2.2
$V_Y^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b, \mathbf{p}^\sharp)$	Section 3.4	F^{no}	Section 2.2
$V_{Y,F_+}^{\mathbb{R}}(\mathcal{D}, \alpha, \beta^{\text{re}}, \beta^{\text{im}}, \mathbf{p}^b, \mathbf{p}^\sharp)$	Section 3.8	$V_Y(D, \alpha, \beta, \mathbf{p}^b)$	Section 3.4
$\mu_\varphi(C)$	Section 3.6	$GW_0(Y, D)$	Section 4
$\mu_\varphi^\pm(C)$	Section 3.8	$N_Y(D, 0, (DE)e_1)$	Section 4

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