ONE-CONNECTIVITY AND FINITENESS OF HAMILTONIAN $S^1$-MANIFOLDS WITH MINIMAL FIXED SETS

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ABSTRACT. Let the circle act effectively in a Hamiltonian fashion on a compact symplectic manifold $(M, \omega)$. Assume that the fixed point set $M^{S^1}$ has exactly two components, $X$ and $Y$, and that $\dim(X) + \dim(Y) + 2 = \dim(M)$. We first show that $X$, $Y$ and $M$ are simply connected. Then we show that, up to $S^1$-equivariant diffeomorphism, there are finitely many such manifolds in each dimension. Moreover, we show that in low dimensions, the manifold is unique in a certain category. We use techniques from both areas of symplectic geometry and geometric topology.

1. INTRODUCTION

Let a Lie group act non-trivially on a manifold $M$. The manifold $M$ may have certain geometric or topological structures which are invariant under the action. For instance, $M$ may have a symplectic structure or a Kähler structure, $M$ may have a certain homotopy type or a certain cohomology ring. When $M$ is a symplectic manifold, and when the Lie group action is Hamiltonian, the moment map provides a key tool for the study of $M$.

One fundamental question in symplectic geometry is which symplectic manifold admits a Hamiltonian group action, or, when a symplectic group action is a Hamiltonian action. This has been answered in various cases. One way of approaching the question is to look at necessary conditions, and to find the topological and geometrical properties such manifolds can have. Furthermore, we can ask how many such manifolds can arise with these given properties. These can be the first steps for further classifications of the manifolds.

Consider a compact connected symplectic manifold $(M, \omega)$ which admits an effective Hamiltonian $T$-action, where $T$ is a connected compact torus. The number $k = \frac{1}{2} \dim(M) - \dim(T)$ is called the complexity of the Hamiltonian $T$-manifold $M$. Complexity zero Hamiltonian manifolds, also called symplectic toric manifolds, are classified [8]. Complexity one Hamiltonian 4-manifolds and complexity one Hamiltonian manifolds whose nonempty symplectic quotients are all 2-dimensional are also classified [1, 2, 18, 21, 22]. All symplectic toric manifolds are Kähler [8], while there exist complexity one Hamiltonian manifolds which do not admit any invariant Kähler structure [40].

Let $(M, \omega)$ be a compact symplectic manifold which admits an effective Hamiltonian $T = S^1$-action with moment map $\phi: M \to \mathbb{R}$. A general classification of such manifolds does not appear tractable. Let us first look at a simple constraint.

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on the dimensions of the connected components of the fixed point set $M^{S^1}$. First of all, each connected component of $M^{S^1}$ is symplectic hence even dimensional. The moment map $\phi$ is a perfect Morse-Bott function whose critical set is exactly $M^{S^1}$. The fact that $M$ is compact and symplectic implies that $b_{2i}(M) \geq 1$ for all $0 \leq 2i \leq \dim(M)$. Using Morse-Bott theory, we may obtain the inequality
\[
\sum_{F \subset M^{S^1}} (\dim(F) + 2) \geq \dim(M) + 2,
\]
where the sum is over all the connected components of $M^{S^1}$. When $M$ has minimal even Betti numbers, i.e., when $b_{2i}(M) = b_{2i}(\mathbb{CP}^n) = 1$ for all $0 \leq 2i \leq 2n = \dim(M)$, we have the equality
\[
(1.1) \quad \sum_{F \subset M^{S^1}} (\dim(F) + 2) = \dim(M) + 2.
\]
But this equality does not imply that the even Betti numbers of $M$ are minimal. For details of these arguments, we refer to [26, Section 4]. Note that the critical set of the moment map has at least two connected components — its minimum and its maximum.

Recent works on compact Hamiltonian $S^1$-manifolds which have minimal even Betti numbers or which satisfy (1.1) include [41, 30, 26, 24, 13, 25]. In these works, the authors consider various cases, for which they show that certain important global invariants — the integral cohomology ring and total Chern class, of the manifold, as well as the circle representations on the normal bundles of the fixed components, are identical to those of some known examples. In several interesting cases, e.g. when the manifold is 6-dimensional, or is Kähler, the manifold can be equivariantly identified in the symplectic or complex categories with some standard examples [30, 24, 25]. Similarly, in another recent work [32], the author studies the integral cohomology ring and total Chern class of compact Hamiltonian $T$-manifolds which are GKM-manifolds with minimal even Betti numbers.

There is also the recent work [15], where the authors study compact Hamiltonian $S^1$-manifolds whose fixed point set $M^{S^1}$ consists of two connected components (but without (1.1)). They give certain description on when two such manifolds are equivariantly diffeomorphic.

In this paper, we restrict our attention to the study of those manifolds satisfying the following assumption:

**Assumption 1.2.** Let $(M, \omega)$ be a $2n$-dimensional compact symplectic manifold equipped with an effective Hamiltonian $S^1$-action such that the fixed point set $M^{S^1}$ consists of two connected components, $X$ and $Y$, satisfying (1.1), which we rewrite as
\[
\dim(X) + \dim(Y) + 2 = \dim(M).
\]

In Section 2 we will see that the Kähler manifolds $\mathbb{CP}^n$, and $\tilde{G}_2(\mathbb{R}^{n+2})$ — the Grassmannian of oriented two-planes in $\mathbb{R}^{n+2}$ with $n \geq 3$ odd, equipped with some standard circle actions, provide two families of standard examples of manifolds satisfying Assumption 1.2.

Suppose Assumption 1.2 holds. In [26], the first author and Tolman determine the integral cohomology rings and total Chern classes of $X$, $Y$ and $M$. In particular, they prove that the integral cohomology ring and total Chern class of $M$
are isomorphic to either those of $\mathbb{C}P^n$, or those of $\tilde{G}_2(\mathbb{R}^{n+2})$ with $n \geq 3$ odd. In this paper, we determine the fundamental groups of $X$, $Y$ and $M$, and using this, we show that in the equivariant smooth category, there exist only finitely many such manifolds. Hence in a sense, our results show that the manifolds satisfying Assumption 1.2 are “very close to” the standard examples $\mathbb{C}P^n$ and $\tilde{G}_2(\mathbb{R}^{n+2})$ with $n$ odd. We use both symplectic and topological techniques, in particular, we use surgery theory for the proof of the finiteness. We hope our results and method will provide insights for further classification of compact Hamiltonian $S^1$-manifolds which have minimal even Betti numbers or which satisfy (1.1).

Now, let us state more precisely our results.

**Theorem 1.** Under Assumption 1.2, the manifolds $M$, $X$ and $Y$ are all 1-connected.

**Theorem 2.** If Assumption 1.2 holds, then

1. both $X$ and $Y$ are homotopy complex projective spaces with standard Pontryagin classes;
2. when the action is semifree, i.e., the action is free outside the fixed point set, then $M$ is a homotopy complex projective space with standard Pontryagin classes.

The study of Hamiltonian $S^1$-manifolds with minimal even Betti numbers was also motivated by the classical Petrie’s conjecture, see [41]. The conjecture states that if a $2n$-dimensional manifold $M$ of the homotopy type of $\mathbb{C}P^n$ admits a non-trivial circle action, then the total Pontryagin class of $M$ agrees with that of $\mathbb{C}P^n$. It is known that Petrie’s conjecture holds when the fixed point set $M^{S^1}$ consists of a small number of connected components, in particular, 2 connected components [45, 48]. Theorem 2 (2) tells us that when $M$ is a compact symplectic manifold with a semi-free, Hamiltonian circle action whose fixed point set consists of two connected components satisfying (1.3), we have both the assumption and the conclusion of Petrie’s conjecture.

Our next result says that in the equivariant smooth category, for each fixed dimension, there are only finitely many manifolds satisfying Assumption 1.2.

**Theorem 3.** In each fixed dimension, up to $S^1$-equivariant diffeomorphism, there are only finitely many manifolds fulfilling Assumption 1.2.

In fact, under Assumption 1.2 if $M$ or one of $X$ and $Y$ is low-dimensional, we can determine $M$ in the equivariant symplectic category. When $X$ or $Y$ is an isolated point, by a theorem of Delzant [3], $M$ is $S^1$-equivariantly symplectomorphic to $\mathbb{C}P^n$ with a standard circle action. In particular, when $\dim(M) = 2$ or 4, $X$ or $Y$ has to be isolated. When $\dim(M) = 6$, then either $X$ or $Y$ is an isolated point, or both $X$ and $Y$ are 2-dimensional 2-spheres by Theorem 1. In the latter case, if the $S^1$ action on $M$ is semifree, by Gonzalez’s work [12], $M$ is $S^1$-equivariantly symplectomorphic to $\mathbb{C}P^3$ with a standard circle action. See Section 6 for more discussion.

**Remark 1.4.** For the manifolds fulfilling Assumption 1.2, we have the following non-equivariant uniqueness in low dimensions:

1. when $\dim M = 2n \leq 6$, $M$ is diffeomorphic to $\mathbb{C}P^n$ if the action is semifree, and is diffeomorphic to $\tilde{G}_2(\mathbb{R}^5)$ if the action is not semifree;
(2) when dim \( M = 10 \) or dim \( M = 14 \) and the action is not semifree, \( M \) is respectively homeomorphic to \( G_2(\mathbb{R}^7) \) or to \( G_2(\mathbb{R}^9) \).

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2. On our results

Let us first give two families of examples of manifolds satisfying Assumption 1.2 then state earlier results on such manifolds and finally give a short outline on our proofs.

Example (A). Given \( n \geq 1 \), let \( \mathbb{CP}^n \) be the complex projective space. It naturally arises as a coadjoint orbit of \( SU(n + 1) \); it inherits a symplectic form \( \omega \) and a Hamiltonian \( SU(n + 1) \) action.

For any \( j \in \{0, \ldots, n-1\} \), there is a semifree Hamiltonian circle action given by
\[
\lambda \cdot [z_0, z_1, \ldots, z_n] = [\lambda z_0, \lambda z_1, \ldots, \lambda z_j, z_{j+1}, \ldots, z_n].
\]
The fixed set consists of two components:
\[
\{[z] \in \mathbb{CP}^n \mid z_k = 0 \ \forall \ k \leq j\} \simeq \mathbb{CP}^{n-j-1}, \text{ and } \{[z] \in \mathbb{CP}^n \mid z_k = 0 \ \forall \ k > j\} \simeq \mathbb{CP}^j
\]
which satisfy (1.3).

Example (B). Given \( n \geq 3 \), let \( \tilde{G}_2(\mathbb{R}^{n+2}) \) denote the Grassmannian of oriented two–planes in \( \mathbb{R}^{n+2} \). This 2\( n \)-dimensional manifold naturally arises as a coadjoint orbit of \( SO(n + 2) \); it inherits a symplectic form \( \omega \) and a Hamiltonian \( SO(n + 2) \) action.

If \( n \) is odd, there is a Hamiltonian circle action on \( \tilde{G}_2(\mathbb{R}^{n+2}) \) induced by the action on \( \mathbb{R}^{n+2} \simeq \mathbb{R} \times \mathbb{C}^{n+1} \) given by
\[
\lambda \cdot (t, z_1, \ldots, z_{(n+1)}) = (t, \lambda z_1, \ldots, \lambda z_{(n+1)}).
\]
The fixed set consists of two components, corresponding to the two orientations on those real two–planes which are complex lines in \( \{0\} \times \mathbb{C}^{n+1} \). Hence the fixed point set consists of two copies of \( \mathbb{CP}(\{0\} \times \mathbb{C}^{n+1}) \simeq \mathbb{CP}^{n-1} \). Moreover, the set of two–planes which lie entirely in \( \{0\} \times \mathbb{C}^{n+1} \) is fixed by \( \mathbb{Z}_2 \). This submanifold, which is symplectomorphic to \( \tilde{G}_2(\mathbb{R}^{n+1}) \), has codimension 2 in \( M \).

The following theorem summarizes the previous results about those manifolds satisfying Assumption 1.2. Note that the cohomology ring and Chern classes of \( M \) in (2.4) are the same as those of \( \mathbb{CP}^n \), and the cohomology ring and Chern classes of \( M \) in (2.5) are the same as those of \( \tilde{G}_2(\mathbb{R}^{n+2}) \), where \( n \geq 3 \) is odd.

Theorem 4. [26] Theorems 1 and 2] If Assumption 1.2 holds, then:

\[ H^*(X; \mathbb{Z}) = \mathbb{Z}[u]/u^i+1 \text{ and } c(X) = (1 + u)^{i+1}, \text{ where } \dim(X) = 2i; \]

\[ H^*(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^j+1 \text{ and } c(Y) = (1 + v)^{j+1}, \text{ where } \dim(Y) = 2j. \]
Moreover, we only have two cases (A) and (B) below, where \( \deg(x) = 2 \) and \( \deg(y) = n + 1 \).

(A) the action is semifree,

\[
H^*(M; \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1}) \quad \text{and} \quad c(M) = (1 + x)^{n+1};
\]

(B) the action is not semifree, \( \dim(X) = \dim(Y) \), the only finite nontrivial stabilizer group is \( \mathbb{Z}_2 \) and the submanifold fixed by \( \mathbb{Z}_2 \) has codimension 2 in \( M \). In this case, \( n \geq 3 \) is odd,

\[
H^*(M; \mathbb{Z}) = \mathbb{Z}[x, y]/(x^{\frac{1}{2}(n+1)} - 2y, y^2) \quad \text{and} \quad c(M) = \frac{(1 + x)^{n+2}}{1 + 2x}.
\]

In both cases, the Chern classes of each normal subbundle of \( X \) and of \( Y \) on which \( S^1 \) acts with a fixed stabilizer group are completely determined.

Now, we give a short outline of the proof of our results.

We prove Theorem 1 in Section 3. We first show that \( \pi_1(X) = \pi_1(Y) = 1 \), and then a simple Seifert-van Kampen argument (or alternatively the main theorem of [23]) shows that \( \pi_1(M) = 1 \).

To prove \( \pi_1(Y) = 1 \), we consider the symplectic reduced space \( M_r = \phi^{-1}(r)/S^1 \) at any regular value \( r \) of the moment map \( \phi \). The space \( M_r \) is a weighted complex projective bundle over both \( X \) and \( Y \) at the same time. We show that the weighted complex projective spaces that actually arise are homeomorphic to a smooth complex projective space. We then show that the inclusion of the fiber of \( M_r \) as a bundle over \( X \) composed with the projection down to \( Y \) induces an isomorphism on top cohomology groups, and this allows us to obtain \( \pi_1(Y) = 1 \). Similarly \( \pi_1(X) = 1 \).

The short Section 4 contains the proof of Theorem 2 which is a consequence of Theorem 1 and (2.2), (2.3), (2.4).

We prove Theorem 3 in Section 5. The idea is to glue a tubular neighborhood of \( X \) and a tubular neighborhood of \( Y \) along a regular level set of the moment map. We prove that the equivariant diffeomorphism types of the tubular neighborhoods are determined up to finite ambiguity, and that there are finitely many essentially different ways of gluing. In particular, in the proof of the latter, we use the surgery exact sequence and related techniques, and here we need \( \dim(M) > 6 \), so we need to treat the 6-dimensional case separately. For the case when \( \dim(M) > 6 \) and the action is semifree, one can adopt K. Wang’s proof [45] which uses gluing along the smooth quotient of a regular level set of the moment map. We will give a proof for the case when \( \dim(M) > 6 \) and the action is not semifree using equivariant gluing along the level set itself; and we give a proof for the case when \( \dim(M) = 6 \).

In the Appendix, we prove the uniqueness results mentioned in Remark 1.4. As we have mentioned in the Introduction, when \( \dim(M) = 2 \) or 4, \( M \) is respectively equivariantly symplectomorphic to \( \mathbb{CP}^1 \) and to \( \mathbb{CP}^2 \). When \( \dim(M) = 6 \), either one of \( X \) and \( Y \) is isolated or both \( X \) and \( Y \) are 2-spheres; when the action is semifree, by Delzant’s theorem and by Gonzalez’s result, \( M \) is equivariantly symplectomorphic to \( \mathbb{CP}^3 \) with a standard action (2.1). For the case of non-semifree actions, we use Kreck’s modified surgery theory and a computation of the relevant bordism group by F. Fang and J. Wang.
3.1. Degree one maps from a simply connected topological manifold.

The proof of Theorem 1 will use the following result which is due to Olum.

**Lemma 3.1.** (Olum) Let $P$ and $X$ be closed oriented topological $n$-manifolds, and assume that $P$ is simply connected. If there exists a map $f : P \to X$ such that $f^* : H^n(X; \mathbb{Z}) \to H^n(P; \mathbb{Z})$ is an isomorphism, then $\pi_1(X) = 1$.

The proof of Lemma 3.1 uses the following lemma whose proof we omit here — it is a routine consequence of standard results about fundamental classes (see for example [42 Sections 16.3 and 16.4]).

**Lemma 3.2.** Let $X$ be a closed orientable topological $n$-manifold and $\pi : \tilde{X} \to X$ the universal covering map. If $\pi_1(X)$ is infinite then $H^n(\tilde{X}; \mathbb{Z}) \otimes \mathbb{Q} = 0$. If $\pi_1(X)$ is finite then $\pi^* : H^n(X; \mathbb{Z}) \to H^n(\tilde{X}; \mathbb{Z})$ is a degree $\pm |\pi_1(X)|$ map.

**Proof of Lemma 3.1.** Since $P$ is simply connected, the map $f$ factors through the universal covering map $\pi : \tilde{X} \to X$, giving us a commutative diagram:

$$
\begin{array}{ccc}
P & \xrightarrow{f} & X \\
\downarrow & & \downarrow \pi \\
\tilde{X} & \xrightarrow{\tilde{f}} & X \\
\end{array}
$$

Taking $H^n$ gives

$$
\begin{array}{ccc}
& & H^n(\tilde{X}; \mathbb{Z}) \\
& \xrightarrow{\tilde{f}^*} & \xrightarrow{\pi^*} \\
\mathbb{Z} \cong H^n(P; \mathbb{Z}) & \to & H^n(X; \mathbb{Z}) \cong \mathbb{Z},
\end{array}
$$

i.e., $f^* = \tilde{f}^* \pi^*$.

If $\pi_1(X)$ is infinite then by Lemma 3.2 $H^n(\tilde{X}; \mathbb{Z}) \otimes \mathbb{Q} = 0$ and so $f^* \otimes \mathbb{Q} = 0$, giving a contradiction. So we know that $\pi_1(X)$ is finite and then again by Lemma 3.2 and using the isomorphisms of $H^n(P; \mathbb{Z})$ and $H^n(X; \mathbb{Z})$ with $\mathbb{Z}$, we get

$$
f^*(1) = \tilde{f}^* \pi^*(1) = \tilde{f}^* (\pm |\pi_1(X)|) = \pm |\pi_1(X)|.
$$

Since $f^*$ is an isomorphism, $|\pi_1(X)| = 1$, hence, $\pi_1(X) = 1$. \qed

3.2. Weighted projective spaces.

Let $S^1$ act on $\mathbb{C}^{k+1}$ with weight vector $w = (w_1, \ldots, w_{k+1})$, where $w_i \in \mathbb{N}$ for all $i$, i.e., let $S^1$ act by

$$(3.3) \quad \lambda \cdot (z_1, \ldots, z_{k+1}) = (\lambda^{w_1} z_1, \ldots, \lambda^{w_{k+1}} z_{k+1})$$

for each $\lambda \in S^1$. This action preserves the unit sphere $S^{2k+1} \subset \mathbb{C}^{k+1}$. The orbifold $\mathbb{C}P_w^k = S^{2k+1}/S^1$ is called a weighted projective space.

When $w = (1, \ldots, 1)$, $\mathbb{C}P^k_w$ is the smooth projective space $\mathbb{C}P^k_w$.

**Lemma 3.4.** The pair consisting of the weighted projective space $\mathbb{C}P^k_w$ with $w = (2, \ldots, 2, 1)$ and its subspace of singular points is homeomorphic to the pair $(\mathbb{C}P^k, \mathbb{C}P^{k-1})$ consisting of the smooth projective space and a projective hyperplane.
We define an equivariant map $S^{2k+1} \to S^{2k+1}$, where the first sphere has weight vector $w = (2, \ldots, 2, 1)$, and the second sphere has weight vector $w = (2, \ldots, 2, 2)$, by the formula
\[
(z_1, \ldots, z_k, z_{k+1}) \mapsto \left(z_1, \ldots, z_k, \frac{z_{k+1}^2}{|z_{k+1}|}\right).
\]
It is easy to check that this map induces a homeomorphism on the quotient spaces. It maps the singular subspace of $\mathbb{CP}_w^k$, that is, all points with vanishing last coordinate, to the hyperplane in $\mathbb{CP}_w$ given as the vanishing set of the last coordinate. □

**Remark 3.5.** In general, a weighted projective space $\mathbb{CP}_w^k$ with $k > 1$ and weight vector $w$ may not be homeomorphic to a smooth projective space. Its integral cohomology may have a twisted ring structure, see [17]. Here and elsewhere in this paper, the cohomology of a weighted projective space is its singular cohomology as a topological space.

### 3.3. The Duistermaat–Heckman theorem and the Euler class of circle bundles.

In this section, we recall the Duistermaat–Heckman Theorem for the case of Hamiltonian circle actions, and we address the Euler class of circle bundles. In particular we explain the Duistermaat–Heckman Theorem when the circle action is described by a Čech cocycle which is a continuous $\mathbb{Z}$-valued function, and its Euler class is the image under the isomorphism $\hat{H}^1(M; \mathbb{R}/\Lambda) \cong H^2(M; \Lambda)$. Sometimes, we will consider the image of the Euler class under the natural map $H^2(M; \Lambda) \to H^2(M; \mathbb{R})$ induced by the inclusion $\Lambda \hookrightarrow \mathbb{R}$.

Let $M = \mathbb{R}/\mathbb{Z}$ act on a connected symplectic manifold $(M, \omega)$ with proper moment map $\phi: M \to \mathbb{R}$. Let $r \in \text{image}(\phi)$ be a fixed regular value. Let $I$ be a connected open interval of regular values of $\phi$ such that $r \in I$. Then $\phi^{-1}(I)$ is $S^1$-equivariantly diffeomorphic to $\phi^{-1}(r) \times I$ (in the latter, the moment map is the projection to $I$). Hence, for any value $a \in I$, $\phi^{-1}(a)$ is $S^1$-equivariantly diffeomorphic to $\phi^{-1}(r)$, and so the symplectic reduced space $M_a = \phi^{-1}(a)/S^1$ (possibly a symplectic orbifold) is diffeomorphic to $M_r = \phi^{-1}(r)/S^1$. This gives us a way to identify the cohomology groups of $M_a$ with those of $M_r$. Since two such trivializations induce homotopic diffeomorphisms, the identification of $H^*(M_a)$ and of $H^*(M_r)$ is canonical.

The finite stabilizer groups on each $\phi^{-1}(a)$, $a \in I$ are the same as those on $\phi^{-1}(r)$. Let $m$ be the least common multiple of the orders of all the finite stabilizer groups on $\phi^{-1}(r)$. Then $Z_r = \phi^{-1}(r)/\mathbb{Z}_m$ is a principal $S^1/\mathbb{Z}_m$-bundle over $M_r$:
\[
(3.6) \quad S^1/\mathbb{Z}_m \to Z_r \to M_r.
\]
This is not always a bundle of smooth manifolds. For our purposes, it is sufficient to consider it as a principal circle bundle in the category of topological spaces. Let $\mathbb{Z}$ be the integral lattice of $S^1$, then $\Lambda' = \frac{1}{m}\mathbb{Z} \subset \mathbb{R}$ is the integral lattice of $S^1/\mathbb{Z}_m$, i.e., $S^1/\mathbb{Z}_m \cong \mathbb{R}/\Lambda'$. Let $e(Z_r) \in H^2(M_r; \Lambda')$ be the Euler class of the bundle (3.6). As a topological invariant, $e(Z_r)$ is constant as a function over $I$, hence we will simply use $e$ to denote this class.
Theorem 5. (Duistermaat–Heckman [10]) Let \( a, b \in I \), where \( I \) is a connected open interval of regular values of a proper moment map \( \phi \) of a Hamiltonian circle action. Let \( \omega_a \) and \( \omega_b \) be respectively the reduced symplectic forms in \( M_a \) and in \( M_b \) and \([\omega_a]\) and \([\omega_b]\) be the cohomology classes they represent. Then in \( H^2(M_c; \mathbb{R}) \) we have
\[
[\omega_b] - [\omega_a] = e \cdot (b - a).
\]
Here, \( e \in H^2(M_c; \mathcal{N}') \) is the Euler class of \( (3.6) \), and we used the canonical identification of \( H^2(M_a) \), \( H^2(M_b) \) and of \( H^2(M_c) \).

For the linear action \( (3.3) \) of \( S^1 \) with weight vector \( w \) on \( \mathbb{C}^n \) with the standard symplectic form, the moment map is
\[
\phi = \frac{1}{2} (w_1|z_1|^2 + \cdots + w_{k+1}|z_{k+1}|^2),
\]
and any nonzero value of \( \phi \) is a regular value. Since \( S^{2k+1} \) is \( S^1 \)-equivariantly diffeomorphic to a regular level set of \( \phi \), \( \mathbb{C}P^k_w \) is a symplectic reduced space at a regular value of \( \phi \). Hence, we are in the situation as described in the Duistermaat–Heckman theorem. In particular we obtain the following lemma.

Lemma 3.7. Let \( S^1 \) act on \( \mathbb{C}^{k+1} \) \((k > 0)\) with weight vector \( w = (w_1, \ldots, w_{k+1}) \), where \( w_i \in \mathbb{N} \) for \( i = 1, \ldots, k + 1 \), and \( \gcd(w_1, \ldots, w_{k+1}) = 1 \). Let \( S^{2k+1} \subset \mathbb{C}^{k+1} \) and let \( \mathbb{C}P^k_w = S^{2k+1}/S^1 \) be the weighted projective space. Let \( m = \text{lcm}(w_1, \ldots, w_{k+1}) \), and let \( e \) be the Euler class of the principal bundle
\[
S^1/Z_m \hookrightarrow S^{2k+1}/Z_m \to \mathbb{C}P^k_w.
\]
Then in \( H^2(\mathbb{C}P^k_w; \mathbb{R}) \) we have
\[
e = \pm \frac{1}{m} t,
\]
where \( t \in H^2(\mathbb{C}P^k_w; \mathbb{Z}) \) is a generator.

Proof. Let us consider first the bundle as a principal \( S^1 = \mathbb{R}/\mathbb{Z} \)-bundle via the isomorphism \( S^1 \to S^1/Z_m \), and compute its Euler class in \( H^2(\mathbb{C}P^k_w; \mathbb{Z}) \). Kawasaki’s computation \cite{17} of the integral cohomology of \( S^{2k+1}/Z_m \) gives \( H^2(S^{2k+1}/Z_m; \mathbb{Z}) = 0 \). In the Gysin sequence
\[
H^0(\mathbb{C}P^k_w, \mathbb{Z}) \to H^2(\mathbb{C}P^k_w, \mathbb{Z}) \to H^2(S^{2k+1}/Z_m; \mathbb{Z})
\]
the first map sends 1 to the Euler class and must be surjective. Thus the Euler class is a generator of \( H^2(\mathbb{C}P^k_w, \mathbb{Z}) \). Considering our original \( S^1/Z_m = \mathbb{R}/\mathcal{N}' \)-bundle, it follows that the Euler class \( e \) of \( (3.8) \) defines a generator in \( H^2(\mathbb{C}P^k_w; \mathcal{N}') \). Here \( \mathcal{N}' \) is the lattice of \( S^1/Z_m \). As elements of \( H^2(\mathbb{C}P^k_w; \mathbb{R}) \), \( e \) is \( \frac{1}{m} \) times a generator \( \pm t \) of \( H^2(\mathbb{C}P^k_w; \mathbb{Z}) \), where \( Z \) is the lattice of \( S^1 \). Hence, we have our conclusion. \( \square \)

3.4. Consequences of our assumptions.

In this section, we look at several essential ingredients for the proof of Theorem 4 which are consequences of our assumptions.

First, as we saw in Theorem 3 the circle action was classified in \cite{26} into two cases. Now we state the two cases in terms of weights of the \( S^1 \) action on the normal bundles of the fixed sets, \( X \) and \( Y \), where \( \phi(X) < \phi(Y) \).
Lemma 3.9. Under Assumption 1.2, $S^1$ acts on the normal bundles $N_X$ of $X$ and $N_Y$ of $Y$ as in the following two cases:

(A) $S^1$ acts on the fiber of $N_X$ with weights $(1, \ldots, 1)$, and acts on the fiber of $N_Y$ with weights $(-1, \ldots, -1)$;

(B) $\dim(X) = \dim(Y)$, and $S^1$ acts on the fiber of $N_X$ with weights $(2, \ldots, 2, 1)$ and acts on the fiber of $N_Y$ with weights $(-2, \ldots, -2, -1)$.

Let $\dim(X) = 2i$ and $\dim(Y) = 2j$. Since $2j + 2 = \dim(M) - \dim(X)$, the normal bundle $N_X$ of $X$ has complex rank $j + 1$. Hence $N_X$ is $S^1$-equivariantly diffeomorphic to a $C^{j+1}$ bundle over $X$, with the circle acting on $N_X$ fiberwise on $C^{j+1}$. By Lemma 3.9, $S^1$ acts on $C^{j+1}$ with either weight vector $w = (1, \ldots, 1)$ or weight vector $w = (2, \ldots, 2, 1)$. Let $r$ be a fixed regular value of $\phi$, then $\phi^{-1}(r)$ is $S^1$-equivariantly diffeomorphic to an $S^{2j+1}$ bundle over $X$, and hence $M_r = \phi^{-1}(r)/S^1$ is diffeomorphic to a $\mathbb{CP}^j$ bundle over $Y$. Similarly, by looking at the normal bundle $N_Y$ of $Y$, $M_r$ is diffeomorphic to a $\mathbb{CP}^j$ bundle over $Y$.

Now, let us consider the $S^1$-orbibundle

$$S^1 \hookrightarrow \phi^{-1}(r) \rightarrow M_r.$$

If we restrict this bundle to the fiber $\mathbb{CP}^j$ of $M_r$, viewed as a bundle over $X$, we have the Hopf fibration or the “weighted Hopf fibration”

$$S^1 \hookrightarrow S^{2j+1} \rightarrow \mathbb{CP}^j.$$

Lemma 3.10 below follows from Lemma 3.7

Lemma 3.10. In the case of Lemma 3.9, let $\dim(Y) = 2j$, let $w = (w_1, \ldots, w_{j+1})$ be the weight vector of the $S^1$ action on $N_X$ and let $m = \text{lcm}(w_1, \ldots, w_{j+1})$. For a fixed value $r \in (\phi(X), \phi(Y))$, let $e$ be the Euler class of the principal bundle

$$(3.11) \quad S^1/\mathbb{Z}_m \hookrightarrow \phi^{-1}(r)/\mathbb{Z}_m \rightarrow M_r.$$

Then

$$e|_{\mathbb{CP}^j} = \pm t \quad \text{in case (A), and} \quad e|_{\mathbb{CP}^j} = \pm \frac{1}{2} t \quad \text{in case (B)},$$

where $t \in H^2(\mathbb{CP}^j; \mathbb{Z})$ is a generator.

We will also need the integral cohomology ring of $Y$ and the value $\phi(Y) - \phi(X)$. See Propositions 6.1 and 6.15 and Lemmas 8.3 and 8.4 in [20].

Lemma 3.12. Under Assumption 1.2, assume in addition that $[\omega] \in H^2(M; \mathbb{R})$ is a primitive integral class. Then

$$H^*(Y; \mathbb{Z}) = \mathbb{Z}[v]/v^{j+1}, \quad \text{where } v = [\omega]_Y \text{ and } 2j = \dim(Y).$$

Furthermore, $\phi(Y) - \phi(X) = 1$ when the action is semifree; and $\phi(Y) - \phi(X) = 2$ otherwise.

3.5. Proof of Theorem 1.

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. First of all, by the assumption on $X$ and $Y$ and the compactness of $M$, $M$ has to be connected.

Next, we show simply connectedness. Since by Theorem 4, we have $H^2(M; \mathbb{Z}) = \mathbb{Z}$, then, modulo rescaling, we may assume that $[\omega]$ is a primitive integral class.
Let $a$ and $b$ be any two values such that $\phi(X) < a < b < \phi(Y)$. Let $\omega_a$ and $\omega_b$ be the reduced symplectic forms in $M_a$ and in $M_b$ respectively. By Theorem 3 (3.13)

\[ [\omega_b] - [\omega_a] = e \cdot (b - a), \]

where $e \in H^2(M_r)$ is the Euler class as in Lemma 3.10. Let $2i = \dim(X)$ and $2j = \dim(Y)$. Since $M_r$ is a bundle over $X$ and is also a bundle over $Y$, let

\[ p_1: M_r \to X \quad \text{and} \quad p_2: M_r \to Y \]

be the projection maps. Let $u = [\omega|_X]$ and $v = [\omega|_Y]$. By continuity of the classes of the reduced symplectic forms, 

\[ p_1^*(u) = \lim_{a \to \phi(X)} [\omega_a] \quad \text{and} \quad p_2^*(v) = \lim_{b \to \phi(Y)} [\omega_b]. \]

Taking limit in (3.13), we obtain (3.14)

\[ p_2^*(v) - p_1^*(u) = e \cdot (\phi(Y) - \phi(X)). \]

Restricting (3.14) to the fiber $\mathbb{CP}^j$ of $M_r$, viewed as a bundle over $X$, we get

\[ p_2^*(v)|_{\mathbb{CP}^j} = e|_{\mathbb{CP}^j} \cdot (\phi(Y) - \phi(X)). \]

By Lemma 3.10 on $e|_{\mathbb{CP}^j}$ and Lemma 3.12 on $\phi(Y) - \phi(X)$, we get

\[ p_2^*(v)|_{\mathbb{CP}^j} = \pm t, \]

where $t \in H^2(\mathbb{CP}^j; \mathbb{Z})$ is a generator. Hence

\[ p_2^*(v^j)|_{\mathbb{CP}^j} = \pm t^j. \]

Let

\[ f = p_2 \circ \iota: \mathbb{CP}^j \to M_r \xrightarrow{p_2} Y. \]

Then (3.15) is

\[ f^*(v^j) = \pm t^j. \]

By Lemma 3.12, $v^j \in H^{2j}(Y; \mathbb{Z})$ is a generator. By Lemmas 3.9 and 3.14 $H^*(\mathbb{CP}^j; \mathbb{Z}) = \mathbb{Z}[t]/t^{j+1}$, in particular, $t^j \in H^{2j}(\mathbb{CP}^j; \mathbb{Z})$ is a generator. Moreover, Lemmas 3.9 and 3.14 also imply that $\pi_1(\mathbb{CP}^j) = 1$. Now, Lemma 3.1 allows us to conclude

\[ \pi_1(Y) = 1. \]

We can similarly prove

\[ \pi_1(X) = 1. \]

Decompose

\[ M = \phi^{-1}(-\infty, r + \epsilon) \cup \phi^{-1}(r - \epsilon, +\infty). \]

Clearly, $\phi^{-1}(-\infty, r + \epsilon)$ deformation retracts to $X$, and $\phi^{-1}(r - \epsilon, +\infty)$ deformation retracts to $Y$. Moreover, the intersection of the two open sets is homotopy equivalent to $\phi^{-1}(r)$, which is connected (we saw that it is diffeomorphic to an $S^{2j+1}$-bundle over $X$). By the Seifert–van Kampen theorem,

\[ \pi_1(M) \cong \pi_1(X) \ast_{\pi_1(\phi^{-1}(r))} \pi_1(Y) = 1. \]

\hfill \Box
Remark 3.17. For the last step of the proof, one can also refer to [23], where the first author proves that, for every Hamiltonian circle action on a connected compact symplectic manifold $(M, \omega)$ one has

$$\pi_1(M) \cong \pi_1(\text{minimum of } \phi) \cong \pi_1(\text{maximum of } \phi).$$

4. Proof of Theorem 2

We need Lemma 4.1 below to prove Theorem 2.

Lemma 4.1. Let $M$ be a compact simply-connected $2n$-dimensional manifold with cohomology ring $H^*(M; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$. Then there is a homotopy equivalence $\tilde{f}: M \to \mathbb{C}P^n$ such that $\tilde{f}^*(a) = x$, where $a \in H^2(\mathbb{C}P^n; \mathbb{Z})$ is a generator.

Proof. Since $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$ space, there is up to homotopy a unique map $f: M \to \mathbb{C}P^\infty$ such that $f^*(t) = x$, where $t \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$ is the standard generator [14, Theorem 4.57]. Since $\dim M = 2n$, we may assume that $f$ factors through $\tilde{f}$:

$$\begin{array}{ccc}
\mathbb{C}P^n & \xrightarrow{i} & \mathbb{C}P^\infty \\
\downarrow \tilde{f} & & \downarrow f \\
M & \xrightarrow{f} & \mathbb{C}P^\infty.
\end{array}$$

Letting $a = i^*(t)$, then $H^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[a]/(a^{n+1})$. Since $\tilde{f}^*i^* = f^*$, it follows that $\tilde{f}^*(a) = x$ and that $\tilde{f}^*$ is an isomorphism. Since all homology groups of $M$ and $\mathbb{C}P^n$ are finitely generated, the universal coefficient theorem shows that $\tilde{f}$ also induces an isomorphism on all (integral) homology groups. Since $M$ and $\mathbb{C}P^n$ are simply connected, by the Whitehead Theorem [14, Corollary 4.33], the map $\tilde{f}$ is a homotopy equivalence. (Here we use that every topological manifold has the homotopy type of a CW-complex. See [5, Theorem V.1.6].) \qed

We recall that a homotopy complex projective space is said to have standard Pontryagin classes if there exists a homotopy equivalence to $\mathbb{C}P^n$ which preserves the Pontryagin classes of the tangent bundles.

Proof of Theorem 2. By Theorem 1, $X$, $Y$ and $M$ are one-connected. The claim for $X$ follows from [2,2] and Lemma 4.1. The claim for $Y$ and for $M$ follows similarly by using [2,3] and [2,4]. In both cases the total Chern class is standard, so the total Pontryagin class is standard as well. \qed

5. Finiteness up to $S^1$-equivariant diffeomorphism — Proof of Theorem 3

In this section, we prove Theorem 3. We only need to restrict attention to the case when $\dim(M) = 2n \geq 6$ since, as mentioned in the Introduction, when $\dim(M) = 2$ or $4$, the manifold $M$ is unique up to equivariant symplectomorphism.
5.1. Reducing the proof of Theorem 3 to an equivariant pseudo-isotopy classification.

In this section, we will see that we can reduce the proof of Theorem 3 to the classification of equivariant pseudo-isotopy classes of equivariant self-diffeomorphisms of the sphere bundle of an extremum of the moment map. We mostly use geometric topology techniques. So we assume that:

- $M$ is a $2n$-dimensional compact smooth manifold with a smooth $S^1$-action,
- the fixed point set has two connected components $X$ and $Y$, with $\dim(X) = 2i$ and $\dim(Y) = 2(n - i - 1)$, and
- there is a Morse-Bott function $\phi: M \to \mathbb{R}$ with critical set $X \cup Y$.

In the following paragraph, we summarize the facts we know and will use about $M, X, Y$ and the circle action. We use $S$ to denote the sphere bundle of the normal bundle of $X$ and we denote by $P$ the quotient $S/S^1$.

The manifold $M$ is simply connected, and $X$ and $Y$ are homotopy complex projective spaces with standard Chern classes (hence also standard Pontryagin classes) (Theorem 1 and Theorem 2). Moreover, by Theorem 3, we have the following two cases.

Case (A), the action is semi-free. In this case, the fiber of the equivariant normal bundle of $X$ is an $(n - i)$-dimensional complex free $S^1$-representation space, so $S^1$ acts on $S$ freely, and the quotient $P$ is a smooth $\mathbb{C}P^{n-i-1}$-bundle over $X$. Moreover, the Chern classes of the normal bundles of $X$ and of $Y$ are uniquely determined.

Case (B), the action is not semifree. In this case, $\dim(X) = \dim(Y) = 2i$, $n = 2i + 1 \geq 3$ is odd. The equivariant normal bundle of $X$ splits into a direct sum $N_1 \oplus N_2$, where the fibers of $N_1$ and of $N_2$ are respectively of complex dimensions 1 and $n - i - 1$. The Chern classes of $N_1$ and of $N_2$ are uniquely determined. The circle acts on the fiber of $N_1$ with weight 1, and acts on the fiber of $N_2$ with weights $(2, \ldots, 2)$ (see Lemma 3.9). So the sphere bundle $S$ has two connected smooth strata: the bottom stratum $S_0 = S(N_2)$ consisting of all points with stabilizer group $\mathbb{Z}_2$, and the top stratum $S \setminus S_0$ consisting of all points with trivial stabilizer group. The quotient $P = S_0/S^1 \cup ((S \setminus S_0)/S^1) = P_0 \cup (P \setminus P_0)$ is a bundle over $X$ with fiber $\mathbb{C}P^{n-i-1}$, By Lemma 3.4 this fiber is homeomorphic to $\mathbb{C}P^{n-i-1}$. More precisely, the pair $(P, P_0)$ fibers over $X$ with fiber pair homeomorphic to $(\mathbb{C}P^{n-i-1}, \mathbb{C}P^{n-i-2})$. So $P$ and $P_0$ are topological projective bundles over a homotopy complex projective space $X$: in particular, $P \setminus P_0$ is a bundle over $X$ with contractible fiber $\mathbb{C}P^{n-i-1} \setminus \mathbb{C}P^{n-i-2}$, so it is homotopy equivalent to $X$. Note that the spaces $P, P_0$ and $P \setminus P_0$ are simply-connected, and $\dim(P_0) = 2n - 4$ and $\dim(P \setminus P_0) = 2n - 2$.

First, we have finiteness up to equivariant diffeomorphism of the tubular neighborhoods of $X$ and $Y$:

**Lemma 5.1.** If Assumption 1 holds, then up to $S^1$-equivariant diffeomorphism, tubular neighborhoods of $X$ and of $Y$ are determined up to finite ambiguity.

**Proof.** Since $X$ and $Y$ are homotopy complex projective spaces with standard Pontryagin classes, by Proposition 1 there are finitely many diffeomorphism types of $X$ and $Y$ if they are not 4-dimensional. If $X$ or $Y$ is 4-dimensional, since it is a symplectic homotopy $\mathbb{C}P^2$ with standard 1st Chern class, by a theorem of Taubes...
Corollary 7.2, it is symplectomorphic to the standard \( \mathbb{CP}^2 \). (This is the only place in Section 5 where we directly use the symplectic structure.)

By Theorem 4, the Chern classes of the normal bundles of \( X \) and of \( Y \) are uniquely determined in the above two cases. Then Lemma A.5 implies that the normal bundles of \( X \) and of \( Y \) are determined up to finite ambiguity. Since the circle acts in the above two ways, we have finitely many possibilities up to equivariant diffeomorphism for the tubular neighbourhoods of \( X \) and \( Y \). □

Since the Morse-Bott function \( \phi \) has no more critical sets, the manifold \( M \) is obtained by gluing the tubular neighbourhoods of \( X \) and of \( Y \). Hence to prove Theorem 3, it is enough to show that there are only finitely many “different” ways to glue the two tubular neighborhoods. A gluing is an equivariant diffeomorphism \( \varphi \) between the sphere bundle \( S_X \) of \( X \) and the sphere bundle \( S_Y \) of \( Y \). The result of the gluing is the manifold \( D_X \cup \varphi D_Y \) obtained from the disjoint union of the disk bundles \( D_X \) and \( D_Y \) of \( X \) and \( Y \), (identified with their tubular neighborhoods) by identifying the two boundaries via \( \varphi \). Hence, to prove Theorem 3 we only need to prove Propositions 5.3 and 5.4 below.

**Definition 5.2.** Let \( Z, Z' \) be two manifolds and let \( \varphi_0, \varphi_1 : Z \to Z' \) be two diffeomorphisms. A pseudo-isotopy between \( \varphi_0 \) and \( \varphi_1 \) is a diffeomorphism \( \Phi : Z \times I \to Z' \times I \) such that \( \Phi(z,0) = (\varphi_0(z),0) \) and \( \Phi(z,1) = (\varphi_1(z),1) \) for \( z \in Z \). If \( Z, Z' \) are \( S^1 \)-manifolds, and \( \varphi_0, \varphi_1 : Z \to Z' \) are two \( S^1 \)-equivariant diffeomorphisms, an \( S^1 \)-equivariant pseudo-isotopy between \( \varphi_0 \) and \( \varphi_1 \) is an \( S^1 \)-equivariant diffeomorphism \( \Phi : Z \times I \to Z' \times I \) such that \( \Phi(z,0) = (\varphi_0(z),0) \) and \( \Phi(z,1) = (\varphi_1(z),1) \) for \( z \in Z \). Here, \( S^1 \) acts on the first factors on \( Z \times I \) and on \( Z' \times I \).

Proposition 5.3 below was essentially proved by K. Wang, and we refer to [45, Proposition 1.1]. The proof applies to both semifree and non-semifree actions.

**Proposition 5.3.** Let \( \varphi_0, \varphi_1 : S_X \to S_Y \) be two equivariantly pseudo-isotopic diffeomorphisms. Then there exists an equivariant diffeomorphism

\[
D_X \cup_{\varphi_0} D_Y \cong D_X \cup_{\varphi_1} D_Y
\]

between the results of the two gluings.

Since any two diffeomorphisms between the sphere bundles differ by a self-diffeomorphism of one of the sphere bundles, we consider instead the group of pseudo-isotopy classes of equivariant self-diffeomorphisms of one of the sphere bundles, say, over \( X \), the minimum of \( \phi \).

**Proposition 5.4.** The group of equivariant pseudo-isotopy classes of equivariant diffeomorphisms \( S \to S \) is finite when \( \dim(M) = 2n \geq 6 \).

We will prove Proposition 5.4 in the following three subsections.

**Remark 5.5.** For us, the existence of a decomposition \( D_X \cup_{\varphi} D_Y \cong M \) is sufficient. Hausmann and Holm show that given the manifold \( M \) with its symplectic structure and circle action, one can control the choices in the tubular neighbourhood embeddings \( D_X \to M, D_Y \to M \) so that the diffeomorphism \( \varphi \) is determined by \( M \) up to isotopy and certain gauge transformations [15, Lemma 4.3].
5.2. The case when \( \dim(M) = 2n > 6 \) and the action is semifree.

The proof of Proposition 5.4 for this case was essentially given in K. Wang’s paper [45]. Because of some minor inaccuracies in Wang’s proof, and since we want to extend the argument to non-semifree actions, we will give a description of his whole argument in this section.

When the \( S^1 \) action is semifree, an equivariant self-diffeomorphism of \( S \) induces a self-diffeomorphism of the smooth quotient \( P \). The map \( \pi : S \to P \) is a principal \( S^1 \)-bundle and gives rise to an Euler class in \( H^2(P) \). A self-diffeomorphism of \( P \) induced by an equivariant diffeomorphism of \( S \) preserves the Euler class.

On the other hand, a diffeomorphism \( P \to P \) which preserves the Euler class can be lifted to an equivariant diffeomorphism \( S \to S \). The lift is not unique, but two such lifts \( f_0, f_1 \) differ by an element \( g \in \text{Maps}(P, S^1) \), i.e., for \( x \in S \), one has \( f_1(x) = f_0(x) \cdot g(\pi(x)) \). Since \( P \) is simply-connected, there is a homotopy \( H : P \times I \to S^1 \) from \( g \) to the constant map. Then \( \psi : S \times I \to S \times I \), defined by \( \psi(x, t) = (f_0(x) \cdot H(\pi(x), t), t) \) is an isotopy between \( f_0 \) and \( f_1 \). Thus a self-diffeomorphism of \( P \) which preserves the Euler class can be lifted to a unique pseudo-isotopy class of equivariant self-diffeomorphisms of \( S \).

In the following statements, we will always assume the circle action to be semifree (\( P \) is smooth), we will however indicate the dimension of the manifold for the statements to hold.

**Lemma 5.6.** Let \( \dim(M) = 2n \). For any \( n \), in the group of self-diffeomorphisms of \( P \) which preserve the Euler class \( e \) of the circle bundle \( S \to P \), the subgroup \( \text{of} \) self-diffeomorphisms inducing the identity on \( H^*(P; \mathbb{Z}) \) has finite index (which equals 1 or 2).

**Proof.** Suppose \( P \) is a projective bundle over a homotopy complex projective space of dimension \( 2i \). Let \( x \in H^2(P; \mathbb{Z}) \) be the image of the generator of \( H^2 \) of the homotopy complex projective space. Then by the theorem of the projective bundle,

\[
H^*(P; \mathbb{Z}) \cong \mathbb{Z}[e, x]/\langle x^{i+1}, p(e, x) \rangle,
\]

where the polynomial \( p \) is defined by \( p(e, x) = \sum_k e^{-i-k}(1)^k c_k x^k \) for some \( c_k \in \mathbb{Z} \). Thus a \( \mathbb{Z} \)-basis of \( H^*(P; \mathbb{Z}) \) is given by the monomials \( e^l x^k \) with \( 0 \leq l \leq n-i-1 \), \( 0 \leq k \leq i \). Any diffeomorphism \( f \) must induce an automorphism of \( H^2(P; \mathbb{Z}) \). Since \( f^*(e) = e \), we have \( f^*(x) = \pm x + \lambda e \) for some \( \lambda \in \mathbb{Z} \). Up to maybe restricting to a subgroup of index 2, we may assume \( f^*(x) = x + \lambda e \).

It follows that in \( H^*(P; \mathbb{Z}) \) we have \( 0 = f^*(x^{i+1}) = f^*(x)^{i+1} = (x + \lambda e)^{i+1} \) and similarly \( 0 = p(e, x + \lambda e) \). In the polynomial ring \( \mathbb{Z}[e, x] \), we obtain the equality

\[
\langle x^{i+1}, p(e, x) \rangle = \langle (x + \lambda e)^{i+1}, p(e, x + \lambda e) \rangle.
\]

We leave it to the reader to prove that this implies \( \lambda = 0 \). (One can consider the cases \( \deg(p) > i + 1 \), \( \deg(p) = i + 1 \), \( \deg(p) < i + 1 \).) It follows that \( f^* = \text{id} \). \( \square \)

**Lemma 5.7.** Let \( \dim(M) = 2n \). For any \( n \), the group of homotopy classes of self-homotopy equivalences of \( P \) inducing the identity on cohomology is finite.

K. Wang refers to an argument of Kahn, but we give a short proof here using obstruction theory. Recall its setting [14, pages 415ff]: Given two CW-complexes \( A, B \), and a map from a subcomplex \( A_0 \) of \( A \) to \( B \), does this map extend to a map.
\( A \to B \)? For simplicity we assume \( B \) is simply-connected. Obstruction theory says such an extension exists if a sequence of obstruction classes

\[
\alpha_j \in H^{j+1}(A_j; \pi_j(B)) \quad \text{for } j = 2, 3, \ldots
\]

vanishes. However, only the first obstruction (the \( \alpha_2 \)) where \( j = 1 \) is minimal such that \( \pi_j(B) \neq 0 \) is well-defined and depends only on the map \( A_0 \to B \); the higher obstructions \( \alpha_j \) are only defined if the lower obstructions vanish, and they depend on choices. Let \( A^{(j)} \) denote the \( j \)-skeleton of \( A \). Then \( \alpha_j \) depends on the choice of an extension of \( A_0 \to B \) to \( A_0 \cup A^{(j-1)} \) which can be extended to \( A_0 \cup A^{(j)} \); if the corresponding obstruction \( \alpha_j \) vanishes, then it can be extended to \( A_0 \cup A^{(j+1)} \).

**Proof.** Let \( \mathcal{H} = \mathcal{H}(\mathbb{P}) \) denote the group of homotopy classes of self-homotopy equivalences of \( \mathbb{P} \) inducing the identity on cohomology. For each \( j \), let \( \mathcal{H}_j \) be the group of those homotopy classes of self-homotopy equivalences \( \varphi: \mathbb{P} \to \mathbb{P} \) inducing the identity on cohomology, such that there exists a homotopy between the restrictions of \( \varphi \) and \( \varphi \) to \( \mathbb{P}^{(j)} \). Obviously \( \mathcal{H}_j \subseteq \mathcal{H}_{j-1} \) for all \( j \), and \( \mathcal{H}_{\dim(\mathbb{P})} \) is the trivial group.

Obstructions for a homotopy between two maps \( \mathbb{P} \to \mathbb{P} \) lie in \( H^1(\mathbb{P}; \pi_j(\mathbb{P})) \). (We have \( A = \mathbb{P} \times I, A_0 = \mathbb{P} \times \partial I, B = \mathbb{P} \) in the notation above.) Since the maps are equal on \( H^1(\mathbb{P}; \mathbb{Z}) \), the first obstruction in \( H^1(\mathbb{P}; \pi_2(\mathbb{P})) \) is 0, and one has a homotopy between the restriction of the maps to \( \mathbb{P}^{(2)} \). Thus \( \mathcal{H} = \mathcal{H}_2 \). The statement of the lemma will follow from the statement that for all \( j > 2 \), the index of \( \mathcal{H}_j \) in \( \mathcal{H}_{j-1} \) is finite, which we will now prove.

For \( j > 2 \) the group \( H^2(\mathbb{P}; \pi_j(\mathbb{P})) \) is finite since the cohomology of \( \mathbb{P} \) is concentrated in even degrees and the other even homotopy groups of \( \mathbb{P} \) are finite. (This uses the long exact sequences in homotopy for \( S^1 \to S^{2k+1} \to \mathbb{CP}^k \) and \( \mathbb{CP}^k \to \mathbb{P} \to \mathbb{CP}^j \).) There is no well-defined map \( \mathcal{H}_{j-1} \to H^2(\mathbb{P}; \pi_j(\mathbb{P})) \), since the obstruction to the existence of a homotopy over \( \mathbb{P}^{(j)} \) depends on the choice of a homotopy over \( \mathbb{P}^{(j-1)} \) which extends to \( \mathbb{P}^{(j-2)} \). However, we can associate to each element \( \varphi \in \mathcal{H}_{j-1} \) the subset \( O_j(\varphi) \subseteq H^2(\mathbb{P}; \pi_j(\mathbb{P})) \) consisting of all obstruction classes for all such choices. Then \( \mathcal{H}_j = \{ \varphi \in \mathcal{H}_{j-1} \mid 0 \in O_j(\varphi) \} \).

Now assume that \( \alpha_2 \in O_j(\varphi_1) \cap O_j(\varphi_2) \), corresponding to homotopies \( h_1 \) respectively \( h_2 \) by the restrictions of \( \varphi_1 \) and \( \varphi_2 \) respectively \( \varphi_2 \). Then the homotopies \( h_1 \) and \( h_2 \) can be combined to a homotopy \( h \) between the restrictions of \( \varphi_1 \) and \( \varphi_2 \) to \( \mathbb{P}^{(j-2)} \) which extends to \( \mathbb{P}^{(j-1)} \), and the obstruction for extending \( h \) to \( \mathbb{P}^{(j)} \) is 0. This follows from an additivity formula for the obstruction classes, see [3, Theorem 4.2.7]. By composing with \( \varphi_1^{-1} \) we see that \( 0 \in O_j(\varphi_1^{-1} \varphi_2) \) and that \( \varphi_1^{-1} \varphi_2 \in \mathcal{H}_j \).

We have shown that if \( \alpha_2 \in O_j(\varphi_1) \cap O_j(\varphi_2) \), then \( \varphi_1 \) and \( \varphi_2 \) are contained in the same coset modulo \( \mathcal{H}_j \). This shows that in \( \mathcal{H}_{j-1} \) there are at most as many cosets modulo \( \mathcal{H}_j \) as elements in \( H^1(\mathbb{P}; \pi_j(\mathbb{P})) \), so that \( \mathcal{H}_j \subseteq \mathcal{H}_{j-1} \) has finite index. \( \Box \)

**Proposition 5.9.** For \( \dim(\mathbb{P}) = 2n - 2 > 4 \), the group \( D_0(\mathbb{P}) \) of pseudo-isotopy classes of self-diffeomorphisms of \( \mathbb{P} \) which are homotopic to the identity is finite.

**Proof.** The argument involves the surgery exact sequence, and is parallel to [16], Sections 1 and 2. For the surgery exact sequence in general the reader may consult [44] or [29].

There is a map from the so-called structure set \( S(\mathbb{P} \times I, \partial) \) to \( D_0(\mathbb{P}) \) (in fact a group homomorphism): an element in the structure set is represented by a homotopy equivalence of manifolds with boundary \( H: K \to \mathbb{P} \times I \) which restricts to a diffeomorphism of the boundaries. Since \( \dim(\mathbb{P}) > 4 \), we can apply the \( s \)-cobordism
theorem: we may assume that $K = \mathbb{P} \times I$, and furthermore we may assume that $H|_{\mathbb{P} \times \{0\}}$ is the identity. Now the map $S(\mathbb{P} \times I, \partial) \to D_0(\mathbb{P})$ is defined by sending $H \mapsto H|_{\mathbb{P} \times \{1\}}$. K. Wang shows that the map is well-defined and surjective [66].

We also need the following surgery obstruction groups: the group $L_{2n-1}(1)$ is trivial, and $L_{2n}(1)$ is cyclic of infinite order or order 2 (depending on whether $n$ is even or odd). Moreover $[\Sigma^2 \mathbb{P}, G/O]$ (respectively $[\Sigma^2 \mathbb{P}, G/O]$) is the group of homotopy classes of maps from the suspension of $\mathbb{P}$ (respectively double suspension) to a certain space $G/O$.

For $\dim(\mathbb{P}) = 2n - 2 > 4$ there is a diagram:

$$
\begin{align*}
[\Sigma^2 \mathbb{P}, G/O] & \twoheadrightarrow L_{2n}(1) \twoheadrightarrow S(\mathbb{P} \times I, \partial) \twoheadrightarrow [\Sigma \mathbb{P}, G/O] \twoheadrightarrow L_{2n-1}(1) = 0 \\
\downarrow & \\
D_0(\mathbb{P}).
\end{align*}
$$

Here the first row is the surgery exact sequence: since the spaces we consider are products with an interval, this is in our case an exact sequence of groups. We use that $\mathbb{P}$ is simply connected (see beginning of Section 5.1).

By a theorem of Novikov [31, Theorem 2.18], every element in the image of $L_{2n}(1) \to S(\mathbb{P} \times I, \partial)$ is of the form

$$
\mathbb{P} \times I \# \Sigma^{2n-1} \xrightarrow{id\#c} \mathbb{P} \times I \# \Sigma^{2n-1} = \mathbb{P} \times I,
$$

where $\#$ is connected sum, $\Sigma^{2n-1}$ is an exotic sphere, and $c: \Sigma^{2n-1} \to S^{2n-1}$ is a homeomorphism. Since the group of exotic spheres is finite in dimension at least 5, the image of $L_{2n}(1)$ in $S(\mathbb{P} \times I, \partial)$ is finite.

We need the following facts about $G/O$: it is an infinite loop space with finite odd homotopy groups, and $\pi_{4k}(G/O) \otimes \mathbb{Q} \cong \mathbb{Q}$. See for example [31, p.215], [29, Theorem 6.48]. Since $\Sigma \mathbb{P}$ has vanishing even reduced cohomology groups, the Atiyah–Hirzebruch spectral sequence shows that $[\Sigma \mathbb{P}, G/O]$ is finite. It follows that $S(\mathbb{P} \times I, \partial)$ and $D_0(\mathbb{P})$ are also finite. \hfill $\square$

**Proof of Proposition 5.4** (for the case when $\dim(M) = 2n > 6$ and the action is semifree.) In this case $S^1$ acts freely on $\mathcal{S}$, and we saw in the beginning of this subsection that pseudo-isotopy classes of equivariant self-diffeomorphisms of $\mathcal{S}$ correspond exactly to pseudo-isotopy classes of self-diffeomorphisms of $\mathbb{P}$ which preserve the Euler class of the bundle $\mathcal{S} \to \mathbb{P}$. Let $\mathcal{D}(\mathbb{P})$ be the group of pseudo-isotopy classes of self-diffeomorphisms of $\mathbb{P}$ which act as identity on $H^*(\mathbb{P}; \mathbb{Z})$. By Lemma 5.6 it suffices to prove that $\mathcal{D}(\mathbb{P})$ is finite. Let $\mathcal{H}(\mathbb{P})$ be the group of homotopy classes of self-homotopy equivalences of $\mathbb{P}$ which act as identity on $H^*(\mathbb{P}; \mathbb{Z})$.

We have a natural map $\varphi: \mathcal{D}(\mathbb{P}) \to \mathcal{H}(\mathbb{P})$ with $\ker(\varphi) = D_0(\mathbb{P})$. By Lemma 5.7 and Proposition 5.3 the groups $\mathcal{H}(\mathbb{P})$ and $\ker(\varphi)$ are finite, thus $\mathcal{D}(\mathbb{P})$ is finite. \hfill $\square$

5.3. **The case when $\dim(M) = 2n > 6$ and the action is not semifree.**

In this section, we prove Proposition 5.4 for the case when $\dim(M) = 2n > 6$ and the action is not semifree. Since $n$ is odd in this case, we have $\dim(M) = 2n \geq 10$. For the case of semifree actions, we worked with maps between the quotients of the group actions, and we used the $s$-cobordism theorem in the proof of Proposition 5.4. For the case of non-semifree actions, the quotients are not canonically smooth manifolds, so it is more convenient not to work with the quotients but...
to consider equivariant maps instead. However the $s$-cobordism theorem does not
generalize to the category of all equivariant maps, but only to the smaller category
of isovariant maps, which we describe below. The appropriate generalization of the
surgery theory argument to the case of several strata in Proposition 5.12 below uses
the Browder–Quinn isovariant surgery exact sequence. The original source for the
isovariant surgery theory is the article by Browder and Quinn [6]; Weinberger [47]
gives an outline of the proofs of the statements; some aspects are discussed in great
detail by Dovermann and Schultz [9].

An equivariant map is isovariant if it preserves stabilizer groups of points; it is
transverse linear isovariant if it is isovariant and is transverse to the lower strata.
Every equivariant diffeomorphism is transverse linear isovariant. (In the one stratum
case, all equivariant maps are transverse linear isovariant.)

In the following, a homotopy equivalence $f : S_1 \to S_2$ is a homotopy equivalence
in the category of transverse linear isovariant maps, i.e., $f$ is transverse linear
isovariant, there exists $g : S_2 \to S_1$ which is transverse linear isovariant, and, there
exist transverse linear isovariant homotopies such that $g \circ f \simeq \text{id}$ and $f \circ g \simeq \text{id}$.

**Remark 5.11.** For the general Browder-Quinn theory, one has to restrict further
to maps whose restrictions to strata of dimension less than 5 are equivariant diffeo-
morphisms, and homotopies whose restrictions to such strata are constant. In our
case when $\dim(M) = 2n \geq 10$, the bottom stratum $S_0$ of $S$ has dimension at least
7, and the bottom stratum $P_0$ of $P$ has dimension at least 6, so these restrictions
do not apply.

Corresponding to this case, our surgery argument which replaces Proposition 5.9
concerns the group $D_0^{S^1}(S)$ of equivariant pseudo-isotopy classes of self-diffeo-
morphisms of $S$ which are transverse linear isovariantly homotopic to the identity.

**Proposition 5.12.** When $\dim(M) = 2n \geq 10$, $n$ is odd, the group $D_0^{S^1}(S)$ is finite.

**Proof.** Note that since $\dim(M) = 2n \geq 10$, we have $\dim(S) = 2n - 1 \geq 9$.

Again we start by defining the various sets occurring in the isovariant surgery
exact sequence: $[\Sigma P, G/O]$ is the group of homotopy classes of maps from the
suspension of $P$ to $G/O$ as before, so it is a finite set as before. The set $S_0^{S^1}(S \times I, \partial)$
consists of homotopy equivalences $(N, \partial N) \to (S \times I, \partial)$ in the category of transverse
linear isovariant maps, modulo concordance. Since $\dim(S) > 5$, we can apply the
isovariant $s$-cobordism theorem [6]: $N$ is equivariantly diffeomorphic to a cylinder,
and the element is represented by a homotopy equivalence $(S \times I, \partial) \to (S \times I, \partial)$
which is the identity on $S \times 0$. So again we get a surjective map $S_0^{S^1}(S \times I, \partial) \to D_0^{S^1}(S)$. The abelian obstruction groups $L_2^{S^1}(S \times I, \partial)$ can be computed to be:
$L_2^{S^1}(S \times I, \partial) \cong \mathbb{Z} \oplus \mathbb{Z}_2$, $L_2^{S^1}(S \times I, \partial) \cong 0$. One uses an exact sequence from [6] which relates them to the $L$-groups of the strata, which are isomorphic to the
usual non-equivariant $L$-groups of the quotients. These quotients $P_0$ and $P \setminus P_0$ are
simply-connected (see the beginning of Section 5.1). In particular there is the short
exact sequence

$$
0 \longrightarrow L_{2n+1}(\Sigma \setminus S_0) \times I, \partial \longrightarrow L_{2n+1}^S(S \times I, \partial) \longrightarrow L_{2n-1}^S(S_0 \times I, \partial) \longrightarrow 0
$$

$$
\begin{array}{c}
\cong \\
L_{2n}(1) \cong \mathbb{Z}_2
\end{array}
\begin{array}{c}
\cong \\
L_{2n-2}(1) \cong \mathbb{Z}.
\end{array}
$$
In the above, we used the fact that \( n \geq 5 \) is odd for our case of non-semifree actions.

The isovariant surgery exact sequence is the exact sequence of groups in the first row of the diagram

\[
\begin{array}{cccc}
[\Sigma^2 \mathbb{P}, G/O] & \xrightarrow{\sigma} & L_{2n+1}^S(\mathbb{S} \times I, \partial) & \longrightarrow S_0^{S^1}(\mathbb{S} \times I, \partial) \\
\text{inclusion} \quad & & \text{inclusion} \\
[\Sigma^2 \mathbb{P}_0, G/O] & \xrightarrow{\sigma} & L_{2n-1}^S(S_0 \times I, \partial) & \longrightarrow D_0^{S^1}(\mathbb{S}).
\end{array}
\]

Finiteness of \( S_0^{S^1}(\mathbb{S} \times I, \partial) \), and thus of \( D_0^{S^1}(\mathbb{S}) \) now follows from the following lemma.

**Lemma 5.13.** The image of \( L_{2n+1}^S(\mathbb{S} \times I, \partial) \) in \( S_0^{S^1}(\mathbb{S} \times I, \partial) \) is finite.

**Proof.** We show that the image of \([\Sigma^2 \mathbb{P}, G/O]\) in \( L_{2n+1}^S(\mathbb{S} \times I, \partial) \) is infinite. Since \( L_{2n+1}^S(\mathbb{S} \times I, \partial) \rightarrow L_{2n-1}^S(S_0 \times I, \partial) \) has finite kernel, it suffices to show that the composition

\[
[\Sigma^2 \mathbb{P}, G/O] \xrightarrow{i^*} [\Sigma^2 \mathbb{P}_0, G/O] \xrightarrow{\sigma} L_{2n-1}^S(S_0 \times I, \partial) \cong \mathbb{Z}
\]

has nontrivial image. We can compute rationally:

\[
[\Sigma^2 \mathbb{P}, G/O] \otimes \mathbb{Q} \cong \bigoplus_j H^3(\Sigma^2 \mathbb{P}; \pi_j G/O \otimes \mathbb{Q}) ,
\]

\[
[\Sigma^2 \mathbb{P}_0, G/O] \otimes \mathbb{Q} \cong \bigoplus_j H^3(\Sigma^2 \mathbb{P}_0; \pi_j G/O \otimes \mathbb{Q}).
\]

The map \( i^* \) corresponds to the map induced by inclusion

\[
\bigoplus_j H^3(\Sigma^2 \mathbb{P}; \pi_j G/O \otimes \mathbb{Q}) \rightarrow \bigoplus_j H^3(\Sigma^2 \mathbb{P}_0; \pi_j G/O \otimes \mathbb{Q}).
\]

Since \( \mathbb{P}_0 \) is a sub-projective bundle of \( \mathbb{P} \), we can apply the Leray–Hirsch theorem to show that \( i^* \) is rationally surjective. The map \( \sigma \) is the map in the ordinary surgery exact sequence for \( \mathbb{P}_0 \), see [5.11]. We saw that this map must be rationally surjective (using Novikov’s theorem). Thus the lemma follows.

Finally Lemmas 5.6 and 5.7 are replaced by the following result.

**Proposition 5.14.** For \( \dim(M) = 2n > 6 \), \( n \) odd, the subgroup of equivariant diffeomorphisms \( \psi: S \rightarrow S \) such that \( \psi \) is transverse linearly isovariantly homotopic to the identity has finite index in the group of all equivariant self-diffeomorphisms of \( S \).

**Proof.** Let \( \varphi: S \rightarrow S \) be any equivariant diffeomorphism. We try to construct a transverse linear isovariant homotopy from \( \varphi \) to id in three steps. We will see that in the first and third step, we need to restrict to those \( \varphi \) which are in a certain subgroup of finite index to construct the homotopy.

**Step 1:** We construct a homotopy on \( S_0 \). There is a restriction homomorphism \( r: \text{Diff}^S(S) \rightarrow \text{Diff}^S(S_0) \) from the group of equivariant self-diffeomorphisms of \( S \) to the group of equivariant self-diffeomorphisms of \( S_0 \). By Lemmas 5.6 and 5.7 the normal subgroup of \( \text{Diff}^S(S_0) \) consisting of those diffeomorphisms which are equivariantly homotopic to the identity has finite index. It follows that its inverse
image under \( r \) has finite index in \( \text{Diff}^r(S) \). For \( \varphi \) contained in this subgroup, we let \( h: S_0 \times I \to S_0 \) be an equivariant homotopy between \( \text{id} \) and \( r(\varphi) \), which we may assume to be constant for \( t \geq \frac{1}{2} \), i.e. \( h(x, t) = \varphi(x) \) for \( x \in S_0 \) and \( t \geq \frac{1}{2} \).

**Step 2:** We extend the homotopy on \( S_0 \) to a transverse linear isovariant map from a tubular neighbourhood of \( S_0 \times I \) to \( S \). This can be done by Lemma 5.15.

**Step 3:** We extend the map from the neighbourhood of \( S_0 \times I \) to a transverse linear isovariant homotopy \( h': S \times I \to S \) such that \( h'(x, 0) = x \) and \( h'(x, 1) = \varphi(x) \).

By Lemma 5.16 there are finitely many obstructions with values in finite groups against doing this. Again these obstructions depend on some choices (for example choices in steps 1 and 2), but an argument similar to the proof of Lemma 5.7 shows that step 3 can be done for \( \varphi \) contained in a subgroup of finite index.

**Lemma 5.15.** Let \( \varphi: S \to S \) be an equivariant diffeomorphism. Let \( h: S_0 \times I \to S_0 \) be an equivariant homotopy from the identity to the restriction of \( \varphi \) to \( S_0 \) such that \( h(x, t) = \varphi(x) \) for \( x \in S_0 \) and \( t \geq \frac{1}{2} \). Then we can extend \( h \) to a transverse linear isovariant map from a tubular neighbourhood of \( S_0 \times I \) in \( S \times I \) to \( S \).

**Proof.** In order to make the notation less complicated, we replace \( I \) by \([0, 2] = I \cup [1, 2]\) and assume that the homotopy \( h: S_0 \times [0, 2] \to S_0 \) is constant for \( t \geq 1 \).

The codimension 2 invariant submanifolds \( S_0 \subset S \) and \( S_0 \times I \subset S \times I \) have equivariant normal bundles with total spaces \( \nu \) and \( \nu \times I \). By definition the fiber of \( \nu \) at \( x \in S_0 \) is \( T_xS/T_xS_0 \). Let \( g \) be an \( S^1 \)-invariant metric on \( S \). Then \( g_t = (1 - t)g + t\varphi^*g \) is an \( S^1 \)-invariant metric on \( S \times \{t\} \). The metric \( g_t \) identifies the fiber of \( \nu \) at \( x \) with the orthogonal complement (w.r.t \( g \)) of \( T_xS_0 \) in \( T_xS \), thus defines a scalar product on the fibers of \( \nu \) (by restriction of \( g \) to this orthogonal complement). We can thus use the disk bundle \( D_\nu^g(\nu) \) of vectors of norm at most \( \epsilon \). We do similarly for \( \nu \times I \), where in the \( t \)-slice we use the metric \( g_t \).

Let \( \epsilon \) be small such that the exponential map of \( g_t \) in the normal direction in each slice gives tubular neighbourhoods \( N^\nu(\nu)(S_0) \) and \( N^\nu_\nu(S_0 \times I) \) (the union of the \( \epsilon \)-neighbourhoods of \( g_t \) in the \( t \)-slices) and identifications \( \Psi: N^\nu_\nu(S_0) \cong D_\nu^g(\nu) \) and \( \psi: N^\nu_\nu(S_0 \times I) \cong D_\nu^g(\nu \times I) \).

Now if we pull back \( \nu \) via \( h: S_0 \times I \to S_0 \), the pull-back is isomorphic as equivariant vector bundle to \( \nu \times I \). The reason is that \( h \) is equivariantly homotopic to \( \text{pr}_1: S_0 \times I \to S_0 \), so that \( h \circ \nu \equiv \text{pr}_1^*\nu = \nu \times I \). Thus we get a bundle map \( \nu \times I \to \nu \) covering \( h \), which we can assume to be equivariant and orthogonal on each fiber, so it restricts to a map \( D_\nu^g(\nu \times I) \to D_\nu^g(\nu) \), which again restricts to an orthogonal map on each fiber. Composing with \( \psi \) and \( \Psi^{-1} \), we get a map \( H: N^\nu_\nu(S_0 \times I) \to N^\nu_\nu(S_0) \) such that \( H(x, 0) = x \) and \( \Psi(H(x, 1)) \) differs from \( \Psi(\varphi(x)) \) by an equivariant orthogonal bundle automorphism of \( \nu \).

This is an orientable 2-dimensional Euclidean vector bundle, so the group of equivariant bundle automorphisms is isomorphic to \( \text{Maps}(S_0/S^1, SO(2)) \). But since \( S_0/S^1 \) is simply-connected, there is a homotopy to the constant map. This produces a homotopy \( N^\nu_{\nu} \circ (S_0) \times [1, 2] \to N^\nu_\nu(S_0) \) from the restriction of \( H \) to \( \varphi \). The union of this map and \( H \) is a transverse linear isovariant map \( f \) from a tubular neighbourhood of \( S_0 \times [0, 2] \) to a tubular neighbourhood of \( S_0 \). This maps boundaries to boundaries and restricts to the identity in the slice \( t = 0 \), and to \( \varphi \) in the slice \( t = 2 \).

**Lemma 5.16.** Let \( \dim(M) = 2n > 6 \), where \( n \) is odd. Let \( N(S_0 \times I) \) be a tubular neighbourhood of \( S_0 \times I \) in \( S \times I \), and \( F: S \times \partial I \cup N(S_0 \times I) \to S \) be a transverse
linear isovariant map. Then there are finitely many obstructions with values in finite groups for extending $F$ to a transverse linear isovariant map $S \times I \to S$.

**Proof.** To ensure isovariance, it suffices to extend the restriction

$$f : (S \times \partial I \cup N(S_0 \times I)) \setminus (S_0 \times I) \to S \setminus S_0$$

of $F$ to an equivariant map $(S \setminus S_0) \times I \to S \setminus S_0$. On the quotients, we need to extend a map

$$((P \times \partial I \cup N(P_0 \times I)) \setminus (P_0 \times I)) \to P \setminus P_0$$

to a map $(P \setminus P_0) \times I \to P \setminus P_0$. By [5,8], the obstructions for the extension lie in

$$H^{j+1}(P \setminus P_0) \to H^{j+1}(P \times I; P \times \partial I \cup N(P_0 \times I); \pi_j(P \setminus P_0))$$

by excision

$$H^{j+1}(P \times I; P \times \partial I \cup P_0 \times I; \pi_j(CP^1))$$

by homotopy invariance

$$H^j(P, P_0; \pi_j(CP^1))$$

by the Künneth theorem.

For $j = 2$ or for odd $j$ or for even $j > \dim(P)$, the obstruction groups vanish since the pair $(P, P_0)$ has vanishing cohomology in these dimensions (recall $\dim(P_0) > \dim(X) = 2i \geq 4$ when $\dim(M) = 2n > 6$, $n$ odd). For $2 < j \leq \dim(P)$ even, the obstruction groups are finite since $\pi_j(CP^1)$ is finite.

It remains to show that we can lift an extension $K : (P \setminus P_0) \times I \to P \setminus P_0$ on the quotients to an equivariant extension of $f$.

There exists a lift of the map $K : (P \setminus P_0) \times I \to P \setminus P_0$ to some equivariant map $\tilde{K} : (S \setminus S_0) \times I \to S \setminus S_0$, since $K$ preserves the Euler class of the circle bundles. This can be seen by restricting to $(P \setminus P_0) \times \{0\}$, where such a lift — namely the restriction of $f$ — exists.

The restriction of $\tilde{K}$ to $(S \times \partial I \cup N(S_0 \times I)) \setminus (S_0 \times I)$ differs from $f$ by a map

$$g \in \text{Maps}\left((P \times \partial I \cup N(P_0 \times I)) \setminus (P_0 \times I), S^1\right),$$

i.e., $f(x,t) = \tilde{K}(x,t) \cdot g(\pi(x,t))$.

Since $(P \times \partial I \cup N(P_0 \times I)) \setminus (P_0 \times I)$ is simply-connected (this can be proved by the Seifert–van Kampen theorem), the map $g$ factors through the universal covering $\mathbb{R} \to S^1$. By the Tietze extension theorem, $g$ extends to a map $G : (P \setminus P_0) \times I \to \mathbb{R} \to S^1$. Now $\tilde{K} : (S \setminus S_0) \times I \to S \setminus S_0$, defined by $\tilde{K}(x, t) = \tilde{K}(x, t) \cdot G(\pi(x, t))$ is another equivariant lift of $K$, and it is the desired extension of $F$ to an equivariant map $(S \setminus S_0) \times I \to S \setminus S_0$. $\square$

**Proof of Proposition [5.4]** (for the case when $\dim(M) = 2n > 6$ and the action is not semifree.) Let $D^{S^1}(S)$ be the group of equivariant pseudo-isotopy classes of equivariant diffeomorphisms of $S$. Let $\mathcal{H}^{S^1}(S)$ be the group of transverse linear isovariant homotopy classes of transverse linear isovariant self-homotopy equivalences of $S$. Again there is a natural map $\varphi : D^{S^1}(S) \to \mathcal{H}^{S^1}(S)$. By Proposition [5.13] we have that $\ker(\varphi) = D^0(S)(S)$ is finite, and Proposition [5.14] shows that $\mathcal{H}^{S^1}(S)$ is finite. Thus $D^{S^1}(S)$ is finite. $\square$
5.4. The case when \( \dim(M) = 2n = 6 \).

In this section, we prove Proposition 5.4 for the case when \( \dim(M) = 2n = 6 \). In this case the preceding surgery arguments do not work. However a pseudo-isotopy classification of diffeomorphisms of 4-manifolds treats diffeomorphisms of 5-dimensional manifolds (the 4-manifold times the interval), and more refined surgery arguments do work in dimension 5. Before we prove the proposition in this case, let us make the following observation.

**Lemma 5.17.*** In the case of a non-semifree action and \( \dim(M) = 2n = 6 \), the pair \((\mathbb{P}, \mathbb{P}_0)\) is homeomorphic to the pair consisting of \( S^2 \times S^2 \) and its diagonal, and the top stratum \( \mathbb{P} \setminus \mathbb{P}_0 \) is diffeomorphic to the total space \( TS^2 \) of the tangent bundle of the 2-sphere.

**Proof.** In this case, the fixed component \( X \) is diffeomorphic to \( \mathbb{C}P^1 \). By [26, Theorem 2], the equivariant normal bundle of \( X \) in \( M \) is the direct sum of two complex line bundles: \( N_1 \oplus N_2 \), where \( S^1 \) acts on the fiber of \( N_1 \) with weight 1 and acts on the fiber of \( N_2 \) with weight 2, and \( c_1(N_1) = u \) and \( c_1(N_2) = 0 \) with \( u \in H^2(X; \mathbb{Z}) \) being a generator. As in Lemma 5.4 there is an equivariant map

\[
N_1 \oplus N_2 \to N_1^{\otimes 2} \oplus N_2, \quad (v_1, v_2) \mapsto (v_1 \otimes v_1, v_2)
\]

between the total spaces of the two bundles, where \( S^1 \) acts on the fiber of \( N_1^{\otimes 2} \oplus N_2 \) with weights \((2, 2)\). It induces a homeomorphism between the pairs \((\mathbb{P}, \mathbb{P}_0)\) and \((\mathbb{P}(N_1^{\otimes 2} \oplus N_2), \mathbb{P}(N_2))\).

Now the total space of the canonical bundle \( N_1 \) is the subspace

\[
\{(x_0, x_1), [y_0 : y_1]) \in \mathbb{C}^2 \times \mathbb{C}P^1 \mid x_0y_1 = x_1y_0\},
\]

similarly the total space of \( N_1^{\otimes 2} \) is

\[
\{(x_0, x_1), [y_0 : y_1]) \in \mathbb{C}^2 \times \mathbb{C}P^1 \mid x_0y_1^2 = x_1y_0^2\},
\]

and finally \( \mathbb{P}(N_1^{\otimes 2} \oplus N_2) \) is homeomorphic to

\[
\Sigma_2 = \{(x_0 : x_1 : x_2), [y_0 : y_1]) \in \mathbb{C}P^2 \times \mathbb{C}P^1 \mid x_0y_1^2 = x_1y_0^2\}.
\]

The subspace \( \mathbb{P}(N_2) \) consists of the points where \( x_0 = x_1 = 0 \). Here we use the same notation as Hirzebruch in [10], who writes down the homeomorphism

\[
\mathbb{C}P^1 \times \mathbb{C}P^1 \to \Sigma_2
\]

\[
([u_0 : u_1], [v_0 : v_1]) \mapsto \left( \frac{v_1^2(-v_1u_0 + v_0u_1)}{|v_0|^2 + |v_1|^2}, \frac{v_0^2(-v_1u_0 + v_0u_1)}{|v_0|^2 + |v_1|^2}; v_0u_0 + v_1u_1, |v_0 : v_1| \right)
\]

Finally we compose with a self-map of \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) sending the diagonal to the points where \( -v_1u_0 + v_0u_1 = 0 \), namely \( ([u_0 : u_1], [v_0 : v_1]) \mapsto ([u_0 : u_1], [\overline{v_0} : \overline{v_1}]) \).

Note that the homeomorphism we constructed is smooth when restricted to each stratum, in particular it defines a diffeomorphism between \( \mathbb{P} \setminus \mathbb{P}_0 \) and \( S^2 \times S^2 \setminus \Delta \), where \( \Delta \) denotes the diagonal. The latter space is diffeomorphic to the total space of \( TS^2 \) by stereographic projection: for each \( x \in S^2 \), stereographic projection gives a bijection between \( S^2 \setminus \{x\} \) and the tangent space to \( S^2 \) at \( x \). Composing with this gives the required diffeomorphism.

\[\square\]
Proof of Proposition 5.4. (for the case when dim($M$) = 2$n$ = 6.) In the case when the action is semifree, $P$ is a smooth simply connected closed 4-manifold. By a theorem of Kreck [20, Theorem 1], all self-diffeomorphisms of $P$ inducing the identity on cohomology are pseudo-isotopic to the identity. Together with Lemma 5.6 this implies that the group of pseudo-isotopy classes of self-diffeomorphisms of $P$ preserving the Euler class of the circle bundle $S \to P$ is finite. Hence so is the group of pseudo-isotopy classes of equivariant self-diffeomorphisms of $S$.

Now we consider the case when the action is not semifree. We prove the proposition in the following steps.

Step 0: We restrict to a subgroup of finite index.

We start with an equivariant self-diffeomorphism $\Phi: S \to S$. The quotient map $\overline{\Phi}: P \to P$ induces self-diffeomorphisms of the strata $P_0$ and $P \setminus P_0$. The strata $P_0$ and $P \setminus P_0$ are both homotopy equivalent to $S^2$, so self-diffeomorphisms of the strata act as $\pm \text{id}$ on $H_2(P_0)$ and $H_2(P \setminus P_0)$. If both signs are positive, then $\overline{\Phi}$ induces the identity on the homology of $P$, since $(P,P_0,P \setminus P_0)$ is homotopy equivalent to $(S^2 \times S^2, \text{diagonal}, \text{antidiagonal})$ by Lemma 5.17. So the subgroup of equivariant self-diffeomorphisms of $S$ such that the quotient map $\overline{\Phi}: P \to P$ induces the identity on homology has index at most 4 in the group of all equivariant self-diffeomorphisms of $S$.

In the following we show that an equivariant self-diffeomorphism $\Phi: S \to S$ in this subgroup is pseudo-isotopic to the identity.

Step 1: We show that the restriction of $\Phi$ to $S_0$ is equivariantly isotopic to the identity.

Let $\varphi: S_0 \to S_0$ be the restriction of $\Phi$, inducing a self-diffeomorphism $\overline{\varphi}: P_0 \to P_0$ on the quotient.

Since $P_0$ is diffeomorphic to $S^2$, by a theorem of Munkres [33], every orientation-preserving self-diffeomorphism $\overline{\varphi}$ is isotopic to the identity. Let $\overline{\varphi}_0: I \times P_0 \to P_0$ be such an isotopy, i.e. $\overline{\varphi}_0 = \text{id}$ and $\overline{\varphi}_1 = \overline{\varphi}$.

The pull-back of the $S^1$-bundle $S_0 \to P_0$ by $\overline{\varphi}_0$ is isomorphic to $I \times S_0$, so we obtain an $S^1$-equivariant map $\overline{\varphi}_t: I \times S_0 \to S_0$ covering $\overline{\varphi}_t$, such that $\overline{\varphi}_0(x) = x$ for all $x$. Since $P_0$ is simply-connected, we may also assume that $\overline{\varphi}_1(x) = \overline{\varphi}(x)$ for all $x$; so we obtain an equivariant isotopy $\overline{\varphi}_t: I \times S_0 \to S_0$ from the identity to the restriction $\varphi$ of $\Phi$.

Step 2: We show that $\Phi$ is equivariantly isotopic to a self-diffeomorphism $\Phi_0: S \to S$ whose restriction to a neighborhood $N(S_0)$ of $S_0$ is the identity.

The equivariant isotopy $\overline{\varphi}_t$ is generated by an equivariant vector field $V(t,x)$ on $S_0$. An equivariant extension of this vector field produces an ambient isotopy $\Phi_t: I \times S \to S$, such that $\Phi_1 = \Phi$, and $\Phi_0$ is the identity on $S_0$. Moreover, by uniqueness of equivariant neighbourhoods up to isotopy (see Theorem VI.2.6 in [2]), we may assume that the restriction of $\Phi_0$ to a small equivariant tubular neighbourhood of $S_0$ is an equivariant vector bundle automorphism. The group of equivariant normal bundle automorphisms of $S_0$ is isomorphic to Maps$(P_0,SO(2))$, and since $P_0$ is simply-connected, each such automorphism is homotopic to the trivial automorphism. It follows that we may assume that $\Phi_0$ is the identity on an equivariant tubular neighbourhood $N(S_0)$ of $S_0$.

Step 3: We show that every self-diffeomorphism $\Phi_0$ of $S$ which restricts to the identity on $N(S_0)$ is equivariantly pseudo-isotopic to the identity of $S$. It suffices to construct an equivariant pseudo-isotopy on $S \setminus N(S_0)$ relative to its boundary.
On $S \setminus N(S^0)$, the circle acts freely, so we can work with quotient spaces. We can assume that the equivariant diffeomorphism $\Phi_0: S \setminus N(S^0) \to S \setminus N(S^0)$ is the identity on a neighbourhood of the boundary, so the induced quotient diffeomorphism
\begin{equation}
\psi = \Phi_0: \mathbb{P} \setminus N(\mathbb{P}_0) \to \mathbb{P} \setminus N(\mathbb{P}_0)
\end{equation}
where $N(\mathbb{P}_0) = N(S^0)/S^1$, is the identity on a neighborhood of the boundary. Moreover, $\psi$ induces identity on homology. Lemma 5.19 below shows that the proof of [20] Theorem 1] for simply-connected closed 4-manifolds can be slightly modified to show that $\psi$ is pseudo-isotopic to the identity.

Again, the pair $(I \times (\mathbb{P} \setminus N(\mathbb{P}_0)), \partial)$ is 1-connected, so the isotopy from $\psi = \Phi_0$ to the identity lifts to an equivariant isotopy on $S \setminus N(S^0)$ between $\Phi_0$ and the identity, constant on the boundary. This finishes the proof of the proposition. □

**Lemma 5.19.** The diffeomorphism $\psi$ in (5.18) is pseudo-isotopic (relative boundary) to the identity.

**Proof.** By Lemma 5.17, the pair $(\mathbb{P} \setminus N(\mathbb{P}_0), \partial)$ is diffeomorphic to $(D(TS^2), S(TS^2))$, the pair of (disk bundle, sphere bundle) of the tangent bundle of $S^2$. Let $Q = D(TS^2)$, which is a simply connected spin 4-manifold with connected boundary. We identify $\psi$ in (5.18) with
\[ \psi: Q \to Q, \]
which is a diffeomorphism restricting to the identity on the boundary, and inducing the identity on homology.

We consider $R = Q \times I$, this has boundary $\partial R = Q \times \{0\} \cup \partial Q \times I \cup Q \times \{1\}$. Given the self-diffeomorphism $f = \psi \cup \text{id} \cup \text{id}$ of $\partial R$, we ask whether it extends to a self-diffeomorphism of $R$. If it does, then this self-diffeomorphism is a pseudo-isotopy from $\psi$ to $\text{id}_Q$ relative to the boundary.

We use $f$ to produce the twisted double $R \cup_f -R$. This is homeomorphic to the union $T_\psi \cup \partial Q \times D^2$, where $T_\psi = Q \times I/(0, x) \sim (1, \psi(x))$ is the mapping torus of $\psi$. Now a Mayer-Vietoris sequence shows that $H^*(T_\psi) = H^*(Q \times S^1)$, and another Mayer-Vietoris sequence shows that $R \cup_f -R$ is spin if $T_\psi$ is (the second Stiefel-Whitney class $w_2$ of the former restricts to $w_2$ of the latter). Now $T_\psi$ is spin since $Q$ is, here one can use the Wang sequence as in [20]. Now the proof proceeds as in [20]: one has a spin nullbordism $W$ of $R \cup_f -R$, which one can turn into a relative $h$-cobordism by surgery.

Note that $H_3(R \cup_f -R) \cong H_2(Q)$, as is seen by the Mayer-Vietoris sequence for $R \cup_f -R = T_\psi \cup \partial Q \times D^2$, also we have $H_3(Q) = H^1(Q, \partial Q) = 0$. These are the two assumptions for the proof of Theorem 1 in [20] which hold trivially for simply-connected closed 4-manifolds. □

6. Discussion and a Question

As we have discussed, under Assumption 1.2 when $X$ or $Y$ is an isolated point, by Delzant’s theorem [8], $M$ is equivariantly symplectomorphic to $\mathbb{C}P^n$ with a standard circle action as in (2.1) with $j = 0$. When $\dim(M) = 6$ and when the action is semifree, by [8] and by a result of Gonzalez [12], $M$ is equivariantly symplectomorphic to $\mathbb{C}P^3$ with a standard circle action.

In the case Delzant studied, the regular symplectic quotients are smooth $\mathbb{C}P^{n-1}$ with a standard symplectic structure. In the case when $\dim(M) = 6$ and the action
is semifree, the regular symplectic quotients are smooth 4-manifolds. Symplectic topological methods for symplectic 4-manifolds give certain nice results on the symplectic structure, so that the rigidity condition of the regular symplectic quotients proposed by Gonzalez [12] is fulfilled. The rigidity property of the regular symplectic quotient grants a unique gluing up to symplectic isotopy of the symplectic tubular neighborhoods of $X$ and of $Y$ along a regular level set of the moment map. When the dimensions of the symplectic quotients are bigger, it is hard to determine if the quotients are rigid.

For the 6-dimensional Hamiltonian $S^1$-manifolds McDuff studied in [30] (see the Introduction), she uses the rigidity criterion and resolution of singularities of the 4-dimensional symplectic quotients to prove uniqueness up to equivariant symplectomorphism of the Hamiltonian $S^1$-manifolds. In her case, the symplectic quotients are symplectic 4-orbifolds which have isolated singularities. In our case, when $\dim(M) = 6$ and when the action is not semifree, the symplectic quotient is $\mathbb{P}_2(\mathbb{C} \oplus H)$, the weighted projective bundle of the direct sum of a trivial and the Hopf bundle over $S^2$. We saw in Lemma 5.17 that it is homeomorphic to $S^2 \times S^2$ with the diagonal as the singular set. The method in [30] does not generalize directly to this case.

We do not know examples other than $\mathbb{CP}^n$ and $\tilde{G}_2(\mathbb{R}^{n+2})$ satisfying Assumption 1.2. The following question is open.

**Question.** If Assumption 1.2 holds, is $M$ $S^1$-equivariantly diffeomorphic or symplectomorphic to $\mathbb{CP}^n$ or to $\tilde{G}_2(\mathbb{R}^{n+2})$?

### Appendix A. Non-equivariant results

In this appendix, we prove the non-equivariant uniqueness results mentioned in Remark 1.4. Due to the close relation of the arguments, we also prove finiteness of the manifold up to non-equivariant diffeomorphism. More precisely, we prove the following theorem.

**Theorem A.** If Assumption 1.2 holds, then

1. up to diffeomorphism there are finitely many such manifolds in each fixed dimension;
2. when $\dim(M) = 2n \leq 6$, $M$ is diffeomorphic to $\mathbb{CP}^n$ if the action is semifree, and is diffeomorphic to $\tilde{G}_2(\mathbb{R}^3)$ if the action is not semifree;
3. when $\dim(M) = 10$ or $\dim(M) = 14$ and the action is not semifree, $M$ is respectively homeomorphic to $\tilde{G}_2(\mathbb{R}^7)$ or to $\tilde{G}_2(\mathbb{R}^9)$.

For the case of low dimensions and semifree actions, as mentioned in the Introduction, the results follow from the literature. For the rest of the proof, we will use the one-connectivity of the manifold given by Theorem 1 and the integral cohomology ring and Chern classes of the manifold given by (2.4) and (2.5) in Theorem 4. The more direct tool we use in this section is Kreck’s modified surgery theory.

#### A.1. The case when the action is semifree.

Theorem A(1) for the case when $\dim(M) \neq 4$ and when the action is semifree follows from Proposition A.1 below and Theorem 2(2). For the cases when $\dim(M) \leq 6$ and when the action is semifree, we saw in the Introduction that $M$ is diffeomorphic to $\mathbb{CP}^n$, where $n \leq 3$. 


Proposition A.1. ([38, 27]) Let $X$ be a homotopy complex projective space with standard Pontryagin classes. If $\dim(X) \neq 4$, there are finitely many diffeomorphism types of $X$.

A.2. The case when the action is not semifree.

In this case a classification up to homotopy equivalence, which is needed for the classical surgery theory used by Sullivan and Little, is not known. We use Kreck’s modified surgery theory [19], which does not need the homotopy type, but the normal $(n - 1)$-type (see Definition A.6) of the manifolds. We determine the normal $(n - 1)$-type of the manifold in Lemma A.7 and then prove the theorem using Kreck’s theory and a computation by F. Fang and J. Wang.

Lemma A.2. The manifold $\tilde{G}_2(R^{n+2})$ is diffeomorphic to a quadratic hypersurface in $\mathbb{CP}^{n+1}$, i.e., the vanishing set of a homogeneous polynomial of degree 2.

Proof. Consider $\mathbb{CP}^{n+1}$ as the set of complex lines in the complexification of $R^{n+2}$. Define $\iota: \tilde{G}_2(R^{n+2}) \to \mathbb{CP}^{n+1}$ such that it maps an oriented plane with oriented orthonormal basis $v, w$ to the complex line spanned by $v + iw$. It is easy to check that this map is well-defined, injective, smooth, and that the image is the vanishing set of the homogeneous quadratic polynomial $\sum z_i^2$. □

Definition A.3. Let $n \geq 3$ be odd. We name $M_G$ to be any smooth, compact, one-connected, almost complex $2n$-manifold with the same integral cohomology ring and total Chern class as those of $\tilde{G}_2(R^{n+2})$.

Let $BO$ be the classifying space of the stable orthogonal group. It is the union of the classifying spaces $BO_k$ of the orthogonal groups, and each $BO_k$ is a union of Grassmann manifolds $G_k(R^{n+k})$ via the natural inclusions $G_k(R^{n+k}) \subseteq G_k(R^{n+k+1})$.

Let $N$ be a closed smooth $n$-manifold. An embedding $i: N \to R^{n+k}$ gives rise to a normal Gauss map $N \to G_k(R^{n+k})$: to each $x \in N$ one assigns the normal space of $i(N)$ at $i(x)$. In this way we also obtain a vector bundle over $N$, the normal bundle of the embedding.

By composition we get the stable Gauss map $N \to BO$. This map is (up to homotopy) independent of the choice of the embedding $i$. This is because the embeddings $i$ and $N \to R^{n+k} \to R^{n+k+1}$ have the same stable normal bundle map, two embeddings into $R^{n+k}$ are isotopic for large $k$, and an isotopy of embeddings induces a homotopy of the stable Gauss maps.

The vector bundles over $N$ corresponding to isotopic embeddings into Euclidean space are isomorphic, and the normal bundle of $N \to R^{n+k} \to R^{n+k+1}$ differs from the normal bundle of $i$ by the addition of a trivial rank 1 bundle. Thus every closed manifold $N$ has a unique stable normal bundle.

Recall that a stable (real or complex) vector bundle is an equivalence class of vector bundles, where we identify bundles which are isomorphic after adding trivial bundles to them. For a compact Hausdorff space $X$, stable real (respectively complex) vector bundles over $X$ form a group $\widetilde{KO}(X)$ (respectively $\widetilde{K}(X)$) under direct sum, the reduced real (complex) $K$-theory of $X$. Stable real bundles over $X$ are classified by homotopy classes of maps to $BO$. The direct sum of stable bundles induces a map $\oplus: BO \times BO \to BO$. 
Lemma A.4. Let $X$ be a compact Hausdorff space with the homotopy type of a finite CW-complex, and assume that $H^*(X;\mathbb{Z})$ is torsion-free and concentrated in even degrees. Then two stable complex vector bundles over $X$ with the same total Chern class are isomorphic.

Proof. The Atiyah–Hirzebruch spectral sequence for reduced complex $K$-theory of $X$ degenerates since both the cohomology groups of $X$ and the coefficients of $K$-theory are zero in odd degrees. It follows that $\tilde{K}^0(X)$ is a free abelian group. Thus the composition $\tilde{K}^0(X) \to \tilde{K}^0(X) \otimes \mathbb{Q} \cong \bigoplus_j \tilde{H}^{2j}(X;\mathbb{Q})$ is injective. Here the isomorphism is given by the Chern character. It follows that two stable vector bundles over $X$ with the same Chern classes are isomorphic. \hfill \Box

With an additional argument which covers the difference between stable and non-stable bundles, K. Wang proves the following result.

Lemma A.5. [15 Proposition 3.1] Let $X$ be a homotopy complex projective space. Then there are at most finitely many complex vector bundles of fixed rank over $X$ which have the same Chern classes.

Definition A.6. Let $N$ be a $2n$-dimensional manifold. The $n$-th Postnikov–Moore factorization of the stable Gauss map $N \to BO$ is a factorization $N \to B \to BO$ such that $N \to B$ is $n$-connected, i.e., an isomorphism on the homotopy groups $\pi_j$ for $j < n$ and surjective on $\pi_n$; and $B \to BO$ is $n$-coconnected, i.e., an isomorphism on the homotopy groups $\pi_j$ for $j > n$ and injective on $\pi_n$.

We may assume that $B \to BO$ is a fibration. It is unique up to fiber homotopy equivalence, and is called the normal $(n-1)$-type of $N$.

Let $H$ be the Hopf bundle on $\mathbb{CP}^\infty$. The virtual vector bundle (formal difference of vector bundles) $H^{\otimes 2} - (n+2)H$ defines a stable vector bundle over each $\mathbb{CP}^N$. The corresponding classifying maps are compatible and induce a (unique) classifying map

$$\xi : \mathbb{CP}^\infty \to BO.$$

The $n$-th Postnikov–Moore factorization of $pt \to BO$ is denoted by $pt \to BO(n+1) \xrightarrow{\xi} BO$. The space $BO(n+1)$ is called the $(n+1)$-connected cover of $BO$.

Lemma A.7. The normal $(n-1)$-type of $M_G$ is $\mathbb{CP}^\infty \times BO(n+1) \xrightarrow{\xi \times p} BO \times BO \xrightarrow{\xi} BO$.

Proof. If $M_G = \tilde{G}_2(\mathbb{R}^{n+2})$, then the statement is [19 Proposition 3]. We need to check that this remains true for all $M_G$.

Let $f : M_G \to \mathbb{CP}^\infty = K(\mathbb{Z}, 2)$ represent the generator $x \in H^2(M_G;\mathbb{Z})$. Note that both $M_G$ and $\mathbb{CP}^\infty$ are simply connected; using the integral cohomology ring structures of these two spaces and the Hurewicz Theorem, we can check that $f$ is $n$-connected.

Let $*$ be any constant map $M_G \to BO(n+1)$. We claim that the normal bundle map $M_G \to BO$ factors up to homotopy through

$$M_G \xrightarrow{(f,*)} \mathbb{CP}^\infty \times BO(n+1) \xrightarrow{\xi \times p} BO \times BO \xrightarrow{\xi} BO.$$  \hfill (A.8)

To prove this, we need to show that the stable normal bundle of $M_G$ is isomorphic to the pullback of the universal bundle via the composition

$$M_G \xrightarrow{f} \mathbb{CP}^\infty \xrightarrow{\xi} BO.$$  \hfill (A.9)
Since \( c(M_G) = (1 + x)^{n+2}(1 + 2x)^{-1} \) (see (2.5)), the stable normal bundle of \( M_G \) has total Chern class \((1 + 2x)(1 + x)^{-(n+2)}\). Now the pullback of the universal bundle via \((A.5)\) is \( f^*(-(n+2)H \oplus H^{\otimes 2})\). Hence the two stable bundles have isomorphic total Chern class (induced by \( f^* \)). By Lemma (A.4) the two stable bundles are isomorphic. It is easy to check that \((A.8)\) is a Postnikov–Moore factorization. □

For every fibration \( B \to BO \), and every \( k \in \mathbb{N} \), there is a bordism group \( \Omega_k(B) \). An element is represented by a closed \( k \)-manifold \( N \), together with a lift of its stable Gauss map along \( B \to BO \), but without any conditions on the homotopy groups. Two such manifolds \( N_0, N_1 \) represent the same element if there is a bordism between them, i.e. a compact \((k+1)\)-manifold \( W \) with \( \partial W = N_0 \cup N_1 \), and \( W \) itself is equipped with a lift of its stable Gauss map to \( B \) which restricts to the lifts of the \( N_i \)'s. By a Pontryagin–Thom construction, the bordism group \( \Omega_k(B) \) is the \( k \)-th homotopy group of the Thom spectrum of the stable vector bundle described by the map \( B \to BO \).

In particular we have the \( 2n \)-th bordism group corresponding to the fibration \( \mathbb{CP}^\infty \times BO(n+1) \to BO \), and every manifold \( M_G \) (together with a choice of lift) defines an element of this bordism group. Following F. Fang and J. Wang [11], we denote this bordism group by \( \Omega^{(n+1)}(\mathbb{CP}^\infty; \xi) \).

In [19], Kreck uses his modified surgery theory to classify manifolds with a given normal \((n-1)\)-type. Kreck uses that two manifolds which are diffeomorphic must in particular represent the same element of the bordism group \( \Omega_k(B) \), where \( B \) is the common normal \((n-1)\)-type. On the other hand, if two manifolds represent the same element in \( \Omega_k(B) \), there is a bordism between these two manifolds (which is additionally equipped with a map to \( B \)), and one can try to modify this bordism by surgery in the interior, so that the new bordism is an \( s \)-cobordism, which would imply that the two manifolds are diffeomorphic. In general it is not possible to perform such modifications, but in certain cases the additional information contained in the map to \( B \) implies that the obstruction to do such surgeries vanishes. We have in particular Proposition (A.10)

**Proposition A.10.** ([19] Corollary 4) Let \( n \geq 3 \) be odd. Two closed, simply connected \( 2n \)-manifolds with the same Euler characteristic and the same normal \((n-1)\)-type \( B \to BO \) are diffeomorphic if they represent the same element in \( \Omega_{2n}(B) \).

Proof of Theorem (A) (for the case of non-semifree actions.) By Lemma (A.7) all \( M_G \) have the same normal \((n-1)\)-type, the normal \((n-1)\)-type of \( G_2(\mathbb{R}^{n+2}) \). By Lemma (A.2) \( G_2(\mathbb{R}^{n+2}) \) is a hypersurface in \( \mathbb{CP}^{n+1} \). Thus we can use Kreck’s modified surgery theory [19] and the computation by F. Fang and J. Wang [11] for the normal \((n-1)\)-type of a hypersurface.

By Proposition (A.10) and Lemma (A.7) two of the manifolds \( M_G \) are diffeomorphic if they represent the same element in \( \Omega_{2n}^{(n+1)}(\mathbb{CP}^\infty; \xi) \). More precisely, such an element is given by the map \( M_G \overset{(f, \xi)}{\to} \mathbb{CP}^\infty \times BO(n+1) \) from \((A.8)\).

All our manifolds \( M_G \) have the same Pontryagin classes. Using the integral cohomology ring structure of \( M_G \) given by \((2.5)\), we can check that

\[(A.11)\]
\[ (f^*(t)^n, [M_G]) = \langle x^n, [M_G] \rangle = 2. \]

In [19, p.745], for complete intersections, Kreck shows that the Pontryagin classes and the total degree determine the element in \( \Omega_{2n}(B) \otimes \mathbb{Q} \). The proof of this
applies to our case, i.e., the Pontryagin classes and the number in \( (A.11) \) (which generalizes the total degree of a complete intersection) determine the element in \( \Omega^{2n+1}_\infty(\mathbb{C}P^\infty; \xi) \otimes \mathbb{Q} \). Hence, the difference between two of the manifolds \( M_G \) in \( \Omega^{2n+1}_\infty(\mathbb{C}P^\infty; \xi) \) is a torsion class. As a consequence, the size of the finite torsion subgroup of this bordism group gives an upper bound on the number of diffeomorphism types of the manifolds \( M_G \). This proves the finiteness result in part (1) of the theorem.

For \( n = 3 \), the above torsion subgroup is trivial. This can be proved in the following way. In this case, the Thom spectrum is \( T\xi \wedge M\text{Spin} \), where \( T\xi \) is the Thom spectrum of \( \xi \), see [34], section 1.4. Thus the bordism group is the sixth spin bordism group of \( T\xi \) and can be computed using the Atiyah–Hirzebruch spectral sequence and the Thom isomorphism in singular homology. (For the differential in the Atiyah–Hirzebruch spectral sequence one needs to know, see [39], lemma on p. 751.) Thus \( M_G \) is unique up to diffeomorphism, proving the 6-dimensional result in part (2) of the theorem. We do not carry out the details of the proof here, since this part of the theorem does also follow from Žubr’s classification of simply-connected 6-manifolds [49].

For \( n = 5 \) and \( n = 7 \), F. Fang and J. Wang prove in [11, Propositions 4.3 and 5.1] that all torsion in \( \Omega^{2n+1}_\infty(\mathbb{C}P^\infty; \xi) \) is in the kernel of the map to \( \Omega^{2\text{Top}(n+1)}_\infty(\mathbb{C}P^\infty; \xi) \). The latter is a bordism group of topological manifolds and plays the corresponding role in a homeomorphism classification of the manifolds. Part (3) of the theorem follows from this.

□

References

ONE-CONNECTIVITY AND FINITENESS OF HAMILTONIAN $S^1$-MANIFOLDS


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