ALGEBRAIC $K$-THEORY OF THE FIRST MORAVA $K$-THEORY

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ABSTRACT. For a prime $p \geq 5$, we compute the algebraic $K$-theory modulo $p$ and $v_1$ of the mod $p$ Adams summand, using topological cyclic homology. On the way, we evaluate its modulo $p$ and $v_1$ topological Hochschild homology. Using a localization sequence, we also compute the $K$-theory modulo $p$ and $v_1$ of the first Morava $K$-theory.

1. Introduction

In this paper we continue the investigation from [AR02] and [Aus10] of the algebraic $K$-theory of topological $K$-theory and related $S$-algebras. Let $\ell_p$ be the $p$-complete Adams summand of connective complex $K$-theory, and let $\ell/p = k(1)$ be the first connective Morava $K$-theory. It has a unique $S$-algebra structure [Ang] Th. A], and we show in Section 2 that $\ell/p$ is an $\ell_p$-algebra (in uncountably many ways), so that $K(\ell/p)$ is a $K(\ell_p)$-module spectrum.

Let $V(1) = S/(p,v_1)$ be the type 2 Smith–Toda complex (see Section 1 below for a definition). It is a homotopy commutative ring spectrum for $p \geq 5$, with a preferred periodic class $v_2 \in V(1)_*$ of degree $2p^2 - 2$. We write $V(1)_*(X) = \pi_*(V(1) \wedge X)$ for the $V(1)$-homotopy of a spectrum $X$. Multiplication by $v_2$ makes $V(1)_*(X)$ a $P(v_2)$-module, where $P(v_2)$ denotes the polynomial algebra over $\mathbb{F}_p$ generated by $v_2$. We denote by $\mathbb{F}_p\{x_1, \ldots, x_n\}$ the $\mathbb{F}_p$-vector space generated by $x_1, \ldots, x_n$, and by $E(x_1, \ldots, x_n)$ the exterior algebra over $\mathbb{F}_p$ generated by $x_1, \ldots, x_n$.

We computed the $V(1)$-homotopy of $K(\ell_p)$ in [AR02] Th. 9.1], showing that it is essentially a free $P(v_2)$-module on $(4p + 4)$ generators. The following is our main result, corresponding to Theorem 7.7 in the body of the paper.

Theorem 1.1. Let $p \geq 5$ be a prime and let $\ell/p = k(1)$ be the first connective Morava $K$-theory spectrum. There is an isomorphism of $P(v_2)$-modules

$$V(1)_*K(\ell/p) \cong P(v_2) \otimes E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{1, \partial \lambda_2, \lambda_2, \partial v_2\}$$

$$\oplus P(v_2) \otimes E(d \log v_1) \otimes \mathbb{F}_p\{t^d v_2 \mid 0 < d < p^2 - p, p \nmid d\}$$

$$\oplus P(v_2) \otimes E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^{dp} \lambda_2 \mid 0 < d < p\}.$$

Here $|\lambda_1| = |\bar{\epsilon}_1| = 2p - 1$, $|\lambda_2| = 2p^2 - 1$, $|v_2| = 2p^2 - 2$, $|d \log v_1| = 1$, $|\partial| = -1$ and $|t| = -2$. This is a free $P(v_2)$-module of rank $(2p^2 - 2p + 8)$ and of zero Euler characteristic.

We prove this theorem by means of the cyclotomic trace map [BHM93] to topological cyclic homology $TC(\ell/p;p)$. Along the way we evaluate $V(1)_*THH(\ell/p)$, where $THH$ denotes topological Hochschild homology, as well as $V(1)_*TC(\ell/p;p)$, see Proposition 1.2 and Theorem 7.6.

Let $L_p$ be the $p$-complete Adams summand of periodic complex $K$-theory, and let $L/p = K(1)$ be the first periodic Morava $K$-theory. The localization cofiber sequence
MOD IN TERMS OF LOCALIZATION AND GALOIS DESCENT, IN THE SAME WAY AS WE UNDERSTAND THE ALGEBRAIC RINGS. THE FIRST TASK IS TO RELATE OF RANK $\ell_p$-THEORY OF RINGS OF INTEGERS IN (LOCAL) NUMBER FIELDS OR MORE GENERAL REGULAR RINGS. THE FIRST TASK IS TO RELATE $K(\ell_p)$ TO THE ALGEBRAIC K-THEORY OF ITS “RESIDUE RINGS” AND “FRACTION FIELD”, FOR WHICH WE EXPECT A DESCRIPTION IN TERMS OF GALOIS COHOMOLOGY TO EXIST, STARTING WITH THE GALOIS THEORY FOR COMMUTATIVE S-ALGEBRAS DEVELOPED BY THE SECOND AUTHOR [ROG08]. THE RESIDUE RINGS OF $\ell_p$ APPEAR TO BE $\ell_p$, $H\mathbb{Z}_p$ AND $H\mathbb{Z}/p$, WHILE THE FRACTION FIELD $\mathbb{F}(\ell_p)$ IS MORE MYSTERIOUS. FOR OUR PURPOSES, ITS ALGEBRAIC K-THEORY $K(\mathbb{F}(\ell_p))$ SHOULD FIT IN A NATURAL LOCALIZATION COFIBRE SEQUENCE OF SPECTRA

$$K(L/p) \rightarrow K(L_p) \rightarrow K(\mathbb{F}(\ell_p)) \rightarrow \Sigma K(L/p).$$

AN OBVIOUS CANDIDATE FOR $\mathbb{F}(\ell_p)$ IS PROVIDED BY THE ALGEBRAIC LOCALIZATION $L_p[p^{-1}] = L\mathbb{Q}_p$, HAVING AS COEFFICIENTS THE GRADED FIELD $\mathbb{Q}_p[v_1^{\pm 1}]$. HOWEVER, BY THE FOLLOWING COROLLARY, THIS IS TOO NAIVE.

COROLLARY 1.3. THE SPECTRA $K(L/p)$, $K(L_p)$ AND $K(L\mathbb{Q}_p)$ CANNOT POSSIBLY FIT IN A COFIBRE SEQUENCE

$$K(L/p) \rightarrow K(L_p) \rightarrow K(L\mathbb{Q}_p) \rightarrow \Sigma K(L/p).$$

Indeed, the above computation implies that $V(1), K(L/p)[v_2^{-1}]$ AND $V(1), K(L_p)[v_2^{-1}]$ ARE NOT ABSTRACTLY ISOMORPHIC, WHILE $V(1), K(L\mathbb{Q}_p)[v_2^{-1}]$ IS ZERO SINCE IT IS AN ALGEBRA OVER $V(1), K(\mathbb{Q}_p)[v_2^{-1}] = 0$. THE LATER EQUALITY FOLLOWS FROM THE COMPUTATION OF THE $p$-PRIMARY HOMOTOPY TYPE OF $K(\mathbb{Q}_p)$ [HM03, TH. D], WHICH SHOWS THAT $V(1), K(\mathbb{Q}_p)$ IS $v_2$-TORSION.

IN CONCLUSION, THE CONJECTURAL FRACTION FIELD $\mathbb{F}(\ell_p)$ APPEARS TO BE A LOCALIZATION OF $L_p$ AWAY FROM $L/p$ LESS DRAMATIC THAN THE ALGEBRAIC LOCALIZATION $L_p[p^{-1}] = L\mathbb{Q}_p$. WE ELABORATE MORE ON THIS ISSUE IN [AR].

THE PAPER IS ORGANIZED AS FOLLOWS. IN SECTION 2 WE FIX OUR NOTATIONS, SHOW THAT $\ell/p$ ADMITS THE STRUCTURE OF AN ASSOCIATIVE $\ell_p$-ALGEBRA, AND GIVE A SIMILAR DISCUSSION FOR $ku/p$
and the periodic versions \( L/p \) and \( KU/p \). Section 3 contains the computation of the mod \( p \) homology of \( THH(\ell/p) \), and in Section 4 we evaluate its \( V(1) \)-homotopy. In Section 5 we show that the \( C_p \)-fixed points and \( C_p \)-homotopy fixed points of \( THH(\ell/p) \) are closely related, and use this to inductively determine their \( V(1) \)-homotopy in Section 6. Finally, in Section 7 we achieve the computation of \( TC(\ell/p;p) \) and \( K(\ell/p) \) in \( V(1) \)-homotopy.

Notations and conventions. Let \( p \) be a fixed prime. We write \( E(x) = \mathbb{F}_p[x]/(x^2) \) for the exterior algebra, \( P(x) = \mathbb{F}_p[x] \) for the polynomial algebra and \( P(x^{p+1}) = \mathbb{F}_p[x, x^{-1}] \) for the Laurent polynomial algebra on one generator \( x \), and similarly for a list of generators. We will also write \( \Gamma(x) = \mathbb{F}_p\{\gamma_i(x) \mid i \geq 0\} \) for the divided power algebra, with \( \gamma_i(x) \cdot \gamma_j(x) = (i, j)\gamma_{i+j}(x) \), where \( (i, j) = (i + j)!/i!j! \) is the binomial coefficient. We use the obvious abbreviations \( \gamma_0(x) = 1 \) and \( \gamma_1(x) = x \). Finally, we write \( P_h(x) = \mathbb{F}_p[x]/(x^h) \) for the truncated polynomial algebra of height \( h \), and recall the isomorphism \( \Gamma(x) \cong P_p(\gamma_p(x) \mid e \geq 0) \) in characteristic \( p \). We write \( X(p) \) and \( X_p \) for the \( p \)-localization and the \( p \)-completion, respectively, of any spectrum or abelian group \( X \). In the spectral sequences (of \( \mathbb{F}_p \)-modules) discussed below, we often determine differentials only up to multiplication by a unit. We use the notation \( d(x) = y \) to indicate that the equation \( d(x) = \alpha y \) holds for some unit \( \alpha \in \mathbb{F}_p \).

2. BASE CHANGE SQUARES OF \( S \)-ALGEBRAS

Let \( p \) be a prime, even or odd for now. Let \( ku \) and \( KU \) be the connective and the periodic complex \( K \)-theory spectra, with homotopy rings \( ku_\ast = \mathbb{Z}[u] \) and \( KU_\ast = \mathbb{Z}[u^{\pm 1}] \), where \( |u| = 2 \). Let \( \ell = BP(1) \) and \( L = E(1) \) be the \( p \)-local Adams summands, with \( \ell_\ast = \mathbb{Z}(p)[v_1] \) and \( L_\ast = \mathbb{Z}(p)[v_1^{\pm 1}] \), where \( |v_1| = 2p - 2 \). The inclusion \( \ell \to ku(p) \) maps \( v_1 \) to \( w^{p-1} \). Alternate notations in the \( p \)-complete cases are \( KU_p = E_1 \) and \( L_p = \widehat{E_1} \). These ring spectra are all commutative \( S \)-algebras, in the sense that each admits a unique \( E_\infty \) ring spectrum structure. See [BR05, p. 692] for proofs of uniqueness in the periodic cases.

Let \( ku/p \) and \( KU/p \) be the connective and periodic mod \( p \) complex \( K \)-theory spectra, with coefficients \( (ku/p)_\ast = \mathbb{Z}[u]/p[u] \) and \( (KU/p)_\ast = \mathbb{Z}/p[u^{\pm 1}] \). These are 2-periodic versions of the first Morava \( K \)-theory spectra \( \ell/p = k(1) \) and \( L/p = K(1) \), with \( (\ell/p)_\ast = \mathbb{Z}/p[v_1] \) and \( (L/p)_\ast = \mathbb{Z}/p[v_1^{\pm 1}] \). Each of these can be constructed as the cofiber of the multiplication by \( p \) map, as a module over the corresponding commutative \( S \)-algebra.

For example, there is a cofiber sequence of \( ku \)-modules \( ku \xrightarrow{p} ku \xrightarrow{\ell} ku/p \to \Sigma ku \).

Let \( HR \) be the Eilenberg–Mac Lane spectrum of a ring \( R \). When \( R \) is commutative, \( HR \) admits a unique associative \( S \)-algebra structure, and when \( R \) is commutative, \( HR \) admits a unique commutative \( S \)-algebra structure. The zeroth Postnikov section defines unique maps of commutative \( S \)-algebras \( \tau : ku \to H\mathbb{Z} \) and \( \tau : \ell \to H\mathbb{Z}(p) \), which can be followed by unique commutative \( S \)-algebra maps to \( H\mathbb{Z}/p \).

The \( ku \)-module spectrum \( ku/p \) does not admit the structure of a commutative \( ku \)-algebra. It cannot even be an \( E_2 \) or \( H_2 \) ring spectrum, since the homomorphism induced in mod \( p \) homology by the resulting map \( \tau : ku/p \to H\mathbb{Z}/p \) of \( H_2 \) ring spectra would not commute with the homology operation \( Q^1(\bar{\gamma}_0) = \bar{\gamma}_1 \) in the target \( H_\ast(\mathbb{Z}/p; \mathbb{F}_p) \) [BMMS85, III.2.3]. Similar remarks apply for \( KU/p \), \( \ell/p \) and \( L/p \). Associative algebra structures, or \( A_\infty \) ring spectrum structures, are easier to come by. The following result is a direct application of the methods of [Laz01, §§9–11]. We adapt the notation of [BJ02, §3] to provide some details in our case.
Proposition 2.1. The $ku$-module spectrum $ku/p$ admits the structure of an associative $ku$-algebra, but the structure is not unique. Similar statements hold for $KU/p$ as a $KU$-algebra, $\ell/p$ as an $\ell$-algebra and $L/p$ as an $L$-algebra.

Proof. We construct $ku/p$ as the (homotopy) limit of its Postnikov tower of associative $ku$-algebras $P^{2m-2} = ku/(p, u^m)$, with coefficient rings $ku/(p, u^m)_* = ku_*/(p, u^m)$ for $m \geq 1$. To start the induction, $P^0 = H\mathbb{Z}/p$ is a $ku$-algebra via $i \circ \pi : ku \to H\mathbb{Z} \to H\mathbb{Z}/p$. Assume inductively for $m \geq 1$ that $P = P^{2m-2}$ has been constructed. We will define $P^{2m}$ by a (homotopy) pullback diagram

$$
\begin{array}{ccc}
P^{2m} & \to & P \\
\downarrow & & \downarrow m_1 \\
P & \to & P \vee \Sigma^{2m+1}H\mathbb{Z}/p
\end{array}
$$

in the category of associative $ku$-algebras. Here

$$
d \in \text{ADer}_{ku}^{2m+1}(P, H\mathbb{Z}/p) \cong \text{THH}_{ku}^{2m+2}(P, H\mathbb{Z}/p)
$$

is an associative $ku$-algebra derivation of $P$ with values in $\Sigma^{2m+1}H\mathbb{Z}/p$, and the group of such can be identified with the indicated topological Hochschild cohomology group of $P$ over $ku$. We recall that these are the homotopy groups (cohomologically graded) of the function spectrum $F_{P \wedge ku, P^{op}}(P, H\mathbb{Z}/p)$. The composite map $pr_2 \circ d : P \to \Sigma^{2m+1}H\mathbb{Z}/p$ of $ku$-modules, where $pr_2$ projects onto the second wedge summand, is restricted to equal the $ku$-module Postnikov $k$-invariant of $ku/p$ in

$$
H_{ku}^{2m+1}(P; \mathbb{Z}/p) = \pi_0 F_{ku}(P, \Sigma^{2m+1}H\mathbb{Z}/p).
$$

We compute that $\pi_*(P \wedge ku P^{op}) = ku_*/(p, u^m) \otimes E(\tau_0, \tau_{1,m})$, where $|\tau_0| = 1$, $|\tau_{1,m}| = 2m+1$ and $E(-)$ denotes the exterior algebra on the given generators. (For $p = 2$, the use of the opposite product is essential here [Ang08 §3].) The function spectrum description of topological Hochschild cohomology leads to the spectral sequence

$$
E_2^{s,t} = \text{Ext}^{s,t}_{\pi_*(P \wedge ku P^{op})}(\pi_*(P), \mathbb{Z}/p) \\
\cong \mathbb{Z}[[y_0, y_{1,m}]] \\
\Rightarrow \text{THH}_{ku}^*(P, H\mathbb{Z}/p),
$$

where $y_0$ and $y_{1,m}$ have cohomological bidegrees $(1, 1)$ and $(1, 2m + 1)$, respectively. The spectral sequence collapses at $E_2 = E_\infty$, since it is concentrated in even total degrees. In particular,

$$
\text{ADer}_{ku}^{2m+1}(P, H\mathbb{Z}/p) \cong \mathbb{F}_p\{y_{1,m}, y_0^{m+1}\}.
$$

Additively, $H_{ku}^{2m+1}(P; \mathbb{Z}/p) \cong \mathbb{F}_p\{Q_{1,m}\}$ is generated by a class dual to $\tau_{1,m}$, which is the image of $y_{1,m}$ under left composition with $pr_2$. It equals the $ku$-module $k$-invariant of $ku/p$. Thus there are precisely $p$ choices $d = y_{1,m} + \alpha y_0^{m+1}$, with $\alpha \in \mathbb{F}_p$, for how to extend any given associative $ku$-algebra structure on $P = P^{2m-2}$ to one on $P^{2m} = ku/(p, u^{m+1})$.

In the limit, we find that there are an uncountable number of associative $ku$-algebra structures on $ku/p = \text{holim}_m P^{2m}$, each indexed by a sequence of choices $\alpha \in \mathbb{F}_p$ for all $m \geq 1$.

The periodic spectrum $KU/p$ can be obtained from $ku/p$ by Bousfield $KU$-localization in the category of $ku$-modules [EKMM97 VIII.4], which makes it an associative $KU$-algebra. The classification of periodic $S$-algebra structures is the same as in the connective case, since the original $ku$-algebra structure on $ku/p$ can be recovered from
that on $KU/p$ by a functorial passage to the connective cover. To construct $\ell/p$ as an associative $\ell$-algebra, or $L/p$ as an associative $L$-algebra, replace $u$ by $v_1$ in these arguments.

By varying the ground $S$-algebra, we obtain the same conclusions about $ku/p$ as a $ku(p)$-algebra or $ku_p$-algebra, and about $\ell/p$ as an $\ell_p$-algebra.

For each choice of $ku$-algebra structure on $ku/p$, the zeroth Postnikov section

$$
\pi: ku/p \to \mathbb{H} \mathbb{Z}/p
$$

is a $ku$-algebra map, with the unique $ku$-algebra structure on the target. Hence there is a commutative square of associative $ku$-algebras

$$
\begin{array}{ccc}
k u & \xrightarrow{i} & ku/p \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathbb{H} \mathbb{Z} & \xrightarrow{i} & \mathbb{H} \mathbb{Z}/p
\end{array}
$$

and similarly in the $p$-local and $p$-complete cases. In view of the weak equivalence $\mathbb{H} \mathbb{Z} \wedge ku \simeq H \mathbb{Z}/p$, this square expresses the associative $\mathbb{H} \mathbb{Z}$-algebra $\mathbb{H} \mathbb{Z}/p$ as the base change of the associative $ku$-algebra $ku/p$ along $\pi: ku \to \mathbb{H} \mathbb{Z}$. Likewise, there is a commutative square of associative $\ell_p$-algebras

$$
\begin{array}{ccc}
\ell_p & \xrightarrow{i} & \ell/p \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathbb{H} \mathbb{Z}_p & \xrightarrow{i} & \mathbb{H} \mathbb{Z}/p
\end{array}
$$

that expresses $\mathbb{H} \mathbb{Z}/p$ as the base change of $\ell/p$ along $\ell_p \to \mathbb{H} \mathbb{Z}_p$, and similarly in the $p$-local case. By omission of structure, these squares are also diagrams of $S$-algebras and $S$-algebra maps.

We end this section by formulating the mod $p$ analogue of the localization cofibre sequence in algebraic $K$-theory

$$
K(\mathbb{Z}_p) \to K(\ell_p) \to K(L_p) \to \Sigma K(\mathbb{Z}_p)
$$

conjectured by the second author and established by Blumberg and Mandell [BM08, p. 157]

**Proposition 2.2.** There is a localization cofibre sequence of spectra

$$
K(\mathbb{Z}/p) \to K(\ell/p) \to K(L/p) \to \Sigma K(\mathbb{Z}/p)
$$

where the first map is the transfer and the second map is induced by the localization $\ell/p \to \ell/p[v_1^{-1}] = L/p$.

**Proof.** The proof of the existence of the localization sequence (2.2) given in [BM08, p. 160–163] and the identification of the transfer map adapt without change to cover the mod $p$ analogue stated in this proposition. Here we use that a finite cell $\ell/p$-module that is $v_1$-torsion has finite homotopy groups, and the non-zero groups are concentrated in a finite range of degrees. \qed
3. **Topological Hochschild homology**

We shall compute the $V(1)$-homotopy of the topological Hochschild homology $THH(-)$ and topological cyclic homology $TC(-;p)$ of the $S$-algebras in diagram (2.1), for primes $p \geq 5$. Passing to connective covers, this also computes the $V(1)$-homotopy of the algebraic $K$-theory spectra appearing in that square. With these coefficients, or more generally, after $p$-adic completion, the functors $THH$ and $TC$ are insensitive to $p$-completion in the argument, so we shall simplify the notation slightly by working with the associative $S$-algebras $\ell$ and $HZ_{(p)}$ in place of $\ell_p$ and $HZ_p$. For ordinary rings $R$ we almost always shorten notations like $THH(HR)$ to $THH(R)$.

The computations follow the strategy of [Bök, BM94, BM95 and HM97] for $HZ/p$ and $HZ$, and of [MS93] and [AR02] for $\ell$. See also [AR05, §8–7] for further discussion of the $THH$-part of such computations. In this section we shall compute the mod $p$ homology of the topological Hochschild homology of $\ell/p$ as a module over the corresponding homology for $\ell$, for any odd prime $p$.

**Remark 3.1.** Our computations are based on comparisons, using the maps displayed in diagram (2.1) above. We will abuse notation and use the same name for classes in the homology or $V(1)$-homotopy of $THH(\ell_p)$, $THH(\ell/p)$, $THH(Z_p)$ or $THH(Z/p)$, when these classes unambiguously correspond to each other under the homomorphisms induced by the maps $i$ and $\pi$ in (2.1). We also use this abuse of notations in later sections for the $V(1)$-homotopy of $TC$, etc.

We write $H_*(-)$ for homology with mod $p$ coefficients. It takes values in graded $A_*$-comodules, where $A_*$ is the dual Steenrod algebra [Mil58 Th 2]. Explicitly (for $p$ odd),

$$A_* = P(\xi_k \mid k \geq 1) \otimes E(\tau_k \mid k \geq 0)$$

with coproduct

$$\psi(\xi_k) = \sum_{i+j=k} \xi_i \otimes \xi_j$$

and

$$\psi(\tau_k) = 1 \otimes \tau_k + \sum_{i+j=k} \tau_i \otimes \tau_j.$$  

Here $\xi_0 = 1$, $\xi_k = 1(\xi_k)$ has degree $2(p^k - 1)$ and $\tau_k = 1(\tau_k)$ has degree $2p^k - 1$, where $1$ is the canonical conjugation [MM65, 8.4]. Then the maps $i$ and the zeroth Postnikov sections $\pi$ of (2.1) induce identifications

$$H_*(HZ_{(p)}) = P(\xi_k \mid k \geq 1) \otimes E(\tau_k \mid k \geq 1)$$

$$H_*(\ell) = P(\xi_k \mid k \geq 1) \otimes E(\tau_k \mid k \geq 2)$$

$$H_*(\ell/p) = P(\xi_k \mid k \geq 1) \otimes E(\tau_0, \tau_k \mid k \geq 2)$$

as $A_*$-comodule subalgebras of $H_*(HZ/p) = A_*$. We often make use of the following $A_*$-comodule coactions

$$\nu(\tau_0) = 1 \otimes \tau_0 + \tau_0 \otimes 1$$

$$\nu(\xi_1) = 1 \otimes \xi_1 + \xi_1 \otimes 1$$

$$\nu(\tau_1) = 1 \otimes \tau_1 + \tau_0 \otimes \xi_1 + \tau_1 \otimes 1$$

$$\nu(\xi_2) = 1 \otimes \xi_2 + \xi_1 \otimes \xi_1 + \xi_2 \otimes 1$$

$$\nu(\tau_2) = 1 \otimes \tau_2 + \tau_0 \otimes \xi_2 + \tau_1 \otimes \xi_1 + \tau_2 \otimes 1.$$
The Bökstedt spectral sequences 

\[ E^2(B) = HH_*(H_*(B)) \implies H_*(THH(B)) \]

for the commutative \( S \)-algebras \( B = H\mathbb{Z}/p, H\mathbb{Z}_\mathbb{F}(p) \) and \( \ell \) begin 

\[
\begin{align*}
E^2(\mathbb{Z}/p) &= A_* \otimes E(\sigma \xi_k \mid k \geq 1) \otimes \Gamma(\sigma \tau_k \mid k \geq 0) \\
E^2(\mathbb{Z}_\mathbb{F}(p)) &= H_*(H\mathbb{Z}_\mathbb{F}(p)) \otimes E(\sigma \xi_k \mid k \geq 1) \otimes \Gamma(\sigma \tau_k \mid k \geq 1) \\
E^2(\ell) &= H_*(\ell) \otimes E(\sigma \xi_k \mid k \geq 1) \otimes \Gamma(\sigma \tau_k \mid k \geq 2).
\end{align*}
\]

Here \( HH_*(H_*(B)) \) denotes the Hochschild homology of the graded \( \mathbb{F}_p \)-algebra \( H_*(B) \). In the above formula we made use of the \( \mathbb{F}_p \)-linear operator \( \sigma: H_*(B) \to HH_1(H_*(B)) \), \( x \mapsto \sigma x \), where \( \sigma x \) is the class represented by \( 1 \otimes x - x \otimes 1 \) in the Hochschild complex. Notice that \( \sigma \) is the restriction of Connes’ operator \( d \) to \( HH_0(H_*(B)) = H_*(B) \), and is a derivation in the sense that 

\[ \sigma(xy) = x\sigma(y) + (-1)^{\|x\|\|y\|}y\sigma(x) \]

for all \( x, y \in H_*(B) \). These spectral sequences are (graded) commutative \( A_* \)-comodule algebra spectral sequences, and there are differentials 

\[ d^{p-1}(\gamma_j \sigma \tau_k) = \sigma \xi_{k+1} \cdot \gamma_{j-p} \sigma \tau_k \]

for \( j \geq p \) and \( k \geq 0 \), see [Bök Lem. 1.3], [Hun96 Th. 1] or [Aus05 Lem. 5.3], leaving 

\[
\begin{align*}
E^\infty(\mathbb{Z}/p) &= A_* \otimes P_p(\sigma \tau_k \mid k \geq 0) \\
E^\infty(\mathbb{Z}_\mathbb{F}(p)) &= H_*(H\mathbb{Z}_\mathbb{F}(p)) \otimes E(\sigma \xi_1) \otimes P_p(\sigma \tau_1) \\
E^\infty(\ell) &= H_*(\ell) \otimes E(\sigma \xi_1, \sigma \xi_2) \otimes P_p(\sigma \tau_2) \end{align*}
\]

The inclusion of 0-simplices \( \eta: B \to THH(B) \) is split for commutative \( B \) by the augmentation \( \epsilon: THH(B) \to B \). Thus there are unique representatives in Bökstedt filtration 1, with zero augmentation, for each of the classes \( x \). There are multiplicative extensions \( (\sigma \tau_k)^p = \sigma \tau_{k+1} \) for \( k \geq 0 \), see [AR05 Prop. 5.9], so 

\[
\begin{align*}
H_*(THH(\mathbb{Z}/p)) &= A_* \otimes P(\sigma \tau_0) \\
H_*(THH(\mathbb{Z}_\mathbb{F}(p))) &= H_*(H\mathbb{Z}_\mathbb{F}(p)) \otimes E(\sigma \xi_1) \otimes P(\sigma \tau_1) \\
H_*(THH(\ell)) &= H_*(\ell) \otimes E(\sigma \xi_1, \sigma \xi_2) \otimes P(\sigma \tau_2)
\end{align*}
\]

as \( A_* \)-comodule algebras. The \( A_* \)-comodule coactions are given by 

\[
\begin{align*}
\nu(\sigma \tau_0) &= 1 \otimes \sigma \tau_0 \\
\nu(\sigma \xi_1) &= 1 \otimes \sigma \xi_1 \\
\nu(\sigma \tau_1) &= 1 \otimes \sigma \tau_1 + \tau_0 \otimes \sigma \xi_1 \\
\nu(\sigma \xi_2) &= 1 \otimes \sigma \xi_2 \\
\nu(\sigma \tau_2) &= 1 \otimes \sigma \tau_2 + \tau_0 \otimes \sigma \xi_2.
\end{align*}
\]

The natural map \( \pi_*: THH(\ell) \to THH(\mathbb{Z}/p) \) induced by \( \pi: \ell \to \mathbb{Z}/p \) takes \( \sigma \xi_2 \) to 0 and \( \sigma \tau_2 \) to \((\sigma \tau_1)^p \). The natural map \( i_*: THH(\mathbb{Z}_\mathbb{F}(p)) \to THH(\mathbb{Z}/p) \) induced by \( i: \mathbb{Z}_\mathbb{F}(p) \to \mathbb{Z}/p \) takes \( \sigma \xi_1 \) to 0 and \( \sigma \tau_1 \) to \((\sigma \tau_0)^p \). 

The Bökstedt spectral sequence for the associative \( S \)-algebra \( B = \ell/p \) begins 

\[ E^2(\ell/p) = H_*(\ell/p) \otimes E(\sigma \xi_k \mid k \geq 1) \otimes \Gamma(\sigma \tau_0, \sigma \tau_k \mid k \geq 2). \]
It is an $A_\ast$-comodule module spectral sequence over the B"okstedt spectral sequence for $\ell$, since the $\ell$-algebra multiplication $\ell \wedge \ell/p \to \ell/p$ is a map of associative $S$-algebras. However, it is not itself an algebra spectral sequence, since the product on $\ell/p$ is not commutative enough to induce a natural product structure on $\text{THH}(\ell/p)$. Nonetheless, we will use the algebra structure present at the $E^2$-term to help in naming classes.

The map $\pi: \ell/p \to H\mathbb{Z}/p$ induces an injection of B"okstedt spectral sequence $E^2$-terms, so there are differentials generated algebraically by

$$d^{p-1}(\gamma_j \sigma \bar{\tau}_k) = \sigma \bar{\xi}_{k+1} \cdot \gamma_{j-p} \sigma \bar{\tau}_k$$

for $j \geq p$, $k = 0$ or $k \geq 2$, leaving

$$E^\infty(\ell/p) = H_\ast(\ell/p) \otimes E(\sigma \bar{\xi}_2) \otimes P_\ast(\sigma \bar{\tau}_0, \sigma \bar{\tau}_k \mid k \geq 2)$$

as an $A_\ast$-comodule module over $E^\infty(\ell)$. In order to obtain $H_\ast(\text{THH}(\ell/p))$, we need to resolve the $A_\ast$-comodule and $H_\ast(\text{THH}(\ell))$-module extensions. This is achieved in Lemma 3.3 below.

The natural map $\pi_*: E^\infty(\ell/p) \to E^\infty(\mathbb{Z}/p)$ is an isomorphism in total degrees $\leq (2p - 2)$ and injective in total degrees $\leq (2p^2 - 2)$. The first class in the kernel is $\sigma \bar{\xi}_2$. Hence there are unique classes

$$1, \bar{\tau}_0, \sigma \bar{\tau}_0, \bar{\tau}_0 \sigma \bar{\tau}_0, \ldots, (\sigma \bar{\tau}_0)^{p-1}$$

in degrees $0 \leq * \leq 2p - 2$ of $H_\ast(\text{THH}(\ell/p))$, mapping to classes with the same names in $H_\ast(\text{THH}(\mathbb{Z}/p))$. More concisely, these are the monomials $\bar{\tau}_i^\ast(\sigma \bar{\tau}_0)^{\delta}$ for $0 \leq \delta \leq 1$ and $0 \leq i \leq p - 1$, except that the degree $(2p - 1)$ case $(\delta, i) = (1, p - 1)$ is omitted. The $A_\ast$-comodule coaction on these classes is given by the same formulas in $H_\ast(\text{THH}(\ell/p))$ as in $H_\ast(\text{THH}(\mathbb{Z}/p))$, cf. (3.1).

There is also a class $\bar{\xi}_1$ in degree $(2p - 2)$ of $H_\ast(\text{THH}(\ell/p))$ mapping to a class with the same name, and same $A_\ast$-coaction, in $H_\ast(\text{THH}(\mathbb{Z}/p))$.

In degree $(2p - 1)$, $\pi_*$ is a map of extensions from

$$0 \to \mathbb{F}_p\{\bar{\xi}_1 \bar{\tau}_0\} \to H_{2p - 1}(\text{THH}(\ell/p)) \to \mathbb{F}_p\{\bar{\tau}_0(\sigma \bar{\tau}_0)^{p-1}\} \to 0$$

to

$$0 \to \mathbb{F}_p\{\bar{\tau}_1, \bar{\xi}_1 \bar{\tau}_0\} \to H_{2p - 1}(\text{THH}(\mathbb{Z}/p)) \to \mathbb{F}_p\{\bar{\tau}_0(\sigma \bar{\tau}_0)^{p-1}\} \to 0.$$ 

The latter extension is canonically split by the augmentation $\epsilon: \text{THH}(\mathbb{Z}/p) \to H\mathbb{Z}/p$, which uses the commutativity of the $S$-algebra $H\mathbb{Z}/p$.

In degree $2p$, the map $\pi_*$ goes from

$$H_{2p}(\text{THH}(\ell/p)) = \mathbb{F}_p\{\bar{\xi}_1 \sigma \bar{\tau}_0\}$$

and the map $\pi_*\sigma_0$ to

$$0 \to \mathbb{F}_p\{\bar{\tau}_0 \bar{\tau}_1\} \to H_{2p}(\text{THH}(\mathbb{Z}/p)) \to \mathbb{F}_p\{\sigma \bar{\tau}_1, \bar{\xi}_1 \sigma \bar{\tau}_0\} \to 0.$$ 

Again the latter extension is canonically split.

**Lemma 3.2.** There is a unique class $y$ in $H_{2p-1}(\text{THH}(\ell/p))$ represented by $\bar{\tau}_1(\sigma \bar{\tau}_0)^{p-1}$ in $E^\infty_{-1, \ast}(\ell/p)$ and mapped by $\pi_*$ to $\bar{\tau}_0(\sigma \bar{\tau}_0)^{p-1} = \bar{\tau}_1$ in $H_\ast(\text{THH}(\mathbb{Z}/p))$.

**Proof.** This follows from naturality of the suspension operator $\sigma$ and the multiplicative relation $(\sigma \bar{\tau}_0)^{p} = \sigma \bar{\tau}_1$ in $H_\ast(\text{THH}(\mathbb{Z}/p))$. A class $y$ in $H_{2p-1}(\text{THH}(\ell/p))$ represented by $\bar{\tau}_0(\sigma \bar{\tau}_0)^{p-1}$ is determined modulo $\bar{\xi}_1 \bar{\tau}_0$. Its image in $H_{2p-1}(\text{THH}(\mathbb{Z}/p))$ thus has the form $\alpha \bar{\tau}_1 + \bar{\tau}_0(\sigma \bar{\tau}_0)^{p-1}$ modulo $\bar{\xi}_1 \bar{\tau}_0$, for some $\alpha \in \mathbb{F}_p$. The suspension $\sigma y$ lies in $H_{2p}(\text{THH}(\ell/p)) = \mathbb{F}_p\{\bar{\xi}_1 \sigma \bar{\tau}_0\}$, so its image in $H_{2p}(\text{THH}(\mathbb{Z}/p))$ is $0$ modulo $\bar{\tau}_0 \bar{\tau}_1$ and $\bar{\xi}_1 \sigma \bar{\tau}_0$. It is also the suspension of $\alpha \bar{\tau}_1 + \bar{\tau}_0(\sigma \bar{\tau}_0)^{p-1}$ modulo $\bar{\xi}_1 \bar{\tau}_0$, which equals...
Lemma 3.3. The classes 
\( E_{\ell/p} \) rank
The graded map\( H_\ast(THH(\ell)) \) maps to \( 0 \) for \( \sigma \) and \( \pi \leq 1 \). This generates all of \( \ell/p \).

Let
\[ H_\ast(THH(\ell)) / (\sigma \xi_1) = H_\ast(\ell) \otimes E(\sigma \xi_2) \otimes P(\sigma \tau_2) \]
denote the quotient algebra of \( H_\ast(THH(\ell)) \) by the ideal generated by \( \sigma \xi_1 \).

Lemma 3.3. The classes
\[ 1, \bar{\tau}_0, \sigma \bar{\tau}_0, \bar{\tau}_0 \sigma \bar{\tau}_0, \ldots, (\sigma \bar{\tau}_0)^{p-1}, \bar{\tau}_0(\sigma \bar{\tau}_0)^{p-1}, \]
in \( E_\infty(\ell/p) \) represent unique homology classes in \( H_\ast(THH(\ell/p)) \), which by abuse of notation will be denoted
\[ 1, \bar{\tau}_0, \sigma \bar{\tau}_0, \bar{\tau}_0 \sigma \bar{\tau}_0, \ldots, (\sigma \bar{\tau}_0)^{p-1}, y, \]
mapping under \( \pi_* \) to classes with the same names in \( H_\ast(THH(\mathbb{Z}/p)) \), except for \( y \), which maps to
\[ \bar{\tau}_0(\sigma \bar{\tau}_0)^{p-1} - \bar{\tau}_1. \]
The graded \( H_\ast(THH(\ell)) \)-module \( H_\ast(THH(\ell/p)) \) is a free \( H_\ast(THH(\ell)) / (\sigma \xi_1) \)-module of rank \( 2p \) generated by these classes in degrees \( 0 \) through \( 2p - 1 \):
\[ H_\ast(THH(\ell/p)) = H_\ast(THH(\ell)) / (\sigma \xi_1) \otimes \mathbb{F}_p \{ 1, \bar{\tau}_0, \sigma \bar{\tau}_0, \bar{\tau}_0 \sigma \bar{\tau}_0, \ldots, (\sigma \bar{\tau}_0)^{p-1}, y \}. \]
The \( A_* \)-comodule coactions are given by
\[ \nu((\sigma \bar{\tau}_0)^i) = 1 \otimes (\sigma \bar{\tau}_0)^i \]
for \( 0 \leq i \leq p - 1 \),
\[ \nu(\bar{\tau}_0(\sigma \bar{\tau}_0)^i) = 1 \otimes \bar{\tau}_0(\sigma \bar{\tau}_0)^i + \bar{\tau}_0 \otimes (\sigma \bar{\tau}_0)^i \]
for \( 0 \leq i \leq p - 2 \), and
\[ \nu(y) = 1 \otimes y + \bar{\tau}_0 \otimes (\sigma \bar{\tau}_0)^{p-1} - \bar{\tau}_0 \otimes \xi_1 - \bar{\tau}_1 \otimes 1. \]

Proof. \( H_\ast(\ell/p) \) is freely generated as a module over \( H_\ast(\ell) \) by \( 1 \) and \( \bar{\tau}_0 \), and the classes \( \sigma \xi_2 \) and \( \sigma \bar{\tau}_2 \) in \( H_\ast(THH(\ell)) \) induce multiplication by the same symbols in \( E_\infty(\ell/p) \), as given in \( (3.2) \). This generates all of \( E_\infty(\ell/p) \) from the \( 2p \) classes \( \bar{\tau}_0^\delta (\sigma \bar{\tau}_0)^i \) for \( 0 \leq \delta \leq 1 \) and \( 0 \leq i \leq p - 1 \).

We claim that multiplication by \( \sigma \xi_1 \) acts trivially on \( H_\ast(THH(\ell/p)) \). It suffices to verify this on the module generators \( \bar{\tau}_0^\delta (\sigma \bar{\tau}_0)^i \), for which the product with \( \sigma \xi_1 \) remains in the range of degrees where the map to \( H_\ast(THH(\mathbb{Z}/p)) \) is injective. The action of \( \sigma \xi_1 \) is trivial on \( H_\ast(THH(\mathbb{Z}/p)) \), since \( d_\ast^{p-1}(\gamma_p \sigma \bar{\tau}_0) = \sigma \xi_1 \) and \( e(\sigma \xi_1) = 0 \), and this implies the claim.

The \( A_* \)-comodule coaction on each module generator, including \( y \), is determined by that on its image under \( \pi_* \). In the latter case, for example, we have
\[ (1 \otimes \pi_*)(\nu(y)) = \nu(\pi_*(y)) = \nu(\bar{\tau}_0(\sigma \bar{\tau}_0)^{p-1} - \bar{\tau}_1) \]
\[ = 1 \otimes \bar{\tau}_0(\sigma \bar{\tau}_0)^{p-1} + \bar{\tau}_0 \otimes (\sigma \bar{\tau}_0)^{p-1} - 1 \otimes \bar{\tau}_1 - \bar{\tau}_0 \otimes \xi_1 - \bar{\tau}_1 \otimes 1 \]
\[ = (1 \otimes \pi_*)(1 \otimes y + \bar{\tau}_0 \otimes (\sigma \bar{\tau}_0)^{p-1} - \bar{\tau}_0 \otimes \xi_1 - \bar{\tau}_1 \otimes 1), \]
and this proves our formula for \( \nu(y) \) since \( 1 \otimes \pi_* \) is injective in this degree.

Remark 3.4. Notice that Lemma 3.3 implies that for different choices of \( \ell \)-module structure on \( \ell/p \), the resulting homology groups \( H_\ast(THH(\ell/p)) \) are (abstractly) isomorphic as graded \( H_\ast(THH(\ell)) \)-modules and \( A_* \)-comodules.
4. Passage to $V(1)$-homotopy

For $p \geq 5$ the Smith–Toda complex $V(1) = S \cup_p e^1 \cup_{\alpha_1} e^{2p-1} \cup_p e^{2p}$ is a homotopy commutative ring spectrum [Smi70, Th 5.1], [Oka84, Ex. 4.5]. It is defined as the mapping cone of the Adams self-map $v_1 : \Sigma^{2p-2}V(0) \to V(0)$ of the mod $p$ Moore spectrum $V(0) = S \cup_p e^1$. Hence there is a cofiber sequence

$$\Sigma^{2p-2}V(0) \xrightarrow{i} V(0) \xrightarrow{i} V(1) \xrightarrow{\beta_1} \Sigma^{2p-1}V(0).$$

There are some choices of orientations involved in fixing such an exact triangle, compare for instance with [HM03, Sect. 2.1]. The composite map $\beta_{1,1} = i_1 j_1 : V(1) \to \Sigma^{2p-1}V(1)$ defines the primary $v_1$-Bockstein homomorphism, acting naturally on $V(1)_*(X)$.

In this section we compute $V(1)_*\text{THH}(\ell/p)$ as a module over $V(1)_*\text{THH}(\ell)$, for any prime $p \geq 5$. The unique ring spectrum map from $V(1)$ to $HZ/p$ induces the identification

$$H_*(V(1)) = E(\tau_0, \tau_1)$$

(no conjugations) as $A_*$-comodule subalgebras of $A_*$, see [Tod71, §4]. Here

$$\nu(\tau_0) = 1 \otimes \tau_0 + \tau_0 \otimes 1$$

$$\nu(\tau_1) = 1 \otimes \tau_1 + \xi_0 \otimes \tau_0 + \tau_1 \otimes 1.$$

A form of the following lemma goes back to [Whi62, p. 271].

**Lemma 4.1.** Let $M$ be any $HZ/p$-module spectrum. Then $M$ is equivalent to a wedge sum of suspensions of $HZ/p$. Hence $H_*(M)$ is a sum of shifted copies of $A_*$ as an $A_*$-comodule, and the Hurewicz homomorphism $\pi_*(M) \to H_*(M)$ identifies $\pi_*(M)$ with the $A_*$-comodule primitives in $H_*(M)$.

**Proof.** The module action map $\lambda : HZ/p \wedge M \to M$ is a retraction, so $\pi_*(M)$ is a direct summand of $\pi_*(HZ/p \wedge M) = H_*(M)$, hence is a graded $\mathbb{Z}/p$-vector space. Choose maps $\alpha : S^n \to M$ that represent a basis for this vector space. The wedge sum of the maps

$$\lambda \circ (1 \wedge \alpha) : \Sigma^n HZ/p = HZ/p \wedge S^n \to M$$

is the desired $\pi_*$-isomorphism $\bigvee \alpha \Sigma^n HZ/p \to M$. \[\square\]

For each $\ell$-algebra $B$, $V(1) \wedge \text{THH}(B)$ is a module spectrum over $V(1) \wedge \text{THH}(\ell)$ and thus over $V(1) \wedge \ell \simeq HZ/p$, so $H_*(V(1) \wedge \text{THH}(B))$ is a sum of copies of $A_*$ as an $A_*$-comodule, by Lemma 4.1. In particular, $V(1) \wedge \text{THH}(B) = \pi_*(V(1) \wedge \text{THH}(B))$ is naturally identified with the subgroup of $A_*$-comodule primitives in

$$H_*(V(1) \wedge \text{THH}(B)) \cong H_*(V(1)) \otimes H_*(\text{THH}(B))$$

with the diagonal $A_*$-comodule coaction. We write $\nu \wedge x$ for the image of $\nu \otimes x$ under this identification, with $\nu \in H_*(V(1))$ and $x \in H_*(\text{THH}(B))$. Let

$$\epsilon_0 = 1 \wedge \tau_0 + \tau_0 \wedge 1$$

$$\epsilon_1 = 1 \wedge \tau_1 + \tau_0 \wedge \xi_1 + \tau_1 \wedge 1$$

$$\lambda_1 = 1 \wedge \sigma \xi_1$$

$$\lambda_2 = 1 \wedge \sigma \xi_2$$

$$\mu_0 = 1 \wedge \sigma \bar{\tau}_0$$

$$\mu_1 = 1 \wedge \sigma \bar{\tau}_1 + \tau_0 \wedge \sigma \bar{\xi}_1$$

$$\mu_2 = 1 \wedge \sigma \bar{\tau}_2 + \tau_0 \wedge \sigma \bar{\xi}_2.$$

(4.1)
These are all $A_*$-comodule primitive, when defined, in $H_*(V(1) \wedge THH(B))$ for $B = \ell, \ell/p, H\mathbb{Z}_p$ or $H\mathbb{Z}/p$ (see Remark 4.1). By a dimension count,
\[
V(1) \wedge THH(\mathbb{Z}/p) = E(\epsilon_0, \epsilon_1) \otimes P(\mu_0)
\]
\[
V(1) \wedge THH((\mathbb{Z}/p) = E(\epsilon_1) \otimes E(\lambda_1) \otimes P(\mu_1)
\]
\[
V(1) \wedge THH(\ell) = E(\lambda_1, \lambda_2) \otimes P(\mu_2)
\]
as commutative $\mathbb{F}_p$-algebras. The map $\pi: \ell \to H\mathbb{Z}(p)$ takes $\lambda_2$ to 0 and $\mu_2$ to $\mu_1^p$. The map $i: H\mathbb{Z}(p) \to H\mathbb{Z}/p$ takes $\lambda_1$ to 0 and $\mu_1$ to $\mu_0^p$. Note that $\mu_2 \in V(1)_{2p^2} THH(\ell)$ was simply denoted $\mu$ in [AR02].

In degrees $\leq (2p - 2)$ of $H_*(V(1) \wedge THH(\ell/p))$ the classes
\[
\mu_0^i := 1 \wedge (\sigma \bar{\tau}_0)^i
\]
for $0 \leq i \leq p - 1$ and
\[
\epsilon_0 \mu_0^i := 1 \wedge \bar{\tau}_0(\sigma \bar{\tau}_0)^i + \tau_0 \wedge (\sigma \bar{\tau}_0)^i
\]
for $0 \leq i \leq p - 2$ are $A_*$-comodule primitive, hence lift uniquely to $V(1) \wedge THH(\ell/p)$.

These map to the classes $\epsilon_0 \mu_0^i$ in $V(1) \wedge THH(\mathbb{Z}/p)$ for $0 \leq \delta \leq 1$ and $0 \leq i \leq p - 1$, except that the degree bound excludes the top case of $\epsilon_0 \mu_0^{p-1}$.

In degree $(2p - 1)$ of $H_*(V(1) \wedge THH(\ell/p))$ we have generators $1 \wedge \xi_1 \bar{\tau}_0$, $\tau_0 \wedge (\sigma \bar{\tau}_0)^{p-1}$, $\tau_0 \wedge \xi_1$, $\tau_1 \wedge 1$ and $1 \wedge y$. These have coactions
\[
\nu(1 \wedge \xi_1 \bar{\tau}_0) = 1 \otimes 1 \wedge \xi_1 \bar{\tau}_0 + \bar{\tau}_0 \otimes 1 \wedge \xi_1 + \xi_1 \otimes 1 \wedge \bar{\tau}_0 + \bar{\xi}_1 \bar{\tau}_0 \otimes 1 \wedge 1
\]
\[
\nu(\tau_0 \wedge (\sigma \bar{\tau}_0)^{p-1}) = 1 \otimes \tau_0 \wedge (\sigma \bar{\tau}_0)^{p-1} + \tau_0 \otimes 1 \wedge (\sigma \bar{\tau}_0)^{p-1}
\]
\[
\nu(\tau_0 \wedge \xi_1) = 1 \otimes \tau_0 \wedge \xi_1 + \tau_0 \otimes 1 \wedge \xi_1 + \xi_1 \otimes \tau_0 \wedge 1 + \bar{\xi}_1 \tau_0 \otimes 1 \wedge 1
\]
\[
\nu(\tau_1 \wedge 1) = 1 \otimes \tau_1 \wedge 1 + \xi_1 \otimes \tau_0 \wedge 1 + \tau_1 \otimes 1 \wedge 1
\]
and
\[
\nu(1 \wedge y) = 1 \otimes 1 \wedge y + \bar{\tau}_0 \otimes 1 \wedge (\sigma \bar{\tau}_0)^{p-1} - \bar{\tau}_0 \otimes 1 \wedge \bar{\xi}_1 - \bar{\tau}_1 \wedge 1 \wedge 1.
\]

Hence the sum
\[
\bar{\epsilon}_1 := 1 \wedge y + \tau_0 \wedge (\sigma \bar{\tau}_0)^{p-1} - \tau_0 \wedge \bar{\xi}_1 - \tau_1 \wedge 1
\]
is $A_*$-comodule primitive. Its image under $\pi_*$ in $H_*(V(1) \wedge THH(\mathbb{Z}/p))$ is
\[
\epsilon_0 \mu_0^{p-1} - \epsilon_1 = 1 \wedge \bar{\tau}_0(\sigma \bar{\tau}_0)^{p-1} + \tau_0 \wedge (\sigma \bar{\tau}_0)^{p-1} - 1 \wedge \bar{\tau}_1 - \tau_0 \wedge \bar{\xi}_1 - \tau_1 \wedge 1.
\]

Let
\[
V(1) \wedge THH(\ell)/(\lambda_1) = E(\lambda_2) \otimes P(\mu_2)
\]
be the quotient algebra of $V(1) \wedge THH(\ell)$ by the ideal generated by $\lambda_1$.

**Proposition 4.2.** The classes
\[
1, \epsilon_0, \mu_0, \epsilon_0 \mu_0, \ldots, \mu_0^{p-1}, \bar{\epsilon}_1 \in H_*(V(1) \wedge THH(\ell/p))
\]
defined in (4.2), (4.3) and (4.4) have unique lifts with same names in $V(1) \wedge THH(\ell/p)$.

The graded $V(1) \wedge THH(\ell/p)$-module $V(1) \wedge THH(\ell/p)$ is a free $V(1) \wedge THH(\ell)/(\lambda_1)$-module generated by these $2p$ classes:
\[
V(1) \wedge THH(\ell/p) = V(1) \wedge THH(\ell)/(\lambda_1) \otimes \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0 \mu_0, \ldots, \mu_0^{p-1}, \bar{\epsilon}_1\}.
\]

The map $\pi_*$ to $V(1) \wedge THH(\mathbb{Z}/p)$ takes $\epsilon_0^0 \mu_0^i$ in degree $0 \leq \delta + 2i \leq 2p - 2$ to $\epsilon_0 \mu_0^i$, and takes $\bar{\epsilon}_1$ in degree $(2p - 1)$ to $\epsilon_0 \mu_0^{p-1} - \epsilon_1$. 
Proof. Additively, this follows by another dimension count, and the description of \( \pi_* \) follows from the definition of the classes in question. It remains to prove that the action of \( V(1)_*\text{THH}(\ell) \) is as claimed.

The action of \( \mu_i^1 \) and \( \lambda_2 \mu_i^2 \) in \( V(1)_*\text{THH}(\ell) \) on the generators

\[
1, \varepsilon_0, \mu_0, \varepsilon_0 \mu_0, \ldots, \mu_0^{p-1}, \varepsilon_1
\]

of \( V(1)_*\text{THH}(\ell/p) \) is non-trivial for all \( i \geq 0 \), since the corresponding statement holds for the images of these classes in \( H_*(V(1) \wedge \text{THH}(\ell)) \) and \( H_*(V(1) \wedge \text{THH}(\ell/p)) \). This follows from Lemma 3.3 and the definition these classes. It remains to show that \( \lambda_1 \) acts trivially on \( V(1)_*\text{THH}(\ell/p) \). For degree reasons, multiplication by \( \lambda_1 \) is zero on all classes except possibly \( \mu_i^1 \) and \( \lambda_2 \mu_i^2 \), for \( i \geq 0 \). Because of the module structure, it suffices to show that \( \lambda_1 = \lambda_1 \cdot 1 = 0 \) in \( V(1)_*\text{THH}(\ell/p) \). This follows from the statement that the image of \( \lambda_1 \) in \( H_*(V(1) \wedge \text{THH}(\ell/p)) \) is equal to \( 1 \wedge \sigma \varepsilon_1 = 0 \), as implied by Lemma 3.3. \( \square \)

5. The \( C_p \)-Tate construction

For the remainder of this paper, let \( p \) be a prime with \( p \geq 5 \). We briefly recall the terminology on equivariant stable homotopy theory used in the sequel, and refer to [GM95, HM97, §1], [HM03, §4] and [AR02, §3] for more details. Let \( C_{p^\infty} \) denote the cyclic group of order \( p^n \), considered as a closed subgroup of the circle group \( S^1 \), and let \( G = S^1 \) or \( C_{p^n} \). For each spectrum \( X \) with \( S^1 \)-action, let \( X_{hG} = EG_+ \wedge_G X \) and \( X^{hG} = F(EG_+, X)^G \) denote its homotopy orbit and homotopy fixed point spectra, as usual. We now write \( X^{hG} = [\tilde{E}G \wedge F(EG_+, X)]^G \) for the \( G \)-Tate construction on \( X \), which was denoted \( t_G(X)^G \) in [GM95] and \( \mathbb{H}(G, X) \) in [HM97, HM03, AR02].

We denote by \( F \) the Frobenius map \( X_{C_{p^n}} \rightarrow X_{C_{p^n-1}} \) given by the inclusion of fixed-point spectra, and by \( V \) the Verschiebung map \( X_{C_{p^n-1}} \rightarrow X_{C_{p^n}} \) given by transfer. We shall also consider the homotopy Frobenius, Tate Frobenius and homotopy Verschiebung maps \( F^h \colon X^{hS^1} \rightarrow X^{hC_{p^n}}, \quad F^h \colon X^{hC_{p^n}} \rightarrow X^{hC_{p^n-1}}, \quad F^t \colon X^{S^1} \rightarrow X^{C_{p^n}} \) and \( V^h \colon X^{hC_{p^n-1}} \rightarrow X^{hC_{p^n}} \).

There are conditionally convergent \( G \)-homotopy fixed point and \( G \)-Tate spectral sequences in \( V(1)_*\)-homotopy for \( X \), with

\[
E^2_{s,t}(G, X) = H_{sp}^{\cdot \cdot}(G; V(1)_t(X)) \Longrightarrow V(1)_{s+t}(X^{hG})
\]

and

\[
\hat{E}^2_{s,t}(G, X) = \hat{H}_{sp}^{\cdot \cdot}(G; V(1)_t(X)) \Longrightarrow V(1)_{s+t}(X^{hG})
\]

Here \( H_{sp}(G; V(1)_*(X)) \) denotes the group cohomology of \( G \) and \( H_{sp}^{\cdot \cdot}(G; V(1)_*(X)) \) the Tate cohomology of \( G \), with coefficients in \( V(1)_*(X) \). Notice that in our case, with \( X = \text{THH}(B) \), the action of \( G \) on \( V(1)_*(X) \) is trivial, since it is the restriction of an \( S^1 \)-action. We write \( H_{sp}^*(C_{p^n}; \mathbb{F}_p) = E(u_n) \otimes P(t) \) and \( \hat{H}_{sp}^{\cdot \cdot}(C_{p^n}; \mathbb{F}_p) = E(u_n) \otimes P(t^{\pm 1}) \) with \( u_n \) in degree 1 and \( t \) in degree 2, see for example [Ben98, Prop. 3.5.5] and [HM03, Lem. 4.2.1]. So \( u_n, t \) and \( x \in V(1)_n(X) \) have bidegree \((-1, 0), (-2, 0) \) and \((0, t) \) in either spectral sequence, respectively. See [HM03, §4.3] for proofs of the multiplicative properties of these spectral sequences. Similarly, we write \( H_{sp}^*(S^1; \mathbb{F}_p) = P(t) \) and \( \hat{H}_{sp}^{\cdot \cdot}(S^1; \mathbb{F}_p) = P(t^{\pm 1}) \). We have morphisms of spectral sequences induced by the homotopy and Tate Frobenii, which on the \( E^2 \)-terms map \( t \) to \( t \) and \( u_n \) to zero.

We are principally interested in the case when \( X = \text{THH}(B) \), with the \( S^1 \)-action given by the cyclic structure [Lod98, Def. 7.1.9], [HM03, §1.2]. It is a cyclotomic spectrum, in
the sense of [HM97 §1], leading to the commutative diagram

\[
\begin{array}{ccc}
THH(B)_{hC_p^n} & \xrightarrow{N} & THH(B)_{C_p^n} \\
\downarrow & & \downarrow \\
THH(B)_{hC_p^n} & \xrightarrow{N^h} & THH(B)_{hC_p^n}
\end{array}
\]

of horizontal cofiber sequences. We abbreviate \( E^2(G, THH(B)) \) to \( E^2(G, B) \), etc. When \( B \) is a commutative \( S \)-algebra, this is a commutative algebra spectral sequence, and when \( B \) is an associative \( A \)-algebra, with \( A \) commutative, then \( E^*(G, B) \) is a module spectral sequence over \( E^*(G, A) \). The map \( R^h \) corresponds to the inclusion \( E^2(G, B) \to E^2(G, B) \) from the second quadrant to the upper half-plane, for connective \( B \).

**Definition 5.1.** We call a homomorphism of graded groups \( k \)-cocomplete if it is an isomorphism in all dimensions greater than \( k \) and injective in dimension \( k \).

In this section we compute \( V(1)_*, THH(\ell/p)^{IC_p} \) by means of the \( C_p \)-Tate spectral sequence in \( V(1)_* \)-homotopy for \( THH(\ell/p) \). In Propositions 5.7 and 5.8 we show that the comparison map \( \hat{\Gamma}_1 : V(1)_*, THH(\ell/p) \to V(1)_*, THH(\ell/p)^{IC_p} \) is \((2p-2)\)-coconnected and can be identified with the algebraic localization homomorphism that inverts \( \mu_2 \).

First we recall the structure of the \( C_p \)-Tate spectral sequence for \( THH(\mathbb{Z}/p) \), with \( V(0) \) - and \( V(1) \)-coefficients. We have \( V(0)_*, THH(\mathbb{Z}/p) = E(0) \otimes P(\mu_0) \), and (with an obvious notation for the case of \( V(0) \)-homotopy) the \( E^2 \)-terms are

\[
\hat{E}^2(C_p, Z/p; V(0)) = E(u_1) \otimes P(t^{±1}) \otimes E(\mu_0) \\
\hat{E}^2(C_p, Z/p) = E(u_1) \otimes P(t^{±1}) \otimes E(\mu_0)
\]

In each \( G \)-Tate spectral sequence we have a first differential

\[
d^2(x) = t \cdot x,
\]

see e.g. [Rog98 §3.3]. We easily deduce \( \sigma \epsilon_0 = \mu_0 \) and \( \sigma \epsilon_1 = \mu_0^p \) from (4.1), so

\[
\hat{E}^3(C_p, Z/p; V(0)) = E(u_1) \otimes P(t^{±1}) \\
\hat{E}^3(C_p, Z/p) = E(u_1) \otimes P(t^{±1}) \otimes E(\mu_0^{p-1} - \epsilon_1).
\]

Thus the \( V(0) \)-homotopy spectral sequence collapses at \( \hat{E}^3 = \hat{E}^\infty \). By naturality with respect to the map \( i_1 : V(0) \to V(1) \), all the classes on the horizontal axis of \( \hat{E}^3(C_p, Z/p) \) are infinite cycles, so also the latter spectral sequence collapses at \( \hat{E}^3(C_p, Z/p) \).

We know from [HM03 Cor. 4.4.2] that the comparison map

\[
\hat{\Gamma}_1 : V(0)_*, THH(\mathbb{Z}/p) \to V(1)_*, THH(\mathbb{Z}/p)^{IC_p}
\]

takes \( \epsilon_0 \mu_0^i \) to \((u_1 t^{±1})^i \), for all \( 0 \leq \delta \leq 1 \), \( i \geq 0 \). In particular, the integral map \( \hat{\Gamma}_1 : \pi_* THH(\mathbb{Z}/p) \to \pi_* THH(\mathbb{Z}/p)^{IC_p} \) is \((-2)\)-coconnected. From this we can deduce the following behavior of the comparison map \( \hat{\Gamma}_1 \) in \( V(1)_* \)-homotopy.

**Lemma 5.2.** The map

\[
\hat{\Gamma}_1 : V(1)_*, THH(\mathbb{Z}/p) \to V(1)_*, THH(\mathbb{Z}/p)^{IC_p}
\]

takes the classes \( \epsilon_0 \mu_0^i \) from \( V(0)_*, THH(\mathbb{Z}/p) \), for \( 0 \leq \delta \leq 1 \) and \( i \geq 0 \), to classes represented in \( \hat{E}^\infty(C_p, Z/p) \) by \((u_1 t^{±1})^i \) (on the horizontal axis). Furthermore, it takes the
class \(\epsilon_0\mu_0^{p-1} - \epsilon_1\) in degree \((2p - 1)\) to a class represented by \(\epsilon_0\mu_0^{p-1} - \epsilon_1\) (on the vertical axis).

**Proof.** The classes \(\epsilon_0\mu_0^{p-1}\) are in the image from \(V(0)\)-homotopy, and we recalled above that they are detected by \((u_1t^{-1})^d t^{-1}\) in the \(V(0)\)-homotopy \(C_p\)-Tate spectral sequence for \(\text{THH}(\mathbb{Z}/p)\). By naturality along \(i_1: V(0) \to V(1)\), they are detected by the same (nonzero) classes in the \(V(1)\)-homotopy spectral sequence \(E^\infty(\mathbb{C}_p, \mathbb{Z}/p)\).

To find the representative for \(\hat{\Gamma}_1(\epsilon_0\mu_0^{p-1} - \epsilon_1)\) in degree \((2p - 1)\), we appeal to the cyclotomic trace map from algebraic \(K\)-theory, or more precisely, to the commutative diagram

\[
\begin{array}{ccc}
K(B) & \xleftarrow{\text{tr}} & \text{THH}(B) \\
\xleftarrow{\text{tr}_1} & & \xrightarrow{\text{tr}} \\
\text{THH}(B) & \xrightarrow{F} & \text{THH}(B)^{C_p} \\
\xleftarrow{\Gamma_1} & & \xrightarrow{R} \\
\text{THH}(B)^{hc_p} & \xrightarrow{R^h} & \text{THH}(B)^{hc_p}.
\end{array}
\]

The Bökstedt trace map \(\text{tr}: K(B) \to \text{THH}(B)\) admits a preferred lift \(\text{tr}_n\) through each fixed point spectrum \(\text{THH}(B)^{C_p}\), which homotopy equalizes the iterated restriction and Frobenius maps \(R^n\) and \(F^n\) to \(\text{THH}(B)\), see [Dum04 §3]. In particular, the \(\sigma\)-operator on \(V(1)\cdot\text{THH}(B)\) is zero on classes in the image of \(\text{tr}\).

In the case \(B = H\mathbb{Z}/p\) we know that \(K(\mathbb{Z}/p)_p \simeq H\mathbb{Z}_p\), so \(V(1)\cdot K(\mathbb{Z}/p) = E(\bar{\epsilon}_1)\), where the \(v_1\)-Bockstein of \(\bar{\epsilon}_1\) is \(-1\). The Bökstedt trace image \(\text{tr}(\bar{\epsilon}_1) \in V(1)\cdot\text{THH}(\mathbb{Z}/p)\) lies in \(\mathbb{F}_p\{\epsilon_1, \epsilon_0\mu_0^{p-1}\}\), has \(v_1\)-Bockstein \(\text{tr}(-1) = -1\) and suspends by \(\sigma\) to 0. Hence

\[
\text{tr}(\bar{\epsilon}_1) = \epsilon_0\mu_0^{p-1} - \epsilon_1.
\]

As we recalled above, the map \(\hat{\Gamma}_1: \pi_*\text{THH}(\mathbb{Z}/p) \to \pi_*\text{THH}(\mathbb{Z}/p)^{C_p}\) is \((-2)\)-coconnected, so the corresponding map in \(V(1)\)-homotopy is at least \((2p - 2)\)-coconnected. Thus it takes \(\epsilon_0\mu_0^{p-1} - \epsilon_1\) to a nonzero class in \(V(1)\cdot\text{THH}(\mathbb{Z}/p)^{C_p}\), represented somewhere in total degree \((2p - 1)\) of \(E^\infty(\mathbb{C}_p, \mathbb{Z}/p)\), in the lower right hand corner of the diagram.

Going down the middle part of the diagram, we reach a class \((\Gamma_1 \circ \text{tr}_1)(\bar{\epsilon}_1)\), represented in total degree \((2p - 1)\) in the left half-plane \(C_p\)-homotopy fixed point spectral sequence \(E^\infty(\mathbb{C}_p, \mathbb{Z}/p)\). Its image under the edge homomorphism to \(V(1)\cdot\text{THH}(\mathbb{Z}/p)\) equals \((F \circ \text{tr}_1)(\epsilon_1) = tr(\bar{\epsilon}_1)\), hence \((\Gamma_1 \circ \text{tr}_1)(\epsilon_1)\) is represented by \(\epsilon_0\mu_0^{p-1} - \epsilon_1\) in \(E^\infty_{0,2p-1}(\mathbb{C}_p, \mathbb{Z}/p)\). Its image under \(R^h\) in the \(C_p\)-Tate spectral sequence is the generator of \(E^\infty_{0,2p-1}(\mathbb{C}_p, \mathbb{Z}/p) = \mathbb{F}_p\{\epsilon_0\mu_0^{p-1} - \epsilon_1\}\), hence that generator is the \(E^\infty\)-representative of \(\hat{\Gamma}_1(\epsilon_0\mu_0^{p-1} - \epsilon_1)\).

The \((2p - 2)\)-connected map \(\pi: \ell/p \to H\mathbb{Z}/p\) induces a \((2p - 1)\)-connected map \(V(1)\cdot K(\ell/p) \to V(1)\cdot K(\mathbb{Z}/p) = E(\bar{\epsilon}_1)\), by [BM94 Prop. 10.9]. We can lift the algebraic \(K\)-theory class \(\bar{\epsilon}_1\) to \(\ell/p\). This lift is not unique, but we fix one choice.

**Definition 5.3.** We call

\[
\epsilon_1^K \in V(1)_{2p-1}K(\ell/p)
\]

a chosen class that maps to the generator \(\bar{\epsilon}_1\) in \(V(1)_{2p-1}K(\mathbb{Z}/p) \cong \mathbb{Z}/p\).

**Lemma 5.4.** The Bökstedt trace \(\text{tr}: V(1)_*K(\ell/p) \to V(1)_*\text{THH}(\ell/p)\) takes \(\epsilon_1^K\) to \(\bar{\epsilon}_1\).
Proof. In the commutative square
\[ V(1)_{*}K(\ell/p) \xrightarrow{\text{tr}} V(1)_{*}THH(\ell/p) \]
the trace image \( \text{tr}(\hat{\epsilon}_1^K) \) in \( V(1)_{*}THH(\ell/p) \) must map under \( \pi_* \) to \( \text{tr}(\hat{\epsilon}_1) = \epsilon_0 \mu_0^{p-1} - \epsilon_1 \) in \( V(1)_{*}THH(\mathbb{Z}/p) \), which by Proposition 4.2 characterizes it as being equal to the class \( \hat{\epsilon}_1 \). Hence \( \text{tr}(\hat{\epsilon}_1^K) = \hat{\epsilon}_1 \).

Next we turn to the \( C_p \)-Tate spectral sequence \( \hat{E}^*(C_p, \ell/p) \) in \( V(1) \)-homotopy for \( THH(\ell/p) \). Its \( E^2 \)-term is
\[
\hat{E}^2(C_p, \ell/p) = E(u_1) \otimes P(t^{\pm 1}) \otimes \mathbb{F}_p \{ 1, \epsilon_0, \mu_0, \epsilon_0 \mu_0, \ldots, \mu_0^{p-1}, \hat{\epsilon}_1 \} \otimes E(\lambda_2) \otimes P(\mu_2) .
\]
We have \( d^2(x) = t \cdot \sigma x \), where
\[
\sigma(\epsilon_0^i \mu_0^{i-1}) = \begin{cases} 
\mu_0^i & \text{for } \delta = 1, 0 < i < p, \\
0 & \text{otherwise}
\end{cases}
\]
is readily deduced from (4.1), and \( \sigma(\hat{\epsilon}_1) = 0 \) since \( \hat{\epsilon}_1 \) is in the image of \( \text{tr} \). Thus
\[
\hat{E}^3(C_p, \ell/p) = E(u_1) \otimes P(t^{\pm 1}) \otimes E(\hat{\epsilon}_1) \otimes E(\lambda_2) \otimes P(t \mu_2) .
\] (5.2)
We prefer to use \( t \mu_2 \) rather than \( \mu_2 \) as a generator, since it represents multiplication by \( v_2 \) (up to a unit factor in \( \mathbb{F}_p \)) in all module spectral sequences over \( E^*(S^1, \ell) \), by [AR02 Prop. 4.8].

To proceed, we shall use that \( \hat{E}^* (C_p, \ell/p) \) is a module over the spectral sequence for \( THH(\ell) \). We therefore recall the structure of the latter spectral sequence, from [AR02 Th. 5.5]. It begins
\[
\hat{E}^2(C_p, \ell) = E(u_1) \otimes P(t^{\pm 1}) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu_2) .
\]
The classes \( \lambda_1, \lambda_2 \) and \( t \mu_2 \) are infinite cycles, and the differentials
\[
d^{2p}(t^{1-p}) = t \lambda_1 \\
d^{2p^2}(p^{-p^2}) = t^p \lambda_2 \\
d^{2p^2 + 1}(u_1 t^{-p^2}) = t \mu_2
\]
leave the terms
\[
\hat{E}^{2p+1}(C_p, \ell) = E(u_1, \lambda_1, \lambda_2) \otimes P(t^{p^2}, t \mu_2) \\
\hat{E}^{2p^2 + 1}(C_p, \ell) = E(u_1, \lambda_1, \lambda_2) \otimes P(t^{p^2 + 2}, t \mu_2) \\
\hat{E}^{2p^2 + 2}(C_p, \ell) = E(\lambda_1, \lambda_2) \otimes P(t^{p^2 + 2})
\]
with \( \hat{E}^{2p^2 + 2} = \hat{E}^\infty \), converging to \( V(1)_{*}THH(\ell)^{C_p} \). The comparison map \( \hat{\Gamma}_1 \) takes \( \lambda_1, \lambda_2 \) and \( \mu_2 \) to \( \lambda_1, \lambda_2 \) and \( t^{-p^2} \) (up to a unit factor in \( \mathbb{F}_p \)), respectively, inducing the algebraic localization map and identification
\[
\hat{\Gamma}_1 : V(1)_{*}THH(\ell) \rightarrow V(1)_{*}THH(\ell)[\mu_2^{-1}] \cong V(1)_{*}THH(\ell)^{C_p} .
\]
Lemma 5.5. In $\hat{E}^*(C_p, \ell/p)$, the class $u_t^{-p}$ supports the nonzero differential
\[ d^{2p^2}(u_t^{-p}) \cong u_t^{p^2-p}\lambda_2, \]
and does not survive to the $E^{\infty}$-term.

Proof. In $\hat{E}^*(C_p, \ell)$, there is such a differential. By naturality along $i: \ell \to \ell/p$, it follows that there is also such a differential in $\hat{E}^*(C_p, \ell/p)$. It remains to argue that the target class is nonzero at the $E^{2p^2}$-term. Considering the $E^3$-term in \([5, 2]\), the only possible source of a previous differential hitting $u_t^{p^2-p}\lambda_2$ is $\bar{\epsilon}_1$, supporting a $d^{2p^2-2p+1}$-differential. But $\bar{\epsilon}_1$ is in an even column and $u_t^{p^2-p}\lambda_2$ is in an odd column. By naturality with respect to the Tate Frobenius map $F^t: THH(\ell/p)^{S^1} \to THH((\ell/p)^{\mu_0})$, any such differential from an even to an odd column must be zero. Indeed, the $S^1$-Tate spectral sequence has $E^2$-term given by $P(t^{\pm1}) \otimes V(1)_*, THH(\ell/p)$, and $F^t$ induces the injective homomorphism that takes $E^2(S^1, \ell/p)$ isomorphically to the even columns of $E^2(C_p, \ell/p)$. Since $E^*(S^1, \ell/p)$ is concentrated in even columns, all differentials of odd length are zero. By naturality, classes of $\hat{E}^*(C_p, \ell/p)$ that lie in the image of $\hat{E}^*(F^t)$ cannot support a differential of odd length; compare with [AR02, Lemma 5.2]. In the present situation, the $d^2$-differential of $\hat{E}^*(C_p, \ell/p)$ leading to \([5, 2]\) is also non-zero in $\hat{E}^*(S^1, \ell/p)$, so that we have
\[ \hat{E}^3(S^1, \ell/p) = P(t^{\pm1}) \otimes E(\epsilon_1) \otimes E(\lambda_2) \otimes P(t\mu_2). \]

By inspection, if the class $\bar{\epsilon}_1 \in \hat{E}^2(C_p, \ell/p)$ survives to $\hat{E}^{2p^2-2p+1}(C_p, \ell/p)$, then it will lie in the image of $\hat{E}^{2p^2-2p+1}(F^t)$. \hfill $\Box$

To determine the map $\tilde{\Gamma}_1$ we use naturality with respect to the map $\pi: \ell/p \to H\mathbb{Z}/p$.

Lemma 5.6. The classes $1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \ldots, \mu_0^{p-1}$ and $\bar{\epsilon}_1$ in $V(1)_*, THH(\ell/p)$ map under $\tilde{\Gamma}_1$ to classes in $V(1)_*, THH(\ell/p)^{\mu_0}$ that are represented in $\hat{E}^\infty(C_p, \ell/p)$ by the permanent cycles $(u_t^{-1})^{4t-1}$ (on the horizontal axis) in degrees $\leq (2p-2)$, and by the permanent cycle $\bar{\epsilon}_1$ (on the vertical axis) in degree $(2p-1)$.

Proof. In the commutative square
\[
\begin{array}{ccc}
V(1)_* THH(\ell/p) & \xrightarrow{\tilde{\Gamma}_1} & V(1)_* THH(\ell/p)^{\mu_0} \\
| & | & | \\
V(1)_* THH(\mathbb{Z}/p) & \xrightarrow{\tilde{\Gamma}_1} & V(1)_* THH(\mathbb{Z}/p)^{\mu_0}
\end{array}
\]
the classes $\epsilon_0^\ell \mu_0^t$ in the upper left hand corner map to classes in the lower right hand corner that are represented by $(u_t^{-1})^{4t-1}$ in degrees $\leq (2p-2)$, and $\bar{\epsilon}_1$ maps to $\epsilon_0\mu_0^{p-1} - \epsilon_1$ in degree $(2p-1)$. This follows by combining Proposition \([1, 2]\) and Lemma \([5, 2]\).

The first $(2p-1)$ of these are represented in maximal filtration (on the horizontal axis), so their images in the upper right hand corner must be represented by permanent cycles $(u_t^{-1})^{4t-1}$ in the Tate spectral sequence $\hat{E}^\infty(C_p, \ell/p)$.

The image of the last class, $\bar{\epsilon}_1$, in the upper right hand corner could either be represented by $\bar{\epsilon}_1$ in bidegree $(0, 2p-1)$ or by $u_t^{-p}$ in bidegree $(2p-1, 0)$. However, the last class supports a differential $d^{2p^2}(u_t^{-p}) \cong u_t^{p^2-p}\lambda_2$, by Lemma \([5, 5]\) above. This only leaves the other possibility, that $\tilde{\Gamma}_1(\bar{\epsilon}_1)$ is represented by $\bar{\epsilon}_1$ in $\hat{E}^\infty(C_p, \ell/p)$. \hfill $\Box$
We proceed to determine the differential structure in $\hat{E}^\ast(C_p,\ell/p)$, making use of the permanent cycles identified above.

**Proposition 5.7.** The $C_p$-Tate spectral sequence in $V(1)$-homotopy for $THH(\ell/p)$ has

$$\hat{E}^3(C_p,\ell/p) = E(u_1,\tilde{e}_1,\lambda_2) \otimes P(t^\pm 1,t\mu_2).$$

It has differentials generated by

$$d^{2p^2-2p+2}(tp^{-p^2} \cdot t^{-i}\tilde{e}_1) \equiv t\mu_2 \cdot t^{-i}$$

for $0 < i < p$, $d^{2p^2}(tp^{-p^2}) \equiv tp\lambda_2$ and $d^{2p^2+1}(u_1t^{-p^2}) \equiv t\mu_2$. The subsequent terms are

$$\hat{E}^{2p^2-2p+3}(C_p,\ell/p) = E(u_1,\lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^\pm p)$$

$$+ E(u_1,\tilde{e}_1,\lambda_2) \otimes P(t^\pm p, t\mu_2)$$

$$\hat{E}^{2p^2+1}(C_p,\ell/p) = E(u_1,\lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^\pm p)$$

$$+ E(u_1,\tilde{e}_1,\lambda_2) \otimes P(t^\pm p, t\mu_2)$$

$$\hat{E}^{2p^2+2}(C_p,\ell/p) = E(u_1,\lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^\pm p)$$

$$+ E(\ell_1,\lambda_2) \otimes P(t^\pm p).$$

The last term can be rewritten as

$$\hat{E}^\infty(C_p,\ell/p) = (E(u_1) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \oplus E(\tilde{e}_1)) \otimes E(\lambda_2) \otimes P(t^\pm p^2).$$

**Proof.** We have already identified the $E^2$- and $E^3$-terms above. The $E^3$-term (5.2) is generated over $E^3(C_p,\ell)$ by an $\mathbb{F}_p$-basis for $E(\tilde{e}_1)$, so the next possible differential is induced by $d^{2p}(t^{1-p}) \equiv t\lambda_1$. But multiplication by $\lambda_1$ is trivial in $V(1)\cdot THH(\ell/p)$, by Proposition [3.1,2] so $E^3(C_p,\ell/p) = E^{2p+1}(C_p,\ell/p)$. This term is generated over $E^{2p+1}(C_p,\ell)$ by $P_p(t^{-1}) \otimes \mathbb{F}(\tilde{e}_1)$. Here $t_1, t^{-1}, \ldots, t^{1-p}$ and $\tilde{e}_1$ are permanent cycles, by Lemma 5.6. Any $d^r$-differential before $d^{2p^2}$ must therefore originate on a class $t^{-i}\tilde{e}_1$ for $0 < i < p$, and be of even length $r$, since these classes lie in even columns. For bidegree reasons, the first possibility is $r = 2p^2 - 2p + 2$, so $E^3(C_p,\ell/p) = E^{2p^2-2p+2}(C_p,\ell/p)$.

Multiplication by $v_2$ acts trivially on $V(1)\cdot THH(\ell)$ and $V(1)\cdot THH(\ell)^G_{C_p}$ for degree reasons, and therefore also on $V(1)\cdot THH(\ell/p)$ and $V(1)\cdot THH(\ell/p)^{G_{C_p}}$ by the module structure. The class $v_2$ maps to $t\mu_2$ in the $S^1$-Tate spectral sequence for $\ell$, as recalled above, so multiplication by $v_2$ is represented by multiplication by $t\mu_2$ in the $C_p$-Tate spectral sequence for $\ell/p$. Applied to the permanent cycles $(u_1t^{-1})^\delta t^{-i}$ in degrees $\leq (2p - 2)$, this implies that the products

$$t\mu_2 \cdot (u_1t^{-1})^\delta t^{-i}$$

must be infinite cycles representing zero, i.e., they must be hit by differentials. In the cases $\delta = 1, 0 \leq i \leq p - 2$, these classes in odd columns cannot be hit by differentials of odd length, such as $d^{2p^2+1}$, so the only possibility is

$$d^{2p^2-2p+2}(tp^{-p^2} \cdot (u_1t^{-1})t^{-i}\tilde{e}_1) \equiv t\mu_2 \cdot (u_1t^{-1})t^{-i}$$

for $0 \leq i \leq p - 2$. By the module structure (consider multiplication by $u_1$) it follows that

$$d^{2p^2-2p+2}(tp^{-p^2} \cdot t^{-i}\tilde{e}_1) \equiv t\mu_2 \cdot t^{-i}$$
for $0 < i < p$. Hence we can compute from (5.2) that
\[
\hat{E}^{2p^2-2p+3}(C_p, \ell/p) = E(u_1) \otimes P(t^{\pm p}) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes E(\lambda_2) \\
\oplus E(u_1) \otimes P(t^{\pm p}) \otimes E(\bar{e}_1) \otimes E(\lambda_2) \otimes P(t\mu_2).
\]
This is generated over $\hat{E}^{2p+1}(C_p, \ell)$ by the permanent cycles $1, t^{-1}, \ldots, t^{1-p}$ and $\bar{e}_1$, so the next differential is induced by $d^{2p^2}(t^{p-p^2}) \cong t^p\lambda_2$. This leaves
\[
\hat{E}^{2p^2+1}(C_p, \ell/p) = E(u_1) \otimes P(t^{\pm p^2}) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes E(\lambda_2) \\
\oplus E(u_1) \otimes P(t^{\pm p^2}) \otimes E(\bar{e}_1) \otimes E(\lambda_2) \otimes P(t\mu_2).
\]
Finally, $d^{2p^2+1}(u_1t^{-p^2}) \cong t\mu_2$ applies, and leaves
\[
\hat{E}^{2p^2+2}(C_p, \ell/p) = E(u_1) \otimes P(t^{\pm p^2}) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes E(\lambda_2) \\
\oplus P(t^{\pm p^2}) \otimes E(\bar{e}_1) \otimes E(\lambda_2).
\]
For bidegree reasons, $\hat{E}^{2p^2+2} = \hat{E}^\infty$. 

**Proposition 5.8.** The comparison map $\tilde{\Gamma}_1$ takes the classes
\[
\bar{e}_0^i\mu_0, \bar{e}_1, \lambda_2 \text{ and } \mu_2 \text{ in } V(1)_{\ast}THH(\ell/p)
\]
to classes in $V(1)_{\ast}THH(\ell/p)|_{C_p}$ represented by
\[
(u_1t^{-1})^d t^{-i}, \bar{e}_1, \lambda_2 \text{ and } t^{-p^2} \text{ in } \hat{E}^\infty(C_p, \ell/p),
\]
up to a unit factor in $\mathbb{F}_p$, respectively. Thus
\[
V(1)_{\ast}THH(\ell/p)|_{C_p} \cong \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \ldots, \mu_0^{p-1}, \bar{e}_1\} \otimes E(\lambda_2) \otimes P(\mu_2^{p+1})
\]
and $\tilde{\Gamma}_1$ induces an identification $V(1)_{\ast}THH(\ell/p)[\mu_2^{-1}] \cong V(1)_{\ast}THH(\ell/p)|_{C_p}$. In particular, $\tilde{\Gamma}_1$ factors as the algebraic localization map and identification
\[
\hat{\Gamma}_1: V(1)_{\ast}THH(\ell/p) \to V(1)_{\ast}THH(\ell/p)[\mu_2^{-1}] \cong V(1)_{\ast}THH(\ell/p)|_{C_p},
\]
and is $(2p-2)$-cocomplete.

**Proof.** The image under $\hat{\Gamma}_1$ of the classes $1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \ldots, \mu_0^{p-1}$ and $\bar{e}_1$ was given in Lemma [5.6] and the action on the classes $\lambda_2$ and $\mu_2$ is given in the proof of [AR02 Th. 5.5]. The structure of $V(1)_{\ast}THH(\ell/p)|_{C_p}$ is then immediate from the $E^\infty$-term in Proposition [5.7]. The top class not in the image of $\hat{\Gamma}_1$ is $\bar{e}_1\lambda_2\mu_2^{-1}$, in degree $(2p-2)$. 

Recall that
\[
TF(B; p) = \text{holim}_n THH(B)^{C_p^n}
\]
\[
TR(B; p) = \text{holim}_n THH(B)^{C_p^n}
\]
are defined as the homotopy limits over the Frobenius and the restriction maps
\[
F, R: THH(B)^{C_p^n} \to THH(B)^{C_p^{n-1}},
\]
respectively.
Corollary 5.9. The comparison maps
\[ \Gamma_n : \text{THH}(\ell/p)^{C_{p^n}} \to \text{THH}(\ell/p)^{hC_{p^n}} \]
\[ \hat{\Gamma}_n : \text{THH}(\ell/p)^{C_{p^n-1}} \to \text{THH}(\ell/p)^{tC_{p^n}} \]
for \( n \geq 1 \), and
\[ \Gamma : TF(\ell/p; p) \to \text{THH}(\ell/p)^{hS^1} \]
\[ \hat{\Gamma} : TF(\ell/p; p) \to \text{THH}(\ell/p)^{tS^1} \]
all induce \((2p - 2)\)-coconnected homomorphisms on \( V(1) \)-homotopy.

Proof. This follows from a theorem of Tsalidis [Tsa98 Th. 2.4] and Proposition 5.8 above, just like in [AR02 Th. 5.7]. See also [BBLNR Ex. 10.2] □

6. Higher fixed points

Let \( n \geq 1 \). Write \( v_p(i) \) for the \( p \)-adic valuation of \( i \). Define a numerical function \( \rho(-) \) by
\[ \rho(2k - 1) = (p^{2k+1} + 1)/(p + 1) = p^{2k} - p^{2k-1} + \cdots - p + 1 \]
\[ \rho(2k) = (p^{2k+2} - p^2)/(p^2 - 1) = p^{2k} + p^{2k-2} + \cdots + p^2 \]
for \( k \geq 0 \), so \( \rho(-1) = 1 \) and \( \rho(0) = 0 \). For even arguments, \( \rho(2k) = r(2k) \) as defined in [AR02 Def. 2.5].

In all of the following spectral sequences we know that \( \lambda_2, t\mu_2 \) and \( \bar{\epsilon}_1 \) are infinite cycles. For \( \lambda_2 \) and \( \bar{\epsilon}_1 \) this follows from the \( C_{p^n} \)-fixed point analogue of diagram (5.1), by [AR02 Prop. 2.8] and Lemma 5.7. For \( t\mu_2 \) it follows from [AR02 Prop. 4.8], by naturality.

Theorem 6.1. The \( C_{p^n} \)-Tate spectral sequence in \( V(1) \)-homotopy for \( \text{THH}(\ell/p) \) begins
\[ \hat{E}^2(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p \{ 1, \epsilon_0, \mu_0, \epsilon_0 \mu_0, \ldots, \mu_0^{p-1}, \bar{\epsilon}_1 \} \otimes P(t^{\pm 1}, \mu_2) \]
and converges to \( V(1) \). This \( \text{THH}(\ell/p)^{C_{p^n}} \). It is a module spectral sequence over the algebra spectral sequence \( \hat{E}^2(C_{p^n}, \ell) \) converging to \( V(1) \). This \( \text{THH}(\ell)^{C_{p^n}} \).

There is an initial \( d^2 \)-differential generated by
\[ d^2(\epsilon_0 \mu_0^{i-1}) = t\mu_0^i \]
for \( 0 < i < p \). Next, there are \( 2n \) families of even length differentials generated by
\[ d^{2\rho(2k-1)}(t^{2k-1} - p^{2k-1} \cdot \bar{\epsilon}_1) \overset{=}{\rightarrow} (t\mu_2)^{\rho(2k-3)} \cdot t^i \]
for \( v_p(i) = 2k - 2 \), for each \( k = 1, \ldots, n \), and
\[ d^{2\rho(2k)}(t^{2k-1} - p^{2k}) \overset{=}{\rightarrow} \lambda_2 \cdot t^{2^{2k-1}} \cdot (t\mu_2)^{\rho(2k-2)} \]
for each \( k = 1, \ldots, n \). Finally, there is a differential of odd length generated by
\[ d^{2\rho(2n) + 1}(u_n \cdot t^{p^{2n}}) \overset{=}{\rightarrow} (t\mu_2)^{\rho(2n-2)+1} \cdot t^{p^{2n}} \]

We shall prove Theorem 6.1 by induction on \( n \). The base case \( n = 1 \) was covered by Proposition 5.7. We can therefore assume that Theorem 6.1 holds for some fixed \( n \geq 1 \), and must prove the corresponding statement for \( n + 1 \). First we make the following deduction.
Corollary 6.2. The initial differential in the $C_{p^n}$-Tate spectral sequence in $V(1)$-homotopy for $\text{THH}(\ell/p)$ leaves

$$\hat{E}^3(C_{p^n}, \ell/p) = E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm 1}, t\mu_2).$$

The next $2n$ families of differentials leave the intermediate terms

$$\hat{E}^{2p(1)+1}(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p})$$

$$\oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p}, t\mu_2)$$

(for $m = 1$),

$$\hat{E}^{2p(2m-1)+1}(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2})$$

$$\oplus \bigoplus_{k=2}^{m-1} E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 2\} \otimes P(t^{\rho(2k-3)}(t\mu_2))$$

$$\oplus \bigoplus_{k=2}^{m-1} E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 1\} \otimes P(t^{\rho(2k-2)}(t\mu_2))$$

$$\oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m-1}}, t\mu_2)$$

for $m = 2, \ldots, n$, and

$$\hat{E}^{2p(2m)+1}(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2})$$

$$\oplus \bigoplus_{k=2}^{m} E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 2\} \otimes P(t^{\rho(2k-3)}(t\mu_2))$$

$$\oplus \bigoplus_{k=2}^{m} E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 1\} \otimes P(t^{\rho(2k-2)}(t\mu_2))$$

$$\oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m}}, t\mu_2)$$

for $m = 1, \ldots, n$. The final differential leaves the $E^{2p(2n)+2} = E^{\infty}$-term, equal to

$$\hat{E}^{\infty}(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2})$$

$$\oplus \bigoplus_{k=2}^{n} E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 2\} \otimes P(t^{\rho(2k-3)}(t\mu_2))$$

$$\oplus \bigoplus_{k=2}^{n} E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 1\} \otimes P(t^{\rho(2k-2)}(t\mu_2))$$

$$\oplus E(\bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2n}}) \otimes P(t^{\rho(2n-2)+1}(t\mu_2)).$$

Proof. The statements about the $E^3$-, $E^{2p(1)+1}$- and $E^{2p(2)+1}$-terms are clear from Proposition 5.7. For each $m = 2, \ldots, n$ we proceed by a secondary induction. The differential

$$d^{2p(2m-1)}(P^{2m-1-p^{2m+i}} \cdot \bar{\epsilon}_1) = (t\mu_2)^{\rho(2m-3)} \cdot t^i$$

for $v_p(i) = 2m - 2$ is non-trivial only on the summand

$$E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m-2}}, t\mu_2)$$
of the $E^{2p(2m-2)+1} = E^{2p(2m-1)}$-term, with homology

$$E(u_n, \lambda_2) \otimes \mathbb{F}_p \{ t^j \mid j \in \mathbb{Z}, \nu_p(j) = 2m - 2 \} \otimes P_{p^{2m-3}}(t\mu_2)$$

$$\oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm 2m-1}, t\mu_2).$$

This gives the stated $E^{2p(2m-1)+1}$-term. Similarly, the differential

$$d^{2p(2m)}(t^{2m-1-p_{2m}}) = \lambda_2 \cdot t^{2m-1} \cdot (t\mu_2)^{p(2m-2)}$$

is non-trivial only on the summand

$$E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm 2m-1}, t\mu_2)$$

of the $E^{2p(2m-1)+1} = E^{2p(2m)}$-term, with homology

$$E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p \{ t^j \lambda_2 \mid j \in \mathbb{Z}, \nu_p(j) = 2m - 1 \} \otimes P_{p^{2m-2}}(t\mu_2)$$

$$\oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm 2m}, t\mu_2).$$

This gives the stated $E^{2p(2m)+1}$-term. The final differential

$$d^{2p(2n)+1}(u_n \cdot t^{2n}) = (t\mu_2)^{p(2n-2)+1}$$

is non-trivial only on the summand

$$E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm 2n}, t\mu_2)$$

of the $E^{2p(2n)+1}$-term, with homology

$$E(\bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm 2n}) \otimes P_{p(2n-2)+1}(t\mu_2).$$

This gives the stated $E^{2p(2n)+2}$-term. At this stage there is no room for any further differentials, since the spectral sequence is concentrated in a narrower horizontal band than the vertical height of the following differentials. □

Next we compare the $C_{p^n}$-Tate spectral sequence with the $C_{p^n}$-homotopy fixed point spectral sequence obtained by restricting the $E^2$-term to the second quadrant ($s \leq 0$, $t \geq 0$). It is algebraically easier to handle the latter after inverting $\mu_2$, which can be interpreted as comparing $THH(\ell/p)$ with its $C_{p^n}$-Tate construction.

In general, there is a commutative diagram

$$
\begin{array}{ccc}
THH(B)^{C_{p^n}} & \xrightarrow{R} & THH(B)^{C_{p^{n-1}}} \\
\gamma_{n-1} & \downarrow & \gamma_n \\
THH(B)^{hC_{p^n}} & \xrightarrow{R_h} & THH(B)^{hC_{p^{n-1}}} \\
\Gamma_1 & \downarrow & \Gamma_n \\
G_{n-1}^{-1} & \downarrow & (\rho_p^*THH(B)^{tC_p})^{hC_{p^{n-1}}}
\end{array}

(6.1)

Here $\rho_p^*THH(B)^{tC_p}$ is a notation for the $S^1$-spectrum obtained from the $S^1/C_{p^n}$-spectrum $THH(B)^{tC_p}$ via the $p$-th root isomorphism $\rho_p^* : S^1 \to S^1/C_{p^n}$, and $G_{n-1}$ is the comparison map from the $C_{p^{n-1}}$-fixed points to the $C_{p^{n-1}}$-homotopy fixed points of $\rho_p^*THH(B)^{tC_p}$, in view of the identification

$$(\rho_p^*THH(B)^{tC_p})^{C_{p^{n-1}}} = THH(B)^{tC_p}.\$$

We are of course considering the case $B = \ell/p$. In $V(1)$-homotopy all four maps with labels containing $\Gamma$ are $(2p - 2)$-cocompact, by Corollary 5.9, so $G_{n-1}$ is at least $(2p-1)$-cocompact. (We shall see in Lemma 6.8 that $V(1)$, $G_{n-1}$ is an isomorphism in all degrees.) By Proposition 5.8 the map $\Gamma_1$ precisely inverts $\mu_2$, so the $E^2$-term of the $C_{p^n}$-homotopy fixed point spectral sequence in $V(1)$-homotopy for $THH(\ell/p)^{tC_p}$ is obtained.
by inverting $\mu_2$ in $E^2(C_{p^n}, \ell/p)$. We denote this spectral sequence by $\mu_2^{-1}E^*(C_{p^n}, \ell/p)$, even though in later terms only a power of $\mu_2$ is present.

**Theorem 6.3.** The $C_{p^n}$-homotopy fixed point spectral sequence $\mu_2^{-1}E^*(C_{p^n}, \ell/p)$ in $V(1)$-homotopy for $THH(\ell/p)^{tC_{p^n}}$ begins

$$\mu_2^{-1}E^2(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p \{ 1, c_0, \mu_0, c_0 \mu_0, \ldots, \mu_0^{p-1}, c_1 \} \otimes P(t, \mu_2^{1})$$

and converges to $V(1)^*\rho_1^{*}THH(\ell/p)^{tC_{p^n}}$, which receives a $(2p-2)$-coconnected map $(\hat{\Gamma}_1)^{hC_{p^n}}$ from $V(1)^*THH(\ell/p)^{hC_{p^n}}$. There is an initial $d^2$-differential generated by

$$d^2(\epsilon_0 \mu_0^{i-1}) = t\mu_0^i$$

for $0 < i < p$. Next, there are $2n$ families of even length differentials generated by

$$d^{2p(2k-1)}(\mu_2^{2k-1} + j \cdot c_1) \simeq (t\mu_2)^{(2k-1)} \cdot \mu_2^j$$

for $v_p(j) = 2k - 2$, for each $k = 1, \ldots, n$, and

$$d^{2p(2k)}(\mu_2^{2k-1} + j \cdot c_1) \simeq \lambda_2 \cdot t^{(2k)} \cdot (t\mu_2)^{(2k)}$$

for each $k = 1, \ldots, n$. Finally, there is a differential of odd length generated by

$$d^{2p(2n)+1}(u_n \cdot \mu_2^{2n}) \simeq (t\mu_2)^{(2n)+1}.$$

**Proof.** The differential pattern follows from Theorem 6.1 by naturality with respect to the maps of spectral sequences

$$\mu_2^{-1}E^*(C_{p^n}, \ell/p) \xleftarrow{\hat{\Gamma}_1^{hC_{p^n}}} E^*(C_{p^n}, \ell/p) \xrightarrow{R^h} \hat{E}^*(C_{p^n}, \ell/p)$$

induced by $\hat{\Gamma}_1^{hC_{p^n}}$ and $R^h$. The first inverts $\mu_2$ and the second inverts $t$, at the level of $E^2$-terms. We are also using that $t\mu_2$, the image of $v_2$, multiplies as an infinite cycle in all of these spectral sequences. \qed

**Corollary 6.4.** The initial differential in the $C_{p^n}$-homotopy fixed point spectral sequence in $V(1)$-homotopy for $THH(\ell/p)^{tC_{p^n}}$ leaves

$$\mu_2^{-1}E^3(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p \{ \mu_0^i \mid 0 < i < p \} \otimes P(\mu_2^{\pm 1})$$

$$\oplus E(u_n, \bar{c}_1, \lambda_2) \otimes P(\mu_2^{\pm 1}, t\mu_2).$$

The next $2n$ families of differentials leave the intermediate terms

$$\mu_2^{-1}E^{2p(2m-1)+1}(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p \{ \mu_0^i \mid 0 < i < p \} \otimes P(\mu_2^{\pm 1})$$

$$\oplus \bigoplus_{k=1}^{m} E(u_n, \lambda_2) \otimes \mathbb{F}_p \{ \lambda_2 \mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 2 \} \otimes P(\mu_2^{(2k-1)}(t\mu_2)$$

$$\oplus \bigoplus_{k=1}^{m-1} E(u_n, \bar{c}_1) \otimes \mathbb{F}_p \{ \lambda_2 \mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 1 \} \otimes P(\mu_2^{(2k)}(t\mu_2)$$

$$\oplus E(u_n, \bar{c}_1, \lambda_2) \otimes P(\mu_2^{\pm 2m-1}, t\mu_2).$$
and
\[ \mu_2^{-1} E^{2\rho(2m)+1}(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p \{ \mu_0^i \mid 0 < i < p \} \otimes P(\mu_2^{\pm 1}) \]
\[ \oplus \bigoplus_{k=1}^m E(u_n, \lambda_2) \otimes \mathbb{F}_p \{ \mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 2 \} \otimes P_{\rho(2k-1)}(t \mu_2) \]
\[ \oplus \bigoplus_{k=1}^m E(u_n, \tilde{\epsilon}_1) \otimes \mathbb{F}_p \{ \lambda_2 \mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 1 \} \otimes P_{\rho(2k)}(t \mu_2) \]
\[ \oplus E(u_n, \tilde{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm 2m}, t \mu_2) \]
for \( m = 1, \ldots, n \). The final differential leaves the \( E^{2\rho(2n)+2} = E^\infty \)-term, equal to
\[ \mu_2^{-1} E^\infty(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p \{ \mu_0^i \mid 0 < i < p \} \otimes P(\mu_2^{\pm 1}) \]
\[ \oplus \bigoplus_{k=1}^n E(u_n, \lambda_2) \otimes \mathbb{F}_p \{ \mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 2 \} \otimes P_{\rho(2k-1)}(t \mu_2) \]
\[ \oplus \bigoplus_{k=1}^n E(u_n, \tilde{\epsilon}_1) \otimes \mathbb{F}_p \{ \lambda_2 \mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2k - 1 \} \otimes P_{\rho(2k)}(t \mu_2) \]
\[ \oplus E(\tilde{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm 2n}) \otimes P_{\rho(2n)+1}(t \mu_2). \]

Proof. The computation of the \( E^3 \)-term from the \( E^2 \)-term is straightforward. The rest
of the proof goes by a secondary induction on \( m = 1, \ldots, n \), very much like the proof of
Corollary 5.2. The differential
\[ d^{2\rho(2m-1)}(\mu_2^{2m-2}) \otimes (t \mu_2)^{\rho(2m-1)} \cdot \mu_2^j \]
for \( v_p(j) = 2m - 2 \) is non-trivial only on the summand
\[ E(u_n, \tilde{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm 2m-2}, t \mu_2) \]
of the \( E^3 = E^{2\rho(1)} \)-term (for \( m = 1 \)), resp. the \( E^{2\rho(2m-2)+1} = E^{2\rho(2m-1)} \)-term (for \( m = 2, \ldots, n \)). Its homology is
\[ E(u_n, \lambda_2) \otimes \mathbb{F}_p \{ \mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2m - 2 \} \otimes P_{\rho(2m-1)}(t \mu_2) \]
\[ \oplus E(u_n, \tilde{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm 2m-1}, t \mu_2), \]
which gives the stated \( E^{2\rho(2m-1)+1} \)-term. The differential
\[ d^{2\rho(2m)}(\mu_2^{2m-2}) \otimes (t \mu_2)^{\rho(2m)} \]
is non-trivial only on the summand
\[ E(u_n, \tilde{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm 2m-1}, t \mu_2) \]
of the \( E^{2\rho(2m-1)+1} = E^{2\rho(2m)} \)-term, leaving
\[ E(u_n, \tilde{\epsilon}_1) \otimes \mathbb{F}_p \{ \lambda_2 \mu_2^j \mid j \in \mathbb{Z}, v_p(j) = 2m - 1 \} \otimes P_{\rho(2m)}(t \mu_2) \]
\[ \oplus E(u_n, \tilde{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm 2m}, t \mu_2). \]
This gives the stated \( E^{2\rho(2m)+1} \)-term. The final differential
\[ d^{2\rho(2n)+1}(u_n \cdot \mu_2^{2n}) \otimes (t \mu_2)^{\rho(2n)+1} \]
Proof of Theorem 6.1. To make the inductive step to $C_{p^{n+1}}$, we use that the first $d^r$-differential of odd length in $\hat{E}^\ast(C_{p^n}, \ell/p)$ occurs for $r = r_0 = 2\rho(2n) + 1$. It follows from [AR02, Lem. 5.2] that the terms $\hat{E}^r(C_{p^n}, \ell/p)$ and $\hat{E}^r(C_{p^{n+1}}, \ell/p)$ are isomorphic for $r \leq 2\rho(2n) + 1$, via the Frobenius map (taking $t^j$ to $t^j t^r$) in even columns and the Verschiebung map (taking $u_i t^j$ to $u_{i+1} t^j$) in odd columns. Furthermore, the differential $d^{2\rho(2n)+1}$ is zero in the latter spectral sequence. This proves the part of Theorem 6.1 for $n + 1$ that concerns the differentials leading up to the term

$$\hat{E}^{2\rho(2n)+2}(C_{p^{n+1}}, \ell/p) = E(u_{n+1}, \lambda_2) \otimes \mathbb{F}_p \{ t^{-i} | 0 < i < p \} \otimes P(t^{\pm p^2})$$

(6.2)

Next we use the following commutative diagram, where we abbreviate $\text{THH}(B)$ to $T(B)$ for typographical reasons:

$$\xymatrix{ (\rho^p \circ T(B) | C_{p^n}) \ar[r]^\Gamma & T(B) | C_{p^n} \ar[r]^\Gamma & T(B) | C_{p^{n+1}} \ar[r]^\Gamma & T(B) | C_{p^{n+1}} \ar[r]^\Gamma & (\rho^p \circ T(B) | C_{p^n}) \ar[d]^F \\
\rho^p \circ T(B) | C_{p^n} \ar[r]^\Gamma & T(B) | C_{p^n} \ar[r]^\Gamma & T(B) | C_{p^n} \ar[r]^\Gamma & T(B) | C_{p^n} \ar[r]^\Gamma & \rho^p \circ T(B) | C_{p^n} \ar[d]^F 
}$$

(6.3)

The horizontal maps all induce $(2p - 2)$-coconnected maps in $V(1)$-homotopy for $B = \ell/p$. Each $F$ is a Frobenius map, forgetting invariance under a $C_{p^n}$-action. Thus the map $\hat{\Gamma}_{n+1}$ to the right induces an isomorphism of $E(\lambda_2) \otimes P(v_2)$-modules in all degrees $*$ > $(2p - 2)$ from $V(1) \ast \text{THH}(\ell/p)^{C_{p^n}}$, implicitly identified to the left with the abutment of $\mu_2^{-1} E^\ast(C_{p^n}, \ell/p)$, to $V(1) \ast \text{THH}(\ell/p)^{C_{p^{n+1}}}$, which is the abutment of $\hat{E}^\ast(C_{p^{n+1}}, \ell/p)$. The diagram above ensures that the isomorphism induced by $\hat{\Gamma}_{n+1}$ is compatible with the one induced by $\hat{\Gamma}_1$. By Proposition 5.3 it takes $\bar{e}_1, \lambda_2$ and $\mu_2$ to $\bar{e}_1, \lambda_2$ and $t^{-p^2}$ up to a unit factor in $\mathbb{F}_p$, respectively, and similarly for monomials in these classes.

We focus on the summand

$$E(u_n, \lambda_2) \otimes \mathbb{F}_p \{ \mu_2^j | j \in \mathbb{Z}, v_p(j) = 2n - 2 \} \otimes P_{\rho(2n-1)}(t \mu_2)$$

in $\mu_2^{-1} E^\infty(C_{p^n}, \ell/p)$, abutting to $V(1) \ast \text{THH}(\ell/p)^{C_{p^n}}$ in degrees > $(2p - 2)$. In the $P(v_2)$-module structure on the abutment, each class $\mu_2^j$ with $v_p(j) = 2n - 2$, $j > 0$, generates a copy of $P_{\rho(2n-1)}(v_2)$, since there are no permanent cycles in the same total degree as


\[ y = (t_{\mu_2})^{e(2n-1)} \cdot \mu_2^i \] that have lower (= more negative) homotopy fixed point filtration. See Lemma 6.5 below for the elementary verification. The \( P(v_2) \)-module isomorphism induced by \( \Gamma_{n+1} \) must take this to a copy of \( P_{\rho(2n-1)}(v_2) \) in \( V(1)_* \text{THH}(\ell/p)^{C_{p^{n+1}}} \), generated by \( t^{p^2 j} \).

Writing \( i = -p^2 j \), we deduce that for \( v_p(i) = 2n, i < 0 \), the infinite cycle \( z = (t_{\mu_2})^{e(2n-1)}. t^i \) must represent zero in the abutment, and must therefore be hit by a differential \( z = d'x \) in the \( C_{p^{n+1}} \)-Tate spectral sequence. Here \( r \geq 2p(2n) + 2 \).

Since \( z \) generates a free copy of \( P(t_{\mu_2}) \) in the \( E^{2p(2n)+2} \)-term displayed in (6.2), and \( d' \) is \( P(t_{\mu_2}) \)-linear, the class \( x \) cannot be annihilated by any power of \( t_{\mu_2} \). This means that \( x \) must be contained in the summand

\[ E(u_{n+1}, \bar{e}_1, \lambda_2) \otimes P(t^{p^{2n}}, t_{\mu_2}) \]

of \( E^{2p(2n)+2} (C_{p^{n+1}}, \ell/p) \). By an elementary check of bidegrees, see Lemma 6.6 below, the only possibility is that \( x \) has vertical degree \( (2p-1) \), so that we have differentials

\[ d^{p(2n+1)}(p^{2n+1} - p^{2n+2+i} \cdot \bar{e}_1) = (t_{\mu_2})^{e(2n-1)} \cdot t^i \]

for all \( i < 0 \) with \( v_p(i) = 2n \). The cases \( i > 0 \) follow by the module structure over the \( C_{p^{n+1}} \)-Tate spectral sequence for \( \ell \). The remaining two differentials,

\[ d^{p(2n+2)}(p^{2n+1} - p^{2n+2}) = \lambda_2 \cdot t^{p^{2n+1}} \cdot (t_{\mu_2})^{e(2n)} \]

and

\[ d^{p(2n+2)}(u_{n+1} \cdot t^{p^{2n+2}}) = (t_{\mu_2})^{e(2n)+1} \]

are also present in the \( C_{p^{n+1}} \)-Tate spectral sequence for \( \ell \), see [AR02 Th. 6.1], hence follow in the present case by the module structure. With this we have established the complete differential pattern asserted by Theorem 6.1. \hfill \Box

**Lemma 6.5.** For \( j \in \mathbb{Z} \) with \( v_p(j) = 2n - 2, \) where \( n \geq 1 \), there are no classes in \( \mu_2^{-1} E^\infty(C_{p^n}, \ell/p) \) in the same total degree as \( y = (t_{\mu_2})^{e(2n-1)} \cdot \mu_2^i \) that have lower homotopy fixed point filtration.

**Proof.** The total degree of \( y \) is \( 2(p^{2n+2} - p^{2n+1} + p - 1) + 2p^2 j \equiv (2p - 2) \mod 2p^{2n} \), which is even.

Looking at the formula for \( \mu_2^{-1} E^\infty(C_{p^n}, \ell/p) \) in Corollary 6.1, the classes of lower filtration than \( y \) all lie in the terms

\[ E(u_n, \bar{e}_1) \otimes F_p \{ \lambda_2 \mu_2^i \mid j \in \mathbb{Z}, v_p(i) = 2n - 1 \} \otimes P_{\rho(2n)}(t_{\mu_2}) \]

and

\[ E(\bar{e}_1, \lambda_2) \otimes P(\mu_2^{3p^{2n}}) \otimes P_{\rho(2n)+1}(t_{\mu_2}) \].

Those in even total degree and of lower filtration than \( y \) are

\[ u_n \lambda_2 \cdot \mu_2^i (t_{\mu_2})^e, \quad \bar{e}_1 \lambda_2 \cdot \mu_2^i (t_{\mu_2})^e \]

with \( v_p(i) = 2n - 1, \rho(2n-1) < e < \rho(2n) \), and

\[ \mu_2^i (t_{\mu_2})^e, \quad \bar{e}_1 \lambda_2 \cdot \mu_2^i (t_{\mu_2})^e \]

with \( v_p(i) \geq 2n, \rho(2n-1) < e < \rho(2n) \).

The total degree of \( u_n \lambda_2 \cdot \mu_2^i (t_{\mu_2})^e \) for \( v_p(i) = 2n - 1 \) is \( (1 - (2p^2 - 1) + 2p^2 i + (2p^2 - 2) e \equiv (2p^2 - 2)(e + 1) \mod 2p^{2n} \). For this to agree with the total degree of \( y \), we must have \( (2p^2 - 2)(e + 1) \mod 2p^{2n} \), so \( e + 1 \equiv 1/(1 + p) \mod p^{2n} \) and \( e \equiv \rho(2n-1) - 1 \mod p^{2n} \). There is no such \( e \) with \( \rho(2n-1) < e < \rho(2n) \).
The total degree of $\tilde{e}_1\lambda_2 \cdot \mu_2^s(t\mu_2)^e$ for $v_p(i) = 2n - 1$ is $(2p - 1) + (2p^2 - 1) + 2p^2i + (2p^2 - 2)e \equiv 2p + (2p^2 - 2)(e + 1)$ mod $2p^{2n}$. To agree with that of $y$, we must have $(2p - 2) \equiv 2p(2p^2 - 2)(e + 1)$ mod $2p^{2n}$, so $e(1) \equiv 1/(1 + p)$ mod $p^{2n}$ and $e \equiv \rho(2n)$ mod $p^{2n}$. There is no such $e$ with $\rho(2n - 1) < e < \rho(2n)$.

The total degree of $\mu_2^s(t\mu_2)^e$ for $v_p(i) \geq 2n$ is $2p^2i + (2p^2 - 2)e \equiv (2p^2 - 2)e$ mod $2p^{2n}$. To agree with that of $y$, we must have $(2p - 2) \equiv (2p^2 - 2)e$ mod $2p^{2n}$, so $e \equiv 1/(1 + p) \equiv \rho(2n - 1)$ mod $p^{2n}$. There is no such $e$ with $\rho(2n - 1) < e < \rho(2n)$.

The total degree of $\tilde{e}_1\lambda_2 \cdot \mu_2^s(t\mu_2)^e$ for $v_p(i) \geq 2n$ is $(2p - 1) + (2p^2 - 1) + 2p^2i + (2p^2 - 2)e$. To agree modulo $2p^{2n}$ with that of $y$, we must have $e \equiv \rho(2n)$ mod $p^{2n}$. The only such $e$ with $\rho(2n - 1) < e \leq \rho(2n)$ is $e = \rho(2n)$. But in that case, the total degree of $\tilde{e}_1\lambda_2 \cdot \mu_2^s(t\mu_2)^e$ is $2p^2i + (2p^2 - 2)(\rho(2n) + 1) = 2(p^{2n+2} + p - 1) + 2p^2i$. To be equal to that of $y$, we must have $2p^2i + 2p^{2n+1} = 2p^2j$, which is impossible for $v_p(i) \geq 2n$ and $v_p(j) = 2n - 2$.

**Lemma 6.6.** For $v_p(i) = 2n$, $n \geq 1$ and $z = (t\mu_2)^{\rho(2n-1)} \cdot t^i$, the only class in

$$E(u_{n+1}, \tilde{e}_1, \lambda_2) \otimes P(t^{2p^{2n}}, t\mu_2)$$

that can support a non-zero differential $d^r(x) = z$ for $r \geq 2\rho(2n) + 2$ is (a unit times)

$$x = t^{2p^{2n} - p^{2n+1} + i} \cdot \tilde{e}_1.$$

**Proof.** The class $z$ has total degree $(2p^2 - 2)\rho(2n - 1) - 2i = 2p^{2n+2} - 2p^{2n+1} + 2p - 2 - 2i \equiv (2p - 2) \mod 2p^{2n}$, which is even, and vertical degree $2p^2\rho(2n - 1)$. Hence $x$ has odd total degree, and vertical degree at most $2p^2\rho(2n - 1) - 2\rho(2n - 1) = 2p^{2n+2} - 2p^{2n+1} - \cdots - 2p^3 - 1$.

This leaves the possibilities

$$u_{n+1} \cdot t^j(t\mu_2)^e, \quad \tilde{e}_1 \cdot t^j(t\mu_2)^e, \quad \lambda_2 \cdot t^j(t\mu_2)^e$$

with $v_p(j) \geq 2n$ and $0 \leq e < p^{2n} - p^{2n-1} - \cdots - p = \rho(2n - 1) - \rho(2n - 2) - 1$, and

$$u_{n+1} \cdot \tilde{e}_1 \lambda_2 \cdot t^j(t\mu_2)^e$$

with $v_p(j) \geq 2n$ and $0 \leq e < p^{2n} - p^{2n-1} - \cdots - p - 1 = \rho(2n - 1) - \rho(2n - 2) - 2$.

The total degree of $x$ must be one more than the total degree of $z$, hence is congruent to $(2p - 2) \mod 2p^{2n}$.

The total degree of $u_{n+1} \cdot t^j(t\mu_2)^e$ is $-1 - 2j + (2p^2 - 2)e \equiv -1 + (2p^2 - 2)e$ mod $2p^{2n}$. To have $(2p - 1) \equiv -1 + (2p^2 - 2)e$ mod $2p^{2n}$, we must have $e \equiv -p/(1 + p) \equiv p^{2n} - p^{2n-1} - \cdots - p \mod p^{2n}$, which does not happen for $e$ in the allowable range.

The total degree of $\lambda_2 \cdot t^j(t\mu_2)^e$ is $(2p^2 - 1) - 2j + (2p^2 - 2)e \equiv (2p^2 - 1) + (2p^2 - 2)e$ mod $2p^{2n}$. To have $(2p - 1) \equiv (2p^2 - 1) + (2p^2 - 2)e$ mod $2p^{2n}$, we must have $e \equiv -p/(1 + p) \equiv \rho(2n - 1)$ mod $p^{2n}$, which does not happen.

The total degree of $u_{n+1} \tilde{e}_1 \lambda_2 \cdot t^j(t\mu_2)^e$ is $-1 + (2p - 1) + (2p^2 - 2) - 2j + (2p^2 - 2)e \equiv (2p - 1) + (2p^2 - 2)(e + 1)$ mod $2p^{2n}$. To have $(2p - 1) \equiv (2p - 1) + (2p^2 - 2)(e + 1)$ mod $2p^{2n}$, we must have $(e + 1) \equiv 0$ mod $p^{2n}$, so $e = 0$ is the only possibility in the allowable range. In that case, a check of total degrees shows that we must have $j = p^{2n+1} - p^{2n+2} + i$. □

**Corollary 6.7.** $V(1)_* THH(\ell/p)^{C_p\eta}$ is finite in each degree.

**Proof.** This is clear by inspection of the $E^\infty$-term in Corollary 6.2 □
Lemma 6.8. The map $G_n$ induces an isomorphism
\[ V(1)_* \text{THH}(\ell/p)^{\text{TC}} \cong V(1)_* (\rho_p^* \text{THH}(\ell/p)^{\mu_0})^{\text{TC}} \]
in all degrees. In the limit over the Frobenius maps $F$, there is a map $G$ inducing an isomorphism
\[ V(1)_* \text{THH}(\ell/p)^{\text{ST}} \cong V(1)_* (\rho_p^* \text{THH}(\ell/p)^{\mu_0})^{\text{ST}}. \]

Proof. As remarked after diagram (6.1), $G_n$ induces an isomorphism in $V(1)$-homotopy above degree $(2p - 2)$. The permanent cycle $t^{-p^{2n+2}}$ in $\hat{E}^{\infty}(C_{p^{n+1}}, \ell)$ acts invertibly on $\hat{E}^{\infty}(C_{p^{n+1}}, \ell/p)$, and its image $G_n(t^{-p^{2n+2}}) = \mu_2^{2n}$ in $\hat{E}^{\infty}(C_{p^n}, \ell)$ acts invertibly on $\mu_2^{-1}E^{\infty}(C_{p^n}, \ell/p)$. Therefore the module action derived from the $\ell$-algebra structure on $\ell/p$ ensures that $G_n$ induces isomorphisms in $V(1)$-homotopy in all degrees. \qed

Theorem 6.9. The isomorphism (6.4) admits the following description at the associated graded level:

(a) The associated graded of $V(1)_* \text{THH}(\ell/p)^{\text{ST}}$ for the $S^1$-Tate spectral sequence is
\[
\hat{E}^{\infty}(S^1, \ell/p) = E(\lambda_2) \otimes \mathbb{F}_p \{ t^{-i} \mid 0 < i < p \} \otimes P(t^{\pm 2})
\]
\[\oplus \bigoplus_{k \geq 2} E(\lambda_2) \otimes \mathbb{F}_p \{ t^j \mid j \in \mathbb{Z}, \nu_p(j) = 2k - 2 \} \otimes P_{\rho(2k-3)}(t \mu_2)\]
\[\oplus \bigoplus_{k \geq 2} E(\bar{\epsilon}_1) \otimes \mathbb{F}_p \{ t^j \lambda_2 \mid j \in \mathbb{Z}, \nu_p(j) = 2k - 1 \} \otimes P_{\rho(2k-2)}(t \mu_2)\]
\[\oplus E(\bar{\epsilon}_1, \lambda_2) \otimes P(t \mu_2).\]

(b) The associated graded of $V(1)_* \text{THH}(\ell/p)^{\text{ST}}$ for the $S^1$-homotopy fixed point spectral sequence maps by a $(2p - 2)$-cocommutative map to
\[
\mu_2^{-1}E^{\infty}(S^1, \ell/p) = E(\lambda_2) \otimes \mathbb{F}_p \{ \mu_0^i \mid 0 < i < p \} \otimes P(\mu_2^{\pm 1})
\]
\[\oplus \bigoplus_{k \geq 1} E(\lambda_2) \otimes \mathbb{F}_p \{ \mu_2^j \mid j \in \mathbb{Z}, \nu_p(j) = 2k - 2 \} \otimes P_{\rho(2k-1)}(t \mu_2)\]
\[\oplus \bigoplus_{k \geq 1} E(\bar{\epsilon}_1) \otimes \mathbb{F}_p \{ \lambda_2 \mu_2^j \mid j \in \mathbb{Z}, \nu_p(j) = 2k - 1 \} \otimes P_{\rho(2k)}(t \mu_2)\]
\[\oplus E(\bar{\epsilon}_1, \lambda_2) \otimes P(t \mu_2).\]

(c) The isomorphism from (a) to (b) induced by $G$ takes $t^{-i}$ to $\mu_0^i$ for $0 < i < p$ and $t^i$ to $\mu_2^i$ for $i + p^2 j = 0$, up to a unit factor in $\mathbb{F}_p$. Furthermore, it takes multiples by $\bar{\epsilon}_1$, $\lambda_2$ or $t \mu_2$ in the source to the same multiples in the target, up to a unit factor in $\mathbb{F}_p$.

Proof. Claims (a) and (b) follow by passage to the limit over $n$ from Corollaries 6.2 and 6.4. Claim (c) follows by passage to the same limit from the formulas for the isomorphism induced by $\hat{\Gamma}_{n+1}$, which were given below diagram (6.3). \qed

7. Topological cyclic homology

By definition, there is a fiber sequence
\[ TC(B; p) \xrightarrow{\pi} TF(B; p) \xrightarrow{R^{-1}} TF(B; p) \rightarrow \Sigma TC(B; p) \]
inducing a long exact sequence

\[ \cdots \to V(1)_*TC(B; p) \to V(1)_*TF(B; p) \xrightarrow{R_*^{-1}} V(1)_*TF(B; p) \xrightarrow{G} \cdots \]  

in \( V(1) \)-homotopy. By Corollary 5.9 there are \((2p - 2)\)-cocompact maps \( \Gamma \) and \( \hat{\Gamma} \) from \( V(1)_*TF(\ell/p; p) \) to \( V(1)_*THH(\ell/p)^{hS^1} \) and \( V(1)_*THH(\ell/p)^{tS^1} \), respectively. We model \( V(1)_*TF(\ell/p; p) \) in degrees \( > (2p - 2) \) by the map \( \hat{\Gamma} \) to the \( S^1 \)-Tate construction. Then, by diagram (6.1), \( R_* \) is modeled in the same range of degrees by the chain of maps below:

\[ V(1)_*THH(B)^{tS^1} \xrightarrow{G} V(1)_*THH(B)^{hS^1} \xrightarrow{(\hat{\Gamma})_*^{hS^1}} V(1)_*(\rho^*_{p}\text{THH}(B)^{tG_p})^{hS^1}. \]

Here \( R^k \) induces a map of spectral sequences

\[ E^*(R^k): E^*(S^1, B) \to \hat{E}^*(S^1, B), \]

(abutting to \( R^k \)), which at the \( E^2 \)-term equals the inclusion that algebraically inverts \( t \). When \( B = \ell/p \), the left hand map \( G \) is an isomorphism by Lemma 6.8, and the middle (wrong-way) map is \((2p - 2)\)-cocompact.

**Proposition 7.1.** In degrees \( > (2p - 2) \), the homomorphism

\[ E^\infty(R^k): E^\infty(S^1, \ell/p) \to \hat{E}^\infty(S^1, \ell/p) \]

maps

(a) \( E(\bar{\epsilon}_1, \lambda_2) \otimes P(t\mu_2) \) identically to the same expression;
(b) \( E(\lambda_2) \otimes \mathbb{F}_p\{\mu_2^{-2}\} \otimes P_{p(2k-1)}(t\mu_2) \) surjectively onto \( E(\lambda_2) \otimes \mathbb{F}_p\{t\} \otimes P_{p(2k-3)}(t\mu_2) \) for each \( k \geq 2, j = dp^{2k-2}, 0 < d < p^2 - p \) and \( p \nmid d; \)
(c) \( E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{\lambda_2\mu_2^{-2}\} \otimes P_{p(2k)}(t\mu_2) \) surjectively onto \( E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t\lambda_2\} \otimes P_{p(2k-2)}(t\mu_2) \) for each \( k \geq 2, j = dp^{2k-1} \) and \( 0 < d < p; \)
(d) the remaining terms to zero.

Notice that in the statements (b) and (c) above, we abuse notation and indentify the components of degree \( > 2p - 2 \) of \( E^\infty(S^1, \ell/p) \) and \( \mu_2^{-1}E^\infty(S^1, \ell/p) \), using Theorem 6.9(b).

**Proof.** Consider the summands of \( E^\infty(S^1, \ell/p) \) and \( \hat{E}^\infty(S^1, \ell/p) \) given in Theorem 6.9. Clearly, the first term \( E(\lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2) \) goes to zero (these classes are hit by \( d^2 \)-differentials), and the last term \( E(\bar{\epsilon}_1, \lambda_2) \otimes P(t\mu_2) \) maps identically to the same term. This proves (a) and part of (d).

For each \( k \geq 1 \) and \( j = dp^{2k-2} \) with \( p \nmid d \), the term \( E(\lambda_2) \otimes \mathbb{F}_p\{\mu_2^{-2}\} \otimes P_{p(2k-1)}(t\mu_2) \) maps to the term \( E(\lambda_2) \otimes \mathbb{F}_p\{t\} \otimes P_{p(2k-3)}(t\mu_2) \), except that the target is zero for \( k = 1 \). In symbols, the element \( \lambda_2^{\bar{\epsilon}_1}\mu_2^{-2j}(t\mu_2)^j \) maps to the element \( \lambda_2^{\bar{\epsilon}_1}(t\mu_2)^j \). If \( d < 0 \), then the \( t \)-exponent in the target is bounded above by \( dp^{2k-2} + \rho(2k - 3) < 0 \), so the target lives in the right half-plane and is essentially not hit by the source, which lives in the left half-plane. If \( d > p^2 - p \), then the total degree in the source is bounded above by \( (2p^2 - 1) - 2dp^{2k} + \rho(2k - 1)(2p^2 - 2) < 2p - 2 \), so the source lives in total degree \( < (2p - 2) \) and will be disregarded. If \( 0 < d < p^2 - p \), then \( \rho(2k - 1) - dp^{2k-2} > \rho(2k - 3) \) and \( -dp^{2k-2} < 0 \), so the source surjects onto the target. This proves (b) and part of (d).

Lastly, for each \( k \geq 1 \) and \( j = dp^{2k-1} \) with \( p \nmid d \), the term \( E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{\lambda_2\mu_2^{-2}\} \otimes P_{p(2k)}(t\mu_2) \) maps to the term \( E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t\lambda_2\} \otimes P_{p(2k-2)}(t\mu_2) \). The target is zero for \( k = 1 \). If \( d < 0 \),
then $dp^{2k-1} + \rho(2k - 2) < 0$ so the target lives in the right half-plane. If $d > p$, then 
$(2p - 1) + (2p^2 - 1) - 2dp^{2k+1} + \rho(2k)(2p^2 - 2) < 2p - 2$, so the source lives in total degree 
$< (2p - 2)$. If $0 < d < p$, then $\rho(2k) - dp^{2k-1} > \rho(2k - 2)$ and $-dp^{2k-1} < 0$, so the source 
surjects onto the target. This proves (c) and the remaining part of (d). \hfill $\Box$

**Definition 7.2.** Let

$$A = E(\varepsilon_1, \lambda_2) \otimes P(\mu_2)$$

$$B_k = E(\lambda_2) \otimes \mathbb{F}_p\{t^{dp^{2k-2}} \mid 0 < d < p^2 - p, p \nmid d\} \otimes P_{\rho(2k-3)}(\mu_2)$$

$$C_k = E(\tilde{\varepsilon}_1) \otimes \mathbb{F}_p\{t^{dp^{2k-1}} \lambda_2 \mid 0 < d < p\} \otimes P_{\rho(2k-2)}(\mu_2)$$

for $k \geq 2$ and let $D$ be the span of the remaining monomials in $\hat{E}^\infty(S^1, \ell/p)$. Let $B = \bigoplus_{k \geq 2} B_k$ and $C = \bigoplus_{k \geq 2} C_k$. Then $\hat{E}^\infty(S^1, \ell/p) = A \oplus B \oplus C \oplus D$.

**Proposition 7.3.** In degrees $> (2p - 2)$, there are closed subgroups $\tilde{A} = E(\varepsilon_1, \lambda_2) \otimes P(v_2)$, $\tilde{B}_k$, $\tilde{C}_k$ and $\tilde{D}$ in $V(1), TF(\ell/p, p)$, represented by the subgroups $A$, $B_k$, $C_k$ and $D$ of $\hat{E}^\infty(S^1, \ell/p)$, respectively, such that the homomorphism $R_* = V(1)_* R$ induced by the restriction map $R$

(a) is the identity on $\tilde{A}$;

(b) maps $\tilde{B}_{k+1}$ surjectively onto $\tilde{B}_k$ for all $k \geq 2$;

(c) maps $\tilde{C}_{k+1}$ surjectively onto $\tilde{C}_k$ for all $k \geq 2$;

(d) is zero on $\tilde{B}_2$, $\tilde{C}_2$ and $\tilde{D}$.

In these degrees, $V(1)_* TF(\ell/p, p) \cong \tilde{A} \oplus \tilde{B} \oplus \tilde{C} \oplus \tilde{D}$, where $\tilde{B} = \prod_{k \geq 2} \tilde{B}_k$ and $\tilde{C} = \prod_{k \geq 2} \tilde{C}_k$.

**Proof.** The proof is the same as the proof of [AR02 Thm. 7.7], except that in the present paper we work with the Tate model $THH(\ell/p)^{hS^1}$ for $TF(\ell/p, p)$, in place of the homotopy fixed point model $THH(\ell/p)^{hS^1}$. The computations are made in $V(1)$-homotopy, and we disregard all classes in total degrees $\leq (2p - 2)$. For example with this convention we write $\mu_2^{-1}E^\infty(S^1, \ell/p) \cong E^\infty(S^1, \ell/p)$, using the same abuse of notation as in Proposition [7.1].

In these terms, the restriction homomorphism $R_*$ is given at the level of $E^\infty$-terms as the composite of the isomorphism

$$G_*: \hat{E}^\infty(S^1, \ell/p) \to \mu_2^{-1}E^\infty(S^1, \ell/p) \cong E^\infty(S^1, \ell/p)$$

and the map

$$E^\infty(R^b): E^\infty(S^1, \ell/p) \to \hat{E}^\infty(S^1, \ell/p).$$

As an endomorphism of $\hat{E}^\infty(S^1, \ell/p)$, this composite $E^\infty(R^b)G_*$ is the identity on $A$, maps $B_{k+1}$ onto $B_k$ and $C_{k+1}$ onto $C_k$ for all $k \geq 2$, and is zero on $B_2$, $C_2$ and $D$, by Theorem [6.9](c) and Proposition [7.7]. The task is to find closed lifts of these groups to $V(1)_* TF(\ell/p, p)$ such that $R_*$ has similar properties.

Let $\tilde{A} = E(\varepsilon_1, \lambda_2) \otimes P(v_2) \subset V(1)_* TF(\ell/p, p)$ be the (degree-wise finite, hence closed) subalgebra generated by the images of the classes $\bar{\varepsilon}_1^k$, $\lambda_2$ and $v_2$ in $V(1)_* K(\ell/p)$. Then $\tilde{A}$ lifts $A$ and consists of classes in the image of the trace map from $V(1)_* K(\ell/p)$. Hence $R_*$ is the identity on $\tilde{A}$.
We fix \( k \geq 2 \) and choose, for all \( n \geq 0 \), a subgroup \( B_k^n \subset B_{k+n} \), as follows. We take
\[
B_k^0 = B_k \cap \ker(E^\infty(R^h)G_*)
\]
\[
= \begin{cases} 
B_2 & \text{for } k = 2, \\
E(\lambda_2) \otimes \bigoplus_{0 < d < p^2 - p, \ p \nmid d} \mathbb{F}_p \{t^{dp^{2k-2}}\} \otimes P_{dp^{2k-3} + p(2k-4)}(t \mu_2) & \text{for } k \geq 3,
\end{cases}
\]
where \( P_a(t) = \mathbb{F}_p \{t^{a c} \mid a \leq c \leq b\} \). We proceed by induction on \( n \) for \( n \geq 1 \), choosing a subgroup \( B_k^n \) of \( B_{k+n} \) mapping isomorphically onto \( B_k^{n-1} \) under \( E^\infty(R^h)G_* \) (such a group exists by Theorem 6.9(c) and Proposition 7.1(b)). We then have
\[
B_k = \bigoplus_{n=0}^{k-2} B_k^n.
\]
By the argument given on top of page 31 of [AR02], we can choose a lift \( \tilde{B}_k^0 \) of \( B_k^0 \) with \( \tilde{B}_k^0 \subset \text{im}(R_a) \cap \ker(R_*) \)
in \( V(1)_*TF(\ell/p; p) \). By induction on \( n \geq 1 \), we choose a lift \( \tilde{B}_k^n \subset \text{im}(R_*) \) of \( B_k^n \) mapping isomorphically onto \( \tilde{B}_k^{n-1} \) under \( R_* \). Such a choice is possible since the image of \( R_* \) on \( V(1)_*TF(\ell/p; p) \) equals the image of its restriction to \( \text{im}(R_*) \), see [AR02, p. 30]. Now
\[
\tilde{B}_k = \bigoplus_{n=0}^{k-2} \tilde{B}_k^n
\]
is a (degreewise finite, hence closed) lift of \( B_k \) with \( R_*(\tilde{B}_2) = 0 \) and \( R_*(\tilde{B}_k) = \tilde{B}_k-1 \) for \( k \geq 3 \).

To construct \( \tilde{C}_k \) we proceed as for \( \tilde{B}_k \) above, starting with \( C_k^0 = C_2 \) and
\[
C_k^0 = C_k \cap \ker(E^\infty(R^h)G_*)
\]
\[
= E(\bar{\lambda}_2) \otimes \bigoplus_{0 < d < p^2 - p, \ p \nmid d} \mathbb{F}_p \{t^{dp^{2k-1}}\lambda_2\} \otimes P_{dp^{2k-3} + p(2k-4)}(t \mu_2)
\]
for \( k \geq 3 \), and using Theorem 6.9(c) and Proposition 7.1(c) to choose \( C_k^n \) for \( n \geq 1 \).

It remains to construct \( \tilde{D} \). By Proposition 7.1(d), the isomorphism \( G_* \) maps \( D \) into \( \ker(E^\infty(R^h)) \). By [AR02, Lem. 7.3] the representatives in \( E^\infty(S^1, \ell/p) \) of the kernel of \( R_*^h \) equal the kernel of \( E^\infty(R^h) \). It follows that the representatives in \( E^\infty(S^1, \ell/p) \) of the kernel of \( R_* \) are mapped isomorphically by \( G_* \) to \( \ker(E^\infty(R^h)) \). Hence we can pick a vector space basis for \( D \), choose a representative in \( \ker(R_*) \subset V(1)_*TF(\ell/p; p) \) of each basis element, and let \( \tilde{D} \subset V(1)_*TF(\ell/p; p) \) be the closure of the vector space spanned by these chosen representatives. This closure is contained in \( \ker(R_*) \) since \( R_* \) is continuous. Hence \( R_* \) is zero on \( \tilde{D} \).

\[\square\]

**Proposition 7.4.** In degrees \( > (2p - 2) \) there are isomorphisms
\[
\ker(R_* - 1) \cong \tilde{A} \oplus \lim_k \tilde{B}_k \oplus \lim_k \tilde{C}_k
\]
\[
\cong E(\bar{\lambda}_2) \otimes P(v_2)
\]
\[
+ E(\lambda_2) \otimes \mathbb{F}_p \{t^d \mid 0 < d < p^2 - p, p \nmid d\} \otimes P(v_2)
\]
\[
+ E(\bar{\lambda}_1) \otimes \mathbb{F}_p \{t^{dp} \lambda_2 \mid 0 < d < p\} \otimes P(v_2)
\]
and \( \text{cok}(R_\ast - 1) \cong \tilde{A} = E(t_1, \lambda_2) \otimes P(v_2) \). Hence there is an isomorphism
\[
V(1), TC(\ell/p; p) \cong E(\partial, \tilde{A}, \lambda_2) \otimes P(v_2)
\]
\[
= E(\lambda_2) \otimes \mathbb{F}_p \{ t^d \mid 0 < d < p^2 - p, p \nmid d \} \otimes P(v_2)
\]
\[
\oplus E(\epsilon_1) \otimes \mathbb{F}_p \{ t^{dp} \lambda_2 \mid 0 < d < p \} \otimes P(v_2)
\]
in these degrees, where \( \partial \) has degree \(-1\) and represents the image of \( 1 \) under the connecting map \( \partial \) in \([7.1]\).

**Proof.** By Proposition [7.3], the homomorphism \( R_\ast - 1 \) is zero on \( \tilde{A} \) and an isomorphism on \( \tilde{D} \). Furthermore, there is an exact sequence
\[
0 \to \lim_{k \to} \tilde{B}_k \to \prod_{k \geq 2} \tilde{B}_k \xrightarrow{\partial_{R-1}} \prod_{k \geq 2} \tilde{B}_k \to \lim_{k \to} \tilde{B}_k \to 0
\]
and similarly for the \( C \)'s. The derived limit on the right vanishes since each \( \tilde{B}_{k+1} \) surjects onto \( \tilde{B}_k \).

Multiplication by \( t \mu_2 \) in each \( B_k \) is realized by multiplication by \( v_2 \) in \( \tilde{B}_k \). Each \( \tilde{B}_k \) is a sum of \( 2(p - 1)^2 \) cyclic \( P(v_2) \)-modules, and since \( \rho(2k - 3) \) grows to infinity with \( k \) their limit is a free \( P(v_2) \)-module of the same rank, with the indicated generators \( t^d \) and \( t^d \lambda_2 \) for \( 0 < d < p^2 - p, p \nmid d \). The argument for the \( C \)'s is practically the same.

The long exact sequence \([7.1]\) yields the short exact sequence
\[
0 \to \Sigma^{-1} \text{cok}(R_\ast - 1) \xrightarrow{\partial} V(1), TC(\ell/p; p) \xrightarrow{\pi} \ker(R_\ast - 1) \to 0,
\]
from which the formula for the middle term follows.

**Remark 7.5.** A more obvious set of \( E(\lambda_2) \otimes P(v_2) \)-module generators for \( \lim_{k \to} \tilde{B}_k \) would be the classes \( t^{dp^2} \) in \( B_2 \cong \tilde{B}_2 \), for \( 0 < d < p^2 - p, p \nmid d \). We have a commutative diagram
\[
\begin{array}{ccc}
TF(\ell/p; p) & \xrightarrow{\hat{f}} & THH(\ell/p)^{tS^1} \\
\downarrow & & \downarrow_{FG} \\
THH(\ell/p)^{C_p} & \xrightarrow{G; \Gamma_p} & (\rho_p^*THH(\ell/p)^{tC_p})^{hC_p}.
\end{array}
\]
Under the left-hand canonical map \( TF(\ell/p; p) \to THH(\ell/p)^{C_p} \), modeled here by \( FG: THH(\ell/p)^{tS^1} \to (\rho_p^*THH(\ell/p)^{tC_p})^{hS^1} \to (\rho_p^*THH(\ell/p)^{tC_p})^{hC_p} \), the class \( t^{dp^2} \) maps to the class \( \mu_2^{-d} \). Since we are only concerned with degrees \( > (2p - 2) \) we may equally well use its \( v_2 \)-power multiple \( (t\mu_2)^d \). \( \mu_2^{-d} = t^d \) as generator, with the advantage that it is in the image of the localization map
\[
THH(\ell/p)^{hC_p} \to (\rho_p^*THH(\ell/p)^{tC_p})^{hC_p}.
\]

Hence the class denoted \( t^d \) in \( \lim_{k \to} \tilde{B}_k \) is chosen so as to map under \( TF(\ell/p; p) \to THH(\ell/p)^{hC_p} \to t^d \) in \( E^\infty(C_p, \ell/p) \). Similarly, the class denoted \( t^{dp} \lambda_2 \) in \( \lim_{k \to} \tilde{C}_k \) is chosen so as to map to \( t^{dp} \lambda_2 \) in \( E^\infty(C_p, \ell/p) \).

The map \( \pi: \ell/p \to \mathbb{Z}/p \) is \( (2p - 2) \)-connected, hence induces \( (2p - 1) \)-connected maps \( \pi_+: K(\ell/p) \to K(\mathbb{Z}/p) \) and \( \pi_+: V(1), TC(\ell/p; p) \to V(1), TC(\mathbb{Z}/p; p) \), by [BM94 Prop. 10.9] and [Dun97 p. 224]. Here \( TC(\mathbb{Z}/p; p) \cong H_{\mathbb{Z}_p} \vee \Sigma^{-1} H_{\mathbb{Z}_p} \) and we have an
There is an isomorphism of \( \text{Theorem 7.6.} \) 
\[
\text{isomorphism } V(1)_*TC(\mathbb{Z}/p; p) \cong E(\partial, \tilde{\epsilon}_1), \text{ so we can recover } V(1)_*TC(\ell/p; p) \text{ in degrees } \leq (2p - 2) \text{ from this map.}
\]

**Theorem 7.6.** There is an isomorphism of \( P(v_2) \)-modules
\[
\text{isomorphism } V(1)_*TC(\ell/p; p) \cong P(v_2) \otimes E(\partial, \tilde{\epsilon}_1, \lambda_2)
\]
\[
\oplus P(v_2) \otimes E(\text{dlog } v_1) \otimes \mathbb{F}_p \{ t^d v_2 \mid 0 < d < p^2 - p, p \nmid d \}
\]
\[
\oplus P(v_2) \otimes E(\tilde{\epsilon}_1) \otimes \mathbb{F}_p \{ t^d \lambda_2 \mid 0 < d < p \}
\]
where \( \text{dlog } v_1 \cdot t^d v_2 = t^d \lambda_2 \). The degrees are \( |\partial| = -1, |\tilde{\epsilon}_1| = |\lambda_1| = 2p - 1, |\lambda_2| = 2p^2 - 1 \), \( |v_2| = 2p^2 - 2, |t| = -2 \) and \( |\text{dlog } v_1| = 1 \).

The notation \( \text{dlog } v_1 \) for the multiplier \( v_2^{-1} \lambda_2 \) is suggested by the relation \( v_1 \cdot \text{dlog } p = \lambda_1 \) in \( V(0)_*TC(\mathbb{Z}(p)|\mathbb{Q}; p) \).

**Proof.** Only the additive generators \( t^d \) for \( 0 < d < p^2 - p, p \nmid d \) from Proposition [7.4] do not appear in \( V(1)_*TC(\ell/p; p) \), but their multiples by \( \lambda_2 \) and positive powers of \( v_2 \) do. This leads to the given formula, where \( \text{dlog } v_1 \cdot t^d v_2 \) must be read as \( t^d \lambda_2 \). □

By [HM97, Th. C] the cyclotomic trace map of [BHM93] induces cofiber sequences
\[
K(B_p)_p \xrightarrow{\text{trc}} TC(B; p)_p \xrightarrow{g} \Sigma^{-1} H\mathbb{Z}_p \to \Sigma K(B_p)_p
\]
for each connective \( S \)-algebra \( B \) with \( \pi_0(B_p) = \mathbb{Z}_p \) or \( \mathbb{Z}/p \), and thus long exact sequences
\[
\cdots \to V(1)_* K(B_p)_p \xrightarrow{\text{trc}} V(1)_*TC(B; p)_p \xrightarrow{g} \Sigma^{-1} E(\tilde{\epsilon}_1) \to \cdots
\]
This uses the identifications \( W(\mathbb{Z}_p)_F \cong W(\mathbb{Z}/p)_F \cong \mathbb{Z}_p \) of Frobenius coinvariants of rings of Witt vectors, and applies in particular for \( B = H\mathbb{Z}(p), H\mathbb{Z}/p, \ell \) and \( \ell/p \).

**Theorem 7.7.** There is an isomorphism of \( P(v_2) \)-modules
\[
\text{isomorphism } V(1)_*K(\ell/p) \cong P(v_2) \otimes E(\tilde{\epsilon}_1) \otimes \mathbb{F}_p \{ t^d v_2 \mid 0 < d < p^2 - p, p \nmid d \}
\]
\[
\oplus P(v_2) \otimes E(\tilde{\epsilon}_1) \otimes \mathbb{F}_p \{ t^d \lambda_2 \mid 0 < d < p \cdot \}
\]
This is a free \( P(v_2) \)-module of rank \( (2p^2 - 2p + 8) \) and of zero Euler characteristic.

**Proof.** In the case \( B = \mathbb{Z}/p, K(\mathbb{Z}/p)_p \cong H\mathbb{Z}_p \) and the map \( g \) is split surjective up to homotopy. So the induced homomorphism to \( V(1)_* \Sigma^{-1} H\mathbb{Z}_p = \Sigma^{-1} E(\tilde{\epsilon}_1) \) is surjective. Since \( \pi: \ell/p \to \mathbb{Z}/p \) induces a \( (2p - 1) \)-connected map in topological cyclic homology, and \( \Sigma^{-1} E(\tilde{\epsilon}_1) \) is concentrated in degrees \( \leq (2p - 2) \), it follows by naturality that also in the case \( B = \ell/p \) the map \( g \) induces a surjection in \( V(1) \)-homotopy. The kernel of the surjection \( P(v_2) \otimes E(\partial, \tilde{\epsilon}_1, \lambda_2) \to \Sigma^{-1} E(\tilde{\epsilon}_1) \) gives the first row in the asserted formula. □

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