

# KÄHLER GROUPS, QUASI-PROJECTIVE GROUPS, AND 3-MANIFOLD GROUPS

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**ABSTRACT.** We prove two results relating 3-manifold groups to fundamental groups occurring in complex geometry. Let  $N$  be a compact, connected, orientable 3-manifold. If  $N$  has non-empty, toroidal boundary, and  $\pi_1(N)$  is a Kähler group, then  $N$  is the product of a torus with an interval. On the other hand, if  $N$  has either empty or toroidal boundary, and  $\pi_1(N)$  is a quasi-projective group, then all the prime components of  $N$  are graph manifolds.

## 1. INTRODUCTION AND MAIN RESULTS

1.1. A classical problem, going back to J.-P. Serre, asks for a characterization of fundamental groups of smooth projective varieties. Still comparatively little is known about this class of groups. For instance, it is not known whether it coincides with the putatively larger class of groups that can be realized as fundamental groups of compact Kähler manifolds, known for short as Kähler groups.

Another, very much studied class of groups is that consisting of fundamental groups of compact, connected, 3-dimensional manifolds, known for short as 3-manifold groups. Recent years have seen the complete validation of the Thurston program for understanding 3-manifolds. This effort, begun with the proof of the Geometrization Conjecture by Perelman, has culminated in the results of Agol [Ag12], Wise [Wi12a] and Przytycki–Wise [PW12] which now give us a remarkably good understanding of 3-manifold groups (see also [AFW12] for more information).

In this context, a natural question arises: Which 3-manifold groups are Kähler groups? This question, first raised by S. Donaldson and W. Goldman in 1989, and independently by Reznikov in 1993 (see [Re02]), has led to a flurry of activity in recent years. In [DS09] the second author and A. Dimca showed that if the fundamental group of a closed 3-manifold is Kähler, then the group is finite. Alternative proofs have since been given by Kotschick [Kot12] and by Biswas, Mj and Seshadri [BMS12], while the analogous question for quasi-Kähler groups was considered in [DPS11].

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1.2. In this paper, we pursue these lines of inquiry in two directions. First, we determine which fundamental groups of 3-manifolds with non-empty, toroidal boundary are Kähler. Note that the 2-torus is a Kähler manifold. We will show that, in fact,  $\mathbb{Z}^2$  is the only fundamental group of a 3-manifold with non-empty, toroidal boundary which is also a Kähler group. More precisely, we will prove the following theorem.

**Theorem 1.1.** *Let  $N$  be a 3-manifold with non-empty, toroidal boundary. If  $\pi_1(N)$  is a Kähler group, then  $N \cong S^1 \times S^1 \times [0, 1]$ .*

If  $N$  is allowed to have non-toroidal boundary components, other Kähler groups can appear. For instance, if  $\Sigma_g$  is a Riemann surface of genus  $g \geq 2$ , then  $\pi_1(\Sigma_g \times [0, 1])$  is certainly a Kähler group. The complete, full-generality classification of Kähler 3-manifold groups is still unknown to us.

1.3. Next, we turn to the question, which 3-manifold groups are quasi-projective, that is, occur as fundamental groups of complements of divisors in a smooth, complex projective variety. Examples of 3-manifolds with fundamental groups that are quasi-projective are given by the exteriors of torus knots and Hopf links, by connected sums of  $S^1 \times S^2$ 's, and by Brieskorn manifolds; we refer to [DPS11, §1.2] for more examples. Note that all these examples are connected sums of graph manifolds.

We will show that this is not a coincidence:

**Theorem 1.2.** *Let  $N$  be a 3-manifold with empty or toroidal boundary. If  $\pi_1(N)$  is a quasi-projective group, then all the prime components of  $N$  are graph manifolds.*

In particular, the fundamental group of a hyperbolic 3-manifold with empty or toroidal boundary is never a quasi-projective group.

In fact, as we shall see in §§7.3 and 8, we can get even finer results. For instance, if a 3-manifold  $N$  as above is not prime, then none of its prime components is virtually fibered. But, at the moment, the complete determination of which 3-manifold groups are quasi-projective groups is still out of reach for us.

1.4. The proofs of Theorems 1.1 and 1.2 rely on some deep results from both complex geometry and 3-dimensional topology. We extract from those results precise qualitative statements about the Alexander polynomials of the manifolds that occur in those settings. Comparing the answers gives us enough information to whittle down the list of fundamental groups that may occur in the opposite settings until we reach the desired conclusions.

Results by Arapura [Ar97], as strengthened in [DPS09] and [ACM10], put strong restrictions on the characteristic varieties of both Kähler manifolds and smooth, quasi-projective varieties. Following the approach from [DPS08], we use those restrictions to conclude that the respective Alexander polynomials are “thin”: their Newton polytopes have dimension at most 1.

Recent results of Agol [Ag08, Ag12], Kahn–Markovic [KM12], Przytycki–Wise [PW12] and Wise [Wi12a, Wi12b] give us a good understanding of fundamental groups of irreducible 3-manifolds which are not graph manifolds. Using those results, we show that every irreducible 3-manifold with empty or toroidal boundary is either not a graph manifold, or it admits a finite cover which has a “thick” Alexander polynomial.

We conclude this paper with a short list of open questions.

**Convention.** All groups are understood to be finitely presented and all manifolds are understood to be compact, orientable and connected, unless otherwise stated.

## 2. ALEXANDER POLYNOMIALS, CHARACTERISTIC VARIETIES, AND THICKNESS

**2.1. Orders of modules.** Let  $H$  be a finitely generated, free abelian group and let  $M$  be a finitely generated module over the group ring  $\mathbb{Z}[H]$ . Since the ring  $\mathbb{Z}[H]$  is Noetherian, there exists a finite presentation

$$(1) \quad \mathbb{Z}[H]^r \xrightarrow{\alpha} \mathbb{Z}[H]^s \longrightarrow M \longrightarrow 0.$$

We can furthermore arrange that  $r \geq s$ , by adding zero columns if necessary.

Given an integer  $k \geq 0$ , the  $k$ -th elementary ideal of  $M$ , denoted by  $E_k(M)$ , is the ideal in  $\mathbb{Z}[H]$  generated by all minors of size  $s - k$  of the matrix  $\alpha$ . The  $k$ -th order of the module  $M$  is a generator of the smallest principal ideal in  $\mathbb{Z}[H]$  containing  $E_k(M)$ ; that is,

$$(2) \quad \text{ord}_{\mathbb{Z}[H]}^k(M) = \text{gcd of all } (s - k) \times (s - k)\text{-minors of } \alpha.$$

Note that the element  $\text{ord}_{\mathbb{Z}[H]}^k(M) \in \mathbb{Z}[H]$  is well-defined up to multiplication by a unit in  $\mathbb{Z}[H]$ ; if the ring  $\mathbb{Z}[H]$  is understood, then we will just write  $\text{ord}^k(M)$ . Furthermore, we will write  $\text{ord}_{\mathbb{Z}[H]}(M) := \text{ord}_{\mathbb{Z}[H]}^0(M)$ .

Denote by  $\text{Tors}_{\mathbb{Z}[H]} M$  the  $\mathbb{Z}[H]$ -torsion submodule of  $M$ . Note that  $\text{ord}^0(M) \neq 0$  if and only if  $M$  is a torsion  $\mathbb{Z}[H]$ -module (see e.g. [Tu01, Remark 4.5]). Furthermore, denote by  $r$  the rank of  $M$  as a  $\mathbb{Z}[H]$ -module. It then follows from [Tu01, Lemma 4.9] that

$$(3) \quad \text{ord}^i(M) = \begin{cases} 0 & \text{if } i < r, \\ \text{ord}^{r-i}(\text{Tors}_{\mathbb{Z}[H]} M) & \text{if } i \geq r. \end{cases}$$

**2.2. The thickness of a module.** As before, let  $H$  be a finitely generated, free abelian group, and let  $M$  be a finitely generated  $\mathbb{Z}[H]$ -module. Write

$$(4) \quad \text{ord}_{\mathbb{Z}[H]}(\text{Tors}_{\mathbb{Z}[H]} M) = \sum_{h \in H} a_h h.$$

**Definition 2.1.** The *thickness* of the  $\mathbb{Z}[H]$ -module  $M$  is the integer

$$(5) \quad \text{th}_{\mathbb{Z}[H]}(M) := \dim \text{span}\{g - h \in H \otimes \mathbb{Q} \mid a_g \neq 0 \text{ and } a_h \neq 0\}.$$

Put differently, the thickness  $\text{th}(M) := \text{th}_{\mathbb{Z}[H]}(M)$  is the dimension of the Newton polyhedron of the Laurent polynomial  $\text{ord}_{\mathbb{Z}[H]}(\text{Tors}_{\mathbb{Z}[H]} M)$ . Note that the definition does not depend on a representative for this polynomial. Later on we will make use of the following lemma.

**Lemma 2.2.** *Let  $H_1, \dots, H_r$  be free abelian groups, and let  $M_i$  be finitely generated modules over  $\mathbb{Z}[H_i]$ . Set  $H := \bigoplus_{i=1}^r H_i$ , and view  $\mathbb{Z}[H_i]$  as subrings of  $\mathbb{Z}[H]$ . Then for  $i = 1, \dots, r$  we have*

$$\text{ord}_{\mathbb{Z}[H]}(\text{Tors}_{\mathbb{Z}[H]}(M_i \otimes_{\mathbb{Z}[H_i]} \mathbb{Z}[H])) = \text{ord}_{\mathbb{Z}[H_i]} \text{Tors}_{\mathbb{Z}[H_i]} M_i,$$

and furthermore

$$\mathrm{th}_{\mathbb{Z}[H]} \left( \bigoplus M_i \otimes_{\mathbb{Z}[H_i]} \mathbb{Z}[H] \right) = \sum \mathrm{th}_{\mathbb{Z}[H_i]}(M_i).$$

*Proof.* It follows easily from the definitions that

$$\mathrm{Tors}_{\mathbb{Z}[H]}(M_i \otimes_{\mathbb{Z}[H_i]} \mathbb{Z}[H]) = (\mathrm{Tors}_{\mathbb{Z}[H_i]} M_i) \otimes_{\mathbb{Z}[H_i]} \mathbb{Z}[H].$$

In particular,

$$\mathrm{ord}_{\mathbb{Z}[H]}(\mathrm{Tors}_{\mathbb{Z}[H]}(M_i \otimes_{\mathbb{Z}[H_i]} \mathbb{Z}[H])) = \mathrm{ord}_{\mathbb{Z}[H_i]} \mathrm{Tors}_{\mathbb{Z}[H_i]} M_i,$$

and so

$$\mathrm{ord}_{\mathbb{Z}[H]} \mathrm{Tors}_{\mathbb{Z}[H]} \left( \bigoplus M_i \otimes_{\mathbb{Z}[H_i]} \mathbb{Z}[H] \right) = \prod \mathrm{ord}_{\mathbb{Z}[H_i]} \mathrm{Tors}_{\mathbb{Z}[H_i]} M_i.$$

The desired conclusion follows at once.  $\square$

**2.3. Alexander polynomials.** Let  $X$  be a connected CW-complex with finitely many 1-cells. We denote by  $\tilde{X}$  the universal cover of  $X$ . Note that  $\pi_1(X)$  canonically acts on  $\tilde{X}$  on the left; we use the natural involution  $g \mapsto g^{-1}$  on  $\pi_1(X)$  to endow  $\tilde{X}$  with a right  $\pi_1(X)$ -action.

Let  $H := H_1(X; \mathbb{Z})/\mathrm{Tors}$  be the maximal torsion-free abelian quotient of  $\pi_1(X)$ . We view  $\mathbb{Z}[H]$  as a left  $\pi_1(X)$ -module via the canonical projection  $\pi_1(X) \twoheadrightarrow H$ . Consider the tensor product  $C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z}[H]$ . This defines a chain complex of  $\mathbb{Z}[H]$ -modules, and we denote its homology groups by  $H_*(X; \mathbb{Z}[H])$ . Most important for our purposes is the *Alexander invariant*,  $A_X = H_1(X; \mathbb{Z}[H])$ .

For each integer  $k \geq 0$ , we define the  $k$ -th *Alexander polynomial* of  $X$  as

$$(6) \quad \Delta_X^k := \mathrm{ord}_{\mathbb{Z}[H]}^k(A_X).$$

The Laurent polynomial  $\Delta_X^k \in \mathbb{Z}[H]$  is well-defined up to multiplication by a unit in  $\mathbb{Z}[H]$ , and only depends on  $\pi_1(X)$ . We write  $\Delta_X := \Delta_X^0$ , and call it the *Alexander polynomial* of  $X$ . If  $\pi$  is a finitely generated group, then we denote by  $\Delta_\pi^k$  the Alexander polynomials of its Eilenberg–MacLane space. Note that  $\Delta_X^k = \Delta_{\pi_1(X)}^k$ .

We denote the thickness of the module  $A_X$  by  $\mathrm{th}(X)$ . It follows from the definitions and formula (3) that

$$(7) \quad \mathrm{th}(X) = \dim(\mathrm{Newt}(\Delta_X^r)),$$

where  $r = \mathrm{rank}_{\mathbb{Z}[H]}(A_X)$ . In particular, if the Alexander invariant is a torsion  $\mathbb{Z}[H]$ -module, then  $\mathrm{th}(X) = \dim(\mathrm{Newt}(\Delta_X))$ .

Let  $\mathbb{C}^*$  be the multiplicative group of non-zero complex numbers. We shall call the connected algebraic group  $\hat{H} = \mathrm{Hom}(H, \mathbb{C}^*)$  the *character torus* of  $X$ . The Laurent polynomial  $\Delta_X^k$  can be viewed as a regular function on  $\hat{H}$ . As such, it defines a hypersurface,

$$(8) \quad V(\Delta_X^k) = \{\rho \in \hat{H} \mid \Delta_X^k(\rho) = 0\}.$$

For instance, if  $K$  is a knot in  $S^3$ , with exterior  $X = S^3 \setminus \nu K$ , then  $\Delta_X$  is the classical Alexander polynomial of the knot, and  $V(\Delta_X^k) \subset \mathbb{C}^*$  is the set of roots of  $\Delta_X$ , of multiplicity at least  $k$ .

**2.4. Homology jump loci.** The *characteristic varieties* of  $X$  are the jump loci for homology with coefficients in the rank 1 local systems defined by characters inside the character torus of  $X$ . For each  $k \geq 1$ , the set

$$(9) \quad \mathcal{V}_k(X) = \{\rho \in \widehat{H} \mid \dim H_1(X, \mathbb{C}_\rho) \geq k\}$$

is a Zariski closed subset of  $\widehat{H}$ . These varieties depend only on the group  $\pi = \pi_1(X)$ , so we will sometimes write them as  $\mathcal{V}_k(\pi)$ . For more details on all this, we refer to [Su11, Su12].

As shown by Hironaka [Hir97], the characteristic varieties coincide with the varieties defined by the Alexander ideals of  $X$ , at least away from the trivial representation. More precisely,

$$(10) \quad \mathcal{V}_k(X) \setminus \{1\} = V(E_{k-1}(A_X)) \setminus \{1\}.$$

The next lemma details the relationship between the hypersurfaces defined by the Alexander polynomials of  $X$  and the characteristic varieties of  $X$ . (The case  $k = 1$  was proved by similar methods in [DPS08, Corollary 3.2].)

**Lemma 2.3.** *For each  $k \geq 1$ , let  $\check{\mathcal{V}}_k(X)$  be the union of all codimension-one irreducible components of  $\mathcal{V}_k(X)$ . Then,*

- (1)  $\Delta_X^{k-1} = 0$  if and only if  $\mathcal{V}_k(X) = \widehat{H}$ , in which case  $\check{\mathcal{V}}_k(X) = \emptyset$ .
- (2) If  $b_1(X) \geq 1$  and  $\Delta_X^{k-1} \neq 0$ , then

$$\check{\mathcal{V}}_k(X) = \begin{cases} V(\Delta_X^{k-1}) & \text{if } b_1(X) \geq 2 \\ V(\Delta_X^{k-1}) \amalg \{1\} & \text{if } b_1(X) = 1. \end{cases}$$

*Proof.* Given an ideal  $\mathfrak{a} \subset \mathbb{Z}[H]$ , let  $\check{V}(\mathfrak{a})$  be the union of all codimension-one irreducible components of the subvariety  $V(\mathfrak{a}) \subset \widehat{H}$  defined by  $\mathfrak{a}$ . As noted in [DPS08, Lemma 3.1], we have that  $V(\gcd(\mathfrak{a})) = \check{V}(\mathfrak{a})$ .

Applying this observation to the ideal  $\mathfrak{a} = E_{k-1}(A_X)$ , and using formula (10), we see that  $V(\Delta_X^{k-1}) = \check{\mathcal{V}}_k(X)$ , at least away from the identity. The desired conclusions follow at once.  $\square$

The next theorem generalizes Proposition 3.7 from [DPS08], which treats the case  $k = 1$  along the same lines. For the sake of completeness, we provide full details.

**Theorem 2.4.** *Suppose  $b_1(X) \geq 2$ . Then  $\Delta_X^{k-1} \doteq \text{const}$  if and only if  $\check{\mathcal{V}}_k(X) = \emptyset$ ; otherwise, the following are equivalent:*

- (1) *The Newton polytope of  $\Delta_X^{k-1}$  is a line segment.*
- (2) *All irreducible components of  $\check{\mathcal{V}}_k(X)$  are parallel, codimension-one subtori of  $\widehat{H}$ .*

*Proof.* The first equivalence follows at once from Lemma 2.3. So let us assume  $\Delta := \Delta_X^{k-1}$  is non-constant, and set  $n = b_1(X)$ .

First suppose (1) holds. Then, in a suitable coordinate system  $(t_1, \dots, t_n)$  on  $\widehat{H} = (\mathbb{C}^*)^n$ , the polynomial  $\Delta$  can be written as  $(t_1 - z_1)^{\alpha_1} \dots (t_n - z_n)^{\alpha_n}$ , for some pairwise distinct, non-zero complex numbers  $z_i$  and positive exponents  $\alpha_i$ . From Lemma 2.3, we conclude that  $\check{\mathcal{V}}_k(X)$  is the (disjoint) union of the parallel subtori  $\{t_1 = z_1\}, \dots, \{t_n = z_n\}$ .

Next, suppose (2) holds. Then again in a suitable coordinate system, we have that  $\check{V}_k(X) = \bigcup_i \{t_1 = z_i\}$ . Now let  $\Delta = f_1^{\beta_1} \cdots f_q^{\beta_q}$  be the decomposition of  $\Delta$  into irreducible factors. Then  $V(\Delta)$  decomposes into irreducible components as  $\bigcup_j \{f_j = 0\}$ . Since the two decompositions into irreducible components of  $\check{V}_k(X) = V(\Delta)$  must agree, we must have that  $\Delta \doteq \prod_i (t_1 - z_i)^{\alpha_i}$ . Hence,  $\text{Newt}(\Delta)$  is a line segment, and we are done.  $\square$

### 3. KÄHLER GROUPS

A *Kähler manifold* is a compact, connected, complex manifold without boundary, admitting a Hermitian metric  $h$  for which the imaginary part  $\omega = \text{im}(h)$  is a closed 2-form. The class of Kähler manifolds, is closed under finite direct products. The main source of examples are smooth, complex projective varieties, such as Riemann surfaces, complex Grassmannians, and abelian varieties.

Now suppose  $\pi$  is a Kähler group, i.e., there is a Kähler manifold  $M$  with  $\pi = \pi_1(M)$ . This condition puts severe restrictions on the group  $\pi$ , besides the obvious fact that  $\pi$  must be finitely presented (we refer to [A–T96] for a comprehensive survey).

For instance, the first Betti number  $b_1(\pi)$  must be even, and all higher-order Massey products of classes in  $H^1(\pi, \mathbb{Q})$  vanish. Furthermore, the group  $\pi$  cannot split as a non-trivial free product, by work of Gromov [Gr89] and Arapura, Bressler, and Ramachandran [ABR92]. Finally, as shown in [DPS09], the only right-angled Artin groups which are also Kähler groups are the free abelian groups of even rank.

The pull-back of a Kähler metric to a finite cover is again a Kähler metric. It follows that the finite cover of a Kähler manifold is also a Kähler manifold. We thus have the following lemma (see also [A–T96, Example 1.10].)

**Lemma 3.1.** *Any finite-index subgroup of a Kähler group is again a Kähler group.*

An analogous result holds for fundamental groups of smooth, complex projective varieties.

The basic structure of the characteristic varieties of Kähler manifolds was described in work of Beauville, Green–Lazarsfeld, Simpson, Campana, and Arapura [Ar97]. We state a simplified version of this result, in the form we need it.

**Theorem 3.2.** *Let  $M$  be a Kähler manifold. Then, for each  $k \geq 1$ , all the positive-dimensional, irreducible components of  $\mathcal{V}_k(M)$  are even-dimensional subtori of the character torus of  $M$ , possibly translated by torsion characters.*

The Alexander polynomial of a Kähler group is highly restricted. The next result sharpens Theorem 4.3(3) from [DPS08], where a similar result is proved in the case when  $\pi$  is the fundamental group of a smooth projective variety, and  $k = 0$ .

**Theorem 3.3.** *If  $\pi$  be a Kähler group. Then, for any  $k \geq 0$ , the polynomial  $\Delta_\pi^k$  is a constant. In particular,  $\text{th}(\pi) = 0$ .*

*Proof.* From Hodge theory, we know that  $b_1(\pi)$  is even. If  $b_1(\pi) = 0$ , there is nothing to prove; so we may as well assume  $b_1(\pi) \geq 2$ .

Now, from Theorem 3.2 we know that all positive-dimensional irreducible components of  $\mathcal{V}_{k+1}(\pi)$  are even-dimensional. Thus, there are no codimension-one components in  $\mathcal{V}_{k+1}(\pi)$ ; in other words,  $\check{\mathcal{V}}_{k+1}(\pi) = \emptyset$ . Finally, Theorem 2.4 implies that  $\Delta_\pi^k$  is constant.  $\square$

## 4. QUASI-PROJECTIVE GROUPS

A manifold  $X$  is said to be a (*smooth*) *quasi-projective variety* if there is a connected, smooth, complex projective variety  $\overline{X}$  and a divisor  $D$  such that  $X = \overline{X} \setminus D$ . (If  $\overline{X}$  is only known to admit a Kähler metric, then  $X$  is said to be a *quasi-Kähler manifold*.) Using resolution of singularities, one may choose the compactification  $\overline{X}$  so that  $D = \overline{X} \setminus X$  is a normal-crossings divisor. An important source of examples is provided by complements of hypersurfaces in  $\mathbb{C}\mathbb{P}^n$ .

Now suppose  $\pi$  is a quasi-projective group, i.e., there is a smooth, quasi-projective variety  $X$  such that  $\pi = \pi_1(X)$ . Again, the group  $\pi$  must be finitely presented, but much weaker restrictions are now imposed on  $\pi$  than in the case of Kähler groups. For instance,  $b_1(\pi)$  can be arbitrary, non-trivial Massey products can occur, and  $\pi$  can split as a non-trivial free product.

Examples of quasi-projective groups include all finitely generated free groups  $F_n$ , which may be realized as  $\pi_1(\mathbb{C}\mathbb{P}^1 \setminus \{n+1 \text{ points}\})$ , and all free abelian groups  $\mathbb{Z}^n$ , which may be realized as  $\pi_1((\mathbb{C}^*)^n)$ . In fact, a right-angled Artin group  $\pi$  is quasi-projective if and only if  $\pi$  is a direct product of free groups (possibly infinite cyclic), see [DPS09].

The analog of Lemma 3.1 holds for quasi-projective groups. Though this is presumably folklore, we could not find an explicit reference in the literature, so we include a proof, kindly supplied to us by Donu Arapura.

**Lemma 4.1.** *Let  $\pi$  be a group, and let  $\tilde{\pi}$  be a finite-index subgroup. If  $\pi$  is quasi-projective, then  $\tilde{\pi}$  is quasi-projective as well.*

*Proof.* Let  $X$  be a smooth quasi-projective variety, and let  $Y \rightarrow X$  be a finite cover. Choose a projective compactification  $\overline{X}$  of  $X$ , and normalize  $\overline{X}$  in the function field of  $Y$  to get  $\overline{Y}$ . Then  $\overline{Y}$  is a normal variety containing  $Y$  as an open subset. Moreover,  $\overline{Y}$  is a projective variety, since the pullback of an ample line bundle under the finite cover  $\overline{Y} \rightarrow \overline{X}$  is again ample. Thus,  $Y$  is a smooth quasi-projective variety, and this finishes the proof.  $\square$

The analogous (and more difficult) result for quasi-Kähler groups is proved in [AN99, Lemma 4.1].

The basic structure of the cohomology support loci of smooth, quasi-projective varieties was established by Arapura [Ar97]. Additional information on the nature of these varieties has been provided in work of Dimca [Di07], Dimca, Papadima and Suciu [DPS08, DPS09], Artal-Bartolo, Cogolludo and Matei [ACM10], and most recently, by Budur and Wang [BW12]. The next theorem summarizes some of those known results, in the form needed here.

**Theorem 4.2.** *Let  $X$  be a smooth, quasi-projective variety, and set  $H = H_1(X, \mathbb{Z})/\text{Tors}$ . Then, for each  $k \geq 1$ , the following hold:*

- (1) *Every irreducible component of  $\mathcal{V}_k(X)$  is of the form  $\rho T$ , where  $T$  is an algebraic subtorus of  $\widehat{H}$ , and  $\rho$  is a torsion element in  $\widehat{H}$ .*
- (2) *If  $\rho_1 T_1$  and  $\rho_2 T_2$  are two such components, then either  $T_1 = T_2$ , or  $T_1 \cap T_2$  is finite.*

*Proof.* Statement (1) is proved in [Ar97] for the wider class of quasi-Kähler manifolds  $X$ , but only for positive-dimensional irreducible components: in that generality, the isolated

points in  $\mathcal{V}_k(X)$  are only known to be unitary characters. Now, if  $X$  is a smooth quasi-projective variety, it is shown in [ACM10, Theorem 1], and also in [BW12, Theorem 1.1], that the isolated points in  $\mathcal{V}_k(X)$  are, in fact, torsion points.

Statement (2) is proved in [DPS08, Theorem 4.2], for  $k = 1$ ; the general case is established in [ACM10, Proposition 6.5].  $\square$

The intersection theory of translated subtori in a complex algebraic torus was worked out by E. Hironaka in [Hir96], and was further developed in [Na09, SYZ13]. In particular, the following lemma holds.

**Lemma 4.3.** *Let  $T_1$  and  $T_2$  be two algebraic subtori in  $(\mathbb{C}^*)^n$ , and let  $\rho_1$  and  $\rho_2$  be two elements in  $(\mathbb{C}^*)^n$ . Then  $\rho_1 T_1 \cap \rho_2 T_2 \neq \emptyset$  if and only if  $\rho_1 \rho_2^{-1} \in T_1 T_2$ , in which case  $\dim(\rho_1 T_1 \cap \rho_2 T_2) = \dim(T_1 \cap T_2)$ .*

In view of this lemma, Theorem 4.2 has the following immediate corollary.

**Corollary 4.4.** *Let  $\pi$  be a quasi-projective group. Then, for each  $k \geq 1$ , the irreducible components of  $\mathcal{V}_k(\pi)$  are (possibly torsion-translated) subtori of the character torus  $\widehat{H}$ . Furthermore, any two distinct components of  $\mathcal{V}_k(\pi)$  meet in at most finitely many (torsion) points.*

The Alexander polynomial of a quasi-projective group must satisfy certain rather restrictive conditions, though not as stringent as in the Kähler case. The next result sharpens Theorem 4.3 from [DPS08], while following a similar approach.

As before, given a finitely-generated group  $\pi$ , let  $H = H_1(\pi; \mathbb{Z})/\text{Tors}$ , written multiplicatively. Given a polynomial  $p(t) = \sum_{i \in \mathbb{Z}} a_i t^i \in \mathbb{Z}[t^{\pm 1}]$ , and an element  $h \in H$ , we put  $p(h) := \sum_{i \in \mathbb{Z}} a_i h^i \in \mathbb{Z}[H]$ .

**Theorem 4.5.** *Let  $\pi$  be a quasi-projective group, and assume  $b_1(\pi) \neq 2$ . Then, for any  $k \geq 0$ , the following hold.*

- (1) *The polynomial  $\Delta_\pi^k$  is either zero, or the Newton polytope of  $\Delta_\pi^k$  is a point or a line segment. In particular,  $\text{th}(\pi) \leq 1$ .*
- (2) *There exists a polynomial  $p(t) \in \mathbb{Z}[t^{\pm 1}]$  of the form  $c \cdot q(t)$ , where  $c \in \mathbb{Z} \setminus \{0\}$  and  $q(t)$  is a product of cyclotomic polynomials, and an element  $h \in H$ , such that  $\Delta_\pi^k = p(h)$ .*

*Proof.* We start with part (1). Set  $n = b_1(\pi)$ . If either  $n = 0$  or  $1$  or  $\Delta_\pi^k$  is zero, there is nothing to prove. So we may as well assume that  $n \geq 3$ , and  $\Delta_\pi^k \neq 0$ .

From Corollary 4.4, we know that all irreducible components of  $\mathcal{V}_{k+1}(\pi)$  are (possibly translated) subtori of  $\widehat{H} = (\mathbb{C}^*)^n$ , meeting in at most finitely many points. Suppose  $\rho_1 T_1$  and  $\rho_2 T_2$  are two components of codimension 1, that are not parallel. Then, by Lemma 4.3,

$$(11) \quad \begin{aligned} \dim(\rho_1 T_1 \cap \rho_2 T_2) &= \dim(T_1 \cap T_2) \\ &= \dim(T_1) + \dim(T_2) - \dim(T_1 T_2) = n - 2 \geq 1, \end{aligned}$$

a contradiction. Thus, all codimension-one components of  $\mathcal{V}_{k+1}(\pi)$  must be parallel subtori. Claim (1) then follows from Theorem 2.4.

We now prove part (2). Using the previous part, we may find a polynomial  $p \in \mathbb{Z}[t]$  such that, after a change of variables if necessary,  $\Delta_\pi^k(t_1, \dots, t_n) = p(t_1)$ . Clearly,  $V(\Delta_\pi^k) =$



$V(p) \times (\mathbb{C}^*)^{n-1}$ . On the other hand, by Lemma 2.3 and Theorem 4.2, the hypersurface  $V(\Delta_\pi^k)$  is a (possibly torsion-translated) subtorus in  $(\mathbb{C}^*)^n$ . Thus, all roots of  $p$  must be roots of unity, and claim (2) follows.  $\square$

## 5. THREE-MANIFOLD GROUPS

**5.1. The Thurston norm and fibered classes.** Throughout this section,  $N$  will be a 3-manifold with either empty or toroidal boundary. Given a surface  $\Sigma$  with connected components  $\Sigma_1, \dots, \Sigma_s$ , we put  $\chi_-(\Sigma) = \sum_{i=1}^s \max\{-\chi(\Sigma_i), 0\}$ .

Let  $\phi \in H^1(N; \mathbb{Z}) = \text{Hom}(\pi_1(N), \mathbb{Z})$  be a non-trivial cohomology class. We say that  $\phi$  is a *fibered class* if there exists a fibration  $p: N \rightarrow S^1$  such that the induced map  $p_*: \pi_1(N) \rightarrow \pi_1(S^1) = \mathbb{Z}$  coincides with  $\phi$ . It is well-known that  $\phi$  is a fibered class if and only if  $n\phi$  is a fibered class, for any non-zero integer  $n$ . We now say that  $\phi \in H^1(N; \mathbb{Q}) \setminus \{0\}$  is fibered if it is a rational multiple of a fibered integral class.

The *Thurston norm* of  $\phi \in H^1(N; \mathbb{Z})$  is defined as

$$(12) \quad \|\phi\|_T = \min \{ \chi_-(\Sigma) \mid \Sigma \text{ a properly embedded surface in } N, \text{ dual to } \phi \}.$$

W. Thurston [Th86] (see also [CC03, Chapter 10]) proved the following results:

- (1)  $\| - \|_T$  defines a norm<sup>1</sup> on  $H^1(N; \mathbb{Z})$ , which can be extended to a norm  $\| - \|_T$  on  $H^1(N; \mathbb{Q})$ .
- (2) The unit norm ball,  $B_T = \{ \phi \in H^1(N; \mathbb{Q}) \mid \|\phi\|_T \leq 1 \}$ , is a rational polyhedron with finitely many sides.
- (3) There exist open, top-dimensional faces  $F_1, \dots, F_k$  of the Thurston norm ball such that  $\{ \phi \in H^1(N; \mathbb{Q}) \mid \phi \text{ fibered} \} = \bigcup_{i=1}^k \mathbb{Q}^+ F_i$ .

Thus, the set of fibered classes form a cone on certain open, top-dimensional faces of  $B_T$ , which we refer to as the *fibered faces* of the Thurston norm ball. The polyhedron  $B_T$  is evidently symmetric in the origin. We say that two faces  $F$  and  $G$  are *equivalent* if  $F = \pm G$ . Note that a face  $F$  is fibered if and only if  $-F$  is fibered.

The Thurston norm is degenerate in general, for instance, for 3-manifolds with homologically essential tori. On the other hand, the Thurston norm of a hyperbolic 3-manifold is non-degenerate, since a hyperbolic 3-manifold admits no homologically essential surfaces of non-negative Euler characteristic.

We denote by  $\mathcal{B}_T$  the dual of the Thurston norm ball of  $N$ , that is,

$$(13) \quad \mathcal{B}_T := \{ p \in H_1(N; \mathbb{Q}) \mid \phi(p) \leq 1 \text{ for all } \phi \in H^1(N; \mathbb{Q}) \text{ with } \|\phi\|_T \leq 1 \}.$$

From the above discussion, we know that  $\mathcal{B}_T$  is a compact convex polyhedron in  $H_1(N, \mathbb{Q})$ . Its vertices are canonically in one-to-one correspondence with the top-dimensional faces of the Thurston norm ball  $B_T$ . (Thurston [Th86] showed that these vertices correspond in fact to integral classes in  $H_1(N; \mathbb{Z})/\text{Tors} \subset H_1(N; \mathbb{Q})$ .) We say that a vertex  $v$  of  $\mathcal{B}_T$  is fibered if it corresponds to a fibered face of  $B_T$ .

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<sup>1</sup>Throughout this paper, we allow norms to be degenerate; that is, we do not distinguish between the notions of norm and seminorm.

**5.2. Quasi-fibered classes.** The Thurston norm and the set of fibered classes ‘behave well’ under going to finite covers. More precisely, the following proposition holds:

**Proposition 5.1.** *Let  $p: N' \rightarrow N$  be a finite cover and let  $\phi \in H^1(N; \mathbb{Q})$ . Then  $\phi$  is fibered if and only if  $p^*(\phi) \in H^1(N'; \mathbb{Q})$  is fibered. Furthermore,*

$$\|p^*(\phi)\|_T = [N' : N] \cdot \|\phi\|_T.$$

*In particular, the map  $p^*: H^1(N; \mathbb{Q}) \rightarrow H^1(N'; \mathbb{Q})$  is, up to the scale factor  $[N' : N]$ , an isometry which maps fibered cones into fibered cones.*

*Proof.* The first statement is an immediate consequence of Stallings’ fibration theorem [St62], while the second statement follows from work of Gabai [Ga83].  $\square$

For future purposes we also introduce the following definition: We say that a class  $\phi \in H^1(N; \mathbb{Q})$  is *quasi-fibered* if  $\phi$  lies on the boundary of a fibered cone of the Thurston norm ball of  $N$ . Note that  $\phi$  is quasi-fibered if and only if  $\phi$  is the limit of a sequence of fibered classes in  $H^1(N; \mathbb{Q})$ . Proposition 5.1, then, has the following immediate corollary.

**Corollary 5.2.** *Let  $p: N' \rightarrow N$  be a finite cover.*

- (1) *If  $\phi \in H^1(N; \mathbb{Q})$  is a quasi-fibered class, then  $p^*(\phi) \in H^1(N'; \mathbb{Q})$  is also quasi-fibered.*
- (2) *Pull-backs of inequivalent faces of the Thurston norm ball of  $N$  lie on inequivalent faces of the Thurston norm ball of  $N'$ .*

**5.3. Thurston norm and Alexander norm.** As before, let  $N$  be a 3-manifold with empty or toroidal boundary. In [Mc02], McMullen defined the *Alexander norm* on  $H^1(N; \mathbb{Q})$ , as follows.

Let  $\Delta_N \in \mathbb{Z}[H]$  be the Alexander polynomial of  $N$ , where  $H = H_1(N; \mathbb{Z})/\text{Tors}$ . We write

$$(14) \quad \Delta_N = \sum_{h \in H} a_h h.$$

Let  $\phi \in H^1(N; \mathbb{Q})$ . If  $\Delta_N = 0$ , then we define  $\|\phi\|_A = 0$ . Otherwise, the Alexander norm of  $\phi$  is defined as

$$(15) \quad \|\phi\|_A := \max \{ \phi(a_h) - \phi(a_g) \mid g, h \in H \text{ with } a_g \neq 0 \text{ and } a_h \neq 0 \}.$$

This evidently defines a norm on  $H^1(N; \mathbb{Q})$ . We now have the following theorem:

**Theorem 5.3.** *Let  $N$  be a 3-manifold with empty or toroidal boundary and such that  $b_1(N) \geq 2$ . Then*

$$(16) \quad \|\phi\|_A \leq \|\phi\|_T \text{ for any } \phi \in H^1(N; \mathbb{Q}).$$

*Furthermore, equality holds for any quasi-fibered class.*

McMullen [Mc02] proved inequality (16), and he also showed that (16) is an equality for fibered classes. It now follows from the continuity of the Alexander norm and the Thurston norm that (16) is an equality for quasi-fibered classes.

The following is now a well-known consequence of McMullen’s theorem:

**Corollary 5.4.** *Let  $N$  be a 3-manifold with empty or toroidal boundary.*

- (1) If  $N$  fibers over the circle, and if the fiber has negative Euler characteristic, then  $\text{th}(N) \geq 1$ .
- (2) If  $N$  has at least two non-equivalent fibered faces, then  $\text{th}(N) \geq 2$ .

*Proof.* As usual, we write  $H = H_1(N; \mathbb{Z})/\text{Tors}$  and we view  $H$  as a subgroup of  $H_1(N; \mathbb{Q}) = H \otimes \mathbb{Q}$ . We write  $\Delta_N = \sum_{h \in H} a_h h$ , and we set

$$(17) \quad C := \text{hull}\{h - g \in H \otimes \mathbb{Q} \mid a_g \neq 0 \text{ and } a_h \neq 0\}.$$

Theorem 5.3 now says that  $C$  is a subset of  $\mathcal{B}_T$ , and that any fibered vertex of  $\mathcal{B}_T$  also belongs to  $C$ . The desired conclusions follow at once.  $\square$

**5.4. The RFRS property.** We conclude this section with two remarkable theorems that have revolutionized our understanding of 3-manifold groups. First, a definition which is due to Agol [Ag08].

**Definition 5.5.** A group  $\pi$  is called *residually finite rationally solvable (RFRS)* if there is a filtration of groups  $\pi = \pi_0 \supset \pi_1 \supset \pi_2 \supset \dots$  such that the following conditions hold:

- (1)  $\bigcap_i \pi_i = \{1\}$ .
- (2) For any  $i$ , the group  $\pi_i$  is a normal, finite-index subgroup of  $\pi$ .
- (3) For any  $i$ , the map  $\pi_i \rightarrow \pi_i/\pi_{i+1}$  factors through  $\pi_i \rightarrow H_1(\pi_i; \mathbb{Z})/\text{Tors}$ .

We say that a group  $\pi$  is *virtually RFRS* if  $\pi$  admits a finite-index subgroup which is RFRS. The following theorem is due to Agol [Ag08] (see also [FK12]).

**Theorem 5.6.** *Let  $N$  be an irreducible 3-manifold such that  $\pi_1(N)$  is virtually RFRS. Let  $\phi \in H^1(N; \mathbb{Q})$  be a non-fibered class. There exists then a finite cover  $p: N' \rightarrow N$  such that  $p^*(\phi) \in H^1(N'; \mathbb{Q})$  is quasi-fibered.*

We can now also formulate the following theorem which is a consequence of the work of Agol [Ag08, Ag12], Liu [Li11], Wise [Wi12a, Wi12b] and Przytycki–Wise [PW11, PW12], building on work of Kahn–Markovic [KM12] and Haglund–Wise [HW08]. We refer to [AFW12, Section 5] for more background and details.

**Theorem 5.7.** *Let  $N$  be an irreducible 3-manifold with empty or toroidal boundary. If  $N$  is not a closed graph manifold, then  $\pi_1(N)$  is virtually RFRS.*

We now obtain the following corollary.

**Corollary 5.8.** *Let  $N$  be an irreducible 3-manifold with empty or toroidal boundary. If  $N$  is not a closed graph manifold, then  $N$  is virtually fibered.*

*Proof.* It follows from Theorem 5.7 that  $\pi_1(N)$  is virtually RFRS. Since  $N$  is not a graph manifold we know that  $N$  is not spherical, i.e.,  $\pi_1(N)$  is not finite. We therefore see that  $\pi_1(N)$  admits a finite-index non-trivial subgroup  $\pi'$  which is RFRS.

Let us denote by  $N'$  the corresponding finite cover of  $N$ . Since  $\pi'$  is RFRS and non-trivial, it follows that  $b_1(\pi') = b_1(N') > 0$ . It now follows from Theorem 5.6 that  $N'$ , and hence  $N$ , admits a finite cover  $M$  which is fibered. (If  $N$  is a graph manifold with non-trivial boundary, then this also follows from [WY97].)  $\square$

We conclude this section with the following result, which is also an immediate consequence of Theorem 5.6 (see e.g. [AFW12, Section 6] for details).

**Corollary 5.9.** *Let  $N$  be an irreducible 3-manifold with empty or toroidal boundary. Suppose  $N$  is neither  $S^1 \times D^2$ , nor  $S^1 \times S^1 \times I$ , nor finitely covered by a torus bundle. Then, for every  $k \in \mathbb{N}$ , there is a finite cover  $N' \rightarrow N$  such that  $b_1(N') \geq k$ .*

## 6. KÄHLER 3-MANIFOLD GROUPS

After these preparations, we are now ready to prove Theorem 1.1. For the reader's convenience, we first recall the statement of that theorem.

**Theorem 6.1.** *Let  $N$  be a 3-manifold with non-empty, toroidal boundary. If  $\pi_1(N)$  is a Kähler group, then  $N \cong S^1 \times S^1 \times I$ .*

*Proof.* Let  $N$  be a 3-manifold such that  $\partial N$  is a non-empty collection of tori, and assume  $\pi_1(N)$  is a Kähler group. As we pointed out in §3, the group  $\pi_1(N)$  cannot be the free product of two non-trivial groups; thus,  $N$  has to be a prime 3-manifold. Since  $N$  has non-empty boundary, we conclude that  $N$  is in fact irreducible.

It now follows from Corollary 5.8 (again using the assumption that  $N$  is not closed) that  $N$  admits a finite cover  $M$  which is fibered. By Lemma 3.1, the group  $\pi_1(M)$  is also a Kähler group. Note that the fiber  $F$  of the fibration  $M \rightarrow S^1$  is a surface with boundary. There are three cases to consider.

If  $\chi(F) = 1$ , then  $M = S^1 \times D^2$ . But this is not possible, since, as we pointed out in §3, the first Betti number of a Kähler group is even.

If  $\chi(F) = 0$ , then  $M = S^1 \times S^1 \times [0, 1]$ , and either  $N = M$ , or  $N$  is the twisted  $I$ -bundle over the Klein bottle. In the latter case,  $b_1(N) = 1$ , again contradicting the assumption that  $\pi_1(N)$  is a Kähler group.

Finally, if  $\chi(F) < 0$ , then it follows from Corollary 5.4 that  $\text{th}(M) \geq 1$ . But this is not possible, since, by Theorem 3.3, we must have  $\text{th}(\pi_1(M)) = 0$ .  $\square$

**Remark 6.2.** Surely there are other ways to prove this theorem. Let us briefly sketch an alternate approach, relying in part on some machinery outside the scope of this paper.

It follows from the work of Agol, Wise and Przytycki–Wise that the fundamental group of an irreducible 3-manifold  $N$  with non-empty boundary admits a finite-index subgroup which is a subgroup of a Coxeter group (see [AFW12] for details). Combining this with [Py12, Theorem A] and [BHMS02, Theorem A], one can show that if  $\pi_1(N)$  is a Kähler group, then either  $\pi$  is finite, or  $\pi$  admits a finite-index subgroup which is a non-trivial direct product of free groups (possibly infinite cyclic). Hence, by the discussion in §3, we must have  $\pi_1(N) = \mathbb{Z}^2$ , and thus  $N = S^1 \times S^1 \times I$ .

## 7. QUASI-PROJECTIVE 3-MANIFOLD GROUPS

In this section, we prove Theorem 1.2 from the Introduction. We will do that in several steps.

**7.1. Fibered faces in finite covers.** We start out with the following proposition.

**Proposition 7.1.** *Let  $N$  be an irreducible 3-manifold with empty or toroidal boundary which is not a graph manifold. Then given any  $k \in \mathbb{N}$ , there exists a finite cover  $M \rightarrow N$  such that the Thurston norm ball of  $M$  has at least  $k$  non-equivalent faces.*

*Proof.* If  $N$  is hyperbolic, then we know from Corollary 5.9 that  $N$  admits finite covers with arbitrarily large Betti numbers. The proposition is then an immediate consequence of the fact that the Thurston norm for a hyperbolic 3-manifold is non-degenerate. We refer to [AFW12, Section 8.4] for a proof of the proposition in the non-hyperbolic case.  $\square$

The following is now a well-known consequence of Proposition 7.1 and Theorem 5.6.

**Theorem 7.2.** *Let  $N$  be an irreducible 3-manifold with empty or toroidal boundary which is not a graph manifold. Then given any  $k \in \mathbb{N}$ , there exists a finite cover  $N' \rightarrow N$  such that the Thurston norm ball of  $N'$  has at least  $k$  non-equivalent fibered faces.*

The proof is basically identical with the proof of [Ag08, Theorem 7.2]. We provide a quick outline of the proof for the reader's convenience.

*Proof.* We pick classes  $\phi_1, \dots, \phi_k$  in  $H^1(N; \mathbb{Q})$  which lie in  $k$  inequivalent faces. Note that  $\pi_1(N)$  is virtually RFRS by Theorem 5.7. For each  $i = 1, \dots, k$  we can therefore apply Theorem 5.6 to the class  $\phi_i$ , obtaining a finite cover  $\tilde{N}_i \rightarrow N$  such that the pull-back of  $\phi_i$  is quasi-fibered.

We now denote by  $p: M \rightarrow N$  the cover corresponding to the subgroup  $\bigcap_{i=1}^k \pi_1(\tilde{N}_i)$ . By Corollary 5.2, the cohomology classes  $p^*(\phi_1), \dots, p^*(\phi_k)$  lie on closures of inequivalent fibered faces of  $M$ . Hence,  $M$  has at least  $k$  inequivalent fibered faces.  $\square$

**7.2. Irreducible 3-manifolds which are not graph manifolds.** Next, we upgrade the statement about the Thurston unit ball from Theorem 7.2 to a statement about the thickness of the Alexander ball.

**Theorem 7.3.** *Let  $N$  be an irreducible 3-manifold with empty or toroidal boundary, and suppose  $N$  is not a graph manifold. There exists then a finite cover  $N' \rightarrow N$  with  $\text{th}(N') \geq 2$  and  $b_1(N') \geq 3$ .*

*Proof.* By Corollary 5.9,  $N$  admits finite covers with arbitrarily large first Betti numbers. We can thus without loss of generality assume that  $b_1(N) \geq 3$ .

By Theorem 7.2, there exists a finite cover  $N' \rightarrow N$  such that the Thurston norm ball of  $N'$  has at least 2 non-equivalent fibered faces. A basic transfer argument shows that  $b_1(N') \geq b_1(N) \geq 3$ . By Corollary 5.4, we have that  $\text{th}(N') \geq 2$ , and we are done.  $\square$

We can now prove Theorem 1.2 in the case that  $N$  is irreducible.

**Theorem 7.4.** *Let  $N$  be an irreducible 3-manifold with empty or toroidal boundary. If  $N$  is not a graph manifold, then  $\pi_1(N)$  is not a quasi-projective group.*

*Proof.* Suppose  $\pi_1(N)$  is a quasi-projective group. By Theorem 7.3, there exists a finite cover  $N' \rightarrow N$  with  $\text{th}(N') \geq 2$  and  $b_1(N') \geq 3$ . By Lemma 4.1,  $\pi_1(N')$  is also a quasi-projective group, which implies by Theorem 4.5 that either  $b_1(N') = 2$  or  $\text{th}(N') \leq 1$ . We have thus arrived at a contradiction.  $\square$

Let us draw an immediate corollary.

**Corollary 7.5.** *If  $N$  is a hyperbolic 3-manifold with empty or toroidal boundary, then  $\pi_1(N)$  is not a quasi-projective group.*

**7.3. Reducible 3-manifolds.** We turn now to the proof of Theorem 1.2 for non-prime 3-manifolds. We start out with a straightforward observation.

Let  $N$  be a 3-manifold, and let  $N = N_1 \# \cdots \# N_s$  be its prime decomposition. Set  $H = H_1(N; \mathbb{Z})/\text{Tors}$  and  $H_i = H_1(N_i; \mathbb{Z})/\text{Tors}$ . The Mayer–Vietoris sequence with  $\mathbb{Z}$ -coefficients shows that the inclusion maps induce an isomorphism  $H_1 \oplus \cdots \oplus H_s \xrightarrow{\cong} H$ .

**Lemma 7.6.** *Let  $N$  be a 3-manifold which admits a decomposition  $N = N_1 \# N_2 \# N_3$ , and identify  $H$  with  $H_1 \oplus H_2 \oplus H_3$  as above. Suppose that  $H_1$  and  $H_2$  are non-zero. For  $i = 1, 2$  denote by  $r_i$  the rank of  $H_1(N_i; \mathbb{Z}[H_i])$ , and denote by  $r$  the rank of  $H_1(N; \mathbb{Z}[H])$ . Then there exists a non-zero polynomial  $f \in \mathbb{Z}[H]$  such that*

$$\Delta_N^r = \Delta_{N_1}^{r_1} \cdot \Delta_{N_2}^{r_2} \cdot f \in \mathbb{Z}[H] = \mathbb{Z}[H_1 \oplus H_2 \oplus H_3].$$

In particular,

$$\text{th}(N) \geq \text{th}(N_1) + \text{th}(N_2).$$

*Proof.* The Mayer–Vietoris sequence for  $N = N_1 \# N_2 \# N_3$  with coefficients in  $\mathbb{Z}[H]$  yields an exact sequence,

$$(18) \quad 0 \longrightarrow H_1(N_1; \mathbb{Z}[H]) \oplus H_1(N_2; \mathbb{Z}[H]) \oplus H_1(N_3; \mathbb{Z}[H]) \longrightarrow H_1(N; \mathbb{Z}[H]) \longrightarrow \mathbb{Z}[H]^2,$$

where the last term corresponds to the 0-th homology of the two gluing spheres. From this, we get a monomorphism

$$(19) \quad \text{Tors}_{\mathbb{Z}[H]} H_1(N_1; \mathbb{Z}[H]) \oplus \text{Tors}_{\mathbb{Z}[H]} H_1(N_2; \mathbb{Z}[H]) \longrightarrow \text{Tors}_{\mathbb{Z}[H]} H_1(N; \mathbb{Z}[H]).$$

Note that the alternating product of orders in an exact sequence of torsion modules is 1 (see e.g. [Hi02, p. 57 and 60]). Consequently, there exists a non-zero polynomial  $f \in \mathbb{Z}[H_3]$  such that

$$(20) \quad \begin{aligned} \text{ord}_{\mathbb{Z}[H]}(\text{Tors}_{\mathbb{Z}[H]} H_1(N; \mathbb{Z}[H])) &= \text{ord}_{\mathbb{Z}[H]}(\text{Tors}_{\mathbb{Z}[H]} H_1(N_1; \mathbb{Z}[H])) \cdot \\ &\quad \text{ord}_{\mathbb{Z}[H]}(\text{Tors}_{\mathbb{Z}[H]} H_1(N_2; \mathbb{Z}[H])) \cdot f. \end{aligned}$$

Now note that  $H_1(N_i; \mathbb{Z}[H]) \cong H_1(N_i; \mathbb{Z}[H_i]) \otimes_{\mathbb{Z}[H_i]} \mathbb{Z}[H]$ , for  $i = 1, 2, 3$ . Applying Lemma 2.2 and formula (3), it follows that

$$(21) \quad \Delta_N^r = \Delta_{N_1}^{r_1} \cdot \Delta_{N_2}^{r_2} \cdot f \in \mathbb{Z}[H].$$

Using Lemma 2.2 again, we conclude that

$$(22) \quad \text{th}(N) = \text{th}(N_1) + \text{th}(N_2) + \text{th}(f) \geq \text{th}(N_1) + \text{th}(N_2),$$

and this completes the proof.  $\square$

We now obtain the following result, which in particular implies Theorem 1.2 for non-prime 3-manifolds.

**Theorem 7.7.** *Let  $N$  be a 3-manifold with empty or toroidal boundary which is not prime. If  $\pi_1(N)$  is a quasi-projective group, then all prime components of  $N$  are closed graph manifolds which are not virtually fibered.*

*Proof.* In light of Corollary 5.8, it suffices to prove the following: If  $N$  is a non-prime 3-manifold with empty or toroidal boundary, and  $N$  has at least one prime component which is virtually fibered, then  $\pi_1(N)$  is not a quasi-projective group.

Recall from Lemma 4.1 that a finite-index subgroup of a quasi-projective group is again quasi-projective. Thus, after going to a finite cover if necessary, we may assume that  $N$  admits a decomposition  $N = N_1 \# N_2$ , where  $N_1$  is fibered and  $N_2 \neq S^3$ .

First suppose that  $N_1$  is a Sol-manifold and  $N_2 = \mathbb{R}P^3$ . A straightforward calculation shows that

$$(23) \quad \Delta_{N_1} = 2(t - \lambda)(t - \lambda^{-1}) \in \mathbb{Z}[H_1(N; \mathbb{Z})/\text{Tors}] = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}],$$

where  $\lambda \neq \pm 1$  is a real number. (The factor of 2 comes from the order of  $H_1(\mathbb{R}P^3; \mathbb{Z}) = \mathbb{Z}_2$ .) It now follows from Theorem 4.5(2) and Lemma 7.6 that  $\pi_1(N)$  is not a quasi-projective group.

Next, suppose that  $N_1$  is not a Sol-manifold. If  $N_1$  is covered by a torus bundle, then the fact that  $N_1$  is not a Sol-manifold implies that  $N_1$  admits a finite cover with  $b_1 \geq 2$  (see e.g. [AFW12] for details). If  $N_1$  is not covered by a torus bundle, then it follows from Corollary 5.9 that  $N_1$  admits a finite cover with  $b_1 \geq 2$ . We can therefore without loss of generality assume that  $b_1(N) \geq 2$ .

Since  $N_2$  is not the 3-sphere, it follows from the Poincaré Conjecture that  $\pi_1(N_2)$  is non-trivial. Furthermore, it follows from the Geometrization Theorem that  $\pi_1(N_2)$  is residually finite (see [He87]). Therefore, there exists an epimorphism  $\pi_1(N_2) \rightarrow G$  onto a non-trivial finite group. Let  $M \rightarrow N$  be the cover corresponding to the epimorphism

$$(24) \quad \pi_1(N) = \pi_1(N_1) * \pi_1(N_2) \longrightarrow \pi_1(N_2) \longrightarrow G.$$

Note that the above homomorphism is trivial on  $\pi_1(N_1)$ ; hence,  $M$  contains at least  $|G|$  prime components which are diffeomorphic to  $N_1$ . Also note that  $b_1(M)$  is the sum of the Betti numbers of the prime components of  $M$ . By the above, we see that  $b_1(M) \geq 4$ . Furthermore, it follows from Lemma 7.6 that  $\text{th}(M)$  is the sum of the thicknesses of the prime components of  $M$ . Since  $\text{th}(N_1) \geq 1$ , we conclude that

$$(25) \quad \text{th}(M) \geq |G| \cdot \text{th}(N_1) \geq |G| \geq 2.$$

It now follows from Theorem 4.5 that  $\pi_1(M)$  is not a quasi-projective group.

Finally, suppose that  $N_1$  is a Sol-manifold but that  $N_2 \neq \mathbb{R}P^3$ . Then it follows from the Geometrization Theorem that  $\pi_1(N_2)$  has more than two elements. Since  $\pi_1(N_2)$  is residually finite there exists an epimorphism  $\pi_1(N_2) \rightarrow G$  onto a finite group with at least three elements. Again, let  $M \rightarrow N$  be the cover corresponding to the epimorphism (24). As above, we note that  $M$  has at least three prime components homeomorphic to  $N_1$ , which

then implies that  $b_1(M) \geq 3$  and  $\text{th}(M) \geq 3$ . Applying again Theorem 4.5, we conclude that  $\pi_1(M)$  is not a quasi-projective group.  $\square$

## 8. OPEN QUESTIONS

We conclude this paper with some open questions. First recall that a Kähler group is never the free product of two non-trivial groups. On the other hand, free groups are quasi-projective groups. The following question is open to the best of our knowledge.

**Question 8.1.** Suppose  $A$  and  $B$  are groups, such that the free product  $A * B$  is a quasi-projective group. Does it follow that  $A$  and  $B$  are already quasi-projective groups?

We showed that if  $N$  is an irreducible 3-manifold with empty or toroidal boundary such that  $\pi_1(N)$  is a quasi-projective group, then  $N$  is a graph manifold. Not all graph manifold groups are quasi-projective groups, though, as the next example (which we already encountered in the proof of Theorem 7.7) shows.

**Example 8.2.** Suppose  $N$  is a torus bundle whose monodromy has eigenvalues  $\lambda$  and  $\lambda^{-1}$ , for some real number  $\lambda > 1$ . Then  $N$  is a Sol manifold, and thus a graph manifold. On the other hand,  $b_1(N) = 1$ , and the Alexander polynomial  $\Delta_N^1$  equals  $(t - \lambda)(t - \lambda^{-1})$ . Hence, by Theorem 4.5(2), the fundamental group of  $N$  is not a quasi-projective group.

**Question 8.3.** For which graph manifolds is the fundamental group a quasi-projective group?

Finally, the case of connected sums of graph manifolds has also not been completely settled. In light of the results in §7.3, we venture the following conjecture.

**Conjecture 8.4.** Let  $N$  be a compact 3-manifold with empty or toroidal boundary. If  $\pi_1(N)$  is a quasi-projective group and if  $N$  is not prime, then  $N$  is the connected sum of spherical 3-manifolds and manifolds which are either diffeomorphic to  $S^1 \times D^2$ ,  $S^1 \times S^1 \times [0, 1]$ , or the 3-torus.

Note that the groups of the prime 3-manifolds listed above are either finite groups, and thus projective, or finitely generated free abelian groups, and thus quasi-projective.

We finish with one more question.

**Question 8.5.** Can the statements we prove here regarding quasi-projective, 3-manifold groups be extended to the analogous statements for quasi-Kähler, 3-manifold groups?

If one could prove Theorem 4.2 for an arbitrary quasi-Kähler manifold, the answer to this question would be yes: all our remaining arguments would then go through in this wider generality.

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