A NOTE ON MAPPING CLASS GROUP ACTIONS ON DERIVED CATEGORIES.

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Abstract. Let X_n be a cycle of n projective lines, and $\mathbb{T}_n$ a symplectic torus with n punctures. Using the theory of spherical twists introduced by Seidel and Thomas [ST], I will define an action of the pure mapping class group of $\mathbb{T}_n$ on $D^b(Coh(X_n))$. The motivation comes from homological mirror symmetry for degenerate elliptic curves, which was studied by the author with Treumann and Zaslow in [STZ].

Contents

1. Introduction.

According to Kontsevich's *Homological Mirror Symmetry* conjecture (from now on HMS, see [K]), given a Calabi-Yau variety X and a symplectic manifold $\tilde{X}$, if X and $\tilde{X}$ are mirror partners, then the derived category of coherent sheaves over X, $D^b(Coh(X))$, should be equivalent to the Fukaya category of $\tilde{X}$, $Fuk(\tilde{X})$. Since $Fuk(\tilde{X})$ is an invariant of the symplectic geometry of $\tilde{X}$, mirror symmetry predicts that the group of symplectic automorphisms of $\tilde{X}$ acts by equivalences on $D^b(Coh(X))$. In [ST] Seidel and Thomas investigate this aspect of HMS by introducing the notions of *spherical object* and *twist functor*, which can be defined for general triangulated categories, and axiomatize the formal homological properties enjoyed by equivalences of the Fukaya category induced by *generalized Dehn twists* (these are special symplectic automorphisms introduced by Seidel, see [S]). Using their theory they are able, in many interesting examples, to give a conjectural description of the equivalences of $D^b(Coh(X))$ which should be mirror to symplectic automorphisms of $\tilde{X}$. I refer the reader to [ST] for a detailed account of this circle of ideas. A brief overview of the relevant definitions will be given in Section 3 below.

Let X_n be a cycle of n projective lines, i.e. a nodal curve of arithmetic genus 1, with n singular points. Well known mirror symmetry heuristics suggest that the mirror of X_n should be a symplectic torus with n punctures, which I shall denote $\mathbb{T}_n$. In the paper [STZ],

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joint with Treumann and Zaslow, we prove a version of HMS for X_n and $\mathbb{T}_n$, by showing that the category of perfect complexes over X_n, $\mathcal{P}\mathrm{erf}(X_n)$, is quasi-equivalent to a certain conjectural model of $Fuk(\mathbb{T}_n)$ which we develop in the paper. See also the recent work [LP] in which the authors prove, with very different techniques, a HMS statement for the case $n = 1$.

Motivated by [STZ], in this paper I explore the consequences of mirror symmetry for the study of auto-equivalences of $D^b(Coh(X_n))$. Recall that the mapping class group of an oriented surface Σ can be described as the group of symplectic automorphisms of Σ, modulo isotopy. The existence of an action of the mapping class group of $\mathbb{T}_n$ on $D^b(Coh(X_n))$ does not follow directly from [STZ], as the model of the Fukaya category considered there is not acted upon, in any obvious way, by symplectomorphisms of $\mathbb{T}_n$.[1] My main result uses the framework of [ST] to construct an action of the (pure) mapping class group of $\mathbb{T}_n$, $\mathrm{PM}(\mathbb{T}_n)$, over $D^b(Coh(X_n))$. In future work, I plan to establish that this action is faithful. It is worth pointing out that the action I will define is, in an appropriate sense, a categorification of the symplectic representation of the mapping class group, which can be recovered by considering the induced action on the *numerical* Grothendieck group of $\mathcal{P}\mathrm{erf}(X_n)$ (see Remark 3.7 below. For a definition of the symplectic representation, the reader can refer to [FM], Chapter 6).

The paper is organized as follows. In Section 2, I give some background on the mapping class group, and then work out a convenient presentation of $\mathrm{PM}(\mathbb{T}_n)$. The proof of the main result, Theorem 3.6, is contained in Section 3. Theorem 3.6 generalizes previous results in [ST] and in [BK], where the authors considered, respectively, the case of a smooth elliptic curve, and of the nodal cubic in $\mathbb{P}^2$ (i.e. the case $n = 1$). Equivalences of $D^b(Coh(X_n))$ were also investigated in [L]. However, as the author in [L] restricts to a subgroup of equivalences satisfying certain homological conditions, which are violated by the spherical twists I shall consider below, there is essentially no overlap between his work and the present project.

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2. The mapping class group of a punctured torus.

In this section I will briefly review some basic facts about the mapping class group, and then give a presentation of the mapping class group of the punctured torus based on [LP]. Also, it will be useful to spell out some relations between mapping classes which were found by Gervais in [G]. For a comprehensive introduction to the mapping class group I refer the reader to [FM].

[1] Note also that the HMS statement in [STZ] involves $\mathcal{P}\mathrm{erf}(X_n)$, rather than the full derived category of X_n. However, $D^b(Coh(X_n))$ has the same group of auto-equivalences of $\mathcal{P}\mathrm{erf}(X_n)$. In fact, any equivalence of $D^b(Coh(X_n))$ gives, by restriction, an equivalence of $\mathcal{P}\mathrm{erf}(X_n)$, and it follows from Lemma 3.3 that this assignment is a bijection.

Let $\Sigma = \Sigma_{g,n,b}$ be a differentiable, oriented surface of genus g, with n marked points, and b boundary components. The mapping class group of Σ, denoted by $\mathrm{M}(\Sigma)$, is the group of isotopy classes of orientation preserving diffeomorphisms of Σ, which send the set of marked points to itself, and restrict to the identity on the boundary components. Note that $\mathrm{M}(\Sigma_{g,n,b})$ is uniquely determined by the parameters g, n, b. The *pure* mapping class group of Σ is the subgroup $\mathrm{PM}(\Sigma) \hookrightarrow \mathrm{M}(\Sigma)$ of mapping classes fixing pointwise the set of marked points. Alternatively, $\mathrm{PM}(\Sigma)$ can be defined as the subgroup of $\mathrm{M}(\Sigma)$ generated by Dehn twists along simple closed curves (for the definiton of Dehn twist, and a proof of this claim, see Chapter 3 and 4 of [FM]). In making the above definitions, marked points on Σ could be interpreted, equivalently, as punctures, and I shall make use freely of both viewpoints in the following.

A surface Σ with n punctures and $b+m$ boundary components can be immersed in a surface with $n+m$ punctures and b boundary components (we can trade m boundary components for m punctures, by gluing a punctured discs along each boundary component we wish to remove). Further, this immersion induces a map of pure mapping class groups. The details can be found in Section 2 of [LP], together with the following lemma which will be useful later.

Lemma 2.1. *Let $(g, r, m) \notin \{(0, 0, 1), (0, 0, 2)\}$, then we have the exact sequence*

$$1 \to \mathbb{Z}^m \to \mathrm{PM}(\Sigma_{g,n,b+m}) \to \mathrm{PM}(\Sigma_{g,n+m,b}) \to 1,$$

where $\mathbb{Z}^m$ stands for the free abelian group of rank m generated by the Dehn twists along the m boundary components we are removing.

Set $\mathbb{T}_n = \Sigma_{1,n,0}$ and $\mathbb{T}_{n,m} = \Sigma_{1,n,m}$. The pure mapping class group $\mathrm{PM}(\mathbb{T}_n)$ is generated by Dehn twists along $n+1$ non-separating simple closed curves. In order to fix ideas, it is convenient to choose explicit representatives for this collection of curves. I will mostly follow the notation of [G], to which I refer for further details. Let $\Lambda = \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$ be the standard integral lattice, let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$, and fix a fundamental domain for the action of Λ, say $[0, 1) \times [0, 1)$. Choose as set of marked points $P = \{p_1 = (\frac{1}{n+1}, \frac{1}{2}), \dots, p_n = (\frac{n}{n+1}, \frac{1}{2})\}$, and identify the index set $\{1 \dots n\}$ with $\mathbb{Z}/n$ endowed with the natural cyclic order.[2] A cyclic order allows us to speak unambiguously about ordered triples. If $i, j, k \in \{1 \dots n\}$ (not necessarily distinct) form an ordered triple, I shall write $i \preccurlyeq j \preccurlyeq k$. If I require i, j, k to be distinct, I will use the symbol $\prec$.

Let α and β_i, $i \in \{1 \dots n\}$ be the following simple closed curves: in the fundamental domain, α is given by $[0, 1) \times \{\frac{1}{3}\}$, and β_i is given by $\{\frac{i}{n+1} - \frac{1}{2(n+1)}\} \times [0, 1)$. It will be important to consider also separating curves $\gamma_{i,j}$ indexed by an ordered pair $i, j \in \{1 \dots n\}$. The loop $\gamma_{i,j}$ can be described as the boundary of a tubular neighborhood of a straight segment σ in $\mathbb{T}$, starting at p_i and ending at p_j, and such that $p_k \in \sigma$ if and only if $i \preccurlyeq k \preccurlyeq j$. A schematic representation of these curves is given in Figure 1.

If μ is a simple closed curve in a differentiable surface Σ, denote T_μ the Dehn twist along it. I will consider $\mathbb{T}_{n-1,1}$ to be the closed subsurface of $\mathbb{T}_n$ obtained by cutting out a small open disc centered in p_n (small means that its boundary should not intersect any of the loops

[2] In order to make sense of the successor operator $\bullet + 1$ on the index set, I will also use the additive structure of $\mathbb{Z}/n$.

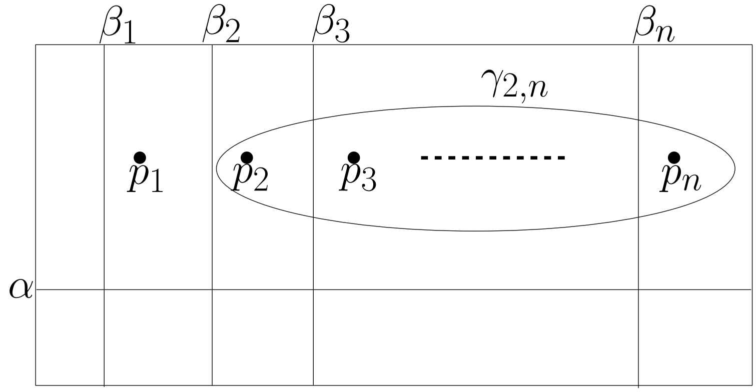

FIGURE 1. The picture above represents the simple closed curves introduced earlier, which are visualized as subsets of the fixed fundamental domain for the action of Λ.

described above). It follows from [LP] that both $\mathrm{PM}(\mathbb{T}_n)$ and $\mathrm{PM}(\mathbb{T}_{n-1,1})$ are generated by Dehn twists T_α, and T_{β_i}, $i \in \{1 \dots n\}$. I will refer to this collection of Dehn twists as *Humphrey generators*, in analogy with Humphrey's set of generators for the mapping class group of a compact surface.

A presentation of $\mathrm{PM}(\mathbb{T}_{n-1,1})$ in terms of Humphrey generators can be read off Proposition 3.3 of [LP]. For the reader's convenience I collect it below.

Proposition 2.2. *The pure mapping class group* $\mathrm{PM}(\mathbb{T}_{n-1,1})$ *is generated by* T_α, *and* T_{β_i}, $i \in \{1 \dots n\}$, *subject to the following relations:*

- *(Braid relations) for every* $i, j \in \{1 \dots n\}$,
  $T_{\beta_i} T_{\beta_j} = T_{\beta_j} T_{\beta_i}$,
  $T_\alpha T_{\beta_i} T_\alpha = T_{\beta_i} T_\alpha T_{\beta_i}$.
- *(Commutativity relations) for every* $i, j, k \in \{1 \dots n\}$, $i \prec j \prec k$,

$$T_{\beta_i}(T_\alpha^{-1} T_{\beta_{k+1}}^{-1} T_{\beta_j}^{-1} T_\alpha^{-1} T_{\beta_k} T_\alpha T_{\beta_j} T_{\beta_{k+1}} T_\alpha) = (T_\alpha^{-1} T_{\beta_{k+1}}^{-1} T_{\beta_j}^{-1} T_\alpha^{-1} T_{\beta_k} T_\alpha T_{\beta_j} T_{\beta_{k+1}} T_\alpha) T_{\beta_i}.$$

An analogous presentation for $\mathrm{PM}(\mathbb{T}_n)$ is described by the following Proposition.

Proposition 2.3. *Let* $i, j \in \{1 \dots n\}$, *and set*

$$A_{i,j} = T_{\beta_{j+1}} T_\alpha T_{\beta_{i+1}}^{-1} T_{\beta_i} T_\alpha^{-1} T_{\beta_{j+1}}^{-1} T_\alpha T_{\beta_i}^{-1} T_{\beta_{i+1}} T_\alpha^{-1} T_{\beta_{i+1}}^{-1} T_{\beta_i}.$$

The pure mapping class group $\mathrm{PM}(\mathbb{T}_n)$ *is generated by* T_α, *and* T_{β_i}, $i \in \{1 \dots n\}$, *subject to the following relations:*

- *Braid relations and Commutativity relations (see Proposition 2.2).*
- *(G-relation)* $(T_\alpha T_{\beta_1})^6 = A_{1,n} A_{2,n} \dots A_{n-1,n}$.

Before giving a proof of Proposition 2.3, it is useful to consider an alternative presentation of $\mathrm{PM}(\mathbb{T}_2)$, which will play a role in the next Section, and is contained in Corollary 2.4 below. Although Corollary 2.4 follows quite easily from Proposition 2.3, rather than discussing the details of this derivation, I refer the reader to [PS] for a direct proof.

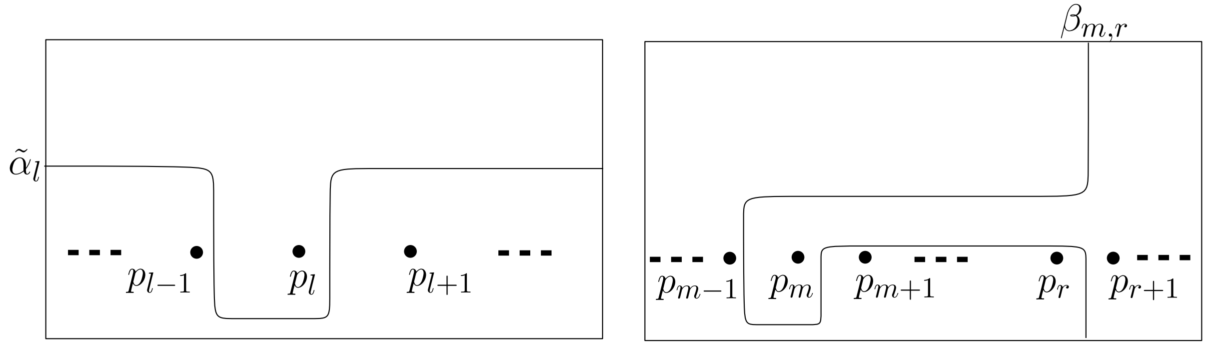

FIGURE 2. Above is a picture of the simple closed curves $\tilde{\alpha}_m$, and $\tilde{\beta}_{n,r}$, which are discussed in the proof of Lemma 2.5.

Corollary 2.4. *The pure mapping class group* $\mathrm{PM}(\mathbb{T}_2)$ *is generated by* T_α*,* T_{β_1} *and* T_{β_2}*, subject to the following relations:*

- *Braid relations*
- *(G_2-relation)* $(T_{\beta_1}T_\alpha T_{\beta_2})^4 = 1$.

The proof of Proposition 2.3 depends on Proposition 2.2, and Lemma 2.1. It follows immediately from the definition of Dehn twist that, for all $i \in \{1 \dots n\}$, $T_{\gamma_{i,i}} = 1$ in $\mathrm{PM}(\mathbb{T}_n)$. In $\mathbb{T}_{n-1,1}$, however, $\gamma_{n,n}$ is isotopic to the boundary component, and therefore $T_{\gamma_{n,n}}$ defines a non trivial mapping class. Lemma 2.1 assures us that the only extra relation needed to obtain $\mathrm{PM}(\mathbb{T}_n)$ from the presentation in Proposition 2.2 is precisely $T_{\gamma_{n,n}} = 1$. What is left to do is finding an expression for $T_{\gamma_{n,n}}$ as a product of Humphrey generators. To achieve this, I need to introduce two more ingredients. The first is the following lemma,

Lemma 2.5. *Let* $i, j \in \{1 \dots n\}$*, and, as in Proposition 2.3, set*

$$A_{i,j} = T_{\beta_{j+1}}T_\alpha T^{-1}_{\beta_{i+1}}T_{\beta_i}T^{-1}_\alpha T^{-1}_{\beta_{j+1}}T_\alpha T^{-1}_{\beta_i}T_{\beta_{i+1}}T^{-1}_\alpha T^{-1}_{\beta_{i+1}}T_{\beta_i},$$

then $T_{\gamma_{i,j}} = A_{i,j}T_{\gamma_{i+1,j}}$.

Proof. Let $\mathbb{T}_{n-1}$ be the torus with $n-1$ punctures obtained from $\mathbb{T}_n$ by filling in the puncture p_i. The Birman exact sequence (see [FM], Theorem 4.6), applied to the inclusion $\mathbb{T}_n \hookrightarrow \mathbb{T}_{n-1}$, yields

$$1 \to \pi_1(\mathbb{T}_{n-1}, p_i) \xrightarrow{\mathcal{P}ush} \mathrm{PM}(\mathbb{T}_n) \xrightarrow{\mathcal{F}orget} \mathrm{PM}(\mathbb{T}_{n-1}) \to 1,$$

where $\pi_1(\mathbb{T}_{n-1}, p_i)$ is the fundamental group of $\mathbb{T}_{n-1}$ with base-point p_i. The names attached to the maps above follow the conventions of Chapter 4 in [FM], to which I refer the reader for further details on the Birman exact sequence.

The key point is that $T_{\gamma_{i,j}}T^{-1}_{\gamma_{i+1,j}}$ lies in the image of the morphism $\mathcal{P}ush$. Figure 2 describes the geometry of two classes of simple closed curves in $\mathbb{T}_n$, called respectively $\tilde{\alpha}_m$, and $\tilde{\beta}_{n,r}$, $m, n, r \in \{1 \dots n\}$. It immediately follows from the definition of $\mathcal{P}ush$ that, in $\mathrm{PM}(\mathbb{T}_n)$,

$$T_{\gamma_{i,j}}T^{-1}_{\gamma_{i+1,j}} = T_{\beta_i}T^{-1}_{\beta_{i+1}}T_{\beta_j}T^{-1}_{\tilde{\beta}_{i,j}}.$$

It is not hard to express $T_{\tilde{\beta}_{i,j}}$ in terms of Humphrey generators. In fact, by simply applying the definition of Dehn twist, one can verify that $\tilde{\beta}_{i,j} = T_{\tilde{\alpha_i}} T_\alpha^{-1}(\beta_j)$, and $\tilde{\alpha_i} = T_{\beta_i}^{-1} T_{\beta_{i+1}}(\alpha)$.[3] Now recall that, if μ and μ' are two simple closed curves in an oriented surface Σ, then $T_{T_\mu(\mu')} = T_\mu T_{\mu'} T_\mu^{-1}$ (this is Fact 3.7 in [FM]). Thus

$$T_{\tilde{\beta}_{i,j}} = T_{\tilde{\alpha_i}} T_\alpha^{-1} T_{\beta_j} T_\alpha T_{\tilde{\alpha_i}}^{-1}, \text{ and } T_{\tilde{\alpha_i}} = T_{\beta_i}^{-1} T_{\beta_{i+1}} T_\alpha T_{\beta_{i+1}}^{-1} T_{\beta_i}.$$

Using this last identity, we can rewrite first $T_{\tilde{\beta}_{i,j}}$, and then $T_{\gamma_{i,j}} T_{\gamma_{i+1,j}}^{-1}$, as a product of Humphrey generators, and this completes the proof of Lemma 2.5. □

The second ingredient is given by a family of relations in the mapping class group, introduced by Gervais in [G] as *star relations*.

Proposition 2.6. *Let $i, j, k \in \{1 \dots n\}$, and $i \preccurlyeq j \preccurlyeq k$. Then*

$$(T_\alpha T_{\beta_i} T_{\beta_j} T_{\beta_k})^3 = T_{\gamma_{i,j}} T_{\gamma_{j,k}} T_{\gamma_{k,i}}.$$

Proof. See Theorem 1 in [G]. □

Note that, when $i = j = k$, one obtains the following 'degenerate' star relations,

$$(T_\alpha T_{\beta_i} T_{\beta_i} T_{\beta_i})^3 = T_{\gamma_{i,i-1}}.$$

Using the braid relations, the product on the LHS of the equality can be rewritten as $(T_\alpha T_{\beta_i})^6$, and therefore Proposition 2.6 yields, for all $i \in \{1, \dots, n\}$, the identity

$$(T_\alpha T_{\beta_i})^6 = T_{\gamma_{i,i-1}}.$$

Let us fix $i \in \{1 \dots n\}$, say $i = 1$. Then the degenerate star identity for $i = 1$ combined with an iterated application of Lemma 2.5 (from which we import the notation), gives the formula

$$(T_\alpha T_{\beta_1})^6 = (A_{1,n} A_{2,n} \dots A_{n-1,n}) T_{\gamma_{n,n}}.$$

Since the $A_{i,j}$-s are defined as a product of T_α and T_{β_i}-s, this yields the sought after expression of $T_{\gamma_{n,n}}$ in terms of Humphrey generators, and concludes the proof of Proposition 2.3.

Lemma 2.7 below is the last result of this Section, and describes a family of identities in $\mathrm{PM}(\mathbb{T}_{n-1,1})$, which will be useful in Section 3.

Lemma 2.7. *If $i \in \{1, \dots, n\}$, then*

$$(T_\alpha T_{\beta_1})^6 (A_{1,n} A_{2,n} \dots A_{n-1,n})^{-1} = (T_\alpha T_{\beta_i})^6 (A_{i,i+n-1} A_{i+1,i+n-1} \dots A_{i+n-2,i+n-1})^{-1}$$

as elements of $\mathrm{PM}(\mathbb{T}_{n-1,1})$.

Before proceeding with the proof of Lemma 2.7, a few comments are in order. Note that the G-relation of Proposition 2.3 depends on the degenerate star identity for $i = 1$. However, because of the evident cyclic symmetry of the problem, in $\mathrm{PM}(\mathbb{T}_n)$ one would have more generally, for any $i \in \{1 \dots n\}$, the identity

$$(T_\alpha T_{\beta_i})^6 = A_{i,i+n-1} A_{i+1,i+n-1} \dots A_{i+n-2,i+n-1}.$$

[3] Note that here, as everywhere in the paper, I am considering curves only up to isotopy.

As a consequence, the following chain of equalities holds in $\mathrm{PM}(\mathbb{T}_n)$,

$$(T_\alpha T_{\beta_1})^6(A_{1,n}A_{2,n}\dots A_{n-1,n})^{-1} = (T_\alpha T_{\beta_i})^6(A_{i,i+n-1}A_{i+1,i+n-1}\dots A_{i+n-2,i+n-1})^{-1} = 1.$$

Lemma 2.7 asserts that, in fact, the first of these two equalities can be lifted to $\mathrm{PM}(\mathbb{T}_{n-1,1})$.

Proof of Lemma 2.7. Consider the element $G' \in \mathrm{PM}(\mathbb{T}_{n-1,1})$ obtained by multiplying the expression on the LHS of the equality, by the inverse of the expression on the RHS, that is

$$G' = (T_\alpha T_{\beta_i})^6(A_{i,i+n-1}A_{i+1,i+n-1}\dots A_{i+n-2,i+n-1})^{-1}((T_\alpha T_{\beta_1})^6(A_{1,n}A_{2,n}\dots A_{n-1,n})^{-1})^{-1}.$$

Also, set $G = (T_\alpha T_{\beta_i})^6(A_{i,i+n-1}A_{i+1,i+n-1}\dots A_{i+n-2,i+n-1})^{-1}$. As I pointed out above, the image of G' in $\mathrm{PM}(\mathbb{T}_n) = \mathrm{PM}(\mathbb{T}_{n-1,1})/\langle G\rangle$ is equal to 1. Since G is central it follows that G' must be a power of G, that is, in $\mathrm{PM}(\mathbb{T}_{n-1,1})$ $G' = G^n$ for some $n \in \mathbb{Z}$. I will show that $n = 0$. This implies that $G' = 1$ in $\mathrm{PM}(\mathbb{T}_{n-1,1})$, and proves Lemma 2.7.

The identity $G' = G^n$ is equivalent to the following,

$$(1)\quad (T_\alpha T_{\beta_i})^6(A_{i,i+n-1}A_{i+1,i+n-1}\dots A_{i+n-2,i+n-1})^{-1} = ((T_\alpha T_{\beta_1})^6(A_{1,n}A_{2,n}\dots A_{n-1,n})^{-1})^{n+1}.$$

Recall that there is a homomorphism $\mathrm{PM}(\mathbb{T}_{n-1,1}) \xrightarrow{\mathcal{F}orget} \mathrm{PM}(\mathbb{T}_{0,1})$,[4] which generalizes the map of the same name appearing in Birman exact sequence (see [FM], Section 9.1 for more details). Since the map $\mathcal{F}orget$ is induced by the inclusion $\mathbb{T}_{n-1,1} \hookrightarrow \mathbb{T}_{0,1}$, and all the β_i-s have identical isotopy class as subsets of $\mathbb{T}_{0,1}$, we have that for all $i, j \in \{1, \dots, n\}$,

$$\mathcal{F}orget(T_{\beta_i}) = \mathcal{F}orget(T_{\beta_j}) =: T_\beta, \text{ and } \mathcal{F}orget(A_{i,j}) = 1.$$

Applying $\mathcal{F}orget$ to both sides of equation (1), yields therefore the identity

$$(T_\alpha T_\beta)^6 = (T_\alpha T_\beta)^{6(n+1)}$$

in $\mathrm{PM}(\mathbb{T}_{0,1})$. As explained by Corollary 7.3 in [FM], there are no torsion elements in the mapping class group of a surface Σ, provided that its boundary set is non-empty. This is indeed the case of $\mathbb{T}_{0,1}$, and thus $(n+1)$ must equal 1, as desired. □

3. The action of $\widetilde{\mathrm{PM}}(\mathbb{T}_n)$ on $D^bCoh(X_n)$.

Let X_n be a cycle of n projective lines over a field κ. That is, X_n is a connected reduced curve with n nodal singularities, such that its normalization $\tilde{X}_n \xrightarrow{\pi} X$ is a disjoint union of n projective lines $D_1, \dots, D_n$, with the property that the pre-image along π of the singular set interesects each D_i in exactly two points. Following the discussion in Section 1 of [ST], the group acting on $D^bCoh(X_n)$ is going to be a suitable central extension of $\mathrm{PM}(\mathbb{T}_n)$, whose elements should be viewed as *graded* symplectic automorphisms of the mirror of X_n, i.e. the torus with n punctures.

Definition 3.1. Define $\widetilde{\mathrm{PM}}(\mathbb{T}_n)$ as the $\mathbb{Z}$-central extension of $\mathrm{PM}(\mathbb{T}_n)$,

$$0 \to \mathbb{Z} \to \widetilde{\mathrm{PM}}(\mathbb{T}_n) \to \mathrm{PM}(\mathbb{T}_n) \to 1,$$

generated by T_α, T_{β_i} $i \in \{1 \dots n\}$, and a central element t, subject to the following relations:

- Braid relations and Commutativity relations, as in Proposition 2.3,
- ($\tilde{G}$-relation) $(T_\alpha T_{\beta_1})^6(A_{1,n}A_{2,n}\dots A_{n-1,n})^{-1} = t^2$.

[4] By $\mathbb{T}_{0,1}$ I mean a symplectic torus with no punctures, and one boundary component.

Remark 3.2. By lifting the G_2-relation of Corollary 2.4 to the central extension, one can give an alternative presentation of $\widetilde{\mathrm{PM}}(\mathbb{T}_2)$ in which the $\tilde{G}$-relation of Definition 3.1 is replaced by the following,

- ($\tilde{G}_2$-relation) $(T_{\beta_1}T_\alpha T_{\beta_2})^4 = t^2$.

The theory of spherical objects was introduced by Seidel and Thomas in [ST]. Given a triangulated category C, under mild assumptions, to any object $\mathcal{E}$ in C such that $Hom^*(\mathcal{E},\mathcal{E})$ is isomorphic to the cohomology of the n-sphere (i.e. a *spherical object*), one can associate an autoequivalence, called *twist*, $T_{\mathcal{E}} : C \to C$.

Let $x_1 \dots x_n \in X_n$ be (closed) smooth points, such that x_i lies on the i-th irreducible component of X_n. It is easy to see that the sheaves $\mathcal{O} = \mathcal{O}_{X_n}$, $\kappa(x_i)$ $i \in \{1 \dots n\}$ in $D^b(Coh(X_n))$ are 1-spherical, and therefore determine twist functors $T_{\mathcal{O}}, T_{\kappa(x_i)}$. These equivalences, together with the *shift* functor, will define the action of $\widetilde{\mathrm{PM}}(\mathbb{T}_n)$ on $D^b(Coh(X_n))$. The main reference for the computations below are [ST] and [BK]. In [BK] the reader can find a detailed treatment of the case $n = 1$, while in [ST] Seidel and Thomas discuss the smooth case, i.e. the action of the mapping class group of a torus with no marked points on the derived category of a smooth elliptic curve.

The following lemma will be extremely useful for computations.

Lemma 3.3. *Let $F : D^b(Coh(X_n)) \to D^b(Coh(X_n))$ be an auto-equivalence of triangulated categories. If*

- *$F(\mathcal{O}) \cong \mathcal{O}$, and*
- *for all $i \in \{1 \dots n\}$, $F(\kappa(x_i)) \cong \kappa(x_i)$,*

then there exists an isomorphisms $f : X_n \to X_n$, such that F is naturally equivalent to $f^* : D^b(Coh(X_n)) \to D^b(Coh(X_n))$.

Proof. Note that X_n is projective, as X_1 is isomorphic to a nodal cubic curve in $\mathbb{P}^2$, X_2 can be embedded as the union of a line and a quadric in $\mathbb{P}^2$, and, if $n \geq 3$, X_n can be embedded as a union of n linear subspaces in $\mathbb{P}^{n-1}$. Consider the line bundle $\mathcal{L} = \mathcal{O}(x_1 + \dots + x_n)$ over X_n. $\mathcal{L}$ is ample (and very ample for $n \geq 3$). Since F preserves $\mathcal{O}$ and $\kappa(x_i)$, it is easy to see that $F(\mathcal{L}^{\otimes m}) \cong \mathcal{L}^{\otimes m}$ for all $m \in \mathbb{Z}$. In fact, $\mathcal{L}^{-1}$ is isomorphic to the kernel of any surjective morphism of sheaves $p : \mathcal{O} \to \bigoplus_{i=1}^{i=n} \kappa(x_i)$. $F(\mathcal{L}^{-1})$ is therefore isomorphic to the (co-)cone of the map

$$F(\mathcal{O})(\cong \mathcal{O}) \stackrel{F(p)}{\to} F(\bigoplus_{i=1}^{i=n} \kappa(x_i))(\cong \bigoplus_{i=1}^{i=n} \kappa(x_i)),$$

where $F(p)$ must be surjective. It follows that $F(\mathcal{L}^{-1}) \cong \mathcal{L}^{-1}$. Similarly $\mathcal{L}$ is isomorphic to the cone of any morphism in $Hom^1(\bigoplus_{i=1}^{i=n} \kappa(x_i), \mathcal{O})$ corresponding, under Serre duality, to a surjective morphism p as above, and thus $F(\mathcal{L}) \cong \mathcal{L}$. Analogous arguments can be made for all the tensor powers of $\mathcal{L}$.

From here, in order to prove the claim, is sufficient to mimic the proof of Theorem 3.1 of [BO] (see also Proposition 6.18 in [Ba], in which the argument from [BO] is applied, as here, in the context of singular algebraic varieties). A brief summary of the argument goes as follows. Note first that the functor F induces a graded automorphism of the homogeneous coordinate algebra $\bigoplus_{m=0}^{m=\infty} H^0(\mathcal{L}^{\otimes m})$, which, up to rescaling, must be given by the pull-back along an automorphism $f : X_n \to X_n$. Call C the full linear sub-category of $D^b(Coh(X_n))$ having as objects $\{\mathcal{L}^{\otimes m}\}_{m\in\mathbb{Z}}$. As explained in [BO], one can define a natural equivalence between the restrictions to C of F and f^*. Further, since X_n is projective, and $\mathcal{L}$ is ample, $\{\mathcal{L}^{\otimes m}\}_{m\in\mathbb{Z}}$ form an ample sequence in the sense of [BO] (for a proof of this, see Proposition 3.18 of [Huy]). The claim then follows from Proposition A.3 of [BO], which implies that the natural equivalence $F \cong f^*$ over C can be extended to the full derived category $D^b(Coh(X_n))$. □

Remark 3.4. Note that if $f : X_n \to X_n$ is an automorphism such that $f(x_i) = x_i$, and $n \geq 3$, then f is the identity. If $n \leq 2$, f may be non-trivial and act as a (non-trivial) permutation on the pre-image of the singular locus in the normalization. However it is immediate to see that f is an involution, i.e. $f^2 = Id$.

Lemma 3.5. *Let $x \in X_n$ be a smooth point, then*

- $T_{\kappa(x)} \cong - \otimes \mathcal{O}(x)$,
- $T_{\mathcal{O}}(\kappa(x)) \cong \mathcal{O}(-x)[1]$,
- $T_{\mathcal{O}}(\mathcal{O}(x)) \cong \kappa(x)$,
- $T_{\mathcal{O}}(\mathcal{O}) \cong \mathcal{O}$.

Proof. The first isomorphism is proved in [ST], Section 3.d. For the other isomorphisms, see Lemma 2.13 in [BK]. □

I am now ready to state the main theorem of this paper.

Theorem 3.6. *The assignment*

- *for all $i \in \{1 \dots n\}$, $T_{\beta_i} \mapsto T_{\kappa(x_i)}$,*
- *$T_\alpha \mapsto T_{\mathcal{O}}$, and*
- *$t \mapsto [1]$,*

defines a weak action of $\widetilde{\mathrm{PM}}(\mathbb{T}_n)$ on $D^b(Coh(X_n))$.

Following [ST], by weak action I mean that this assignment defines a homorphism between $\widetilde{\mathrm{PM}}(\mathbb{T}_n)$, and the group of autoequivalences of $D^b(Coh(X_n))$ modulo natural isomorphism of functors. The action defined in Theorem 3.6 depends on the choice of $x_1, \dots, x_n$. However, the action is unique up to conjugation. Note that there is a natural $(\mathbb{C}^*)^n$-action on X_n, with the property that the i-th copy of $\mathbb{C}^*$ acts by multiplication on the i-th component of X_n. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$, and let $m_\lambda : X_n \to X_n$ be the associated automorphism. Then one can show that, for all $i \in \{1, \dots, n\}$, $(m_\lambda^*)T_{\kappa(x_i)}(m_\lambda^*)^{-1} = T_{k(\lambda_i x_i)}$, and $(m_\lambda^*)T_{\mathcal{O}}(m_\lambda^*)^{-1} = T_{\mathcal{O}}$.

Proof of Theorem 3.6. I will show that Theorem 3.6 gives a well-defined homomorphism, by checking that the relations in Definiton 3.1 hold.

Braid relations. For all $i, j \in \{1, \dots, n\}$, $\mathcal{O}$, $\kappa(x_i)$ and $\kappa(x_j)$ form an A_2-configuration, in the language of [ST]. The fact that such a collection of spherical twists satisfies the braid relations is proved in Proposition 2.13 of [ST].

Commutativity relations. By Lemma 2.11 of [ST], if $\mathcal{E}_1, \mathcal{E}_2$ are spherical objects, then $T_{\mathcal{E}_2}T_{\mathcal{E}_1}T_{\mathcal{E}_2}^{-1} \cong T_{T_{\mathcal{E}_2}(\mathcal{E}_1)}$. It follows that, in order to prove the Commutativity relations, is sufficient to show that, for every $i, j, k \in \{1 \dots n\}$, $i \prec j \prec k$,

$$T_{\mathcal{O}}^{-1}T_{\kappa(x_{k+1})}^{-1}T_{\kappa(x_j)}^{-1}(T_{\mathcal{O}}^{-1}T_{\kappa(x_k)}T_{\mathcal{O}})T_{\kappa(x_j)}T_{\kappa(x_{k+1})}T_{\mathcal{O}}(\kappa(x_i)) \cong \kappa(x_i) \Leftrightarrow$$

$$T_{\mathcal{O}}^{-1}T_{\kappa(x_{k+1})}^{-1}T_{\kappa(x_j)}^{-1}(T_{\kappa(x_k)}T_{\mathcal{O}}T_{\kappa(x_k)}^{-1})T_{\kappa(x_j)}T_{\kappa(x_{k+1})}T_{\mathcal{O}}(\kappa(x_i)) \cong \kappa(x_i) \Leftrightarrow$$ [5]

$$T_{\mathcal{O}}T_{\kappa(x_k)}^{-1}T_{\kappa(x_j)}T_{\kappa(x_{k+1})}T_{\mathcal{O}}(\kappa(x_i)) \cong T_{\kappa(x_k)}^{-1}T_{\kappa(x_j)}T_{\kappa(x_{k+1})}T_{\mathcal{O}}(\kappa(x_i)).$$

By Lemma 3.5 $T_{\kappa(x_k)}^{-1}T_{\kappa(x_j)}T_{\kappa(x_{k+1})}T_{\mathcal{O}}(\kappa(x_i)) \cong \mathcal{O}(-x_i + x_j - x_k + x_{k+1})[1]$. Thus, I need to show that $T_{\mathcal{O}}(\mathcal{O}(-x_i + x_j - x_k + x_{k+1})) \cong \mathcal{O}(-x_i + x_j - x_k + x_{k+1})$. Proposition 2.12 of [ST] states that if $\mathcal{E}_1$, $\mathcal{E}_2$ are spherical objects such that $Hom^i(\mathcal{E}_1, \mathcal{E}_2) = 0$ for all i, then $T_{\mathcal{E}_1}(\mathcal{E}_2) \cong \mathcal{E}_2$. The Commutativity relations reduce therefore to the claim: for all $i, j, k \in \{1 \dots n\}$, $i \prec j \prec k$,

$$H^0(\mathcal{O}(-x_i + x_j - x_k + x_{k+1})) = H^1(\mathcal{O}(-x_i + x_j - x_k + x_{k+1})) = 0.$$

This follows from Theorem 2.2 of [DGK], which gives a general formula for computing the cohomology groups of indecomposable vector bundles over a cycle of projective lines.

$\tilde{G}$-relation. Assume first that $n \geq 3$. I will handle separately the case $n = 2$, for which I will use the alternative $\tilde{G}_2$-relation of Remark 3.2 (for the case $n = 1$, the reader should refer to [BK]). Let $i, j \in \{1 \dots n\}$, and define

$$E_{i,j} = T_{\kappa(x_{j+1})}T_{\mathcal{O}}T_{\kappa(x_{i+1})}^{-1}T_{\kappa(x_i)}T_{\mathcal{O}}^{-1}T_{\kappa(x_{j+1})}^{-1}T_{\mathcal{O}}T_{\kappa(x_i)}^{-1}T_{\kappa(x_{i+1})}T_{\mathcal{O}}^{-1}T_{\kappa(x_{i+1})}^{-1}T_{\kappa(x_i)}.$$

I need to prove that $(T_{\mathcal{O}}T_{k(x_1)})^6 \cong (E_{1,n}E_{2,n} \dots E_{n-1,n})[2]$. After Lemma 3.3 and Remark 3.4, it is sufficient to check the $\tilde{G}$-relation on $\mathcal{O}$, and $\kappa(x_i)$ for all $i \in \{1 \dots n\}$. In fact, in view of Lemma 2.7, it is enough to evaluate the $\tilde{G}$-relation on $\mathcal{O}$ and $\kappa(x_1)$, as, for any $k \in \{1 \dots n\}$,

$$(T_{\mathcal{O}}T_{\kappa(x_1)})^6(\kappa(x_k)) \cong (E_{1,n}E_{2,n} \dots E_{n-1,n})[2](\kappa(x_k)) \Leftrightarrow$$

$$(T_{\mathcal{O}}T_{\kappa(x_k)})^6(\kappa(x_k)) \cong (E_{i,i+n-1}E_{i+1,i+n-1} \dots E_{i+n-2,i+n-1})[2](\kappa(x_k)),$$

and, by the cyclic symmetry of the problem, the latter identity is proved in exactly the same way as the claim that the $\tilde{G}$-relation holds for $\kappa(x_1)$.

• $\tilde{G}$-relation on $\mathcal{O}$. Simply by keeping track of the isomorphisms collected in Lemma 3.5, one can see that

$$(T_{\mathcal{O}}T_{\kappa(x_1)})^6(\mathcal{O}) \cong (T_{\mathcal{O}}T_{\kappa(x_1)})^4(\mathcal{O}(-x_1)[1]) \cong (T_{\mathcal{O}}T_{\kappa(x_1)})^2(\kappa(x_1)[1]) \cong \mathcal{O}[2].$$

On the other hand, I will show that, for all $i, j \in \{1, \dots, n\}$, $E_{i,j}(\mathcal{O}) \cong \mathcal{O}$, and therefore

$$(E_{1,n}E_{2,n} \dots E_{n-1,n})[2](\mathcal{O}) \cong \mathcal{O}[2],$$

as expected. Note first that if $x, y \in X_n$ are smooth points lying on different connected components, then $T_{\mathcal{O}}(\mathcal{O}(x-y)) \cong \mathcal{O}(x-y)$. This again follows from Theorem 2.2 of [DGK], but can also be checked directly using the braid relations. Using this isomorphism, and Lemma 3.5, it is easy to check that

$$E_{i,j}(\mathcal{O}) = T_{\kappa(x_{j+1})}T_{\mathcal{O}}T_{\kappa(x_{i+1})}^{-1}T_{\kappa(x_i)}T_{\mathcal{O}}^{-1}T_{\kappa(x_{j+1})}^{-1}T_{\mathcal{O}}T_{\kappa(x_i)}^{-1}T_{\kappa(x_{i+1})}T_{\mathcal{O}}^{-1}T_{\kappa(x_{i+1})}^{-1}T_{\kappa(x_i)}(\mathcal{O}) \cong$$

$$T_{\kappa(x_{j+1})}T_{\mathcal{O}}T_{\kappa(x_{i+1})}^{-1}T_{\kappa(x_i)}T_{\mathcal{O}}^{-1}T_{\kappa(x_{j+1})}^{-1}T_{\mathcal{O}}T_{\kappa(x_i)}^{-1}T_{\kappa(x_{i+1})}(\mathcal{O}(x_i - x_{i+1})) \cong$$

$$T_{\kappa(x_{j+1})}T_{\mathcal{O}}T_{\kappa(x_{i+1})}^{-1}T_{\kappa(x_i)}T_{\mathcal{O}}^{-1}T_{\kappa(x_{j+1})}^{-1}(\mathcal{O}) \cong$$

$$T_{\kappa(x_{j+1})}T_{\mathcal{O}}T_{\kappa(x_{i+1})}^{-1}(\kappa(x_{j+1})[-1]) \cong \mathcal{O}.$$

[5] The isomorphism $T_{\mathcal{O}}^{-1}T_{\kappa(x_k)}T_{\mathcal{O}} \cong T_{\kappa(x_k)}T_{\mathcal{O}}T_{\kappa(x_k)}^{-1}$ follows immediately from the braid relations.

- $\tilde{G}$-relation on $\kappa(x_1)$. As before, it is enough to apply Lemma 3.5 to see that

$$(T_{\mathcal{O}}T_{\kappa(x_1)})^6(\kappa(x_1)) \cong (T_{\mathcal{O}}T_{\kappa(x_1)})^4(\mathcal{O}[1]) \cong (T_{\mathcal{O}}T_{\kappa(x_1)})^2(\mathcal{O}(-x_1)[2]) \cong \kappa(x_1)[2].$$

Further, for all $i \in \{1,\dots,n-1\}$, $E_{i,n}(\kappa(x_1)) \cong \kappa(x_1)$. In fact,

$$E_{i,n}(\kappa(x_1)) = T_{\kappa(x_1)}T_{\mathcal{O}}T^{-1}_{\kappa(x_{i+1})}T_{\kappa(x_i)}T^{-1}_{\mathcal{O}}T^{-1}_{\kappa(x_1)}T_{\mathcal{O}}T^{-1}_{\kappa(x_i)}T_{\kappa(x_{i+1})}T^{-1}_{\mathcal{O}}T^{-1}_{\kappa(x_{i+1})}T_{\kappa(x_i)}(\kappa(x_1)) \cong$$
$$T_{\kappa(x_1)}T_{\mathcal{O}}T^{-1}_{\kappa(x_{i+1})}T_{\kappa(x_i)}T^{-1}_{\mathcal{O}}T^{-1}_{\kappa(x_1)}T_{\mathcal{O}}T^{-1}_{\kappa(x_i)}T_{\kappa(x_{i+1})}(\mathcal{O}(x_1)).$$

Now,

$$T_{\kappa(x_1)}T_{\mathcal{O}}T^{-1}_{\kappa(x_{i+1})}T_{\kappa(x_i)}T^{-1}_{\mathcal{O}}T^{-1}_{\kappa(x_1)}T_{\mathcal{O}}T^{-1}_{\kappa(x_i)}T_{\kappa(x_{i+1})}(\mathcal{O}(x_1)) \cong \kappa(x_1) \Leftrightarrow$$
$$T^{-1}_{\kappa(x_{i+1})}T_{\kappa(x_i)}(T^{-1}_{\mathcal{O}}T^{-1}_{\kappa(x_1)}T_{\mathcal{O}})T^{-1}_{\kappa(x_i)}T_{\kappa(x_{i+1})}(\mathcal{O}(x_1)) \cong \mathcal{O}(x_1) \Leftrightarrow^{6}$$
$$T^{-1}_{\kappa(x_{i+1})}T_{\kappa(x_i)}(T_{\kappa(x_1)}T^{-1}_{\mathcal{O}}T^{-1}_{\kappa(x_1)})T^{-1}_{\kappa(x_i)}T_{\kappa(x_{i+1})}(\mathcal{O}(x_1)) \cong \mathcal{O}(x_1) \Leftrightarrow$$
$$T^{-1}_{\mathcal{O}}T^{-1}_{\kappa(x_1)}T^{-1}_{\kappa(x_i)}T_{\kappa(x_{i+1})}(\mathcal{O}(x_1)) \cong \mathcal{O}(x_{i+1}-x_i) \Leftrightarrow$$
$$\mathcal{O}(x_{i+1}-x_i) \cong T_{\mathcal{O}}(\mathcal{O}(x_{i+1}-x_i)).$$

As I pointed out above, this last isomorphism can be proved using the braid relations. Thus

$$(E_{1,n}E_{2,n}\dots E_{n-1,n})[2](\kappa(x_1)) \cong \kappa(x_1)[2],$$

and this concludes the proof of Theorem 3.6 for the case $n \geq 3$.

The case $n = 2$. Note that there are isomorphisms

- $(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})^2(\mathcal{O}) \cong \mathcal{O}[1]$, and
- $(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})^2(\kappa(x_1)) \cong \kappa(x_2)[1]$, $(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})^2(\kappa(x_2)) \cong \kappa(x_1)[1]$.

Let us check this for $\kappa(x_1)$:

$$(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})(\kappa(x_1)) \cong (T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})(\mathcal{O}[1]) \cong \kappa(x_2)[1].$$

Consider an involution $\sigma : X_2 \to X_2$ such that $\sigma(x_1) = x_2$, and $\sigma(x_2) = x_1$. It follows from Remark 3.4 that there is an isomorphism $f : X_2 \to X_2$, and a natural equivalence $(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})^2 \cong f^*\sigma^*[1]$. As σ and f commute, by taking the square of this natural equivalence, one gets

$$(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})^4 \cong (f^*\sigma^*[1])(f^*\sigma^*[1]) \cong (f^*)^2(\sigma^*)^2[2] \cong [2].$$

In view of Remark 3.2, this implies that the action of $\widetilde{\mathrm{PM}}(\mathbb{T}_2)$ on $D^b(Coh(X_2))$ is well defined, and proves the case $n = 2$ of Theorem 3.6. □

Remark 3.7. It follows from results in Appendix D of [B], that the action defined by Theorem 3.6 is, in an appropriate sense, a categorification of the symplectic representation of the mapping class group. Denote $K^{num}(X_n)$ the quotient of $K_0(\mathcal{P}\mathrm{erf}(X_n))$ by the radical of the Euler form (see Appendix D of [B] for further details). The Euler form induces a non-degenerate skew-symmetric form on $K^{num}(X_n)$, and there is an isomorphism of symplectic lattices $K^{num}(X_n) \cong H_1(\mathbb{T}_n, \mathbb{Z})(\cong \mathbb{Z}^{n+1})$ (here, $\mathbb{T}_n$ denotes the torus with the n marked points removed). Note that the induced action of $\tilde{\mathrm{PM}}(\mathbb{T}_n)$ on $K^{num}(X_n)$ factors through $\mathrm{PM}(\mathbb{T}_n) \oplus \mathbb{Z}_2$. Bodnarchuk's computations imply that the resulting action of

[6] The isomorphism $T^{-1}_{\mathcal{O}}T^{-1}_{\kappa(x_1)}T_{\mathcal{O}} \cong T_{\kappa(x_1)}T^{-1}_{\mathcal{O}}T^{-1}_{\kappa(x_1)}$ is obtained from the one in Footnote 5, by taking inverses on both sides of the "≅" sign.

$\mathrm{PM}(\mathbb{T}_n)$ on $K^{num}(X_n)$ is isomorphic to the standard symplectic representation of $\mathrm{PM}(\mathbb{T}_n)$ over $H_1(\mathbb{T}_n, \mathbb{Z})$.

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