

## On the derived category of $\overline{M}_{0,n}$

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*ABSTRACT.* Using Keel’s presentation and Orlov’s theorem, we give an inductive description of the derived category of moduli spaces of  $n$ -pointed stable curves of genus zero and some full exceptional collections in it. The detailed calculations are given for  $\overline{M}_{0,6}$ .

### 0. Introduction

**0.1. Summary.** Let  $f : X \rightarrow Y$  be a monoidal transformation of smooth projective schemes with smooth center  $Z \subset Y$ . Then  $D^b(\text{Coh } X)$  in a definite sense can be described in terms of  $D^b(\text{Coh } Y)$  and  $D^b(\text{Coh } Z)$ : cf. [Or1].

We apply this strategy to the calculation of the derived categories of moduli spaces  $\overline{M}_{0,n}$ . Since there are several methods of representing  $\overline{M}_{0,n}$  as the result of a sequence of blow-ups with controlled bases and centers (see e.g. [Ke] and [Ka]), one may later ask about the interaction of the results obtained etc.

In this note we focus on the representation discovered and used by Keel in [Ke]. Its remarkable feature is that  $\overline{M}_{0,n+1}$  is obtained from  $\overline{M}_{0,n} \times \overline{M}_{0,4}$  by iterated blow-ups of manifolds isomorphic to  $\overline{M}_{0,p+1} \times \overline{M}_{0,q+1}$ ,  $p + q = n$ . This produces a straightforward inductive description if one complements the tools used in the computation a version of “Künneth formula” for the derived categories. The simplest useful form is furnished by Proposition 2.1.18 in [Bö] constructing exceptional collections on  $V \times W$  from those on  $V$  and  $W$ . A much more sophisticated version is contained in [BoLaLu]; cf. also [Kel].

This approach might eventually furnish a concise and functorial description of quivers useful for computations of (and in)  $D^b(\text{Coh } \overline{M}_{0,n})$ , but we could not achieve this goal so far.

We imagine two directions of future work on this subject.

(A) We might expect that replacing motives of  $\overline{M}_{0,S}$  by their derived (or enhanced derived) categories, we can get a better understanding of their quantum cohomology: cf. [MaS].

(B) The moduli spaces  $\overline{M}_{0,S}$  are rigid as objects of commutative geometry. Ample experience, including Gauss’  $q$ -numbers and quantum groups, shows that non-commutative (“quantum”) deformations of rigid objects are especially interesting. The beautiful recent paper [KeTe] shows how to represent  $\overline{M}_{0,S}$  as the projective

spectrum of a Koszul graded ring of sections of its *log*-canonical bundle. It would be interesting to study the deformations of the derived category of  $\overline{M}_{0,S}$  in terms of the deformations of this coordinate ring, as well as in terms of Bondal's quivers [Bo].

See [AuKaOr] for a detailed discussion of the case of del Pezzo surfaces, containing in particular  $\overline{M}_{0,5}$ .

Noncommutative deformations of general  $\overline{M}_{0,S}$ , together with structural correspondences between them, that we envision, might become a sophisticated counterpart of the theory of quantum groups.

**0.1. Plan of the paper.** The first section summarizes the main theorem due to Orlov ([Or1]) allowing one to reconstruct the derived category of a blow-up. We supplement it by some computations in the derived category, stressing, in particular, the codimension two blow-ups, that will appear in the applications to  $\overline{M}_{0,S}$ . The second section describes Keel's tower from [Ke]. In the third section the inductive construction based on the results of Orlov and Keel is elaborated. It is complemented by a new result (Proposition 3.5 and Corollary 3.6) allowing us in certain circumstances to get full exceptional collections consisting of locally free sheaves. Finally, sec. 4 gives an application of this construction to  $\overline{M}_{0,6}$ .

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## 1. Orlov's theorem

**1.1. Exceptional collections.** Let  $K$  be a ground field;  $D$  a  $K$ -linear triangulated category;  $T$  its shift functor;  $E, F \in \text{Ob } D$ . Following [Bo], we consider the graded complex of  $K$ -vector spaces with trivial differential

$$\text{Hom}_D^\bullet(E, F) := \bigoplus_{k \in \mathbf{Z}} \text{Hom}_D^k(E, F)[-k], \quad \text{Hom}_D^k(E, F) := \text{Hom}_D(E, T^k F).$$

Remind the following definitions.

(i) An object  $E$  is called *exceptional one*, if  $\text{Hom}_D^\bullet(E, E)$  consists of  $K \cdot \text{id}_E$  put in degree zero.

(ii) A family of exceptional objects  $(E_\alpha)$  indexed by a totally ordered set of subscripts  $\alpha$  is called *an exceptional collection* (or an exceptional sequence), if

$$\text{Hom}_D^\bullet(E_\alpha, E_\beta) = 0 \quad \text{for } \alpha > \beta. \quad (1.1)$$

It is called *full*, if it generates  $D$ .

(iii) An exceptional sequence  $(E_\alpha)$  as above is called *strong*, if for any  $\alpha < \beta$  the complex  $\mathrm{Hom}_D^\bullet(E_\alpha, E_\beta)$  has at most one non-vanishing component which is then of degree zero.

**1.2. Blow-ups.** Let  $X$  be a smooth projective variety over  $K$ ,  $Y \subset X$  its smooth closed subvariety,  $j : Y \rightarrow X$  the respective embedding. We consider the diagram describing the monoidal transformation of  $X$  with center  $Y$ :

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{i} & \tilde{X} \\ \downarrow \pi & & \downarrow q \\ Y & \xrightarrow{j} & X \end{array} \quad (1.2)$$

Let  $\mathcal{N}$  be the sheaf of sections of the normal bundle to  $Y$  in  $X$ . The exceptional divisor  $\tilde{Y}$  in  $\tilde{X}$  is canonically isomorphic to the relative projective spectrum of the symmetric algebra  $S_{\mathcal{O}_Y}(\mathcal{N}^t)$  of the dual sheaf of sections of  $\mathcal{N}$ . The rank  $c \geq 2$  of  $\mathcal{N}$  equals to the codimension of  $Y$ ; fibers of  $\pi$  are  $\mathbf{P}^{c-1}$ . This projective bundle carries the standard invertible relative sheaves  $\mathcal{O}_\pi(l)$ .

From the general definitions, we obtain the canonical exact sequences

$$0 \rightarrow \mathcal{N}^t \rightarrow j^*(\Omega_X^1) \rightarrow \Omega_Y^1 \rightarrow 0 \quad (1.3)$$

and

$$0 \rightarrow \pi^*(\Omega_Y^1) \rightarrow \Omega_{\tilde{Y}}^1 \rightarrow \Omega_{\tilde{Y}/Y}^1 \rightarrow 0. \quad (1.4)$$

The dual exact sequence to that in [Huy], p. 252, reads

$$0 \rightarrow \Omega_{\tilde{Y}/Y}^1(1) \rightarrow \pi^*(\mathcal{N}^t) \rightarrow \mathcal{O}_\pi(1) \rightarrow 0. \quad (1.5)$$

**1.3. From exceptional collections on  $X$  and  $Y$  to those on  $\tilde{Y}$  and  $\tilde{X}$ .** Generally, for a variety  $Z$  we denote by  $D(Z)$  the bounded derived category  $D^b(\mathrm{Coh} Z)$ .

Let  $(E_\alpha), \alpha \in A$ , be an exceptional collection in  $D(X)$  and  $(F_\beta), \beta \in B$ , be an exceptional sequence in  $D(Y)$ . Consider the following sequence of objects in  $D(\tilde{X})$ , that is indexed by the set of pairs  $(\beta, l)$  for  $\beta \in B, -c + 1 \leq l \leq -1$  and pairs  $(\alpha, l = 0)$  for  $\alpha \in A$ :

$$Ri_*(L\pi^*(F_\beta) \otimes \mathcal{O}_\pi(-c + 1)), \dots, Ri_*(L\pi^*(F_\beta) \otimes \mathcal{O}_\pi(-1)), Lq^*(E_\alpha). \quad (1.6)$$

It is ordered as suggested in (1.6): in each group with fixed  $l$  according to the order of  $A$  or  $B$ , and groups are ordered by increasing  $l$ .

**1.4. Proposition.** (i) *The sequence (1.6) is exceptional.*

(ii) *If  $(E_\alpha)$  is full in  $D(X)$  and  $(F_\beta)$  is full in  $D(Y)$ , then (1.6) is full in  $D(\tilde{X})$ .*

This is the Corollary 4.4 in [Or1] (notice that our notation for the blow-up diagram (1.2) differs from Orlov's notation and is closer to the Huybrechts notation in [Huy], Ch. 11).

In the remaining part of this section we produce some formulas for morphism spaces between remaining pairs of objects in (1.6).

**1.5. Auxiliary results.** (a) *Adjunction/duality formula.* For a morphism  $f : U \rightarrow V$  of smooth varieties, define  $\dim f := \dim U - \dim V$  and put

$$\omega_f := \omega_U \otimes f^*(\omega_V^{-1}). \quad (1.7)$$

Then for  $F \in D(U), E \in D(V)$  we have functorial isomorphisms ([Huy], p. 87)

$$\mathrm{Hom}_{D(V)}(Rf_*(F), E) \cong \mathrm{Hom}_{D(U)}(F, Lf^*(E) \otimes \omega_f[\dim f]) \quad (1.8)$$

(tensor products by an invertible or more general locally free sheaf here and below need not be derived).

(b) *The sheaf  $\omega_{\tilde{Y}}$ .* We will prove that

$$\omega_{\tilde{Y}} \cong \pi^* \circ j^*(\omega_X) \otimes \mathcal{O}_\pi(-c). \quad (1.9)$$

In fact, from (1.4) and (1.3) we get

$$\omega_{\tilde{Y}} \cong \pi^*(j^*(\omega_X) \otimes \det \mathcal{N}) \otimes \omega_{\tilde{Y}/Y}. \quad (1.10)$$

From (1.5) we obtain

$$\pi^*(\det \mathcal{N}) \cong \omega_{\tilde{Y}/Y}^{-1} \otimes \mathcal{O}_\pi(-c). \quad (1.11)$$

Substituting (1.11) to (1.10), we get (1.9).

(c) *The sheaf  $\omega_i$ .* We wish to prove that

$$\omega_i \cong \mathcal{O}_\pi(-1), \quad \omega_i[\dim i] \cong \mathcal{O}_\pi(-1)[-1]. \quad (1.12)$$

In fact, in the notation of diagram (1.2), we have

$$\omega_{\tilde{X}} \cong q^*(\omega_X) \otimes \mathcal{O}_{\tilde{X}}((c-1)\tilde{Y}), \quad i^*(\mathcal{O}_{\tilde{X}}(\tilde{Y})) \cong \mathcal{O}_\pi(-1), \quad (1.13)$$

where we write  $\tilde{Y}$  in place of the exceptional divisor  $i(\tilde{Y})$ : cf. [Huy], p. 252.

From (1.7) and (1.13) we find that

$$\omega_i = \omega_{\tilde{Y}} \otimes i^*(\omega_{\tilde{X}}^{-1}) \cong \omega_{\tilde{Y}} \otimes i^* \circ q^*(\omega_X^{-1}) \otimes \mathcal{O}_\pi(c-1). \quad (1.14)$$

Since  $i^* \circ q^* = \pi^* \circ j^*$ , substituting (1.9) into (1.14), we find (1.12).

(d) *The sheaf  $\omega_\pi$ .* From (1.11) we can calculate  $\omega_\pi := \omega_{\tilde{Y}} \otimes \pi^*(\omega_Y^{-1}) \cong \omega_{\tilde{Y}/Y}$ :

$$\omega_\pi \cong \det \pi^*(\mathcal{N})^{-1} \otimes \mathcal{O}_\pi(-c)$$

and

$$\omega_\pi[\dim \pi] \cong \pi^*(\det \mathcal{N})^{-1} \otimes \mathcal{O}_\pi(-c)[c-1]. \quad (1.15)$$

(e) *The sheaf  $\omega_j$ .* We have

$$\omega_j \cong \det \mathcal{N}, \quad \omega_j[\dim j] \cong \det \mathcal{N}[-c]. \quad (1.16)$$

This follows directly from (1.3).

(f) *The sheaf  $\omega_q$ .* Finally, from (1.7) and (1.13) we get

$$\omega_q \cong \mathcal{O}_{\tilde{X}}((c-1)\tilde{Y}) \cong \omega_q[\dim q].$$

**1.6. Calculations of Hom's.** Below we will need some formulas for the morphism spaces between objects of the sequence (1.6) that do not follow directly from the exceptionality of this sequence, i. e.  $\mathrm{Hom}_{D(\tilde{X})}^\bullet(G, H)$  where  $G, H$  are objects of (1.6), indexed respectively by  $(\beta_1, l_1)$  and  $(\beta_2, l_2)$  with  $l_1 < l_2$ , or else by  $(\beta, l), l < 0$ , and  $(\alpha, 0)$ . In fact, as Orlov has proved, for other pairs the respective Hom's come from  $Y, X$  or vanish.

Moreover, in our applications to Keel's tower, all centers of consecutive blow-ups have codimension  $c = 2$ . In this case the only relevant value of  $l$  is  $l = -1$  so that we need to consider only the second option.

Here we take two arbitrary objects  $E \in D(X)$  and  $F \in D(Y)$ .

**1.7. Proposition.** *There are functorial in  $E, F$  isomorphisms*

$$\mathrm{Hom}_{D(\tilde{X})}(Ri_*(L\pi^*(F) \otimes \mathcal{O}_\pi(l)), Lq^*(E)) \cong \quad (1.17)$$

$$\mathrm{Hom}_{D(Y)}(F \otimes S_{\mathcal{O}_Y}^{-l-1}(\mathcal{N}))[1], Lj^*(E)) \cong \quad (1.18)$$

$$\mathrm{Hom}_{D(X)}(Rj_*(F \otimes S_{\mathcal{O}_Y}^{-l-1}(\mathcal{N}) \otimes \det \mathcal{N})[1-c], E). \quad (1.19)$$

**Proof.** Applying (1.8) to  $i : \tilde{Y} \rightarrow \tilde{X}$  in place of  $f$ , we see that (1.17) is isomorphic to

$$\mathrm{Hom}_{D(\tilde{Y})}(L\pi^*(F) \otimes \mathcal{O}_\pi(l), Li^* \circ Lq^*(E) \otimes \omega_i[-1]). \quad (1.20)$$

Replacing here  $\omega_i$  by (1.12) and  $Li^* \circ Lq^*$  by  $L\pi^* \circ Lj^*$ , we rewrite (1.20) as

$$\mathrm{Hom}_{D(\tilde{Y})}(L\pi^*(F) \otimes \mathcal{O}_\pi(l), L\pi^* \circ Lj^*(E) \otimes \mathcal{O}_\pi(-1)[-1]). \quad (1.21)$$

Multiply both arguments of (1.21) by the same invertible sheaf  $\pi^*(\det \mathcal{N})^{-1} \otimes \mathcal{O}_\pi(1-c)$  and then apply to them the same shift  $[c]$ , without changing  $\mathrm{Hom}$ . Using (1.15) to rewrite the new second argument, we see that the result will be

$$\mathrm{Hom}_{D(\tilde{Y})}(L\pi^*(F) \otimes \pi^*(\det \mathcal{N})^{-1} \otimes \mathcal{O}_\pi(l+1-c)[c], L\pi^* \circ Lj^*(E) \otimes \omega_\pi[\dim \pi]) \quad (1.22)$$

We can rewrite (1.22) using the adjunction formula (1.8) for  $\pi$ :

$$\mathrm{Hom}_{D(Y)}(R\pi_* [L\pi^*(F \otimes (\det \mathcal{N})^{-1}) \otimes \mathcal{O}_\pi(l+1-c)] [c], Lj^*(E)) \quad (1.23)$$

Now apply the projection formula for  $\pi$  (cf. [Huy], (3.11) on p. 83) to the first argument:

$$\begin{aligned} R\pi_* [L\pi^*(F \otimes (\det \mathcal{N})^{-1}) \otimes \mathcal{O}_\pi(l+1-c)] [c] &\cong \\ F \otimes (\det \mathcal{N})^{-1} \otimes^L R\pi_*(\mathcal{O}_\pi(l+1-c))[c] & \end{aligned}$$

and remark that the complex  $R\pi_*(\mathcal{O}_\pi(l+1-c))$  for  $1-c \leq l \leq -1$  is quasi-isomorphic to  $R^{c-1}\pi_*(\mathcal{O}_\pi(l+c-1))[1-c]$ . Hence (1.23) will become

$$\mathrm{Hom}_{D(Y)}(F \otimes (\det \mathcal{N})^{-1} \otimes R^{c-1}\pi_*(\mathcal{O}_\pi(l+1-c))[1], Lj^*(E)). \quad (1.24)$$

Finally, relative Serre's duality and (1.5) imply that in the considered range of  $l$  we have

$$R^{c-1}\pi_*(\mathcal{O}_\pi(l+1-c)) \cong S_{\mathcal{O}_Y}^{-l-1}(\mathcal{N}) \otimes \det \mathcal{N}.$$

Thus, we proved (1.18).

Multiplying both arguments by  $\det \mathcal{N}$ , shifting by  $-c$  and applying (1.16), we get (1.19).

This finishes the proof.

*Remarks.* (i) For  $l = -1$ ,  $c = 2$ , (1.18) becomes simply

$$\mathrm{Hom}_{D(Y)}(F[1], Lj^*(E)). \quad (1.25)$$

This is the only case that must be considered, when  $c = 2$ . We will use this formula in sec. 3 and 4.

(ii) We could have started a proof of Prop. 1.7 by applying formula (1.8) to  $q$ , rather than to  $i$ , thus following the diagram (1.8) clockwise rather than counter-clockwise.

**1.8. Reconstruction of the derived category of a blow-up.** For simplicity, we continue considering the case of codimension 2 blow-up. The main result of [Or1] in this case establishes the canonical semiorthogonal composition of  $D(X)$ , whose components are admissible subcategories:  $(\tilde{D}(Y)_{-1}, D(X)_0)$ , where  $D(X)_0 := Rq^*(D(X))$  and  $\tilde{D}(Y)_{-1}$  consists of objects  $Ri_*(L\pi^*(F) \otimes \mathcal{O}_\pi(-1))$  where  $F \in \mathrm{Ob} D(Y)$ .

Together with the formula (1.17)  $\cong$  (1.19) for  $c = 2, l = -1$ , this semiorthogonal decomposition allows us to reconstruct  $D(\tilde{X})$ , however, not directly, but only through the mediation of *DG-categories*, say, of the canonical enhancements of all relevant derived categories (cf. [BoKa] and [LuOr]).

The relevant construction on the level of DG-categories is described in [Ta], Definition 3.57, under the name “upper triangular DG-categories”. Here we briefly describe its relation to the (two terms) semiorthogonal decompositions of triangular categories, as was explained to us in [Or2].

Generally, let  $\mathbf{A}$  be a linear pretriangulated DG-category, and  $\mathcal{A} := H^0(\mathbf{A})$  its homotopy category with its natural triangulated structure. Assume that  $\mathcal{A}$  admits a semiorthogonal decomposition  $(\mathcal{E}, \mathcal{F})$  as above, in the case of blow-up. Lift  $\mathcal{F}, \mathcal{E}$  to the subcategories  $\mathbf{F}, \mathbf{E}$  of  $\mathbf{A}$ , and consider a pair of objects  $F, E$  of the respective categories. The bifunctor given on objects by  $(F, E) \mapsto \mathrm{Hom}_{\mathbf{A}}(F, E)$  with values in complexes of linear spaces can be considered as DG-functor  $\Phi : \mathbf{E} \rightarrow \mathbf{Mod}\text{-}\mathbf{F}$ , “the upper right corner of the upper triangular DG-category” in the sense of [Ta].

Conversely, given a triple  $(\mathbf{F}, \mathbf{E}, \Phi)$  as above, one can explicitly construct a new DG-category  $\mathbf{B}$ . Its objects are triples  $(F, E, q)$  where  $q \in \Phi(E)(F)^1$ . Moreover, morphisms constitute the complex

$$\mathrm{Hom}_{\mathbf{B}}((F, E, q), (F', E', q')) := \mathrm{Hom}_{\mathbf{F}}(F, F') \oplus \mathrm{Hom}_{\mathbf{E}}(E, E') \oplus \Phi(E', F)$$

with pretty obvious composition rule.

Now, the principle result is that  $\mathbf{B}$  is quasiequivalent to  $\mathbf{A}$ .

This fact, together with uniqueness of enhancements and explicit description (1.25) of the bifunctor  $\Phi$  in the blow-up case, furnishes the reconstruction of the triangulated category.

## 2. Keel's tower

**2.1. Notation: combinatorics of marks.** Let  $S$  be a finite set,  $\text{card } S = n \geq 3$ . We will call an *inductive structure on  $S$*  the choice of a three-element subset  $P \subset S$ . We will sometimes denote  $\Sigma := S \setminus P$  so that  $S = \Sigma \sqcup P$ .

Recall that boundary strata of  $\overline{M}_{0,S}$  are bijectively numbered by the (isomorphism classes of) stable  $S$ -marked trees. Such a tree describes the dual combinatorial type of the curve parametrized by the generic point of the respective stratum. The number of edges of such a tree equals the codimension of the stratum.

In particular, boundary divisors, that is, trees with one edge, are determined by *unordered 2-partitions*  $S = S_1 \sqcup S_2$ , stable in the sense that  $|S_i| \geq 2$ . Furthermore, codimension two strata are determined by *3-partitions*  $S = S_1 \sqcup S_2 \sqcup S_3$  in which the middle term  $S_2$  is uniquely defined, whereas  $S_1$  and  $S_3$  can be interchanged. Stability condition here means that  $|S_1|, |S_3| \geq 2, |S_2| \geq 1$ .

Whenever an inductive structure  $P$  is chosen on  $S$ , and  $|S| \geq 4$ , we may and will *order each stable 2-partition* by the condition  $|S_1 \cap P| \leq 1$ , and for  $|S| \geq 4$  we will *order each stable 3-partition* by the condition  $|(S_1 \cup S_2) \cap P| \leq 1$ .

**2.2. Keel's blow-down.** Now, assuming  $n := |S| \geq 4$  and given an inductive structure on  $S$ , consider a one-point set  $\{\bullet\}$  disjoint from  $S$  and the diagram of two forgetful morphisms, forgetting respectively sections marked by  $\Sigma$  and the one marked by  $\bullet$ :

$$\begin{array}{ccc} \overline{M}_{0,S \sqcup \{\bullet\}} & \xrightarrow{f_\Sigma} & \overline{M}_{0,P \sqcup \{\bullet\}} \\ & & \downarrow f_{\{\bullet\}} \\ & & \overline{M}_{0,\Sigma \sqcup P} \end{array}$$

Notice that  $\overline{M}_{0,P \sqcup \{\bullet\}}$  is  $\mathbf{P}^1$  endowed with three boundary points. They are canonically marked by unordered partitions of  $P \sqcup \{\bullet\}$  into two parts of cardinality 2. Such a partition, in turn, is determined by an element  $p$  of  $P$  (one part is  $\{\bullet, p\}$ ) or a two-element subset of  $P$  (the part not containing  $\bullet$ ).

We summarize below some results of [Ke], showing, in particular, that the map (inverse to)

$$(f_{\{\bullet\}}, f_\Sigma) : \overline{M}_{0,S \sqcup \{\bullet\}} \rightarrow \overline{M}_{0,S} \times \overline{M}_{0,P \sqcup \{\bullet\}} \quad (2.1)$$



is a composition of blow-ups of smooth codimension two subvarieties isomorphic to boundary divisors of  $\overline{M}_{0,S}$ . These blow ups naturally form a sequence of  $n-3$  steps  $b_k : B_{k,S} \rightarrow B_{k-1,S}$ ,  $k = 2, \dots, n-2$ . At each step, a union of pairwise disjoint connected smooth submanifolds of codimension two is blown up.

In order to bridge our notation with Keel's, the reader should have in mind the following case:

$$S := \{1, \dots, n\}, \quad \bullet := n+1, \quad P := \{1, 2, 3\}, \quad \Sigma := \{4, \dots, n\}. \quad (2.2)$$

**2.3. Exceptional divisors of  $(f_{\{\bullet\}}, f_{\Sigma})$ .** In our notation, Lemma 1 of [Ke] establishes that exceptional divisors of the morphism  $(f_{\{\bullet\}}, f_{\Sigma})$  are exactly all boundary divisors of  $\overline{M}_{0,S \sqcup \{\bullet\}}$  corresponding to the stable 2-partitions of  $S \sqcup \{\bullet\}$ , satisfying the following condition:

(\*) *The part containing  $\bullet$  contains no more than one element of  $P$  and has cardinality  $\geq 3$ .*

As an independent check, the reader can convince oneself that the number of such partitions coincides with the difference of ranks of the Picard groups

$$\mathrm{rk} \mathrm{Pic} \overline{M}_{0,S \sqcup \{\bullet\}} - \mathrm{rk} \mathrm{Pic} (\overline{M}_{0,S} \times \overline{M}_{0,P \sqcup \{\bullet\}}) = 2^{n-1} - n - 1.$$

Let  $\overline{\sigma}$  be a partition of  $S \sqcup \{\bullet\}$  satisfying (\*) above, and let  $\sigma$  be the respective partition of  $S$  obtained by deleting  $\bullet$ . Obviously, we have

$$f_{\bullet}(D_{\overline{\sigma}}) = D_{\sigma}. \quad (2.3)$$

We will call the cardinality of the second part of  $\overline{\sigma}$  (and of  $\sigma$ ) *the height* of  $D_{\overline{\sigma}}$ .

**2.4. Keel's tower.** The main result of [Ke], sec. 1, can now be stated in the following way.

The morphism (2.1) can be represented as a product of blowing-downs

$$\overline{M}_{0,S \sqcup \{\bullet\}} =: B_{n-2,S} \rightarrow B_{n-3,S} \rightarrow \dots \rightarrow B_{1,S} := \overline{M}_{0,S} \times \overline{M}_{0,P \sqcup \{\bullet\}} \quad (2.4)$$

satisfying the following conditions:

(i) Image of any exceptional divisor  $D_{\overline{\sigma}}$  of height  $h$  remains a divisor in  $B_{h,S}$ , but becomes a closed subscheme of codimension 2 in  $B_{h-1,S}, \dots, B_{1,S}$ . The product of the subsequent arrows, followed by the projection  $B_{1,S} \rightarrow \overline{M}_{0,S}$  identifies this subscheme with  $D_{\sigma}$ .

(ii) Each morphism  $B_{h+1,S} \rightarrow B_{h,S}$  is the blow up of the disjoint union of those subschemes in  $B_{h,S}$  that are images of exceptional divisors of height  $h + 1$ . Hence connected components of the center of the respective blow up are isomorphic to  $\overline{M}_{0,p+1} \times \overline{M}_{0,q+1}$ ,  $p, q \leq n - 2$ .

### 3. Inductive construction of semiorthogonal decompositions and complete exceptional collections.

**3.1. The inductive step I: functoriality in  $S$ .** In order to calculate  $D^b(\overline{M}_{0,S \sqcup \{\bullet\}})$  assuming that the derived categories of the respective modular spaces of smaller dimension are already known, we will apply D. Orlov's results summarized in sec. 1 to Keel's tower.

More precisely, Keel's tower depends on the choice of an *inductive structure* on  $S$  in the sense of 2.1. In order to be able to induce a given inductive structure on the subsets of  $S$ , we will adopt here the following convention, essentially returning us to the Keel's choice (2.2).

*$S$  is totally ordered, and for  $|S| \geq 3$ ,  $P \subset S$  consists of the first three elements of  $S$ .*

The inductive structure induced on subsets of  $S$  is defined then by the induced order.

With this conditions, one easily sees that a bijection of two sets of marks compatible with their respective orders lifts to a unique isomorphism of Keel's towers.

**3.2. The inductive step II: Keel's blow-ups.** Our "inductive leap" from  $S$  to  $S \sqcup \{\bullet\}$  breaks down into the sequence of small inductive steps corresponding to the consecutive floors of the Keel tower (2.4). They will allow us to obtain an inductive description of a class of semiorthogonal decompositions of  $D^b(\text{Coh } \overline{M}_{0,S})$ . Concretely, in view of Orlov's theorem, each floor  $p_k : B_{k+1,S} \rightarrow B_{k,S}$  provides a semiorthogonal decomposition of  $D^b(\text{Coh } B_{k+1,S})$  of the form

$$D^b(\text{Coh } B_{k+1,S}) : (\tilde{D}(Y_{k,S})_{-1}, D(B_{k,S})_0) \quad (3.1)$$

To be more precise, we recapitulate Orlov's results in this context. Consider the following specializations of the blow-up diagram (1.2):

$$\begin{array}{ccc} \hat{Y}_{k,S} & \xrightarrow{i_k} & B_{k+1,S} \\ \downarrow \pi_k & & \downarrow q_k \\ Y_{k,S} & \xrightarrow{j_k} & B_{k,S} \end{array} \quad (3.2)$$

in which  $i_k : \widehat{Y}_{k,S} \rightarrow B_{k+1,S}$  is the embedding of the exceptional divisor of  $q_k$ . Since  $q_k$  blows up codimension two submanifolds,  $\pi_k : \widehat{Y}_{k,S} \rightarrow Y_{k,S}$  is a fibration with fiber  $\mathbf{P}^1$ . Let  $\mathcal{O}(-1)$  be the respective relative sheaf on  $\widehat{Y}_{k,S}$ . Then  $D^b(\text{Coh } \widehat{Y}_{k,S})$  has the semiorthogonal decomposition

$$D^b(\text{Coh } \widehat{Y}_{k,S}) : (\pi_k^*(D^b(\text{Coh } Y_{k,S})) \otimes \mathcal{O}(-1), q_k^*(D^b(\text{Coh } B_{k,S}))) \quad (3.3)$$

Now, the subcategory  $\widetilde{D}(Y_{k,S})_{-1}$  in (3.1) is defined as

$$\widetilde{D}(Y_{k,S})_{-1} := Ri_{k*}[\pi_k^*(D^b(\text{Coh } Y_{k,S})) \otimes \mathcal{O}(-1)] \quad (3.4)$$

whereas  $D(B_{k,S})_0$  in (3.1) is

$$D(B_{k,S})_0 := Lq_k^*(D^b(\text{Coh } B_{k,S})). \quad (3.5)$$

Moreover, since  $q_k$  blows up a disjoint union of smooth submanifolds  $Y_\sigma$ ,  $\sigma = (S_1, S_2)$ , each of the subcategories  $\widetilde{D}(Y_{k,S})_{-1}$  admits an *orthogonal* decomposition

$$\widetilde{D}(Y_{k,S})_{-1} : (\widetilde{D}(Y_\sigma)_{-1} \mid \text{card } S_2 = k + 2) \quad (3.6)$$

Finally, we have a canonical identification

$$Y_\sigma = \overline{M}_{0,S_1 \sqcup \{\bullet_\sigma\}} \times \overline{M}_{0,S_2 \sqcup \{\bullet_\sigma\}} \quad (3.7)$$

where  $\bullet_\sigma$  corresponds to the intersection point of two components. Therefore  $D^b(\text{Coh } Y_\sigma)$  is generated by an external product  $\boxtimes$  of any two exceptional collections generating respectively  $\overline{M}_{0,S_1 \sqcup \{\bullet_\sigma\}}$  and  $\overline{M}_{0,S_2 \sqcup \{\bullet_\sigma\}}$ .

This ‘‘K nneth formula’’ for derived and DG–categories is known on various levels of generality. Ch. B hning ([B ]) establishes it for collections of locally free sheaves; the proof generalizes to complexes of locally free sheaves. In [BoLaLu] the DG case is treated.

The base  $B_{1,S}$  has a similar decomposition (cf. (2.4)). Thus, Keel’s tower provides a tool to generate semi–orthogonal decompositions and exceptional collections for  $D^b(\overline{M}_{0,n+1})$  from similar objects for  $D^b(\overline{M}_{0,m})$ ,  $m \leq n$ .

**3.3. Exceptional collections for small  $n$ .** On  $\overline{M}_{0,4} \cong \mathbf{P}^1$  the standard full strong exceptional collection is  $(\mathcal{O}(-1), \mathcal{O})$ .

If we represent  $\overline{M}_{0,5}$  as a blow–up  $p : \overline{M}_{0,5} \rightarrow \mathbf{P}^2$  at four points, and denote by  $l_i$ ,  $i = 1, \dots, 4$  the respective exceptional divisors, then for any choice  $(F_0, F_1, F_2)$

of a full strong exceptional collection on  $\mathbf{P}^2$ , e. g.  $F_i = \mathcal{O}(i - 2)$ , Orlov's theorem will provide a full strong exceptional collection on  $\overline{M}_{0,5}$  of the form

$$(\mathcal{O}_{l_1}(-1), \dots, \mathcal{O}_{l_d}(-1), p^*F_0, p^*F_1, p^*F_2) \quad (3.8)$$

In [KarN] it was shown that a sequence of mutations turns such a sequence into a full exceptional collection of locally free sheaves (this trick actually works for all del Pezzo surfaces). Taking its  $\boxtimes$  with the standard collection on  $\mathbf{P}^1$  and then using Keel's blow-up, one gets an analog of (3.8) for  $\overline{M}_{0,6}$ .

In the remaining part of this section we will show that under certain assumptions, one can find a sequence of mutations, turning the collection thus obtained into a collection of locally free sheaves. In the next section we will demonstrate that this procedure can be applied to  $\overline{M}_{0,6}$ .

**3.4. Preparation.** Consider a blow-up diagram (1.2)

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{i} & \tilde{X} \\ \downarrow \pi & & \downarrow q \\ Y & \xrightarrow{j} & X \end{array}$$

For a general blow up  $R^0q_*\mathcal{O}_{\tilde{X}} \simeq \mathcal{O}_X$  and  $R^iq_*\mathcal{O}_{\tilde{X}} = 0$  (see [Or1], proof of Lemma 4.1). Hence the projection formula and the Leray spectral sequence give  $H^i(\tilde{X}, q^*\mathcal{E}) = H^i(X, \mathcal{E})$  for any locally free sheaf  $\mathcal{E}$ .

From now on we assume that  $j(Y)$  is of codimension two in  $X$ . If  $F_1, \dots, F_r$  (resp.  $E_1, \dots, E_n$ ) is a full exceptional collection of locally free sheaves on  $Y$  (resp. on  $X$ ), then Orlov's collection (1.6)

$$Ri_*(\pi^*F_1 \otimes \mathcal{O}_\pi(-1)), \dots, Ri_*(\pi^*F_r \otimes \mathcal{O}_\pi(-1)), q^*E_1, \dots, q^*E_n \quad (3.9)$$

is a full exceptional collection on  $\tilde{X}$ .

**3.5. Proposition.** *Let  $E_1, \dots, E_n$  be a full exceptional collection of locally free sheaves on  $X$ . Assume moreover that  $j^*E_1, \dots, j^*E_r$  form a full exceptional collection on  $Y$ . If  $Y$  is of codimension 2 in  $X$ , then the collection*

$$q^*E_1, q^*E_1(\tilde{Y}), \dots, q^*E_r, q^*E_r(\tilde{Y}), q^*E_{r+1}, \dots, q^*E_n \quad (3.10)$$

*is a full exceptional collection on  $\tilde{X}$ .*

Here and below we use notation of the type  $q^*L(\tilde{Y})$  as a shorthand for  $q^*L \otimes \mathcal{O}_{\tilde{X}}(i(\tilde{Y}))$ .

**Proof.** The strategy of our proof is simple. We start with the exceptional collection (3.9), with  $F_a := q^*E_a$  for  $1 \leq a \leq r$ , and show that it can be transformed into (3.10) by an explicit sequence of mutations (cf. [Bo], [Kuz]).

More precisely, for  $1 \leq a \leq r$ , put  $A_a := Ri_*(\pi^*j^*E_a \otimes \mathcal{O}_\pi(-1))$ ,  $B_a := q^*E_a$ .

We will first check that

$$\mathrm{Hom}^\bullet(A_b, B_a) = 0 \quad \text{for } b > a. \quad (3.11)$$

so that the right mutation of such an exceptional pair simply reduces to the permutation  $(A_b, B_a) \mapsto (B_a, A_b)$ . This shows, that we may consecutively move  $A_r$  in (3.9) to the right, until it reaches the position directly to the left of  $B_r$ ; then move  $A_{r-1}$  to the right, until it reaches the position directly to the left of  $B_{r-1}$ ; and so on. The result will be the exceptional collection  $(A_1, B_1; A_2, B_2; \dots; A_r, B_r; B_{r+1}, \dots, B_n)$ .

Second, we will check that

$$R_{B_a}(A_a) \cong B_a(\tilde{Y}) \quad (3.12)$$

Thus additional  $r$  right mutations will transform the latter collection into

$$(B_1, B_1(\tilde{Y})), \dots, B_r, B_r(\tilde{Y}), B_{r+1}, \dots, B_n)$$

that is, to (3.10).

*Proof of (3.11).* Consider the isomorphism (1.17)  $\cong$  (1.25) written for  $F := j^*E_b$ ,  $E := E_a[i]$  where  $i$  is an arbitrary shift and  $b > a$ . Its left hand side will represent one of the components of  $\mathrm{Hom}^\bullet(A_b, B_a)$ . Hence it suffices to prove that the right hand side vanishes. But it is simply  $\mathrm{Hom}_{D(Y)}(j^*(E_b)[1], j^*E_a[i])$ ,  $1 \leq a \leq b \leq r$ , and all these groups vanish, because we assumed that  $j^*E_1, \dots, j^*E_r$  is an exceptional collection.

*Proof of (3.12).* Consider now the case  $a = b$ . First of all, recall that  $R_{B_a}(A_a)$  is defined as the cone  $C(\alpha_a)$  of the canonical morphism in  $D(\tilde{X})$

$$\alpha_a : A_a \rightarrow \mathrm{Hom}^\bullet(A_a, B_a)^t \otimes B_a \quad (3.13)$$

where  $t$  means linear dual in the category of graded linear spaces. In order to calculate  $\mathrm{Hom}^\bullet(A_a, B_a)$ , we note that  $A_a$  fits into exact sequence  $0 \rightarrow q^*E_a \rightarrow$

$q^*E_a(\tilde{Y}) \rightarrow A_a \rightarrow 0$  (cf. (1.13)), and thus  $A_a$  is quasi-isomorphic to its projective resolution

$$\mathcal{A}_a : 0 \rightarrow q^*E_a \rightarrow q^*E_a(\tilde{Y}) \rightarrow 0 \quad (3.14)$$

(with the first term in degree  $-1$ ).

Since  $A_a$  is exceptional,  $\text{Hom}^\bullet(\mathcal{A}_a, B_a)$  is spanned by the canonical isomorphism  $\mathcal{A}_a \rightarrow B_a[1]$  which is identity in degree  $-1$ , and the morphism (3.13) can be represented by the morphism of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & q^*E_a & \xrightarrow{g} & q^*E_a(\tilde{Y}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & q^*E_a & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

The cone of it is the complex

$$0 \longrightarrow q^*E_a \xrightarrow{(id, -g)} q^*E_a \oplus q^*E_a(\tilde{Y}) \longrightarrow 0,$$

where  $q^*E_a \oplus q^*E_a(\tilde{Y})$  is in degree  $-1$ . There exists a short exact sequence of sheaves

$$0 \longrightarrow q^*E_a \xrightarrow{(id, -g)} q^*E_a \oplus q^*E_a(\tilde{Y}) \xrightarrow{\psi} q^*E_a(\tilde{Y}) \longrightarrow 0,$$

where  $\psi(v, w) = g(v) + w$ . Therefore, the cone is quasi-isomorphic to the complex with one non-zero term  $q^*E(\tilde{Y})$  placed in degree  $-1$ , i.e.  $q^*E_a(\tilde{Y})[1]$ . Hence  $R_{B_a}(A_a)[1] \cong q^*E_a(\tilde{Y})[1]$  and finally  $R_{B_a}(A_a) \cong B_a(\tilde{Y})$ .

This completes the proof of Proposition 3.5.

**3.6. Corollary.** *Let  $E_1, \dots, E_n$  be a full exceptional collection of locally free sheaves on  $X$ . Assume moreover that for some  $s \leq r$ ,  $j^*E_s, \dots, j^*E_r$  form a full exceptional collection on  $Y$ . If  $Y$  is of codimension 2 in  $X$ , then the collection*

$$q^*E_1(\tilde{Y}), \dots, q^*E_{s-1}(\tilde{Y}), q^*E_s, q^*E_s(\tilde{Y}), \dots, q^*E_r, q^*E_r(\tilde{Y}), q^*E_{r+1}, \dots, q^*E_n$$

*is a full exceptional collection on  $\tilde{X}$ .*

**Proof.** First, let us recall a general fact. Let  $\mathcal{D}$  be a triangulated category with a Serre functor  $S: \mathcal{D} \rightarrow \mathcal{D}$  and  $\mathcal{A}$  an admissible subcategory (cf. [BoKa1], [Kuz]). In

this situation we have two semi-orthogonal decompositions  $\langle \mathcal{A}, {}^\perp \mathcal{A} \rangle$  and  $\langle \mathcal{A}^\perp, \mathcal{A} \rangle$ . By [BoKa1], Proposition 3.6, we obtain that

$$R_{\perp \mathcal{A}}(\mathcal{A}) = S^{-1}(\mathcal{A}), \quad (3.13)$$

$$L_{\mathcal{A}^\perp}(\mathcal{A}) = S(\mathcal{A}). \quad (3.14)$$

Our proof of corollary 3.6 will consist of applications of formulas (3.13),(3.14) and proposition 3.5.

Let  $\mathcal{D} = D(X)$ ,  $\mathcal{A} = \langle E_1, \dots, E_{s-1} \rangle$ ,  ${}^\perp \mathcal{A} = \langle E_s, \dots, E_n \rangle$  and the Serre functor is  $S_X = \cdot \otimes \omega_X[\dim X]$ . Applying formula (3.13) we get a full exceptional collection

$$E_s, \dots, E_n, S_X^{-1}(E_1), \dots, S_X^{-1}(E_{s-1}).$$

To this collection we can apply proposition 3.5 and we get a full exceptional collection on  $\tilde{X}$

$$q^* E_s, q^* E_s(\tilde{Y}), \dots, q^* E_r, q^* E_r(\tilde{Y}), q^* E_{r+1}, \dots, q^* E_n, Lq^* \circ S_X^{-1}(E_1), \dots, Lq^* \circ S_X^{-1}(E_{s-1}).$$

Let now  $\mathcal{D} = D(\tilde{X})$ ,  $\mathcal{A} = \langle Lq^* S_X^{-1}(E_1), \dots, Lq^* S_X^{-1}(E_{s-1}) \rangle$  and the Serre functor is  $S_{\tilde{X}} = \cdot \otimes \omega_{\tilde{X}}[\dim \tilde{X}]$ . Applying formula (3.14) and using that

$$S_{\tilde{X}} \circ Lq^* \circ S_X^{-1} \simeq Lq^* \otimes \mathcal{O}_{\tilde{X}}(\tilde{Y})$$

we get the desired statement.

#### 4. Example: moduli space $\overline{M}_{0,6}$

**4.1. Preparation: moduli space  $\overline{M}_{0,5}$ .** Let  $S = \{1, 2, 3, 4, 5\}$ ,  $S' = \{1, 2, 3, 4\}$  and  $P = \{1, 2, 3\}$ . Consider the Keel's tower

$$\begin{array}{c} B_{2,S'} = \overline{M}_{0,S} \\ \downarrow q_{1,S'} \\ B_{1,S'} = \overline{M}_{0,S'} \times \overline{M}_{0,P \sqcup \{5\}} \end{array}$$

The map  $q_{1,S'}$  contracts 3 boundary divisors  $D_{\sigma_i}$  corresponding to unordered partitions

$$\sigma_1 = (5, 1, 4|2, 3), \quad \sigma_2 = (5, 2, 4|1, 3), \quad \sigma_3 = (5, 3, 4|2, 1).$$

Let us identify  $\overline{M}_{0,S'}$  with  $\overline{M}_{0,P \sqcup \{5\}}$  using the bijection of labels identical on  $\{1, 2, 3\}$  and mapping 4 to 5. Then images of  $D_{\sigma_i}$  become three points on the diagonal, corresponding to the partitions with one part  $\{i, 4\}$ , resp.  $\{i, 5\}$ . Let us imagine  $\overline{M}_{0,P \sqcup \{5\}}$  as the horizontal axis  $\mathbf{P}^1$ , and  $\overline{M}_{0,S'}$  as the vertical one. Let  $H_i$ , resp.  $V_i$ , be the horizontal, resp. vertical fiber, passing through the image of  $D_{\sigma_i}$ .

Now denote by  $\tilde{H}_i$  and  $\tilde{V}_i$  the proper transforms of  $H_i$ , resp.  $V_i$ , in  $\overline{M}_{0,S}$ , and let  $\tilde{Z}$  be the proper transform of the diagonal  $Z$ . Divisors  $D_{\sigma_i}$ ,  $\tilde{H}_i$ ,  $\tilde{V}_i$  and  $\tilde{Z}$  are all isomorphic to  $\mathbf{P}^1$  and have self-intersection  $(-1)$ . There are precisely 10 such curves on  $\overline{M}_{0,S}$ .

Let  $F_0, F_1$  and  $G_0, G_1$  be full exceptional collections of locally free sheaves on  $\overline{M}_{0,S'}$  and  $\overline{M}_{0,P \sqcup \{5\}}$  respectively. It is well known that  $F_i, G_i$  invertible sheaves. We know that

$$F_0 \boxtimes G_0, F_1 \boxtimes G_0, F_0 \boxtimes G_1, F_1 \boxtimes G_1$$

is a full exceptional collection on  $\overline{M}_{0,S'} \times \overline{M}_{0,P \sqcup \{5\}}$ . Denote it as

$$L_0, L_1, L_2, L_3. \tag{4.1}$$

Consider the decomposition  $q_{1,S'} = f_1 \circ f_2 \circ f_3$  where  $f_i$  contracts only  $D_{\sigma_i}$ . Each  $f_i$  is a blow-up of a surface at a point. Applying corollary 3.6 to the blow-up  $f_3$  and using restriction of  $L_0$  as a full exceptional collection in  $D(pt)$  we obtain a full exceptional collection on the resulting surface

$$f_3^* L_0, f_3^* L_0(D_{\sigma_3}), f_3^* L_1, f_3^* L_2, f_3^* L_3.$$



Continuing in the same way and always using restriction of the first element as a full exceptional collection in  $D(pt)$  we obtain a full exceptional collection on  $\overline{M}_{0,S}$

$$q_{1,S'}^* L_0, q_{1,S'}^* L_0(D_{\sigma_1}), q_{1,S'}^* L_0(D_{\sigma_2}), q_{1,S'}^* L_0(D_{\sigma_3}), q_{1,S'}^* L_1, q_{1,S'}^* L_2, q_{1,S'}^* L_3. \quad (4.2)$$

Of course, this is just an example of a full exceptional collection on  $\overline{M}_{0,S}$ . If we used restrictions of other elements of (4.1) we would have obtained a different answer.

**4.2. Preparation: moduli space  $\overline{M}_{0,6}$ .** Here we will introduce convenient notations which will be used later to write down a full exceptional collection on  $\overline{M}_{0,6}$ .

Let  $S = \{1, 2, 3, 4, 5\}$ ,  $\{\bullet\} = \{6\}$  and  $P = \{1, 2, 3\}$ . Consider the Keel's tower

$$\begin{array}{c} B_{3,S} = \overline{M}_{0,S \sqcup \{6\}} \\ \downarrow q_{2,S} \\ B_{2,S} \\ \downarrow q_{1,S} \\ B_{1,S} = \overline{M}_{0,S} \times \overline{M}_{0,P \sqcup \{6\}} \end{array} \quad (4.3)$$

The map  $q_{1,S} \circ q_{2,S}$  contracts 10 boundary divisors  $E_i$ ,  $1 \leq i \leq 10$ . At the height 3 level it contracts 7 boundary divisors corresponding to the following unordered partitions

$$E_4 \leftrightarrow (6, 1, 4|5, 2, 3), \quad E_6 \leftrightarrow (6, 2, 4|5, 1, 3), \quad E_8 \leftrightarrow (6, 3, 4|5, 2, 1),$$

$$E_5 \leftrightarrow (6, 1, 5|4, 2, 3), \quad E_7 \leftrightarrow (6, 2, 5|4, 1, 3), \quad E_9 \leftrightarrow (6, 3, 5|4, 2, 1),$$

and  $E_{10} \leftrightarrow (6, 4, 5|1, 2, 3)$ . At the height 2 level it contracts images under  $q_{2,S}$  of 3 boundary divisors corresponding to the following unordered partitions

$$E_1 \leftrightarrow (6, 1, 4, 5|2, 3), \quad E_2 \leftrightarrow (6, 2, 4, 5|1, 3), \quad E_3 \leftrightarrow (6, 3, 4, 5|2, 1).$$

The divisors  $E_1, E_2, E_3$  are pairwise disjoint and  $E_4, \dots, E_{10}$  are pairwise disjoint as well. We list below all nonempty intersections

$$E_1 \cdot E_4 = P_1, \quad E_1 \cdot E_5 = Q_1,$$

$$\begin{aligned}
E_2 \cdot E_6 &= P_2, & E_2 \cdot E_7 &= Q_2, \\
E_3 \cdot E_8 &= P_3, & E_3 \cdot E_9 &= Q_3, \\
E_1 \cdot E_{10} &= R_1; & E_2 \cdot E_{10} &= R_2; & E_3 \cdot E_{10} &= R_3,
\end{aligned}$$

where  $P_i, Q_i, R_i$  are isomorphic to  $\mathbf{P}^1$  and all intersections are transversal.

Let  $L_0, \dots, L_6$  be an exceptional collection of invertible sheaves on  $\overline{M}_{0,S}$  and  $G_0, G_1$  an exceptional collection of invertible sheaves on  $\overline{M}_{0,P \sqcup \{6\}}$ . From them we construct a collection on  $\overline{M}_{0,S} \times \overline{M}_{0,P \sqcup \{6\}}$

$$L_0 \boxtimes G_0, \dots, L_6 \boxtimes G_0, L_0 \boxtimes G_1, \dots, L_6 \boxtimes G_1. \quad (4.4)$$

We will need some assumptions on the collection  $L_0, \dots, L_6$ . We use notations for  $\overline{M}_{0,S'}$  introduced earlier in section 4.1.

*Assumption 1.*  $L_1, L_2$  restricted to  $D_{\sigma_1}$  form a full exceptional collection on  $D_{\sigma_1}$ ;  $L_2, L_3$  restricted to  $D_{\sigma_2}$  form a full exceptional collection  $D_{\sigma_2}$ ;  $L_3, L_4$  restricted to  $D_{\sigma_3}$  form a full exceptional collection  $D_{\sigma_3}$ .

*Assumption 2.*  $L_0, L_1$  restrict to a full exceptional collection on  $\tilde{H}_1$  and  $\tilde{V}_1$ . The same holds for  $L_1, L_2$  on  $\tilde{H}_2, \tilde{V}_2$  and  $L_2, L_3$  on  $\tilde{H}_3, \tilde{V}_3$  and  $L_5, L_6$  on  $\tilde{Z}$ .

These assumptions are satisfied, for example, for

$$\mathcal{O}, \mathcal{O}(D_{\sigma_1}), \mathcal{O}(D_{\sigma_2}), \mathcal{O}(D_{\sigma_3}), q_{1,S'}^* \mathcal{O}(0, 1), q_{1,S'}^* \mathcal{O}(1, 0), q_{1,S'}^* \mathcal{O}(1, 1),$$

where we used notations of section 4.1 and identification of  $\overline{M}_{0,S'} \times \overline{M}_{0,P \sqcup \{5\}}$  with  $\mathbf{P}^1 \times \mathbf{P}^1$ .

**4.3. Collection.** Similar to section 4.1, in view of Keel's tower (4.3), consecutively applying corollary 3.6 to full exceptional collection (4.4) one can obtain a full exceptional collection on  $\overline{M}_{0,6}$ . We start with listing its elements, and afterwards give some indications about how it was obtained.

Let  $q = q_{1,S} \circ q_{2,S}$  and  $E_{\geq i} = \sum_{k=i}^{k=10} E_k$ . The exceptional collection is

$$\begin{aligned}
& q^*(L_0 \boxtimes G_0)(E_{\geq 1}), \\
& q^*(L_1 \boxtimes G_0)(E_{\geq 2}), q^*(L_1 \boxtimes G_0)(E_{\geq 1}), \\
& q^*(L_2 \boxtimes G_0)(E_{\geq 2}), q^*(L_2 \boxtimes G_0)(E_1 + E_{\geq 3}), q^*(L_2 \boxtimes G_0)(E_{\geq 1}),
\end{aligned}$$

$$\begin{aligned}
& q^*(L_3 \boxtimes G_0)(E_{\geq 3}), q^*(L_3 \boxtimes G_0)(E_2 + E_{\geq 4}), q^*(L_3 \boxtimes G_0)(E_{\geq 2}), \\
& q^*(L_4 \boxtimes G_0)(E_{\geq 4}), q^*(L_4 \boxtimes G_0)(E_{\geq 3}), \\
& q^*(L_5 \boxtimes G_0)(E_{\geq 4}), \\
& q^*(L_6 \boxtimes G_0)(E_{\geq 4}), \\
& q^*(L_0 \boxtimes G_1)(E_{\geq 5}), q^*(L_0 \boxtimes G_1)(E_4 + E_{\geq 6}), q^*(L_0 \boxtimes G_1)(E_{\geq 4}), \\
& q^*(L_1 \boxtimes G_1)(E_{\geq 6}), q^*(L_1 \boxtimes G_1)(E_{\geq 5}), q^*(L_1 \boxtimes G_1)(E_4 + E_{\geq 7}), q^*(L_1 \boxtimes G_1)(E_4 + E_6 + E_{\geq 8}), \\
& q^*(L_1 \boxtimes G_1)(E_4 + E_{\geq 6}), \\
& q^*(L_2 \boxtimes G_1)(E_{\geq 8}), q^*(L_2 \boxtimes G_1)(E_{\geq 7}), q^*(L_2 \boxtimes G_1)(E_6 + E_{\geq 9}), q^*(L_2 \boxtimes G_1)(E_6 + E_8 + E_{10}), \\
& q^*(L_2 \boxtimes G_1)(E_6 + E_{\geq 8}), \\
& q^*(L_3 \boxtimes G_1)(E_{10}), q^*(L_3 \boxtimes G_1)(E_{\geq 9}), q^*(L_3 \boxtimes G_1)(E_8 + E_{10}), \\
& q^*(L_4 \boxtimes G_1)(E_{10}), \\
& q^*(L_5 \boxtimes G_1), q^*(L_5 \boxtimes G_1)(E_{10}), \\
& q^*(L_6 \boxtimes G_1), q^*(L_6 \boxtimes G_1)(E_{10}).
\end{aligned}$$

Below we describe the algorithm that was used to obtain this collection.

**4.3.1 Algorithm, step I.** Due to Assumption 1, the restriction of the pair

$$L_1 \boxtimes G_0, L_2 \boxtimes G_0$$

to  $q(E_1)$  is a full exceptional collection. The same holds for pairs  $L_2 \boxtimes G_0, L_3 \boxtimes G_0$  on  $q(E_2)$  and  $L_3 \boxtimes G_0, L_4 \boxtimes G_0$  on  $q(E_3)$ .

Represent  $q_{1,S} = f_1 \circ f_2 \circ f_3$  as a composition of three blow-ups (cf. section 4.1) in such a way that the (preimage of)  $q(E_i)$  is blown up at the  $i$ -th step for  $1 \leq i \leq 3$ . At the first step we use the pair  $L_1 \boxtimes G_0, L_2 \boxtimes G_0$  to apply Corollary 3.6. At the second step we use the pair

$$f_1^*(L_2 \boxtimes G_0)(f_2 \circ f_3 \circ q_{2,S}(E_1)), f_1^*(L_3 \boxtimes G_0),$$

which restricts to a full exceptional collection on  $f_2 \circ f_3 \circ q_{2,S}(E_2)$  because  $q(E_1)$  and  $q(E_2)$  are disjoint and  $L_2 \boxtimes G_0, L_3 \boxtimes G_0$  restricts to an exceptional collection on  $q(E_2)$  as was pointed out above.

One proceeds similarly at the third step and obtains the following exceptional collection on  $B_{2,S}$

$$\begin{aligned}
& q_{1,S}^*(L_0 \boxtimes G_0)(E'_1 + E'_2 + E'_3), \\
& q_{1,S}^*(L_1 \boxtimes G_0)(E'_2 + E'_3), q_{1,S}^*(L_1 \boxtimes G_0)(E'_1 + E'_2 + E'_3), \\
& q_{1,S}^*(L_2 \boxtimes G_0)(E'_2 + E'_3), q_{1,S}^*(L_2 \boxtimes G_0)(E'_1 + E'_3), q_{1,S}^*(L_2 \boxtimes G_0)(E'_1 + E'_2 + E'_3), \\
& q_{1,S}^*(L_3 \boxtimes G_0)(E'_3), q_{1,S}^*(L_3 \boxtimes G_0)(E'_2), q_{1,S}^*(L_3 \boxtimes G_0)(E'_2 + E'_3), \\
& q_{1,S}^*(L_4 \boxtimes G_0), q_{1,S}^*(L_4 \boxtimes G_0)(E'_3), \\
& q_{1,S}^*(L_5 \boxtimes G_0), q_{1,S}^*(L_6 \boxtimes G_0), \\
& q_{1,S}^*(L_0 \boxtimes G_1), q_{1,S}^*(L_1 \boxtimes G_1), \dots,
\end{aligned}$$

where  $E'_i = q_{2,S}(E_i)$ .

**4.3.2 Algorithm, step II.** Due to Assumption 2, the restriction of the pair

$$L_0 \boxtimes G_1, L_1 \boxtimes G_1$$

to  $q_{2,S}(E_4)$  and  $q_{2,S}(E_5)$  gives full exceptional collections on them. The same holds for the pair  $L_1 \boxtimes G_1, L_2 \boxtimes G_1$  on  $q_{2,S}(E_6)$  and  $q_{2,S}(E_7)$ ; for  $L_2 \boxtimes G_1, L_3 \boxtimes G_1$  on  $q_{2,S}(E_8)$  and  $q_{2,S}(E_9)$ ; for  $L_5 \boxtimes G_1, L_6 \boxtimes G_1$  on  $q_{2,S}(E_{10})$ .

Represent  $q_{2,S} = g_4 \circ \dots \circ g_{10}$  as the composition of blow-downs  $g_i$  of  $E_i$  for  $4 \leq i \leq 10$ . To apply Corollary 3.6 to a single blow-up one needs to choose an exceptional pair. We always choose pairs related to those described in the beginning of this section (in fact, they are pull-backs of those followed by a twist with  $\mathcal{O}(D)$ , where  $D$  is a divisor disjoint from the exceptional divisor considered at this step).

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