ON ADJUNCTIONS FOR FOURIER-MUKAI TRANSFORMS

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Abstract. We show that the adjunction counits of a Fourier-Mukai transform $\Phi\colon D(X_1) \to D(X_2)$ arise from maps of the kernels of the corresponding Fourier-Mukai transforms. In a very general setting of proper separable schemes of finite type over a field we write down these maps of kernels explicitly – facilitating the computation of the twist (the cone of an adjunction counit) of Φ. We also give another description of these maps, better suited to computing cones if the kernel of Φ is a pushforward from a closed subscheme $Z \subset X_1 \times X_2$. Moreover, we show that we can replace the condition of properness of the ambient spaces X_1 and X_2 by that of Z being proper over them and still have this description apply as is. This can be used, for instance, to compute spherical twists on non-proper varieties directly and in full generality.

1. Introduction

The bounded derived category $D(X)$ of coherent sheaves on a variety X had long been recognized as a crucial invariant of X which holds a wealth of information about its geometry. In order to work conveniently with functors between the derived categories of two varieties the language of Fourier-Mukai transforms was developed by Mukai, Bondal and Orlov, Bridgeland and many others. In brief, we can define a functor $D(X_1) \to D(X_2)$ by specifying an object in the derived category of $D(X_1 \times X_2)$. A morphism between such defining objects induces a natural transformation between the functors. In this paper we write down the adjunction counit of a general Fourier-Mukai transform in this language — as morphisms of defining objects.

Let X_1 and X_2 be a pair of smooth projective varieties. We have the following commutative diagram:

$$\begin{tikzcd}
 & & X_1 \times X_2 \times X_1 \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & & \\
 & X_1 \times X_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & X_1 \times X_1 \arrow[dll, "\tilde{\pi}_1"'] \arrow[drr, "\tilde{\pi}_2"] & X_2 \times X_1 \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\
X_1 & & X_2 & & X_1
\end{tikzcd} \tag{1.1}$$

Let $E \in D(X_1 \times X_2)$. The *Fourier-Mukai transform from X_1 to X_2 with kernel E* is the functor

$$\Phi_E(-) = \pi_{2*}\left(E \otimes \pi_1^*(-)\right). \tag{1.2}$$

Here and throughout the paper all the functors are derived unless mentioned otherwise. It is well-known (e.g. [BO95], Lemma 1.2) that the left adjoint of Φ_E is the Fourier-Mukai transform from $D(X_2)$ to $D(X_1)$ with kernel $E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})$ where $\pi_1^!(\mathcal{O}_{X_1}) = \pi_2^*(\omega_{X_2})[\dim X_2]$. Denote this adjoint by Φ_E^{ladj}. A composition of Fourier-Mukai transforms is again a Fourier-Mukai transform ([Muk81], Prop. 1.3). In particular, $\Phi_E^{\text{ladj}}\Phi_E$ is the Fourier-Mukai transform $D(X_1) \to D(X_1)$ with kernel

$$Q = \pi_{13*}\left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^* \pi_1^!(\mathcal{O}_{X_1})\right). \tag{1.3}$$

On the other hand, the identity functor Id is the Fourier-Mukai transform $D(X_1) \to D(X_1)$ with kernel $\mathcal{O}_\Delta = \Delta_* \mathcal{O}_{X_1}$ where Δ is the diagonal inclusion $X_1 \hookrightarrow X_1 \times X_1$.

Consider now the left adjunction counit

$$\Phi_E^{\text{ladj}}\Phi_E \to \mathrm{Id}. \tag{1.4}$$

In general, morphisms between Fourier-Mukai kernels map neither injectively nor surjectively to natural transformations between the Fourier-Mukai transforms. Thus there is no *a priori* reason for (1.4) to come from some morphism $Q \to \mathcal{O}_\Delta$. In this paper we construct explicitly a natural choice of such morphism, working in a much greater generality of separated schemes of finite type over a field.

The principal application is to compute, and even define, *spherical twists*. These are an important class of auto-equivalences of the derived category $D(X)$ of a variety X. They are first examples of genuinely derived auto-equivalences, in a sense that they are neither shifts, nor come from auto-equivalences of the underlying abelian category $\operatorname{Coh} X$. In brief, *a spherical twist* is an auto-equivalence of $D(X)$ produced from a *spherical object* in $D(X)$ or, more generally, *a spherical functor* $D(Y) \to D(X)$. Spherical objects were introduced by Seidel and Thomas in [ST01] as mirror symmetry analogues of Lagrangian spheres on a symplectic manifold. Their defining properties ensure that the twist by a spherical object is an auto-equivalence of $D(X)$. This was generalised in [Ann07] to exact functors between triangulated categories in such a way that Seidel-Thomas spherical objects are precisely the (Fourier-Mukai kernels of) spherical functors $D(\operatorname{Spec}\, k) \to D(X)$, where k is the base field.

Taking the twist of a functor is completely general and does not in itself rely on the fact that the functor is spherical. The ideal definition would be the following:

"Definition": Let C_1 and C_2 be triangulated categories. Let S be an exact functor $C_1 \to C_2$ which has a right (resp. left) adjoint R (resp. L). The *twist* (resp. the *dual co-twist*) of S is the functor $T_S \colon C_2 \to C_2$ (resp. $F'_S \colon C_1 \to C_1$) which is the functorial cone of the adjunction counit $SR \to \mathrm{Id}$ (resp. $LS \to \mathrm{Id}$).

The problem with this definition is the well-known fact that cones in triangulated categories are not functorial. The cone of a morphism between two objects is uniquely defined (up to an isomorphism), but a cone of a morphism between two functors might not exist or might not be unique. This is usually fixed by restricting to a setting where the cone of a morphism of functors is well-defined, cf. [Ann07], §1. One way is to consider only the functors which are Fourier-Mukai transforms and only the natural transformations which come from morphisms of Fourier-Mukai kernels. But then to define a twist of a Fourier-Mukai transform we need a natural choice of the morphism of Fourier-Mukai kernels underlying the corresponding adjunction counit, while to compute the twist we need an efficient way of computing the cone of this morphism. This paper addresses both of these issues.

The construction of the natural morphism of Fourier-Mukai kernels underlying the adjunction counit of a general Fourier-Mukai transform is carried out in Section 3. Thanks to the recent advances in Grothendieck duality machinery summarised in Section 2 we can work with separated schemes of finite type over a field and with derived categories $D_{qc}(-)$ of unbounded complexes with quasi-coherent cohomology. So let X_1 and X_2 be two separated schemes of finite type, E a perfect object of $D(X_1 \times X_2)$ and Φ_E the Fourier-Mukai transform $D(X_1) \to D(X_2)$ with kernel E. Let X_2 be proper, so that the left adjoint Φ_E^{ladj} of Φ_E is again a Fourier-Mukai transform. Then the left adjunction counit $\Phi_E^{\mathrm{ladj}} \Phi_E \to \mathrm{Id}$ is induced by the morphism $Q = \pi_{13*}\left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^* \pi_1^!(\mathcal{O}_{X_1})\right) \to \mathcal{O}_\Delta$ which roughly is the composition of the following:

$$\pi_{13*}\Big(\text{The adjunction unit } \mathrm{Id} \to \Delta_{13*}\Delta_{13}^* \text{ for the diagonal } X_1 \times X_2 \xrightarrow{\Delta_{13}} X_1 \times X_2 \times X_1\Big) \tag{1.5}$$

$$\Delta_* \pi_{1*}\left(\text{The evaluation map } E \otimes E^\vee \to \mathcal{O}_{X_1 \times X_2} \text{ on } X_1 \times X_2\right) \tag{1.6}$$

$$\Delta_*\left(\text{The adjunction counit } \pi_{1*}\pi_1^!(\mathcal{O}_{X_1}) \to \mathcal{O}_{X_1}\right) \tag{1.7}$$

For the precise formulas see Theorem 3.1. When X_1 is also proper Φ_E, Φ_E^{ladj} and (1.5)-(1.7) restrict to the full subcategories of $D_{qc}(-)$ consisting of bounded complexes with coherent cohomologies. If X_2 is smooth $\pi_1^!(\mathcal{O}_{X_1}) = \pi_2^*(\omega_{X_2})[\dim X_2]$ as before. Theorem 3.2 give the analogous result for the right adjunction counit.

This allows us to define the twist and the dual co-twist of any Fourier-Mukai transform. Section 4 deals with the issue of computing them. Anyone trying to compute the cone of the decomposition (1.5)-(1.7) will find it ill-suited to the task if the support of E has high codimension in $X_1 \times X_2$. We give an example in Section 4.1 with E the structure sheaf $\mathcal{O}_Z$ of a complete intersection subscheme Z in $X_1 \times X_2$ of codimension $d > 0$ which satisfies certain transversality conditions. Then morphisms (1.5) and (1.6) both have huge cones with non-zero cohomologies in all degrees from $-d$ to 0. However these two cones mostly annihilate each other and the cone of composition (1.5)-(1.6) is actually quite small. This suggests an alternative decomposition of (1.5)-(1.6) better suited to computing cones, cf. (4.4).

In the rest of Section 4 we make this into a general argument. The key idea is to take the decomposition (1.5)-(1.7) obtained in Section 3 and apply to it the base change for Künneth maps. If E is a pushforward of an object from a closed subscheme $Z \overset{\iota_Z}{\hookrightarrow} X_1 \times X_2$, then the evaluation map $E \otimes E^{\vee} \rightarrow \mathcal{O}_{X_1 \times X_2}$ involves the derived self-intersection of Z inside $X_1 \times X_2$. In precise terms, it involves the Künneth map (see Section 4.3 for the definition) for the fiber square σ_Δ depicted on the left in (1.8):

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & Z \arrow[d, "\iota_Z"] & & Z' \arrow[r] \arrow[d] & X_1 \times Z \arrow[d, "\iota_{Z23}"] \\
Z \arrow[r, "\iota_Z"'] & X_1 \times X_2 & & Z \times X_1 \arrow[r, "\iota_{Z12}"'] & X_1 \times X_2 \times X_1
\end{tikzcd}$$

$$\sigma_\Delta : \quad \xleftarrow{\text{Restriction to } X_1 \times X_2 \xrightarrow{\Delta_{13}} X_1 \times X_2 \times X_1} \quad \sigma : \tag{1.8}$$

Thus in (1.5)-(1.6) we first restrict fiber square σ to the diagonal $X_1 \times X_2$ in $X_1 \times X_2 \times X_1$ which turns it into σ_Δ and then we do the Künneth map on σ_Δ. Given two subschemes, the cone of the Künneth map for the fiber square of their intersection reflects, roughly, how far this intersection is from transverse. In σ_Δ we have the self-intersection of Z in $X_1 \times X_2$ which is the opposite of transverse. This suggests first doing the Künneth map on σ, as the intersection of $Z \times X_1$ with $X_1 \times Z$ in $X_1 \times X_2 \times X_1$ may be more transverse, and then restricting to the diagonal Z in Z'.

Write π_{Z1} for the composition $Z \overset{\iota_Z}{\hookrightarrow} X_1 \times X_2 \xrightarrow{\pi_1} X_1$. In Prop. 4.4 we prove that Künneth maps commute with arbitrary base change. Then in Theorem 4.1 we show that the composition (1.5)-(1.7) is isomorphic to roughly the following (cf. Theorem 4.1 for precise formulas):

$$\pi_{13*} \left(\text{The Künneth map for } \sigma\right) \tag{1.9}$$

$$\pi_{13*}\iota_{Z'*} \left(\text{The adjunction unit } \mathrm{Id} \rightarrow \Delta'_* \Delta'^* \text{ for the diagonal } Z \xrightarrow{\Delta'} Z'\right) \tag{1.10}$$

$$\Delta_* \pi_{Z1*} \left(\text{The evaluation map for } E \text{ on } Z\right) \tag{1.11}$$

$$\Delta_* \left(\text{The adjunction counit } \pi_{Z1*}\pi^!_{Z1}(\mathcal{O}_{X_1}) \rightarrow \mathcal{O}_{X_1}\right) \tag{1.12}$$

This is our preferred decomposition of morphism $Q \rightarrow \mathcal{O}_\Delta$. Theorem 4.2 gives the analogous statement for the right adjunction counit.

One advantage of decomposition (1.9)-(1.12) is that most of the morphisms in it can become isomorphisms under fairly reasonable assumptions on E and Z. Indeed, while the Künneth map for square σ_Δ is never an isomorphism unless Z is the whole of $X_1 \times X_2$, the Künneth map for σ is an isomorphism whenever the intersection of $Z \times X_1$ with $X_1 \times Z$ in $X_1 \times X_2 \times X_1$ is transverse. The evaluation map for E on Z is an isomorphism whenever E is a line bundle or any invertible object of $D(Z)$. The adjunction counit in (1.12) is an isomorphism whenever $Z \xrightarrow{\pi_{Z1}} X_1$ is such that $\pi_{Z1*}\mathcal{O}_Z = \mathcal{O}_{X_1}$, e.g. Z is a blowup of X_1 or a Fano fibration over it. This allows for a number of scenarios where the twist or the dual co-twist of Φ_E can be written down fairly easily, as we demonstrate in Cor. 4.5.

Another advantage of decomposition (1.9)-(1.12) is that it moves the action away from ambient spaces $X_1 \times X_2 \times X_1$ and $X_1 \times X_2$ to their subschemes Z' and Z. This allows us to replace the assumption of X_2 being proper by the assumption of Z being proper over X_1 and X_2 (see Theorem 4.1). Something to be appreciated by those who want to do spherical twists on non-compact varieties, e.g. total spaces of cotangent bundles of projective varieties.

Finally, in Section 5 we give an example of an explicit computation using Theorem 4.1. We consider the naive derived category transform induced by a Mukai flop. This transform is not an equivalence - it was proved by Namikawa in [Nam03] by direct comparison of Hom spaces. We demonstrate how its dual co-twist can be computed quickly and efficiently by our methods.

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2. Preliminaries

Let k be an algebraically closed field of characteristic 0. The level of generality we choose to work at in the main body of this paper is that of separated schemes of finite type over k. These assumptions are necessary for the Grothendieck duality machinery which ensures that the direct image functor in the definition of a Fourier-Mukai transform has a right adjoint. Without them we cannot expect a general Fourier-Mukai transform to have a right and a left adjoint.

Some of the auxiliary results we prove along the way hold in a greater generality than the one above. We would like to think of these results as being of potential interest to others who find themselves in an unfortunate situation of having to show a complicated diagram of derived functors to commute. We try therefore to state these results in maximal generality they hold at.

By a ringed space we always mean a commutative ringed space. By a *concentrated* map of schemes we mean a map which is quasi-compact and quasi-separated. A scheme X is said to be *concentrated* if it is concentrated over Spec $\mathbb{Z}$. If Y is a concentrated scheme, then a map $X \to Y$ is concentrated if and only if X is concentrated [GD64, §1.2].

We make frequent use of a notion of a *perfect* map of schemes $X \xrightarrow{f} Y$, cf. [Ill71a, §4]. For maps of finite type between noetherian schemes f is perfect if and only if it is of finite Tor-dimension, i.e. the derived functor of f^* is cohomologically bounded.

Given an adjoint pair of functors (F, G), by the *right adjoint with respect to* F of some natural transformation $FH_1 \to H_2$, we mean the natural transformation $H_1 \to GH_2$ induced by the adjunction. Similarly, by *the left adjoint with respect to* G of some $H_1 \to GH_2$ we mean the $FH_1 \to H_2$ induced by the adjunction.

Throughout the paper we employ a variety of greek letters to denote an assortment of natural maps which exist between compositions of standard derived functors. These are defined at length over the course of Sections 2.1-2.3, but for the convenience of our readers we have also compiled a brief index:

α_f	the projection formula $f_*A \otimes B \to f_*(A \otimes f^*B)$	(2.28)
β_f	$\mathrm{Id} \to f_*f^*$	(2.22)
γ_f	$f^*f_* \to \mathrm{Id}$	(2.22)
δ_f	the sheafified Grothendieck duality $f_*\,\mathbf{R}\mathcal{H}om_X(A, f^\times B) \to \mathbf{R}\mathcal{H}om_Y(f_*A, B)$	(2.31)
ϵ_f	$f_*f^\times \to \mathrm{Id}$	(2.30)
$\zeta_{g,f}$	$f^*g^* \xrightarrow{\sim} (g \circ f)^*$	(2.18)
$\eta_{g,f}$	$(g \circ f)_* \xrightarrow{\sim} g_*f_*$	(2.17)
$\theta_{A,B}$	$A \longrightarrow \mathbf{R}\mathcal{H}om_X\left(\mathbf{R}\mathcal{H}om_X(A,B), B\right)$	(2.14)
θ_E	$E \to E^{\vee\vee}$	(2.15)
κ_f	$f_*A \otimes f_*B \to f_*(A \otimes B)$	(2.27)

κ_σ	the Künneth map $f_{1*}(A_1) \otimes f_{2*}(A_2) \to h_*\left(g_1^*(A_1) \otimes g_2^*(A_2)\right)$	(4.9)
λ_f	$\mathrm{Id} \to f^\times f_*$	(2.30)
μ_σ	the base change $g^*f_* \to f'_*g'^*$	(2.34)
ν_f	$f^*(A \otimes B) \xrightarrow{\sim} f^*(A) \otimes f^*B$	(2.24)
ξ	$\mathbf{R}\mathcal{H}om_X(A,B) \otimes C \to \mathbf{R}\mathcal{H}om_X(A, B \otimes C)$	(2.10)
ξ_E	$E^\vee \otimes (-) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_X(E, -)$	(2.12)
ρ	$(A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$	(2.7)
τ_f	$f_*\,\mathbf{R}\mathcal{H}om_X(f^*A, B) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_Y(A, f_*B)$	(2.21)
υ_A	$\mathbf{R}\mathcal{H}om(A \otimes B, C) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om(B, \mathbf{R}\mathcal{H}om(A, C))$	(2.8)
χ_f	$f^\times A \otimes f^*B \to f^\times(A \otimes B)$	(2.32)

2.1. **Derived categories and derived functors.** Let X be a scheme or a ringed space. We denote by $D(\mathcal{O}_X\text{-}\mathbf{Mod})$ the unbounded derived category of the abelian category $\mathcal{O}_X$-$\mathbf{Mod}$. We denote by $D_{\mathrm{qc}}(X)$ (resp. $D(X)$) the full subcategory of $\mathcal{O}_X$-$\mathbf{Mod}$ consisting of complexes with quasi-coherent (resp. bounded and coherent) cohomology. We denote by $D_{\mathrm{perf}}(X)$ the full subcategory of $D(X)$ consisting of the objects which are locally quasi-isomorphic to a bounded complex of free $\mathcal{O}_X$-modules of finite rank.

For a reference text on derived categories and derived functors we recommend [Har66], for the traditional approach, and [Lip09], for a more modern approach. One should also mention the expositions in [KS06] and [Nee01]. A key feature of the modern approach is that thanks to the results of [Spa88] we can now work freely with unbounded complexes. The authors of this paper adhere to a general principle that wherever possible general results on derived functors and isomorphisms between them should first be proved in the setting of $D_{\mathrm{qc}}(-)$, and then shown to restrict to the usual setting of $D(-)$ where applicable.

All the functors in this paper are assumed to be derived, unless specifically mentioned otherwise. With two exceptions listed below we suppress all the usual $\mathbf{R}$'s and $\mathbf{L}$'s and use the same notation for the derived functor as for its abelian category counterpart. Below we summarize basic facts about the derived functors we make use of.

Let X be a ringed space. The derived tensor product functor exists as a functor

$$(-) \otimes (-) \colon D(\mathcal{O}_X\text{-}\mathbf{Mod}) \times D(\mathcal{O}_X\text{-}\mathbf{Mod}) \to D(\mathcal{O}_X\text{-}\mathbf{Mod}).$$

and always restricts to a functor $D_{\text{qc}}(X) \times D_{\text{qc}}(X) \to D_{\text{qc}}(X)$ [Lip09, §2.5]. For X a locally noetherian scheme and for $A \in D_{\text{perf}}(X)$ the functor $A \otimes -$ restricts to a functor $D(X) \to D(X)$ [Har66, Prop. II.4.3]. Similarly, for any $n \in \mathbb{Z}$ the derived tensor product functor in n variables $(-) \otimes \cdots \otimes (-)$ exists as a functor from the product of n copies of $D(\mathcal{O}_X\text{-}\mathbf{Mod})$ into $D(\mathcal{O}_X\text{-}\mathbf{Mod})$ [Lip09, §2.5.9].

The derived functor of the functor $\operatorname{Hom}_X(-,-)$ of taking the global Hom space between two $\mathcal{O}_X$-modules exists as a functor

$$\mathbf{R}\operatorname{Hom}_X(-,-)\colon D(\mathcal{O}_X\text{-}\mathbf{Mod})^{\text{opp}} \times D(\mathcal{O}_X\text{-}\mathbf{Mod}) \to D(\Gamma(\mathcal{O}_X)\text{-}\mathbf{Mod}),$$

see [Lip09, §2.4]. We make an exception and do not supress '$\mathbf{R}$' here in order to differentiate the object $\mathbf{R}\operatorname{Hom}_X(A,B)$ in $D(\Gamma(\mathcal{O}_X)\text{-}\mathbf{Mod})$ from the morphism space $\operatorname{Hom}_{D(X)}(A,B)$. Similarly, the derived functor of the sheafified Hom functor $\mathcal{H}om_X(-,-)$ exists as a functor

$$\mathbf{R}\mathcal{H}om_X(-,-)\colon D(\mathcal{O}_X\text{-}\mathbf{Mod})^{\text{opp}} \times D(\mathcal{O}_X\text{-}\mathbf{Mod}) \to D(\mathcal{O}_X\text{-}\mathbf{Mod})$$

We do not suppress '$\mathbf{R}$' here to emphasize the relation with $\mathbf{R}\operatorname{Hom}_X$. If X is a locally noetherian scheme, then for any $A \in D(X)$ the functor $\mathbf{R}\mathcal{H}om_X(A,-)$ restricts to a functor $D^+_{qc}(X) \to D_{qc}(X)$ [Har66, Prop. II.3.3]. Here $D^+_{qc}(X)$ is the subcategory of $D_{qc}(X)$ consisting of complexes with bounded below cohomology. If X is a noetherian scheme and A is perfect the functor $\mathbf{R}\mathcal{H}om_X(A,-)$ restricts to a functor $D_{\text{qc}}(X) \to D_{\text{qc}}(X)$ and then to a functor $D(X) \to D(X)$ [AIL10, Lemma 1.4.6].

Let now Y be another ringed space, and let $f\colon X \to Y$ be a map of ringed spaces.

The derived direct image functor exists as a functor

$$f_*(-)\colon D(\mathcal{O}_X\text{-}\mathbf{Mod}) \to D(\mathcal{O}_Y\text{-}\mathbf{Mod}),$$

cf. [Lip09, §3.1]. When f is a concentrated map of schemes f_* restricts to a functor $D_{\text{qc}}(X) \to D_{\text{qc}}(Y)$ [Lip09, Prop. 3.9.2]. If X and Y are noetherian and f is proper[1] then f_* restricts to a functor $D(X) \to D(Y)$ [Ill71a, Théorème 2.2.1].

The derived inverse image functor exists as a functor

$$f^*(-)\colon D(\mathcal{O}_Y\text{-}\mathbf{Mod}) \to D(\mathcal{O}_X\text{-}\mathbf{Mod}),$$

cf. [Lip09, §3.1]. When f is a concentrated map of schemes f^* restricts to a functor $D_{\text{qc}}(Y) \to D_{\text{qc}}(X)$ [Lip09, Prop. 3.9.1]. If X and Y are locally noetherian and f is perfect, then f^* restricts to a functor $D(Y) \to D(X)$ [Har66, Prop. II4.4].

2.2. **Adjunctions and dualities for derived functors.** Let X be a ringed space. For any $A \in D(\mathcal{O}_X\text{-}\mathbf{Mod})$ the functor

$$A \otimes (-)\colon D(\mathcal{O}_X\text{-}\mathbf{Mod}) \to D(\mathcal{O}_X\text{-}\mathbf{Mod})$$

is left adjoint to functor

$$\mathbf{R}\mathcal{H}om_X(A,-)\colon D(\mathcal{O}_X\text{-}\mathbf{Mod}) \to D(\mathcal{O}_X\text{-}\mathbf{Mod}),$$

cf. [Lip09, Prop. 2.6.1].

For any $A \in D(\mathcal{O}_X\text{-}\mathbf{Mod})$ denote by $A^\vee$ the object $\mathbf{R}\mathcal{H}om_X(A,\mathcal{O}_X) \in D(\mathcal{O}_X\text{-}\mathbf{Mod})$. There is a natural morphism $A \to A^{\vee\vee}$ which is an isomorphism for any $A \in D_{\text{perf}}(X)$ [Ill71b, Prop. 7.2]. So $(-)^\vee$ restricts to a self-inverse category equivalence $D_{\text{perf}}(X) \to D_{\text{perf}}(X)^{opp}$ giving us *the duality functor for perfect complexes.*

For any $A \in D_{\text{perf}}(X)$ there is a canonical isomorphism $A^\vee \otimes (-) \simeq \mathbf{R}\mathcal{H}om_X(A,-)$, see §2.3(3), so

$$A \otimes (-)\colon D(\mathcal{O}_X\text{-}\mathbf{Mod}) \to D(\mathcal{O}_X\text{-}\mathbf{Mod})$$

is both the left and the right adjoint of functor

$$A^\vee \otimes (-)\colon D(\mathcal{O}_X\text{-}\mathbf{Mod}) \to D(\mathcal{O}_X\text{-}\mathbf{Mod}).$$

Let now Y be another ringed space and let $f\colon X \to Y$ be a map of ringed spaces. Then functor

$$f^*(-)\colon D(\mathcal{O}_Y\text{-}\mathbf{Mod}) \to D(\mathcal{O}_X\text{-}\mathbf{Mod})$$

is left adjoint to functor

$$f_*(-)\colon D(\mathcal{O}_X\text{-}\mathbf{Mod}) \to D(\mathcal{O}_Y\text{-}\mathbf{Mod}),$$

cf. [Lip09, Prop. 3.2.1].

[1] In a non-noetherian world one can work with a more general notion of *a quasi-proper scheme map,* cf. [Lip09, §4.3].

Suppose now that X and Y are concentrated schemes and let $f\colon X \to Y$ be a scheme map. Then the functor

$$f_*(-)\colon D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$$

has a right adjoint which we denote as

$$f^\times(-)\colon D_{\mathrm{qc}}(Y) \to D_{qc}(X),$$

cf. [Lip09, Theorem 4.1] or [Nee96, §4].

To state the rest of the Grothendieck duality results in their full presently known generality we would have to introduce a number of notions (pseudo-coherence, quasi-properness, etc.) which are only meaningfully different from well-established ones in non-noetherian context. Since the main bulk of this paper deals with schemes of finite type over a field, we prefer to state these results for noetherian schemes only and refer the reader to [Lip09, §4] for a more general story.

So let X and Y be noetherian schemes and let $f\colon X \to Y$ be a separated scheme map of finite type. Adjunction $(f_*, f^\times)$ induces a natural morphism $\delta_f\colon f_* \mathbf{R}\mathcal{H}om_X(A, f^\times B) \to \mathbf{R}\mathcal{H}om_Y(f_*A, B)$, see §2.3(10), often referred to as *the sheafified Grothendieck duality morphism*. For δ_f to be an isomorphism we need $f^\times$ to commute with restriction to open sets of Y [Lip09, §4.6]. When f is proper $f^\times$ commutes with Tor-independent base change for all objects in $D^+_{qc}(Y)$ and so δ_f is an isomorphism for all $A \in D_{qc}(X)$ and $B \in D^+_{qc}(Y)$ [Lip09, §4.4]. If f is also perfect, then $f^\times$ commutes with Tor-independent base change for all of $D_{qc}(Y)$ and so δ_f is an isomorphism for all $A, B \in D_{qc}(Y)$ [Lip09, Theorem 4.7.4]. Moreover, the natural map $\chi_f\colon f^\times(A) \otimes f^*(B) \xrightarrow{\sim} f^\times(A \otimes B)$, cf. §2.3(11), is an isomorphism for all $A, B \in D_{qc}(X)$ [Nee96, §5].

By a result of Nagata any separated map of finite type between noetherian schemes decomposes as an open immersion followed by a proper map ([Nag62], or [Voj07] for a more modern exposition). So to make $(-)^\times$ commute with flat base change we can try and modify its behaviour over open immersions. Indeed, there is a unique way to paste $(-)^\times$ over proper maps with $(-)^*$ over étale maps in a way compatible with étale base change of $(-)^\times$ (see [Lip09], Theorem 4.8.1 for more detail). The result is the pseudo-functor $(-)^!$, *Deligne's twisted inverse image pseudo-functor,* which associates to any finite-type separated map $f\colon X \to Y$ of noetherian schemes a functor $f^!\colon D^+_{qc}(Y) \to D^+_{qc}(X)$ with a number of nice properties:

1. $f^! = f^\times|_{D^+_{qc}}$ when f is proper and $f^! = f^*|_{D^+_{qc}}$ when f is étale.
2. For any f functor $f^!$ commutes with Tor-independent base change [Lip09, Theorem 4.8.3].
3. For perfect f functor $f^!$ restricts to a functor $D(Y) \to D(X)$ [AIL10, Remark 2.1.5].
4. There exists, as explained in [Lip09, §4.9.1], for all $A \in D^+_{qc}(X)$ a natural morphism
$$f^!(\mathcal{O}_Y) \otimes f^*(A) \to f^!(A). \tag{2.1}$$
If f is perfect then (2.1) is an isomorphism [Lip09, Theorem 4.9.4] and the morphism
$$f^*(A) \to \mathbf{R}\mathcal{H}om_X\left(f^!(\mathcal{O}_Y), f^!(A)\right) \tag{2.2}$$
right adjoint to (2.1) with respect to $f^!(\mathcal{O}_Y) \otimes (-)$ is also an isomorphism [AIL10, Lemma 2.1.10].
5. If f is a regular immersion of codimension n, then $f^!(\mathcal{O}_Y) = \omega_{X/Y}[-n]$ where $\omega_{X/Y}$ is the top wedge power of the normal bundle $\mathcal{N}_{X/Y}$ [Har66, Cor. III.7.3].
6. If f is smooth of relative dimension n, then $f^!(\mathcal{O}_Y) = \omega_{X/Y}[n]$ where $\omega_{X/Y}$ is the top wedge power of the sheaf $\Omega^1_{X/Y}$ of relative differentials [Ver69, Theorem 3].

When f is both perfect and proper, then $f^! = f^\times|_{D^+_{qc}}$ *and* all the above properties of $f^!$ apply to the whole of $f^\times\colon D_{qc} \to D_{qc}$. We do not therefore distinguish between $f^!$ and $f^\times$ when f is perfect and proper.

If f is proper the RHS of (2.2), as a functor in A, has left adjoint $f_*\left(f^!\mathcal{O}_Y \otimes (-)\right)$. If f is also perfect we denote this functor by $f_!$ and the fact that (2.2) is an isomorphism implies immediately that $f_!\colon D_{qc}(X) \to D_{qc}(Y)$ is the left adjoint of $f^*\colon D_{qc}(Y) \to D_{qc}(X)$ and the adjunction counit $f_! f^* \to \mathrm{Id}$ is the composition

$$f_! f^*(\text{-}) = f_*(f^!(\mathcal{O}_Y) \otimes f^*(\text{-})) \xrightarrow{(2.1)} f_* f^!(\text{-}) \xrightarrow{\text{adj. counit}} \mathrm{Id}\,.$$

Finally, let X be a separated scheme of finite type over a field k and let $\pi_k\colon X \to k$ be the structure morphism. The functor $\mathbf{R}\mathcal{H}om_X\left(-, \pi_k^! k\right)$ restricts to a self-inverse category equivalence $D(X) \to D(X)^{opp}$, *the global* [2] *Grothendieck duality functor* $D_{X/k}$. For any separated finite-type map $f\colon X \to Y$ between two schemes of finite-type over k, the duality $D_{\bullet/k}$ interchanges f^* and $f^!$ [Lip09, Prop. 4.10.1]. For proper f the dual of f_* under $D_{\bullet/k}$ is f_* itself - this is precisely the sheafified Grothendieck duality isomorphism.

2.3. **Standard relations between derived functors.** There exists a number of well-known morphisms and isomorphisms between compositions of the derived functors listed in Sections 2.1 and 2.2. Here we compile for the convenience of the reader a list of such elementary relations employed throughout this paper.

For a number of these morphisms of derived functors we say below that they are compatible with the corresponding natural morphisms for sheaves. For full detail on this the reader should consult the reference we quote for each result, but roughly we mean the following. A natural transformation of compositions of derived functors

$$\mathbf{R}\, f_1 \circ \cdots \circ \mathbf{R}\, f_n \to \mathbf{R}\, g_1 \circ \cdots \circ \mathbf{R}\, g_m \tag{2.3}$$

is said to be compatible with a natural transformation of compositions of the underlying abelian category functors

$$f_1 \circ \cdots \circ f_n \to g_1 \circ \cdots \circ g_m \tag{2.4}$$

if the following diagram commutes

$$\begin{tikzcd} Q \circ f_1 \circ \cdots \circ f_n \arrow[r, "(2.4)"] \arrow[d] & Q \circ g_1 \circ \cdots \circ g_m \arrow[d] \\ \mathbf{R}\, f_1 \circ \cdots \circ \mathbf{R}\, f_n \circ Q \arrow[r, "(2.3)"] & \mathbf{R}\, g_1 \circ \cdots \circ \mathbf{R}\, g_m \circ Q \end{tikzcd} \tag{2.5}$$

where Q denotes localisation functor from each chain homotopy category to the corresponding derived category and the vertical arrows are composed from the natural transformations $Q \circ f_i \to \mathbf{R}\, f_i \circ Q$ and $Q \circ g_i \to \mathbf{R}\, g_i \circ Q$ that $\mathbf{R}\, f_i$ and $\mathbf{R}\, g_i$ come equipped with by the definition of a right derived functor. Compositions of left-derived functors are treated analogously.

(1) *Commutativity and associativity of tensor product.* Let X be a ringed space. Then for any $A, B, C \in D(\mathcal{O}_X\text{-}\mathbf{Mod})$ there exist unique natural isomorphisms

$$A \otimes B \xrightarrow{\sim} B \otimes A \tag{2.6}$$

and

$$\rho\colon (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes B \otimes C \xrightarrow{\sim} A \otimes (B \otimes C) \tag{2.7}$$

which are functorial in A, B and C and which are compatible with the corresponding natural isomorphisms for sheaves [Lip09, §2.5.7 and §2.5.9].

(2) *Sheafified $(A \otimes (-),\ \mathbf{R}\,\mathcal{H}om(A, -))$ adjunction.* Let X be a ringed space. Then for any $A, B, C \in D(\mathcal{O}_X\text{-}\mathbf{Mod})$ there exist unique natural isomorphism

$$\upsilon_A\colon\ \mathbf{R}\,\mathcal{H}om_X\,(A \otimes B, C) \xrightarrow{\sim} \mathbf{R}\,\mathcal{H}om_X\,(B, \mathbf{R}\,\mathcal{H}om_X\,(A, C)) \tag{2.8}$$

compatible with the corresponding natural isomorphism for sheaves [Lip09, Prop. 2.6.1].

Applying the derived global sections functor to (2.8) produces the adjunction isomorphism for the pair $(A \otimes -, \mathbf{R}\,\mathcal{H}om_X(A, -))$. We call its counit the *evaluation map* of A and denote it by

$$\mathrm{ev}_A\colon\ A \otimes \mathbf{R}\,\mathcal{H}om_X(A, -) \to \mathrm{Id}\,. \tag{2.9}$$

An important instance is the morphism $A \otimes A^\vee \xrightarrow{\mathrm{ev}_A} \mathcal{O}_X$ obtained by applying ev_A to $\mathcal{O}_X$.

[2] I.e. over a point. One can obtain duality theories on X relative to any separated, finite-type map $\pi_S\colon X \to S$ with S noetherian, but only after restricting to objects of $D(X)$ perfect over S (see [Ill71a], Cor. 4.9.2 etc.). Since the objects perfect over a point are precisely the complexes with bounded and coherent cohomologies, the global duality works for all of $D(X)$.

(3) *Perfect objects and* $\mathbf{R}\mathcal{H}om$. Let X be a ringed space. For any $A, B, C \in D(\mathcal{O}_X\text{-}\mathbf{Mod})$ define

$$\xi\colon \mathbf{R}\mathcal{H}om_X(A,B)\otimes C \longrightarrow \mathbf{R}\mathcal{H}om_X(A, B\otimes C) \tag{2.10}$$

to be the right adjoint with respect to $A \otimes (-)$ of the composition

$$A\otimes(\mathbf{R}\mathcal{H}om_X(A,B)\otimes C) \xrightarrow{\rho^{-1}} (A\otimes \mathbf{R}\mathcal{H}om_X(A,B))\otimes C \xrightarrow{\mathrm{ev}_A} B\otimes C. \tag{2.11}$$

If either of C or A belong to $D_{\mathrm{perf}}(X)$, then ξ is an isomorphism [AIL10, Lemma 1.4.6]. In particular, for any $E \in D_{\mathrm{perf}}(X)$ we have an isomorphism

$$\xi_E\colon E^{\vee}\otimes(-) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_X(E,-) \tag{2.12}$$

of functors $D(\mathcal{O}_X\text{-}\mathbf{Mod}) \to D(\mathcal{O}_X\text{-}\mathbf{Mod})$.

The adjunction $(E\otimes -, \mathbf{R}\mathcal{H}om_X(E,-))$ induces via ξ_E an adjunction $(E\otimes -, E^{\vee}\otimes -)$ whose adjunction co-unit we also denote by ev_E:

$$E\otimes(E^{\vee}\otimes -) \xrightarrow{\xi_E} E\otimes \mathbf{R}\mathcal{H}om_X(E,-) \xrightarrow{\mathrm{ev}_E} \mathrm{Id}\,. \tag{2.13}$$

(4) $\mathcal{O}_X$*-reflexivity for perfect objects.* Let X be a ringed space. For any $A, B \in D(\mathcal{O}_X\text{-}\mathbf{Mod})$ define

$$\theta_{A,B}\colon A \longrightarrow \mathbf{R}\mathcal{H}om_X(\mathbf{R}\mathcal{H}om_X(A,B), B) \tag{2.14}$$

to be the right adjoint with respect to $\mathbf{R}\mathcal{H}om_X(A,B)\otimes(-)$ of

$$A\otimes \mathbf{R}\mathcal{H}om_X(A,B) \xrightarrow{\mathrm{ev}_A} B.$$

If $B = \mathcal{O}_X$ the resulting morphism

$$\theta_A\colon A \to A^{\vee\vee} \tag{2.15}$$

an isomorphism for all $A \in D_{\mathrm{perf}}(X)$ [AIL10, Prop 1.4.4].

Let $E \in D_{\mathrm{perf}}$. The adjunction $(E^{\vee}\otimes -, E^{\vee\vee}\otimes -)$ induces via the isomorphism $E \xrightarrow{\theta_E} E^{\vee\vee}$ an adjunction $(E^{\vee}\otimes -, E\otimes -)$ whose adjunction co-unit we denote by $\mathrm{ev}_{E^{\vee}}$:

$$E^{\vee}\otimes(E\otimes -) \xrightarrow{\theta_E} E^{\vee}\otimes(E^{\vee\vee}\otimes -) \xrightarrow{\mathrm{ev}_{E^{\vee}}} \mathrm{Id}\,. \tag{2.16}$$

(5) *Pseudofunctoriality of direct and inverse image.* Let X, Y, Z be ringed spaces and $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps of ringed spaces. There exist unique isomorphisms

$$\eta_{g,f}\colon (g\circ f)_* \xrightarrow{\sim} g_* f_* \quad \text{of functors } D(\mathcal{O}_X\text{-}\mathbf{Mod}) \to D(\mathcal{O}_Z\text{-}\mathbf{Mod}) \tag{2.17}$$

and

$$\zeta_{g,f}\colon f^* g^* \xrightarrow{\sim} (g\circ f)^* \quad \text{of functors } D(\mathcal{O}_Z\text{-}\mathbf{Mod}) \to D(\mathcal{O}_X\text{-}\mathbf{Mod}) \tag{2.18}$$

which are compatible with the corresponding natural isomorphisms for sheaves. These isomorphisms give $(-)_*$ and $(-)^*$ the structures of a covariant and a contravariant pseudofunctor over the category of ringed spaces [Lip09, §3.6]. Specifically, for any map $X \xrightarrow{f} Y$ of ringed spaces we have

$$\eta_{\mathrm{Id},f} = \eta_{f,\mathrm{Id}} = \mathrm{Id} \qquad \text{and} \qquad \zeta_{\mathrm{Id},f} = \zeta_{f,\mathrm{Id}} = \mathrm{Id} \tag{2.19}$$

and for any maps $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ of ringed spaces the following diagrams commute

$$\begin{tikzcd} (h\circ g\circ f)_* \arrow[r, "\eta_{h\circ g, f}"] \arrow[d, "\eta_{h, g\circ f}"'] & (h\circ g)_* f_* \arrow[d, "\eta_{h,g}"] \\ h_*(g\circ f)_* \arrow[r, "h_*\eta_{g,f}"'] & h_* g_* f_* \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} f^*g^*h^* \arrow[r, "f^*\zeta_{h,g}"] \arrow[d, "\zeta_{g,f}"'] & f^*(h\circ g)^* \arrow[d, "\zeta_{h\circ g, f}"] \\ (g\circ f)^* h^* \arrow[r, "\eta_{h, g\circ f}"'] & (h\circ g\circ f)^* \end{tikzcd} \;. \tag{2.20}$$

We write $\eta_{h,g,f}$ for the corresponding isomorphism $(h\circ g\circ f)_* \xrightarrow{\sim} h_* g_* f_*$ and $\zeta_{h,g,f}$ for the corresponding isomorphism $f^*g^*h^* \xrightarrow{\sim} (h\circ g\circ f)^*$.

(6) *Sheafified (f^*, f_*) adjunction.* Let X, Y be ringed spaces and let $X \xrightarrow{f} Y$ be a map of ringed spaces. For any $A \in D(\mathcal{O}_Y\text{-}\mathbf{Mod})$ and $B \in D(\mathcal{O}_X\text{-}\mathbf{Mod})$ there exists a unique bifunctorial isomorphism

$$\tau_f \colon f_* \mathbf{R}\mathcal{H}om_X(f^*A, B) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_Y(A, f_*B) \tag{2.21}$$

compatible with the corresponding natural isomorphism for sheaves [Lip09, Prop. 3.2.3].

Applying the derived global sections functor to (2.21) produces an adjunction isomorphism for the pair (f^*, f_*). We denote its unit and counit by

$$\beta_f \colon \mathrm{Id} \to f_* f^* \quad \text{and} \quad \gamma_f \colon f^* f_* \to \mathrm{Id}\,. \tag{2.22}$$

The adjunction (f^*, f_*) is compatible with pseudofunctoriality in the following sense. Let $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ be maps of ringed spaces, then the following diagrams commute:

$$\begin{tikzcd} \mathrm{Id} \arrow[r, "\beta_g"] \arrow[drr, "\beta_{g\circ f}"'] & g_* g^* \arrow[r, "g_*\beta_f"] & g_* f_* f^* g^* \arrow[d, "\eta^{-1}_{g,f}\circ(g_* f_* \zeta_{g,f})"] \\ & & (g\circ f)_* (g\circ f)^* \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} f^* g^* g_* f_* \arrow[r, "f^*\gamma_g"] \arrow[d, "\zeta_{g,f}\circ(f^* g^* \eta^{-1}_{g,f})"'] & f^* f_* \arrow[r, "\gamma_f"] & \mathrm{Id}, \\ (g\circ f)^* (g\circ f)_* \arrow[urr, "\gamma_{g\circ f}"'] & & \end{tikzcd} \tag{2.23}$$

see [Lip09, §3.6] for more details.

(7) *Monoidal functor structure for inverse image.* Let X, Y be ringed spaces and let $X \xrightarrow{f} Y$ be a map of ringed spaces. For any $A, B \in D(\mathcal{O}_Y\text{-}\mathbf{Mod})$ there exists a unique isomorphism

$$\nu_f \colon f^*(A \otimes B) \xrightarrow{\sim} f^*(A) \otimes f^* B \tag{2.24}$$

functorial in A and B which is compatible with the corresponding natural isomorphism for sheaves [Lip09, Prop. 3.2.4(i)]. It is worth noting that as a natural transformation of functors in B isomorphism ν_f is conjugate to τ_f in sense of [Mac98, §IV.7].

Map ν_f is compatible with the associativity of the tensor product in the following sense. Let $X \xrightarrow{f} Y$ be a map of ringed spaces. Then the following diagram

$$\begin{tikzcd} f^*\left((A\otimes B)\otimes C\right) \arrow[r, "\nu_f"] \arrow[d, "f^*\rho"'] & f^*\left(A\otimes B\right)\otimes f^*C \arrow[r, "\nu_f\otimes\mathrm{Id}"] & (f^*A\otimes f^*B)\otimes f^*C \arrow[d, "\rho"'] \\ f^*\left(A\otimes (B\otimes C)\right) \arrow[r, "\nu_f"'] & f^*A\otimes f^*\left(B\otimes C\right) \arrow[r, "\mathrm{Id}\otimes\nu_f"'] & f^*A\otimes (f^*B\otimes f^*C) \end{tikzcd} \tag{2.25}$$

commutes for any $A, B, C \in D(\mathcal{O}_Y\text{-}\mathbf{Mod})$ [Lip09, §3.4].

Map ν_f is compatible with pseudofunctoriality in the following sense. Let $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ be maps of ringed spaces. Then the following diagram commutes

$$\begin{tikzcd} f^*g^*\left(A\otimes B\right) \arrow[r, "f^*\nu_g"] \arrow[d, "\zeta_{g,f}"'] & f^*\left(g^*A\otimes g^*B\right) \arrow[r, "\nu_f"] & f^*g^*A\otimes f^*g^*B \arrow[d, "\zeta_{g,f}\otimes\zeta_{g,f}"] \\ (g\circ f)^*\left(A\otimes B\right) \arrow[rr, "\nu_{g\circ f}"'] & & (g\circ f)^*A\otimes (g\circ f)^*B \end{tikzcd} \tag{2.26}$$

for all $A, B \in D(\mathcal{O}_Z\text{-}\mathbf{Mod})$ [Lip09, §3.6].

(8) *Monoidal functor structure for direct image.* Let X, Y be ringed spaces and let $X \xrightarrow{f} Y$ be a map of ringed spaces. For any $A, B \in D(\mathcal{O}_X\text{-}\mathbf{Mod})$ define morphism

$$\kappa_f \colon f_*A \otimes f_*B \to f_*(A \otimes B), \tag{2.27}$$

functorial in A and B, to be the right adjoint with respect to f^* of the composition

$$f^*(f_*A \otimes f_*B) \xrightarrow{\nu_f} f^*f_*A \otimes f^*f_*B \xrightarrow{\gamma_f\otimes\gamma_f} A \otimes B.$$

Map κ_f is compatible with the associativity of the tensor product and with pseudofunctoriality in a way analogous to map ν_f [Lip09, §3.4 and §3.6].

(9) *Projection formula.* Let X, Y be ringed spaces and let $X \xrightarrow{f} Y$ be a map of ringed spaces. For any $A \in D(\mathcal{O}_X\text{-}\mathbf{Mod})$ and $B \in D(\mathcal{O}_Y\text{-}\mathbf{Mod})$ define the projection formula morphism

$$\alpha_f \colon f_* A \otimes B \to f_*(A \otimes f^* B) \tag{2.28}$$

to be the right adjoint with respect to f^* of the composition

$$f^*(f_* A \otimes B) \xrightarrow{\nu_f} f^* f_* A \otimes f^* B \xrightarrow{\gamma_f \otimes \mathrm{Id}} A \otimes f^* B.$$

If X and Y are concentrated schemes, then α_f is an isomorphism for all $A \in D_{qc}(X)$ and $B \in D_{qc}(Y)$ [Lip09, Prop. 3.9.4].

The projection formula is compatible with pseudofunctoriality in the following sense. Let $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ be maps of ringed spaces. Then the following diagram

$$\begin{tikzcd}
A \otimes g_* f_* B \arrow[r, "\alpha_g"] & g_*\left(g^* A \otimes f_* B\right) \arrow[r, "g_* \alpha_f"] & g_* f_*\left(f^* g^* A \otimes B\right) \arrow[d, "\simeq"', "g_* f_*(\zeta_{f,g} \otimes \mathrm{Id})"] \\
A \otimes (g \circ f)_* B \arrow[u, "\mathrm{Id} \otimes \eta_{f,g}", "\simeq"'] \arrow[r, "\alpha_{g \circ f}"'] & (g \circ f)_*\left((g \circ f)^* A \otimes B\right) \arrow[r, "\simeq", "\eta_{g \circ f}"'] & g_* f_*\left((g \circ f)^* A \otimes B\right)
\end{tikzcd} \tag{2.29}$$

commutes for any $A \in D(\mathcal{O}_Z\text{-}\mathbf{Mod})$ and $B \in D(\mathcal{O}_X\text{-}\mathbf{Mod})$ [Lip09, Prop. 3.7.1].

(10) *The sheafified Grothendieck duality morphism.* Let X and Y be concentrated schemes and let $X \xrightarrow{f} Y$ be a map of schemes. Denote the unit and counit of the $(f_*, f^\times)$ adjunction by

$$\lambda_f \colon \mathrm{Id} \to f^\times f_* \quad \text{and} \quad \epsilon_f \colon f_* f^\times \to \mathrm{Id}\,. \tag{2.30}$$

The $(f_*, f^\times)$ adjunction is compatible with pseudofunctoriality, in the sense that the analogues of diagrams (2.23) for δ_f and λ_f also commute, see [Lip09, Cor. 4.1.2] for more details.

Define for any $A \in D_{qc}(X)$ and $B \in D_{qc}(Y)$ *the sheafified Grothendieck duality morphism*

$$\delta_f \colon f_* \mathbf{R}\mathcal{H}om_X(A, f^\times B) \to \mathbf{R}\mathcal{H}om_Y(f_* A, B) \tag{2.31}$$

to be the composition

$$f_* \mathbf{R}\mathcal{H}om_X(A, f^\times B) \xrightarrow{\gamma_f} f_* \mathbf{R}\mathcal{H}om_X\left(f^* f_* A, f^\times B\right) \xrightarrow{\tau_f} \mathbf{R}\mathcal{H}om\left(f_* A, f_* f^\times B\right) \xrightarrow{\epsilon_f} \mathbf{R}\mathcal{H}om\left(f_* A, B\right).$$

When X and Y are Noetherian and f is proper δ_f is an isomorphism for all $A \in D_{\mathrm{qc}}(X)$ and $B \in D^+_{\mathrm{qc}}(Y)$ [Lip09, Theorem. 4.4.1]. If, in addition to the above, f is perfect, δ_f is an isomorphism for all $A \in D_{\mathrm{qc}}(X)$ and $B \in D_{\mathrm{qc}}(X)$ [Lip09, Theorem 4.7.4].

(11) Let X, Y be concentrated schemes and let $X \xrightarrow{f} Y$ be a map of schemes. For any $A \in D_{qc}(X)$ and $B \in D_{qc}(Y)$ define morphism

$$\chi_f \colon f^\times A \otimes f^* B \to f^\times(A \otimes B) \tag{2.32}$$

functorial in A and B to be the right adjoint with respect to f_* of the composition

$$f_*(f^\times A \otimes f^* B) \xrightarrow{\alpha_f^{-1}} f_* f^\times A \otimes B \xrightarrow{\epsilon_f \otimes \mathrm{Id}} A \otimes B$$

where α_f^{-1} is the inverse of the projection formula isomorphism. When f is proper and perfect χ_f is an isomorphism [Lip09, Exercise 4.7.3.4(a)].

(12) *Base change.* Let σ be a commutative square

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"'] & Y
\end{tikzcd} \tag{2.33}$$

of ringed spaces. We define the base change morphism

$$\mu_\sigma \colon g^* f_* \to f'_* g'^* \tag{2.34}$$

to be the right adjoint with respect to f'^* of the composition

$$f'^*g^*f_* \xrightarrow{\zeta_{g,f'}} (g\circ f')^*f_* = (f\circ g')^*f_* \xrightarrow{\zeta^{-1}_{f,g'}} g'^*f^*f_* \xrightarrow{\gamma_f} g'^*$$

or, equivalently [Lip09, Prop. 3.7.2], the left adjoint with respect to g_* of the composition

$$f_* \xrightarrow{\beta_{g'}} f_*g'_*g'^* \xrightarrow{\eta^{-1}_{f,g'}} (f\circ g')_*g'^* = (g\circ f')_*g'^* \xrightarrow{\eta_{g,f'}} g_*f'_*g'^*.$$

This defines μ_σ as a morphism of functors $D(\mathcal{O}_X\text{-}\mathbf{Mod}) \to D(\mathcal{O}'_Y\text{-}\mathbf{Mod})$. When σ is a square of concentrated schemes the base change map restricts to a morphism of functors $D_{qc}(X) \to D_{qc}(Y')$.

We use σ^T to denote the transposed square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd} \tag{2.35}$$

In particular, we denote by μ_{σ^T} the base change map $f^*g_* \to f'^*g'_*$ for σ^T.

If the restriction of μ_σ to complexes with quasi-coherent cohomology is an isomorphism, then σ is said to be *independent*. A fiber-square of concentrated schemes is independent if and only if it is Tor-*independent*, i.e. for any $x \in X$ and $y' \in Y'$ such that $f(x) = g(y') = y \in Y$ we have

$$\mathrm{Tor}^i_{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y'}) = 0 \quad \text{for all } i > 0, \tag{2.36}$$

cf. [Lip09, Theorem 3.10.3]. In particular, a fiber-square of concentrated schemes is independent if f or g are flat. Another good reference for the above material is [Kuz06, §2.4], where the proofs are carried out via computations with the underlying Fourier-Mukai kernels.

2.4. **Further relations.** To prove our main results in Section 3 we need three technical results which we could not find in the literature. The first two state that the projection formula commutes with certain adjunction units and counits of the direct image functor.

Lemma 2.1. *Let $X \xrightarrow{g} Y \xrightarrow{f} Z$ be maps of ringed spaces. Let $A \in D(\mathcal{O}_Y\text{-}\mathbf{Mod})$ and $B \in D(\mathcal{O}_Z\text{-}\mathbf{Mod})$. Then the following diagram commutes:*

$$\begin{tikzcd} f_*A\otimes B \arrow[rr, "f_*\beta_g\otimes\mathrm{Id}"] \arrow[d, "\alpha_f"'] & & f_*g_*g^*A\otimes B \arrow[d, "f_*\alpha_g\circ\alpha_f"] \\ f_*\left(A\otimes f^*B\right) \arrow[r, "f_*\beta_g"'] & f_*g_*g^*\left(A\otimes f^*B\right) \arrow[r, "f_*g_*\nu_g"'] & f_*g_*\left(g^*A\otimes g^*f^*B\right). \end{tikzcd} \tag{2.37}$$

Proof. By functoriality of α_f it suffices to show that the square

$$\begin{tikzcd} f_*\left(A\otimes f^*B\right) \arrow[r, "f_*(\beta_g\otimes f^*\,\mathrm{Id})"] \arrow[d, "f_*\beta_g"'] & f_*\left(g_*g^*A\otimes f^*B\right) \arrow[d, "f_*\alpha_g"] \\ f_*g_*g^*\left(A\otimes f^*B\right) \arrow[r, "f_*g_*\nu_g"'] & f_*g_*\left(g^*A\otimes g^*f^*B\right) \end{tikzcd}$$

commutes. This square is the image under f_* of the square

$$\begin{tikzcd} A\otimes f^*B \arrow[r, "\beta_g\otimes\mathrm{Id}"] \arrow[d, "\beta_g"'] & g_*g^*A\otimes f^*B \arrow[d, "\alpha_g"] \\ g_*g^*\left(A\otimes f^*B\right) \arrow[r, "g_*\nu_g"'] & g_*\left(g^*A\otimes g^*f^*B\right). \end{tikzcd} \tag{2.38}$$

To show that (2.38) commutes we show that its left adjoint with respect to g_* commutes. By definition of α_g its left adjoint with respect to g_* is $(\gamma_g \otimes \mathrm{Id}) \circ \nu_g$. So the left adjoint with respect to g_* of (2.38) is

$$\begin{tikzcd}
g^*(A \otimes f^*B) \arrow[r, "g^*(\beta_g \otimes \mathrm{Id})"] \arrow[drr, "\nu_g"'] & g^*(g_*g^*A \otimes f^*B) \arrow[r, "\nu_g"] & g^*g_*g^*A \otimes g^*f^*B \arrow[d, "\gamma_g \otimes \mathrm{Id}"] \\
& & g^*A \otimes g^*f^*B
\end{tikzcd}$$

and by functoriality of ν_g it suffices to show that the following composition is the identity morphism:

$$g^*A \otimes g^*f^*B \xrightarrow{g^*\beta_g \otimes \mathrm{Id}} g^*g_*g^*A \otimes g^*f^*B \xrightarrow{\gamma_g \otimes \mathrm{Id}} g^*A \otimes g^*f^*B.$$

Rewrite it as $(g^*\beta_g \circ \gamma_g) \otimes \mathrm{Id}$. Since β_g and γ_g are the unit and the counit of the adjunction (g^*, g_*), the morphism $g^*A \xrightarrow{g^*\beta_g \circ \gamma_g} g^*A$ is the identity morphism. The result follows. $\square$

Lemma 2.2. *Let X, Y, Z be concentrated schemes and $X \xrightarrow{g} Y \xrightarrow{f} Z$ be scheme maps. Let $A \in D_{qc}(Y)$ and $B \in D_{qc}(Z)$. Then the following diagram commutes:*

$$\begin{tikzcd}
f_*g_*g^\times A \otimes B \arrow[rr, "f_*\epsilon_g \otimes \mathrm{Id}"] \arrow[d, "f_*\alpha_g \circ \alpha_f"'] & & f_*A \otimes B \arrow[d, "\alpha_f"] \\
f_*g_*(g^\times A \otimes g^*f^*B) \arrow[r, "f_*g_*\chi_g"'] & f_*g_*g^\times(A \otimes f^*B) \arrow[r, "f_*\epsilon_g"'] & f_*(A \otimes f^*B).
\end{tikzcd} \tag{2.39}$$

Proof. The proof is analogous to that of Lemma 2.1. By functoriality of α_f it suffices to show that the image under f_* of

$$\begin{tikzcd}
g_*g^\times A \otimes f^*B \arrow[d, "\alpha_g"'] \arrow[drr, "\epsilon_g \otimes \mathrm{Id}"] & & \\
g_*(g^\times A \otimes g^*f^*B) \arrow[r, "g_*\chi_g"'] & g_*g^\times(A \otimes f^*B) \arrow[r, "\epsilon_g"'] & A \otimes f^*B.
\end{tikzcd}$$

commutes. Since α_g is an isomorphism, this is equivalent to the diagram

$$\begin{tikzcd}
g_*g^\times A \otimes f^*B \arrow[drr, "\epsilon_g \otimes \mathrm{Id}"] & & \\
g_*(g^\times A \otimes g^*f^*B) \arrow[u, "\alpha_g^{-1}"] \arrow[r, "g_*\chi_g"'] & g_*g^\times(A \otimes f^*B) \arrow[r, "\epsilon_g"'] & A \otimes f^*B.
\end{tikzcd}$$

commuting. But as ϵ_g is the adjunction counit, the composition $\epsilon_g \circ g_*\chi_g$ is the left adjoint of χ_g with respect to $g^\times$. By the definition of χ_g this left adjoint is precisely $(\epsilon_g \otimes \mathrm{Id}) \circ \alpha_g^{-1}$. The result follows. $\square$

The third result shows that for a perfect object E the adjunction co-units for $E \otimes (-)$ commute with the associativity of the tensor product:

Lemma 2.3. *Let X be a ringed space. Then for any $A \in D(\mathcal{O}_X\text{-}\mathbf{Mod})$ and $E \in D_{perf}(X)$ the following diagrams commute*

$$\begin{tikzcd}
E \otimes (E^\vee \otimes A) \arrow[r, "\mathrm{ev}_E"] \arrow[d, "\rho^{-1}"'] & A \arrow[d, "\mathrm{Id}"] \\
(E \otimes E^\vee) \otimes A \arrow[r, "\mathrm{ev}_E(\mathcal{O}_X) \otimes \mathrm{Id}"'] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(E^\vee \otimes E) \otimes A \arrow[r, "\mathrm{ev}_E(\mathcal{O}_X) \otimes \mathrm{Id}"] \arrow[d, "\rho"'] & A \arrow[d, "\mathrm{Id}"] \\
E^\vee \otimes (E \otimes A) \arrow[r, "\mathrm{ev}_{E^\vee}"'] & A.
\end{tikzcd} \tag{2.40}$$

Proof. The adjunction counit $E \otimes (E^\vee \otimes A) \xrightarrow{\mathrm{ev}_E} A$ was defined as the composition

$$E \otimes (E^\vee \otimes A) \xrightarrow[\sim]{\mathrm{Id} \otimes \xi_E} E \otimes \mathbf{R}\mathcal{H}om(E, A) \xrightarrow{\mathrm{ev}_E} A\,.$$

Therefore its right adjoint with respect to $E \otimes (-)$ is isomorphism ξ_E. But isomorphism ξ_E was defined to be the right adjoint with respect to $E \otimes (-)$ of the composition

$$E \otimes (E^\vee \otimes A) \xrightarrow{\rho^{-1}} (E \otimes E^\vee) \otimes A \xrightarrow{\mathrm{ev}_E(\mathcal{O}_X)\otimes \mathrm{Id}} A.$$

Therefore the left diagram commutes.

For the right diagram, recall that by its definition the adjunction co-unit $E^\vee \otimes (E \otimes A) \xrightarrow{\mathrm{ev}_{E^\vee}} \mathrm{Id}$ is

$$E^\vee \otimes (E \otimes A) \xrightarrow[\sim]{(\mathrm{Id} \otimes \theta_E)\otimes \mathrm{Id}} E^\vee \otimes (E^{\vee\vee} \otimes A) \xrightarrow{\mathrm{ev}_{E^\vee}} A\,.$$

Since the left diagram commutes and ρ is functorial, we can rewrite the composition above as

$$E^\vee \otimes (E \otimes A) \xrightarrow{\rho^{-1}} (E^\vee \otimes E) \otimes A \xrightarrow[\sim]{\theta_E} (E^\vee \otimes E^{\vee\vee}) \otimes A \xrightarrow{\mathrm{ev}_{E^\vee}(\mathcal{O}_X)} A\,.$$

To show that the right diagram commutes it now remains only to show that

$$E^\vee \otimes E \xrightarrow[\sim]{\mathrm{Id} \otimes \theta_E} E^\vee \otimes E^{\vee\vee} \xrightarrow{\mathrm{ev}_{E^\vee}(\mathcal{O}_X)} \mathcal{O}_X$$

is the map $E^\vee \otimes E \xrightarrow{\mathrm{ev}_E(\mathcal{O}_X)} \mathcal{O}_X$. The right adjoint of the composition above with respect to $E^\vee \otimes (-)$ is just the map $E \xrightarrow{\theta_E} E^{\vee\vee}$. But θ_E was defined as the right adjoint with respect to $E^\vee \otimes (-)$ of $E^\vee \otimes E \xrightarrow{\mathrm{ev}_E(\mathcal{O}_X)} \mathcal{O}_X$. The claim follows. $\square$

Define a morphism

$$\mathrm{ev}_E \colon E^\vee \otimes E \otimes (-) \longrightarrow \mathrm{Id} \tag{2.41}$$

to be the composition

$$E^\vee \otimes E \otimes (-) \xrightarrow[\sim]{\rho} (E^\vee \otimes E) \otimes (-) \xrightarrow{\mathrm{ev}_E(\mathcal{O}_X)\otimes \mathrm{Id}} \mathrm{Id}\,.$$

By Lemma 2.3 the canonical isomorphisms identifying $E^\vee \otimes E \otimes -$ with $E^\vee \otimes (E \otimes -)$ and $E \otimes (E^\vee \otimes -)$ identify (2.41) with the adjunction counits for the adjunctions $(E^\vee \otimes -, E \otimes -)$ and $(E \otimes -, E^\vee \otimes -)$, respectively. We thus abuse notation by speaking of (2.41) as "the adjunction counit" for these two adjunctions.

3. Adjunction morphisms for Fourier-Mukai transforms

3.1. **Compact case.** Let X_1 and X_2 be a pair of separable schemes of finite type over an algebraically closed field k of characteristic 0 with X_2 proper. We have the following commutative diagram

$$\begin{tikzcd}
 & & X_1 \times X_2 \times X_1 \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & & \\
 & X_1 \times X_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & X_1 \times X_1 \arrow[dll, "\tilde{\pi}_1"] \arrow[drr, "\tilde{\pi}_2"'] & X_2 \times X_1 \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\
X_1 & & X_2 & & X_1
\end{tikzcd} \tag{3.1}$$

All the morphisms in it are separated and of finite-type. They are also flat, and therefore perfect. Moreover, morphisms π_1 and π_{13} are proper.

Definition 3.1. Let E be a perfect object of $D(X_1 \times X_2)$. The `Fourier-Mukai transform` Φ_E `from` X_1 `to` X_2 `with kernel` E is the functor $D_{qc}(X_1) \to D_{qc}(X_2)$ given by

$$\Phi_E(-) = \pi_{2*}\left(E \otimes \pi_1^*(-)\right).$$

By the adjunctions described in Section 2.2 functor Φ_E has both left and right adjoints. The left adjoint Φ_E^{ladj} is isomorphic to the Fourier-Mukai transform from X_2 to X_1 with kernel $E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})$. The composition $\Phi_E^{\mathrm{ladj}}\Phi_E$ is then isomorphic [Muk81, Prop 1.3] to the Fourer-Mukai transform from X_1 to X_1 with kernel

$$Q = \pi_{13*}\left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^*\pi_1^!(\mathcal{O}_{X_1})\right).$$

Let now Δ denote the diagonal inclusion $X_1 \hookrightarrow X_1 \times X_1$ and, by abuse of notation, let it also denote the induced inclusion $X_1 \times X_2 \hookrightarrow X_1 \times X_2 \times X_1$, so that there is the following fiber square:

$$\begin{tikzcd} X_1 \times X_2 \arrow[r, hook, "\Delta"] \arrow[d, "\pi_1"'] & X_1 \times X_2 \times X_1 \arrow[d, "\pi_{13}"] \\ X_1 \arrow[r, hook, "\Delta"'] & X_1 \times X_1 \end{tikzcd} \tag{3.2}$$

The identity functor Id is isomorphic to the Fourier-Mukai transform from X_1 to X_1 with kernel $\Delta_* \mathcal{O}_{X_1}$. We now state the main result of this section:

Theorem 3.1. *Let X_1 and X_2 be two separable schemes of finite type over k with X_2 proper. Let E be a perfect object of $D(X_1 \times X_2)$ and Φ_E be a Fourier-Mukai transform from $D_{qc}(X_1)$ to $D_{qc}(X_2)$ defined by E.*

The adjunction counit $\gamma_E\colon \Phi_E^{ladj}\Phi_E \to \mathrm{Id}$ is isomorphic to the morphism of Fourier-Mukai transforms $D_{qc}(X_1) \to D_{qc}(X_1)$ induced by the following morphism of their kernels:

$$Q = \pi_{13*}\left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^*\pi_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{\pi_{13*}\beta_\Delta} \pi_{13*}\Delta_*\Delta^*\left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^*\pi_1^!(\mathcal{O}_{X_1})\right) \tag{3.3}$$

$$\pi_{13*}\Delta_*\Delta^*\left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^*\pi_1^!(\mathcal{O}_{X_1})\right) \simeq \Delta_*\pi_{1*}\left(E \otimes E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})\right) \tag{3.4}$$

$$\Delta_*\pi_{1*}\left(E \otimes E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{\Delta_*\pi_{1*}\,\mathrm{ev}_E} \Delta_*\pi_{1*}\left(\pi_1^!(\mathcal{O}_{X_1})\right) \tag{3.5}$$

$$\Delta_*\pi_{1*}\pi_1^!(\mathcal{O}_{X_1}) \xrightarrow{\Delta_*\epsilon_{\pi_1}} \Delta_*\mathcal{O}_{X_1} \tag{3.6}$$

where (3.4) is composed of isomorphism $\nu_\Delta\colon \Delta^*(-\otimes-) \xrightarrow{\sim} \Delta^*(-)\otimes\Delta^*(-)$ and of pseudofunctoriality isomorphisms corresponding to the identities $\pi_{13}\circ\Delta = \Delta\circ\pi_1$ and $\pi_{12}\circ\Delta = \pi_{23}\circ\Delta = \mathrm{Id}$.

We first need the following crucial lemma:

Lemma 3.2. *Let σ be the fiber square*

$$\begin{tikzcd} X_1 \times X_2 \times X_1 \arrow[r, "\pi_{12}"] \arrow[d, "\pi_{23}"'] & X_1 \times X_2 \arrow[d, "\pi_2"] \\ X_1 \times X_2 \arrow[r, "\pi_2"'] & X_2. \end{tikzcd} \tag{3.7}$$

Then the following diagram of functors commutes:

$$\begin{tikzcd} \pi_2^*\pi_{2*} \arrow[r, "\gamma_{\pi_2}"] \arrow[d, "\mu_\sigma"', "\simeq"] & \mathrm{Id} \arrow[d, "\eta_{\pi_{23},\Delta}\circ\zeta^{-1}_{\pi_{12},\Delta}", "\simeq"'] \\ \pi_{23*}\pi_{12}^* \arrow[r, "\pi_{23*}\beta_\Delta"'] & \pi_{23*}\Delta_*\Delta^*\pi_{12}^*. \end{tikzcd} \tag{3.8}$$

Proof. It suffices to show that the right adjoints with respect to π_2^* of the composition

$$\pi_2^*\pi_{2*} \xrightarrow{\mu} \pi_{23*}\pi_{12}^* \xrightarrow{\pi_{23*}\beta_\Delta} \pi_{23*}\Delta_*\Delta^*\pi_{12}^* \tag{3.9}$$

and of the composition

$$\pi_2^*\pi_{2*} \xrightarrow{\gamma_{\pi_2}} \mathrm{Id} \xrightarrow{\eta_{\pi_{23},\Delta}\circ\zeta^{-1}_{\pi_{12},\Delta}} \pi_{23*}\Delta_*\Delta^*\pi_{12}^* \tag{3.10}$$

coincide. By the definition of morphism μ_σ the right adjoint with respect to π_2^* of (3.9) is

$$\pi_{2*} \xrightarrow{\pi_{2*}\beta_{\pi_{12}}} \pi_{2*}\pi_{12*}\pi_{12}^* \xrightarrow{\eta_{\pi_2,\pi_{23}}\circ\eta^{-1}_{\pi_2,\pi_{12}}} \pi_{2*}\pi_{23*}\pi_{12}^* \xrightarrow{\pi_{2*}\pi_{23*}\beta_\Delta} \pi_{2*}\pi_{23*}\Delta_*\Delta^*\pi_{12}^*$$

which by functoriality of $\eta_{\pi_2,\pi_{23}} \circ \eta^{-1}_{\pi_2,\pi_{12}}$ is the same as

$$\pi_{2*} \xrightarrow{\pi_{2*}\beta_{\pi_{12}}} \pi_{2*}\pi_{12*}\pi_{12}^* \xrightarrow{\pi_{2*}\pi_{12*}\beta_\Delta} \pi_{2*}\pi_{12*}\Delta_*\Delta^*\pi_{12}^* \xrightarrow{\eta_{\pi_2,\pi_{23}}\circ\eta^{-1}_{\pi_2,\pi_{12}}} \pi_{2*}\pi_{23*}\Delta_*\Delta^*\pi_{12}^*. \tag{3.11}$$

By pseudofunctoriality of the direct image, cf. (2.20), the morphism of functors

$$\pi_{2*}\pi_{12*}\Delta_* \xrightarrow{\eta_{\pi_2,\pi_{23}}\circ\eta^{-1}_{\pi_2,\pi_{12}}} \pi_{2*}\pi_{23*}\Delta_*$$

is the same as the morphism of functors

$$\pi_{2*}\pi_{12*}\Delta_* \xrightarrow{\pi_{2*}(\eta_{\pi_{23},\Delta}\circ\eta^{-1}_{\pi_{12},\Delta})} \pi_{2*}\pi_{23*}\Delta_*$$

and we can therefore rewrite (3.11) as

$$\pi_{2*}\left(\mathrm{Id} \xrightarrow{\beta_{\pi_{12}}} \pi_{12*}\pi_{12}^* \xrightarrow{\pi_{12*}\beta_\Delta} \pi_{12*}\Delta_*\Delta^*\pi_{12}^* \xrightarrow{\eta_{\pi_{23},\Delta}\circ\eta^{-1}_{\pi_{12},\pi_\Delta}} \pi_{23*}\Delta_*\Delta^*\pi_{12}^*\right). \tag{3.12}$$

By the compatibility of β with pseudofunctoriality as per diagram (2.23) we can rewrite (3.12) as

$$\pi_{2*}\left(\mathrm{Id} \xrightarrow{\beta_{\pi_{12}\circ\Delta}} (\pi_{12}\circ\Delta)_*(\pi_{12}\circ\Delta)^* \xrightarrow{\eta_{\pi_{12},\Delta}\circ\zeta^{-1}_{\pi_{12},\Delta}} \pi_{12*}\Delta_*\Delta^*\pi_{12}^* \xrightarrow{\eta_{\pi_{23},\Delta}\circ\eta^{-1}_{\pi_{12},\Delta}} \pi_{23*}\Delta_*\Delta^*\pi_{12}^*\right).$$

Cancelling out $\eta^{-1}_{\pi_{12},\Delta} \circ \eta_{\pi_{12},\Delta}$ and noting that $\beta_{\pi_{12}\circ\Delta} = \mathrm{Id}$ since $\pi_{12}\circ\Delta = \mathrm{Id}$ yields

$$\pi_{2*}\left(\mathrm{Id} \xrightarrow{\eta_{\pi_{23},\Delta}\circ\zeta^{-1}_{\pi_{12},\Delta}} \pi_{23*}\Delta_*\Delta^*\pi_{12}^*\right).$$

which is clearly the right adjoint of (3.10) with respect to π_2^*. The result follows. □

Proof of Theorem 3.1. Set

$$Q' = \pi_{23}^*\left(\pi_1^!\mathcal{O}_{X_1} \otimes E^\vee\right) \otimes \pi_{12}^*E$$

so that $Q = \pi_{13*}Q'$. Since $\pi_{12}\circ\Delta = \pi_{23}\circ\Delta = \mathrm{Id}$ we have a natural isomorphism

$$\Delta^*Q' \xrightarrow{\nu_\Delta} \Delta^*\pi_{23}^*\left(\pi_1^!\mathcal{O}_{X_1}\otimes E^\vee\right)\otimes\Delta^*\pi_{12}^*E \xrightarrow{\zeta_{\pi_{23},\Delta}\otimes\zeta_{\pi_{12},\Delta}} \pi_1^!\mathcal{O}_{X_1}\otimes E^\vee\otimes E. \tag{3.13}$$

We therefore define a morphism

$$\Delta^*Q' \xrightarrow{(3.13)} \pi_1^!\mathcal{O}_{X_1}\otimes E^\vee\otimes E \xrightarrow{\mathrm{ev}_E} \pi_1^!\mathcal{O}_{X_1}. \tag{3.14}$$

Let us write the morphism of functors induced by the morphism $Q \xrightarrow{(3.3)-(3.6)} \Delta_*\mathcal{O}_{X_1}$ of FM-kernels as:

$$\tilde\pi_{2*}\left(\pi_{13*}Q'\otimes\tilde\pi_1^*(-)\right) \xrightarrow{\beta_\Delta} \tilde\pi_{2*}\left(\pi_{13*}\Delta_*\Delta^*Q'\otimes\tilde\pi_1^*(-)\right) \tag{3.15}$$

$$\tilde\pi_{2*}\left(\pi_{13*}\Delta_*\Delta^*Q'\otimes\tilde\pi_1^*(-)\right) \xrightarrow{\eta_{\Delta,\pi_1}\circ\eta^{-1}_{\pi_{13},\Delta}} \tilde\pi_{2*}\left(\Delta_*\pi_{1*}\Delta^*Q'\otimes\tilde\pi_1^*(-)\right) \tag{3.16}$$

$$\tilde\pi_{2*}\left(\Delta_*\pi_{1*}\Delta^*Q'\otimes\tilde\pi_1^*(-)\right) \xrightarrow{(3.14)} \tilde\pi_{2*}\left(\Delta_*\pi_{1*}\pi_1^!\mathcal{O}_{X_1}\otimes\tilde\pi_1^*(-)\right) \tag{3.17}$$

$$\tilde\pi_{2*}\left(\Delta_*\pi_{1*}\pi_1^!\mathcal{O}_{X_1}\otimes\tilde\pi_1^*(-)\right) \xrightarrow{\epsilon_{\pi_1}} \tilde\pi_{2*}\left(\Delta_*\mathcal{O}_{X_1}\otimes\tilde\pi_1^*(-)\right) \tag{3.18}$$

On the other hand, Φ_E is the composition of functors π_1^*, $E\otimes(-)$ and π_{2*}. Each of these functors has a left adjoint, these adjoints are $\pi_{1*}(\pi_1^!\mathcal{O}_{X_1}\otimes -)$, $E^\vee\otimes(-)$ and π_2^*, respectively. Therefore, the adjunction counit $\Phi_E^{\mathrm{ladj}}\Phi_E \to \mathrm{Id}$ is the composition of the three corresponding adjunction counits:

$$\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes E^\vee\otimes\pi_2^*\pi_{2*}\left(E\otimes\pi_1^*(-)\right)\right) \xrightarrow{\gamma_{\pi_2}} \pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes E^\vee\otimes E\otimes\pi_1^*(-)\right) \tag{3.19}$$

$$\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes E^\vee\otimes E\otimes\pi_1^*(-)\right) \xrightarrow{\mathrm{ev}_E} \pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes\pi_1^*(-)\right) \tag{3.20}$$

$$\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes\pi_1^*(-)\right) \xrightarrow{\epsilon_{\pi_1}\circ\chi_{\pi_1}} \mathrm{Id} \tag{3.21}$$

The claim of the theorem is that the composition (3.19)-(3.21) is isomorphic to the composition (3.15)-(3.18).

Let us clarify some terminology. We say that two morphisms of functors $f \to g$ and $f' \to g'$ are isomorphic if there exist connecting isomorphisms $f \xrightarrow{\sim} f'$ and $g \xrightarrow{\sim} g'$ such that the diagram

$$\begin{tikzcd} f \arrow[r] \arrow[d, "\sim"'] & g \arrow[d, "\sim"] \\ f' \arrow[r] & g' \end{tikzcd} \tag{3.22}$$

commutes. Clearly it is an equivalence relation on the set of all morphisms between all functors between two given categories. In particular, it is transitive.

If we further have a morphism of functors $g \to h$ which is isomorphic to a morphism of functors $g'' \to h''$ then $f \to g \to h$ is isomorphic to $f' \to g' \xrightarrow{\sim} g'' \to h''$, where the connecting isomorphism $g' \xrightarrow{\sim} g''$ is the composition of the inverse of the connecting isomorphism $g \xrightarrow{\sim} g'$ with the connecting isomorphism $g \xrightarrow{\sim} g''$.

Our strategy therefore is to consecutively replace the morphisms which compose (3.19)-(3.21) by isomorphic ones until we obtain (3.15)-(3.18). However, every time we replace a composant by an isomorphic one, we introduce a new connecting isomorphism. In the end we have to compose a long chain of these isomorphisms (each composed of natural isomorphisms detailed in §2.3) and simplify the result. It is a mechanical exercise in pseudofunctoriality of direct and inverse image and the associativity of tensor product. To present it in full detail would be very tedious, the end result being always obvious from the start. This had long been lamented in the literature, cf. [Har66, §II.6]. To keep the focus on the substance of a proof we only state the final result of each such computation of a connecting isomorphism, unless something non-trivial is involved. For our most meticulous readers (and our most inquisitive referees) we have included in the Appendix an unabbreviated proof, where all such computations are carried out in full detail.

We begin with morphism (3.19). By Lemma 3.2 it is isomorphic to

$$\pi_{1*}\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1} \otimes \pi_{23*}\pi_{12}^*\left(E \otimes \pi_1^*(-)\right)\right) \xrightarrow{\beta_\Delta} \pi_{1*}\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1} \otimes \pi_{23*}\Delta_*\Delta^*\pi_{12}^*\left(E \otimes \pi_1^*(-)\right)\right). \tag{3.23}$$

By Lemma 2.1 morphism (3.23) is further isomorphic to

$$\pi_{1*}\pi_{23_*}\left(Q' \otimes \pi_{12}^*\pi_1^*(-)\right) \xrightarrow{\nu_\Delta \circ \beta_\Delta} \pi_{1*}\pi_{23_*}\Delta_*\left(\Delta^* Q' \otimes \Delta^*\pi_{12}^*\pi_1^*(-)\right). \tag{3.24}$$

Finally, since $\pi_1 \circ \pi_{23} = \tilde{\pi}_2 \circ \pi_{13}$ and $\pi_1 \circ \pi_{12} = \tilde{\pi}_1 \circ \pi_{13}$, see diagram (3.1), the corresponding pseudofunctoriality isomorphisms imply that (3.24) is isomorphic to

$$\tilde{\pi}_{2*}\pi_{13*}\left(Q' \otimes \pi_{13}^*\tilde{\pi}_1^*(-)\right) \xrightarrow{\nu_\Delta \circ \beta_\Delta} \tilde{\pi}_{2*}\pi_{13*}\Delta_*\left(\Delta^* Q' \otimes \Delta^*\pi_{13}^*\tilde{\pi}_1^*(-)\right) \tag{3.25}$$

We proceed to morphism (3.20) which is induced by the adjunction counit $\pi_1^! \mathcal{O}_{X_1} \otimes E^\vee \otimes E \to \pi_1^! \mathcal{O}_{X_1}$. By its definition morphism (3.14) is isomorphic to this adjunction counit, and so (3.20) is isomorphic to

$$\pi_{1*}\left(\Delta^* Q' \otimes \pi_1^*(-)\right) \xrightarrow{(3.14)} \pi_{1*}\left(\pi_1^!(\mathcal{O}_{X_1}) \otimes \pi_1^*(-)\right) \tag{3.26}$$

As $\tilde{\pi}_2 \circ \Delta = \tilde{\pi}_1 \circ \Delta = \mathrm{Id}$ by pseudofunctoriality (3.26) is isomorphic to

$$\tilde{\pi}_{2*}\Delta_*\pi_{1*}\left(\Delta^* Q' \otimes \pi_1^*\Delta^*\tilde{\pi}_1^*(-)\right) \xrightarrow{(3.14)} \tilde{\pi}_{2*}\Delta_*\pi_{1*}\left(\pi_1^! \mathcal{O}_{X_1} \otimes \pi_1^*\Delta^*\tilde{\pi}_1^*(-)\right). \tag{3.27}$$

Finally, the same pseudofunctoriality isomorphisms imply that (3.21) is isomorphic to

$$\tilde{\pi}_{2*}\Delta_*\pi_{1*}\left(\pi_1^! \mathcal{O}_{X_1} \otimes \pi_1^*\Delta_*\tilde{\pi}_1^*(-)\right) \xrightarrow{\epsilon_{\pi_1} \circ \chi_{\pi_1}} \tilde{\pi}_{2*}\Delta_*\Delta^*\tilde{\pi}_1^*(-). \tag{3.28}$$

We have now shown that (3.19), (3.20) and (3.21) are isomorphic to (3.25), (3.27) and (3.28), respectively. Next, we compute the connecting isomorphisms. The isomorphism connecting (3.25) to (3.27) works out to be the pseudofunctoriality isomorphism

$$\tilde{\pi}_{2*}\pi_{13*}\Delta_*\left(\Delta^* Q' \otimes \Delta^*\pi_{13}^*\tilde{\pi}_1^*(-)\right) \xrightarrow{\eta_{\Delta,\pi_1} \circ \eta^{-1}_{\pi_{13},\Delta}} \tilde{\pi}_{2*}\Delta_*\pi_{1*}\left(\Delta^* Q' \otimes \pi_1^*\Delta^*\tilde{\pi}_1^*(-)\right). \tag{3.29}$$

and the isomorphism connecting (3.27) to (3.28) works out to be the identity.

We can now conclude that the adjunction counit $\Phi_E^{\text{ladj}}\Phi_E \to \text{Id}$, being the composition of (3.19), (3.20) and (3.21), is isomorphic to the composition of (3.25), (3.29), (3.27) and (3.28). The claim of the theorem then follows from the fact that the following diagram commutes:

$$\begin{tikzcd}
\tilde{\pi}_{2*}\left(\pi_{13*}Q' \otimes \tilde{\pi}_1^*(-)\right) \arrow[r, "\sim"] \arrow[d, "(3.15)"'] & \tilde{\pi}_{2*}\pi_{13*}\left(Q' \otimes \pi_{13}^*\tilde{\pi}_1^*(-)\right) \arrow[d, "(3.25)"] \\
\tilde{\pi}_{2*}\left(\pi_{13*}\Delta_*\Delta^*Q' \otimes \tilde{\pi}_1^*(-)\right) \arrow[r, "\sim"] \arrow[d, "(3.16)"'] & \tilde{\pi}_{2*}\pi_{13*}\Delta_*\left(\Delta^*Q' \otimes \Delta^*\pi_{13}^*\tilde{\pi}_1^*(-)\right) \arrow[d, "(3.29)"] \\
\tilde{\pi}_{2*}\left(\Delta_*\pi_{1*}\Delta^*Q' \otimes \tilde{\pi}_1^*(-)\right) \arrow[r, "\sim"] \arrow[d, "(3.17)"'] & \tilde{\pi}_{2*}\Delta_*\pi_{1*}\left(\Delta^*Q' \otimes \pi_1^*\Delta^*\tilde{\pi}_1^*(-)\right) \arrow[d, "(3.27)"] \\
\tilde{\pi}_{2*}\left(\Delta_*\pi_{1*}\pi_1^!\mathcal{O}_{X_1} \otimes \tilde{\pi}_1^*(-)\right) \arrow[r, "\sim"] \arrow[d, "(3.18)"'] & \tilde{\pi}_{2*}\Delta_*\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1} \otimes \pi_1^*\Delta^*\tilde{\pi}_1^*(-)\right) \arrow[d, "(3.28)"] \\
\tilde{\pi}_{2*}\left(\Delta_*\mathcal{O}_{X_1} \otimes \tilde{\pi}_1^*(-)\right) \arrow[r, "\sim"] & \tilde{\pi}_{2*}\Delta_*\Delta^*\tilde{\pi}_1^*(-)
\end{tikzcd} \tag{3.30}$$

where the horizontal isomorphisms are all due to the projection formula. To see that diagram (3.30) indeed commutes, observe that its topmost square commutes by Lemma 2.1, the middle two commute by functoriality and the lowermost square commutes by Lemma 2.2. □

An identical proof yields an analogous result for the right adjunction counit:

Theorem 3.2. *Let X_1 and X_2 be two separable schemes of finite type over k with X_2 proper. Let E be a perfect object of $D(X_1 \times X_2)$ and Ψ_E be a Fourier-Mukai transform from $D_{qc}(X_2)$ to $D_{qc}(X_1)$ defined by E.*

The adjunction counit $\gamma'_E\colon \Psi_E\Psi_E^{radj} \to \text{Id}$ is isomorphic to the morphism of Fourier-Mukai transforms $D_{qc}(X_1) \to D_{qc}(X_1)$ induced by the following morphism of objects of $D(X_1 \times X_1)$:

$$\tilde{Q} = \pi_{13*}\left(\pi_{12}^*E^\vee \otimes \pi_{23}^*E \otimes \pi_{12}^*\pi_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{\pi_{13*}\beta_\Delta} \pi_{13*}\Delta_*\Delta^*\left(\pi_{12}^*E^\vee \otimes \pi_{23}^*E \otimes \pi_{12}^*\pi_1^!(\mathcal{O}_{X_1})\right) \tag{3.31}$$

$$\pi_{13*}\Delta_*\Delta^*\left(\pi_{12}^*E^\vee \otimes \pi_{23}^*E \otimes \pi_{12}^*\pi_1^!(\mathcal{O}_{X_1})\right) \simeq \Delta_*\pi_{1*}\left(E \otimes E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})\right) \tag{3.32}$$

$$\Delta_*\pi_{1*}\left(E \otimes E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{\Delta_*\pi_{1*}\,\text{ev}_E} \Delta_*\pi_{1*}\left(\pi_1^!(\mathcal{O}_{X_1})\right) \tag{3.33}$$

$$\Delta_*\pi_{1*}\pi_1^!(\mathcal{O}_{X_1}) \xrightarrow{\Delta_*\epsilon_{\pi_1}} \Delta_*\mathcal{O}_{X_1}. \tag{3.34}$$

where (3.32) is composed of isomorphism $\nu_\Delta\colon \Delta^*(- \otimes -) \xrightarrow{\sim} \Delta^*(-) \otimes \Delta^*(-)$ and of pseudofunctoriality isomorphisms corresponding to the identities $\pi_{13} \circ \Delta = \Delta \circ \pi_1$ and $\pi_{12} \circ \Delta = \pi_{23} \circ \Delta = \text{Id}$.

3.2. **Non-compact case.** In practice, one often has to deal with cases when neither X_1 nor X_2 are proper. A common way to deal with such situations is to restrict to the full subcategories of objects with proper support. However, with a bit of care it is still possible to work in full generality.

So let X_1 and X_2 be any two separable schemes of finite type over k, not necessarily proper, and let E be a perfect object of $D(X_1 \times X_2)$. We want to write down the left adjoint Φ_E^{ladj} of $\Phi_E = \pi_{2*}(E \otimes \pi_1^*(-))$, but since π_1 is not necessarily a proper morphism, the left adjoint to π_1^* does not necessarily exist.

To construct Φ_E^{ladj}, we compactify X_2 - that is, we choose an open immersion $j\colon X_2 \hookrightarrow \bar{X}_2$ with $\bar{X}_2$ proper over k, cf. [Nag62], or [Voj07] for a more modern exposition. We shall abuse the notation by using j to also denote immersions $X_1 \times X_2 \to X_1 \times \bar{X}_2$ and $X_1 \times X_2 \times X_1 \to X_1 \times \bar{X}_2 \times X_1$ where it causes no confusion. For any such compactified product space we shall denote by $\bar{\pi}_i$ and $\bar{\pi}_{ij}$ projections onto corresponding factors. Also, write $\bar{E}$ for j_*E.

We have the following commutative diagram:

$$\begin{tikzcd} & X_1 \times \bar{X}_2 \arrow[dl, "\bar{\pi}_1"'] \arrow[dr, "\bar{\pi}_2"] & X_1 \times X_2 \arrow[l, hook', "j"'] \arrow[dr, "\pi_2"] & \\ X_1 & & \bar{X}_2 & X_2. \arrow[l, hook', "j"] \end{tikzcd} \tag{3.35}$$

Lemma 3.3. *Let $E \in D_{perf}(X_1 \times X_2)$. There is an isomorphism of functors $D_{qc}(X_1) \to D_{qc}(\bar{X}_2)$*

$$\Phi_{\bar{E}} \xrightarrow{\sim} j_* \Phi_E. \tag{3.36}$$

Its left adjoint with respect to j_* is an isomorphism of functors $D_{qc}(X_1) \to D_{qc}(X_2)$

$$j^* \Phi_{\bar{E}} \xrightarrow{\sim} \Phi_E. \tag{3.37}$$

Proof. For the first claim, we set (3.36) to be

$$\Phi_{\bar{E}} = \bar{\pi}_{2*}\left(j_* E \otimes \bar{\pi}_1^*(-)\right) \xrightarrow{\alpha_j} \bar{\pi}_{2*} j_* \left(E \otimes j^* \bar{\pi}_1^*(-)\right) \xrightarrow{\eta_{j,\pi_2} \circ \eta^{-1}_{\bar{\pi}_2, j} \circ \zeta_{\bar{\pi}_1, j}} j_* \pi_{2*}\left(E \otimes \pi_1^*(-)\right) = j_* \Phi_E.$$

For the second claim: (3.37) is the composition of the image of (3.36) under j^* with the adjunction counit $\gamma_j : j^* j_* \Phi_E \to \Phi_E$. And γ_j is an isomorphism since j is an open immersion [GD60, *Prop. 9.4.2*]. □

We now need the following key lemma:

Lemma 3.4. *Let X be a concentrated scheme and let $U \xrightarrow{j} X$ be an open immersion. Let $D^j_{qc}(X)$ be the full subcategory of $D_{qc}(X)$ formed by the images of the objects of $D_{qc}(U)$ under j_*. Let $D^j(X)$ and $D^j_{perf}(X)$ be the full subcategories of $D^j_{qc}(X)$ consisting of complexes with bounded and coherent cohomology and of perfect complexes. Then:*

1. *Functors j_* and j^* restrict to mutually inverse equivalences between $D^j_{qc}(X)$ and $D_{qc}(U)$.*
2. *For any $A \in D_{qc}(X)$ functors $A \otimes (-)$ and $\mathbf{R}\mathcal{H}om_X(A, -)$ restrict to functors $D^j_{qc}(X) \to D^j_{qc}(X)$ and are identified by j^* with $j^* A \otimes (-)$ and $\mathbf{R}\mathcal{H}om_U(j^* A, -)$.*
3. *Let $X' \xrightarrow{f} X$ be a concentrated map and consider the following base change diagram:*

$$\sigma : \begin{tikzcd} U' \arrow[r, "j'"] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ U \arrow[r, "j"] & X \end{tikzcd} \tag{3.38}$$

   The functors f_* and f^* restrict to functors between $D^{j'}_{qc}(X')$ and $D^j_{qc}(X)$ and are identified by the equivalences j^* and j'^* with functors g_* and g^*.
4. *Let X be Noetherian. The equivalence j^* identifies $D^j(X)$ and $D^j_{perf}(X)$ with $D^{cls}(U)$ and $D^{cls}_{perf}(U)$, the full subcategories of $D(U)$ and $D_{perf}(U)$ consisting of objects whose support is closed in X.*
5. *Let X be Noetherian. For any $A \in D^+_{qc}(X)$ functor $\mathbf{R}\mathcal{H}om_X(-, A)$ restricts to a functor $D^j(X) \to D^j_{qc}(X)$ and the equivalence j^* identifies it with $\mathbf{R}\mathcal{H}om_U(-, j^* A)$.*

Proof. Since j is an open immersion, the adjunction co-unit $j^* j_* \xrightarrow{\gamma_j} \mathrm{Id}$ is an isomorphism. It follows that j_* is fully faithful, so its restriction to a functor $D_{qc}(U) \to D_j(X)$ is tautologically an equivalence. It also follows that j^* is the inverse equivalence to j_*. This settles claim (1).

For claim (2), it follows from the projection formula isomorphism

$$A \otimes j_*(-) \xrightarrow{\alpha_j} j_*(j^* A \otimes -)$$

that $A \otimes (-)$ restricts to a functor $D^j(X) \to D_j(X)$ and that this restriction is identified by j^* with

$$j^* A \otimes (-) \colon D_{qc}(U) \to D_{qc}(U).$$

The assertion for the functor $\mathbf{R}\,\mathcal{H}om_X(A,-)$ follows similarly from the sheafified adjunction isomorphism

$$j_*\,\mathbf{R}\,\mathcal{H}om_U(j^*A,-)\xrightarrow{\tau_j}\mathbf{R}\,\mathcal{H}om_X(A,j_*(-)).$$

The claim (3) follows in the same way from the pseudo-functoriality isomorphism $f_*j'_*\xrightarrow{\eta_{j,g}\circ\eta^{-1}_{f,j'}}j_*g_*$ and the flat base change isomorphism $f^*j_*\xrightarrow{\mu_\sigma}j'_*g^*$.

For claim (4), first note that j is an open immersion of Notherian schemes and thus perfect. Now let A be any object of $D^j(X)$ and let $B=j^*A$ so that $A=j_*B$. Since j is perfect, B lies in $D(U)$. We have $\operatorname{Supp}_U B=(\operatorname{Supp}_X A)\cap U$ and we need to check that this set is closed in X. Since $A\in D(X)$, we know that $\operatorname{Supp}_X A$ is closed in X and any point $p\in X$ lies in $\operatorname{Supp}_X A$ if and only if $\iota_p^*A\neq 0$, where ι_p is the inclusion map. On the other hand, for any $p\in X\setminus U$ we have $\iota_p^*A=\iota_p^*j_*B=0$ by the base change formula. Hence $\operatorname{Supp}_X A\subset U$, so $\operatorname{Supp}_U B=\operatorname{Supp}_X A$ and hence closed in X. We conclude that $B\in D^{cls}(U)$ as required.

Conversely, let $B\in D^{cls}(U)$ and let $A=j_*B$. Since $B\in D(U)$ we can find a fat enough closed subscheme $Z\xrightarrow{k}U$ with the underlying set $\operatorname{Supp}_U B$ to ensure that $B\simeq k_*C$ for some $C\in D(Z)$. Since $\operatorname{Supp}_U B$ is closed in X, the composite map $Z\xrightarrow{j\circ k}X$ is a closed immersion. We conclude that $A=j_*B\simeq j_*k_*C\simeq(j\circ k)_*C$ lies in $D(X)$, as required.

We have now shown that j^* identifies $D^j(X)$ with $D^{cls}(U)$. Finally, any inverse image functor takes perfect complexes to perfect complexes [Ill71b, Cor. 4.19.1], therefore j^* takes $D^j_{perf}(X)$ to $D^{cls}_{perf}(U)$. Conversely, let A be a perfect object in $D^{cls}(U)$, then it is, in particular, of finite Tor-dimension. As j is perfect, j_*A is also of finite Tor-dimension [Ill71a, Cor. 3.7.2]. Since we already know that $j_*A\in D(X)$, we conclude that j_*A is perfect. Thus j^* identifies $D^j_{perf}(X)$ with $D^{cls}_{perf}(U)$. This settles claim (4).

For claim (5), take any $B\in D^j(X)$. Then, as before, we can find a closed immersion $Z\xrightarrow{k}U$ and an object $C\in D(Z)$ such that $B=(j\circ k)_*C$. We then have a functorial isomorphism

$$\mathbf{R}\,\mathcal{H}om_X((j\circ k)_*C,A)\xrightarrow{\eta_{j,k}\circ\delta_{j\circ k}}j_*k_*\,\mathbf{R}\,\mathcal{H}om_Z\left(C,(j\circ k)^!A\right)$$

which shows that functor $\mathbf{R}\,\mathcal{H}om_X(-,A)$ restricts to a functor $D^j(X)\to D^j_{qc}(X)$. Finally, this restriction is identified by j^* with $\mathbf{R}\,\mathcal{H}om_U(-,j^*A)$ because j is an open immersion and hence the natural morphism

$$j^*\,\mathbf{R}\,\mathcal{H}om_X(B,A)\to\mathbf{R}\,\mathcal{H}om_X(j^*B,j^*A)$$

is an isomorphism [AIL10, Lemma 2.1.7]. □

Corollary 3.5. *The Fourier-Mukai transform*

$$\Phi_E\colon\quad D_{qc}(X_1)\to D_{qc}(X_2)$$

has a left adjoint Φ_E^{ladj}*, and this adjoint is isomorphic to the Fourier-Mukai transform*

$$\Psi_{E^\vee\otimes\pi_1^!(\mathcal{O}_{X_1})}\colon\quad D_{qc}(X_2)\to D_{qc}(X_1).$$

If $\operatorname{Supp}_{X_1\times X_2}E$ *is proper over* X_1 *and* X_2*, then* Φ_E *and* Φ_E^{ladj} *restrict to functors between* $D(X_1)$ *and* $D(X_2)$*.*

Proof. We only prove the first claim, as the assertion about the restriction to $D(X_1)$ and $D(X_2)$ is standard. By Lemma 3.3 functor $\Phi_{\bar{E}}$ is isomorphic to $j_*\Phi_E$. Hence it restricts to a functor $D_{qc}(X_1)\to D^j_{qc}(\bar{X}_2)$. Thus, by the same lemma, Φ_E is isomorphic to the composition

$$D_{qc}(X_1)\xrightarrow{\Phi_{\bar{E}}}D^j_{qc}(\bar{X}_2)\xrightarrow{j^*}D_{qc}(X_2).$$

By Lemma 3.4(1) the functor $D^j_{qc}(\bar{X}_2)\xrightarrow{j^*}D_{qc}(X_2)$ is an equivalence whose inverse is the functor j_*. Therefore Φ_E has a left adjoint Φ_E^{ladj} isomorphic to $\Phi_{\bar{E}}^{ladj}j_*$, that is to

$$\bar{\pi}_{1*}\left(\bar{E}^\vee\otimes\bar{\pi}_1^!(\mathcal{O}_{X_1})\otimes\bar{\pi}_2^*j_*(-)\right).$$

By Lemma 3.4(2)-(5) this is further isomorphic to

$$\bar{\pi}_{1*}j_*\left(E^\vee\otimes j^*\bar{\pi}_1^!(\mathcal{O}_{X_1})\otimes\pi_2^*(-)\right).$$

Since $\pi_1=\bar{\pi}_1\circ j$, the claim now follows by the pseudofunctoriality of $(-)_*$ and $(-)^!$. □

The isomorphism $\Phi_{\bar{E}} \xrightarrow{(3.36)} j_* \Phi_E$ of functors induces the unique isomorphism

$$\Phi_{\bar{E}}^{ladj} \xrightarrow{\sim} \Phi_E^{ladj} j^* \tag{3.39}$$

of their left adjoints $D_{qc}(\bar{X}_2) \to D_{qc}(X_1)$ which makes the diagram

$$\begin{tikzcd} \Phi_{\bar{E}}^{ladj}\Phi_{\bar{E}} \arrow[d, "(3.39)\circ(3.36)"', "\sim"] \arrow[drr, "\gamma_{\bar{E}}"] & & \\ \Phi_E^{ladj} j^* j_* \Phi_E \arrow[r, "\sim", "\gamma_j"'] & \Phi_E^{ladj}\Phi_E \arrow[r, "\gamma_E"'] & \mathrm{Id} \end{tikzcd} \tag{3.40}$$

of functors $D_{qc}(X_1) \to D_{qc}(X_1)$ commute. Therefore the adjunction co-unit $\Phi_E^{ladj}\Phi_E \xrightarrow{\gamma_E} \mathrm{Id}$ is isomorphic to the adjunction co-unit $\Phi_{\bar{E}}^{ladj}\Phi_{\bar{E}} \xrightarrow{\gamma_{\bar{E}}} \mathrm{Id}$. The standard Fourier-Mukai kernel of $\Phi_{\bar{E}}^{ladj}\Phi_{\bar{E}}$ is

$$\bar{Q} = \bar{\pi}_{13*}\left(\bar{\pi}_{12}^* \bar{E} \otimes \bar{\pi}_{23}^* \bar{E}^\vee \otimes \bar{\pi}_{23}^* \bar{\pi}_1^!(\mathcal{O}_{X_1})\right)$$

and Theorem 3.1 supplies us with the morphism $\bar{Q} \to \Delta_* \mathcal{O}_{X_1}$ which induces $\Phi_{\bar{E}}^{ladj}\Phi_{\bar{E}} \xrightarrow{\gamma_{\bar{E}}} \mathrm{Id}$. We obtain:

Proposition 3.6. *The adjunction counit $\gamma_E \colon \Phi_E^{ladj}\Phi_E \to \mathrm{Id}$ is isomorphic to the morphism of Fourier-Mukai transforms $D_{qc}(X_1) \to D_{qc}(X_1)$ induced by the morphism $\bar{Q} \to \Delta_* \mathcal{O}_{X_1}$ of Theorem 3.1.*

As a non-essential aside, the standard Fourier-Mukai kernel of $\Phi_E^{\mathrm{ladj}}\Phi_E$ itself is

$$Q = \pi_{13*}\left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^* \pi_1^!(\mathcal{O}_{X_1})\right)$$

The functors $\Phi_E^{\mathrm{ladj}}\Phi_E$ and $\Phi_{\bar{E}}^{\mathrm{ladj}}\Phi_{\bar{E}}$ are isomorphic, but it does not a priori mean that Q and $\bar{Q}$ are isomorphic. However, it is easy to check that they are — we leave the details as an exercise for the reader.

4. An alternative description for the pushforward kernels

Whenever E is direct image of an object from the derived category of some subscheme of $X_1 \times X_2$ the decomposition of the morphism $Q \to \mathcal{O}_\Delta$ given in Theorem 3.1 is usually very poorly suited for computing cones. We first illustrate this in Section 4.1 with an example where E is the structure sheaf of a global complete intersection subscheme and so everything can be worked out explicitly using Koszul-type resolutions. For a general closed subscheme of $X_1 \times X_2$ such a resolution does not exist and a different approach is needed. But with an insight obtained from Section 4.1 we set up some general machinery in Sections 4.2 and 4.3 which we then apply in Section 4.4 to obtain a better description of the morphism $Q \to \mathcal{O}_\Delta$ for E being a pushforward from an arbitrary closed subscheme.

4.1. **The global complete intersection example.** Let X_1 and X_2 be a pair of smooth varieties over k with X_2 proper. Let $\mathcal{N}$ be a vector bundle of rank d on $X_1 \times X_2$ and let s be a regular global section of $\mathcal{N}$. Let Z be the zero-locus of s in $X_1 \times X_2$, it is a closed subscheme of codimension d and normal bundle $\mathcal{N}|_Z$. Let $Z \times X_1$ and $X_1 \times Z$ be Tor-independent in $X_1 \times X_2 \times X_1$, i.e. the derived tensor product $\mathcal{O}_{Z\times X_1} \otimes \mathcal{O}_{X_1 \times Z}$ is $\mathcal{O}_{Z'}$ where $Z' = (Z \times X_1) \cap (X_1 \times Z)$. We can rewrite the first two morphisms in the decomposition of Theorem 3.1 for $E = \mathcal{O}_Z$ as the images under $\pi_{13*}\left(- \otimes \pi_{23}^* \pi_2^! \mathcal{O}_{X_1}\right)$ of the following morphism in $D_{qc}(X_1 \times X_2 \times X_1)$:

$$\pi_{12}^* \mathcal{O}_Z \otimes \pi_{23}^* \mathcal{O}_Z^\vee \xrightarrow{\beta_\Delta} \Delta_*\left(\mathcal{O}_Z \otimes \mathcal{O}_Z^\vee\right) \xrightarrow{\Delta_* \operatorname{ev}_{\mathcal{O}_Z}} \Delta_* \mathcal{O}_{X_1 \times X_2}. \tag{4.1}$$

Note that by the flat base change for the twisted inverse image pseudofunctor (see §2.2) the object $\pi_1^! \mathcal{O}_{X_1}$ is just the shifted line bundle $\pi_2^* \omega_{X_2}[\dim X_2]$.

The structure sheaf $\mathcal{O}_Z$ has a global Koszul resolution on $X_1 \times X_2$

$$\wedge^d \mathcal{N}^\vee \to \wedge^{d-1} \mathcal{N}^\vee \to \cdots \to \mathcal{N}^\vee \to \mathcal{O}_{X_1 \times X_2} \tag{4.2}$$

whose differential maps are defined in the usual way by valuations at s. In particular, they all vanish along Z. Dualizing the Koszul complex, we see immediately that $\mathcal{O}_Z^\vee$ is isomorphic to $\mathcal{O}_Z \otimes \wedge^d \mathcal{N}[-d]$ in $D(X_1 \times X_2)$.

We have $\pi_{12}^{-1}(Z) = Z \times X_1$ and $\pi_{23}^{-1}(Z) = X_1 \times Z$. So $\pi_{12}^* \mathcal{O}_Z \simeq O_{Z \times X_1}$, while $\pi_{23}^* \mathcal{O}_Z^\vee = \mathcal{O}_{X_1 \times Z} \otimes \pi_{23}^*(\wedge^d \mathcal{N})[-d]$. Thus $\pi_{12}^* \mathcal{O}_Z \otimes \pi_{23}^* \mathcal{O}_Z^\vee$, the first term in (4.1), equals $\mathcal{O}_{Z\times X_1} \otimes \mathcal{O}_{X_1 \times Z} \otimes \pi_{23}^*\left(\wedge^d \mathcal{N}\right)[-d]$. By assumption $Z \times X_1$ and $X_1 \times Z$ are Tor-independent, and $\pi_{23}^* \wedge^d \mathcal{N}[-d]$ is a line bundle, so we conclude that the first term in (4.1) equals $(\pi_{23}^* \wedge^d \mathcal{N})|_{Z'}[-d]$.

On the other hand, $\Delta_* (\mathcal{O}_Z \otimes \mathcal{O}_Z^\vee)$, the second term in (4.1), is isomorphic to the image under Δ_* of the restriction of the dual of the complex (4.2) to Z. Since all the differentials vanish along Z, this equals

$$\Delta_* \left(\mathcal{O}_Z \xrightarrow{0} \mathcal{N}|_Z \xrightarrow{0} \dots \xrightarrow{0} \wedge^d \mathcal{N}|_Z \right) = \bigoplus_{i=0}^{d} \wedge^i \mathcal{N}|_{\Delta(Z)}[-i], \tag{4.3}$$

where $\Delta(Z)$ is the image of Z under $X_1 \times X_2 \xrightarrow{\Delta} X_1 \times X_2 \times X_1$.

Thus the decomposition (4.1) is not practical from the point of view of computing cones. Its first map goes from $(\pi_{23}^* \wedge^d \mathcal{N})|_{Z'}[-d]$, a single sheaf sitting in the degree d, to $\bigoplus_{i=0}^d \wedge^i \mathcal{N}|_{\Delta(Z)}[-i]$, a huge complex with non-zero cohomologies in all degrees from 0 to d. Its second map goes from this huge complex to $\mathcal{O}_{\Delta(X_1 \times X_2)}$, a single sheaf sitting in the degree 0. We get two huge cones with non-zero cohomologies in all degrees from 0 to d which almost entirely annihilate each other when we take the cone of the map between them.

In the rest of this section we prove, in a much more general setting, that there exists a more economical decomposition than (4.1). Applied to the case at hand, our result will tell us that the decomposition (4.1) filters through the summand $\wedge^d \mathcal{N}|_{\Delta(Z)}[-d]$ of $\bigoplus_{i=0}^d \wedge^i \mathcal{N}|_{\Delta(Z)}[-i]$, and can be written simply as:

$$(\pi_{23}^* \wedge^d \mathcal{N})|_{Z'}[-d] \xrightarrow{Z' \to \Delta(Z)} \wedge^d \mathcal{N}|_{\Delta(Z)}[-d] \simeq \Delta_* \mathcal{O}_Z^\vee \xrightarrow{\Delta_* \left(\mathcal{O}_{X_1 \times X_2} \to \mathcal{O}_Z\right)^\vee} \Delta_* \mathcal{O}_{X_1 \times X_2}. \tag{4.4}$$

The cones of these two maps are small compared to those in (4.1) and easy to compute.

4.2. **A decomposition of the evaluation map.** Let $Y \xrightarrow{f} X$ be a map of concentrated schemes.

Proposition 4.1. *For any $E \in D(\mathcal{O}_Y\text{-}\mathbf{Mod})$ the following diagram commutes*

$$\begin{tikzcd}
f_* E \otimes \mathbf{R}\mathcal{H}om\,(f_* E, \mathcal{O}_X) \arrow[ddddr, "ev_{f_* E}"'] & f_* E \otimes f_* \mathbf{R}\mathcal{H}om\,(E, f^\times \mathcal{O}_X). \arrow[l, "\mathrm{Id} \otimes \delta_f"'] \arrow[d, "\kappa_f"] \\
& f_* \left(E \otimes \mathbf{R}\mathcal{H}om\,(E, f^\times \mathcal{O}_X)\right) \arrow[d, "ev_E"] \\
& f_* f^\times \mathcal{O}_X \arrow[d, "\epsilon_f"] \\
& \mathcal{O}_X
\end{tikzcd} \tag{4.5}$$

Proof. Let us show that the right adjoint of (4.5) with respect to $f_* E \otimes (-)$ commutes. The result in [Lip09, Prop. 3.2.4(ii)] tells what is the right adjoint of $f_* E \otimes f_*(-) \xrightarrow{\kappa_f} f_*(E \otimes -)$ with respect to $f_* E \otimes (-)$. It follows immediately that the right adjoint of

$$f_* E \otimes f_* \mathbf{R}\mathcal{H}om\,(E, -) \xrightarrow{\kappa_f} f_* \left(E \otimes \mathbf{R}\mathcal{H}om\,(E, -)\right) \xrightarrow{\mathrm{ev}_E} f_* (-)$$

with respect to $f_* E \otimes (-)$ is

$$f_* \mathbf{R}\mathcal{H}om\,(E, -) \xrightarrow{\gamma_f} f_* \mathbf{R}\mathcal{H}om\,(f^* f_* E, -) \xrightarrow{\tau_f} \mathbf{R}\mathcal{H}om\,(f_* E, f_* -).$$

Therefore the right adjoint of the composition $\epsilon_f \circ \mathrm{ev}_E \circ \kappa_f$ in (4.5) is

$$f_* \mathbf{R}\mathcal{H}om\,\left(E, f^\times \mathcal{O}_X\right) \xrightarrow{\gamma_f} f_* \mathbf{R}\mathcal{H}om\,\left(f^* f_* E, f^\times \mathcal{O}_X\right) \xrightarrow{\tau_f} \mathbf{R}\mathcal{H}om\,\left(f_* E, f_* f^\times \mathcal{O}_X\right) \xrightarrow{\epsilon_f} \mathbf{R}\mathcal{H}om\,(f_* E, \mathcal{O}_X).$$

By definition this is just the sheafified Grothendieck duality morphism

$$f_* \mathbf{R}\mathcal{H}om\,\left(E, f^\times \mathcal{O}_X\right) \xrightarrow{\delta_f} \mathbf{R}\mathcal{H}om\,(f_* E, \mathcal{O}_X).$$

So is clearly the right adjoint of the composition $\mathrm{ev}_{f_* E} \circ (\mathrm{Id} \otimes \delta_f)$ in (4.5). The claim follows. $\square$

4.3. **Künneth maps and the base change.** Let $Y \xrightarrow{f} X$ be a map of concentrated schemes. Morphism $\kappa_f\colon f_*(-)\otimes f_*(-) \to f_*(-\otimes -)$ can be interpreted as the Künneth map of the commutative square:

$$\begin{tikzcd} Y \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] & Y \arrow[d, "f"] \\ Y \arrow[r, "f"'] & X \end{tikzcd} \tag{4.6}$$

We recall the basics on the Künneth map, cf. [Lip09, §3.10]:

Definition 4.2. Let

$$\sigma\colon \quad \begin{tikzcd} Z \arrow[r, "g_2"] \arrow[d, "g_1"'] & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "f_1"'] & X \end{tikzcd} \tag{4.7}$$

be a commutative square of concentrated schemes. Setting $h = f_1 \circ g_1 = f_2 \circ g_2$ define the *Künneth map* to be the bifunctorial morphism

$$\kappa_\sigma\colon\ f_{1*}(A_1)\otimes f_{2*}(A_2) \to h_*\left(g_1^*(A_1)\otimes g_2^*(A_2)\right) \qquad A_i \in D(Y_i) \tag{4.8}$$

which is the composition

$$\begin{aligned} f_{1*}(A_1)\otimes f_{2*}(A_2) \xrightarrow{\beta_h} h_*h^*\left(f_{1*}(A_1)\otimes f_{2*}(A_2)\right) \xrightarrow{\nu_h} h_*\left(h^*f_{1*}(A_1)\otimes h^*f_{2*}(A_2)\right) \xrightarrow{\zeta^{-1}_{f_1,g_1}\otimes\zeta^{-1}_{f_2,g_2}} \\ \xrightarrow{\zeta^{-1}_{f_1,g_1}\otimes\zeta^{-1}_{f_2,g_2}} h_*\left(g_1^*f_1^*f_{1*}(A_1)\otimes g_2^*f_2^*f_{2*}(A_2)\right) \xrightarrow{\gamma_{f_1}\otimes\gamma_{f_2}} h_*\left(g_1^*(A_1)\otimes g_2^*(A_2)\right) \end{aligned} \tag{4.9}$$

with β_h being the adjunction unit $\mathrm{Id}_X \to h_*h^*$ and γ_{f_i} being the adjunction counits $f_i^*f_{i*} \to \mathrm{Id}_{Y_i}$.

A commutative square is called *Künneth-independent* if its Künneth map is a bifunctorial isomorphism. For fiber squares of concentrated schemes this notion of independence is equivalent to several others:

Proposition 4.3 ([Lip09], Theorem 3.10.3)**.** *Let*

$$\sigma\colon \quad \begin{tikzcd} Z = Y_1\times_X Y_2 \arrow[r, "g_2"] \arrow[d, "g_1"'] & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "f_1"'] & X \end{tikzcd} \tag{4.10}$$

be a fiber square of concentrated schemes, then the following are equivalent:

(1) *σ is independent, i.e. the base change map $\mu_\sigma\colon f_1^*f_{2*} \to g_{1*}g_2^*$ is a functorial isomorphism.*
(2) *σ is Künneth-independent.*
(3) *σ is* Tor*-independent, i.e. for any pair of points $y_1 \in Y_1$ and $y_2 \in Y_2$ with $f_1(y_1) = f_2(y_2) = x \in X$ we have*

$$\mathrm{Tor}^i_{\mathcal{O}_{X,x}}\left(\mathcal{O}_{Y_1,y_1}, \mathcal{O}_{Y_2,y_2}\right) = 0 \text{ for all } i > 0. \tag{4.11}$$

What we saw in Section 4.1 is a special case of a very general base change statement for Künneth maps:

Proposition 4.4. *Let*

$$\sigma\colon \quad \begin{tikzcd} Z \arrow[r, "g_2"] \arrow[d, "g_1"'] & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "f_1"'] & X \end{tikzcd} \tag{4.12}$$

be a commutative square of concentrated schemes and set $h = f_1 \circ g_1 = f_2 \circ g_2$*. Let* $u\colon X' \to X$ *be any morphism and let* σ' *be the fiber product of* σ *with* X' *over* X*, that is - the outer square* (Z', Y_1', Y_2', X') *in the commutative diagram*

$$\begin{tikzcd}
Z' \arrow[rrr, "g_2'"] \arrow[dr, "u"] \arrow[ddd, "g_1'"'] & & & Y_2' \arrow[dl, "u"'] \arrow[ddd, "f_2'"] \\
& Z \arrow[r, "g_2"] \arrow[d, "g_1"'] & Y_2 \arrow[d, "f_2"] & \\
& Y_1 \arrow[r, "f_1"'] & X & \\
Y_1' \arrow[ur, "u"] \arrow[rrr, "f_1'"'] & & & X' \arrow[ul, "u"']
\end{tikzcd} \tag{4.13}$$

where $Y_i' = Y_i \times_{X, f_i, u} X'$*,* $Z' = Z \times_{X,h,u} X' = Z \times_{Y_i, g_i, u} Y_i'$ *and the four squares between* σ' *and* σ *are the corresponding fiber squares. Let also* $h' = f_1' \circ g_1' = f_2' \circ g_2'$*. Finally, to unburden the notation, write*

- η_{f_1} *for the pseudofunctoriality isomorphism* $f_{1*}u_* \xrightarrow{\eta_{u,f_1'} \circ \eta^{-1}_{f_1,u}} u_* f'_{1*}$*.*
- ζ_{f_1} *for the pseudofunctoriality isomorphism* $u^* f_1^* \xrightarrow{\zeta^{-1}_{u,f_1'} \circ \zeta_{f_1,u}} f_1'^* u^*$
- μ_{f_1} *for the base change map* $u^* f_{1*} \to f'_{1*} u^*$ *of the corresponding fiber square.*

and analogously for f_2, g_1, g_2 *and* h*.*

Then for any objects $A_1 \in D(Y_1)$ *and* $A_2 \in D(Y_2)$*:*

(1) *The following diagram commutes in* $D(X')$*:*

$$\begin{tikzcd}
u^*\left(f_{1*}(A_1) \otimes f_{2*}(A_2)\right) \arrow[r, "u^*\kappa_\sigma"] \arrow[d, "(\mu_{f_1} \otimes \mu_{f_2}) \circ \nu_u"'] & u^* h_* \left(g_1^*(A_1) \otimes g_2^*(A_2)\right) \arrow[d, "h'_*((\zeta_{g1} \otimes \zeta_{g2}) \circ \nu_u) \circ \mu_h"] \\
f'_{1*}(u^*A_1) \otimes f'_{2*}(u^*A_2) \arrow[r, "\kappa_{\sigma'}"'] & h'_*\left(g_1'^*(u^*A_1) \otimes g_2'^*(u^*A_2)\right)
\end{tikzcd} \tag{4.14}$$

(2) *The following diagram commutes in* $D(X)$*:*

$$\begin{tikzcd}
f_{1*}(A_1) \otimes f_{2*}(A_2) \arrow[r, "\kappa_\sigma"] \arrow[d, "\beta_u"'] & h_*\left(g_1^*(A_1) \otimes g_2^*(A_2)\right) \arrow[d, "h_*\beta_u"] \\
u_*u^*(f_{1*}(A_1) \otimes f_{2*}(A_2)) \arrow[d, "u_*\left((\mu_{f_1} \otimes \mu_{f_2}) \circ \nu_u\right)"'] & h_*u_*u^*\left(g_1^*(A_1) \otimes g_2^*(A_2)\right) \arrow[d, "u_*h'_*((\zeta_{g1} \otimes \zeta_{g2}) \circ \nu_u) \circ \eta_h"] \\
u_*(f'_{1*}(u^*A_1) \otimes f'_{2*}(u^*A_2)) \arrow[r, "u_*\kappa_{\sigma'}"'] & u_*h'_*\left(g_1'^*(u^*A_1) \otimes g_2'^*(u^*A_2)\right)
\end{tikzcd} \tag{4.15}$$

Proof. By definition, the right adjoint of the base change map $u^*h_* \xrightarrow{\mu_h} h'_*u^*$ with respect to u^* is the composition $h_* \xrightarrow{h_*\beta_u} h_*u_*u^* \xrightarrow{\eta_h} u_*h'_*u^*$. It follows that the diagram (4.15) is the right adjoint of the diagram (4.14) with respect to u_*, so we only need to prove that (4.15) commutes.

Let $B \xrightarrow{m} h_*u_*C$ be any morphism between some $B \in D(X)$ and some $C \in D(Z')$. Let l be the left adjoint $u^*h^*B \to C$ of m with respect to h_*u_*. By compatibility of the inverse image/direct image adjunction with pseudofunctoriality, the left adjoint with respect to $u_*h'_*$ of the composition $B \xrightarrow{m} h_*u_*C \xrightarrow{\eta_h} u_*h'_*C$ is the composition $h'^*u^*B \xrightarrow{\zeta_h^{-1}} u^*h^*B \xrightarrow{l} C$. Hence the left adjoint with respect to $u_*h'_*$ of the upper-right half

$$f_{1*}A_1 \otimes f_{2*}A_2 \xrightarrow{\kappa_\sigma} h_*\left(g_1^*A_1 \otimes g_2^*A_2\right) \xrightarrow{\nu_u \circ h_*\beta_u} h_*u_*\left(u^*g_1^*A_1 \otimes u^*g_2^*A_2\right) \xrightarrow{\eta_h \circ h_*u_*(\zeta_{g_1} \otimes \zeta_{g_2})} u_*h'_*\left(g_1'^*u^*A_1 \otimes g_2'^*u^*A_2\right)$$

of (4.15) is the composition of $h'^* u^* \left(f_{1*}A_1 \otimes f_{2*}A_2\right) \xrightarrow{\zeta_h^{-1}} u^* h^* \left(f_{1*}A_1 \otimes f_{2*}A_2\right)$ with the left adjoint of

$$f_{1*}A_1 \otimes f_{2*}A_2 \xrightarrow{\kappa_\sigma} h_* \left(g_1^* A_1 \otimes g_2^* A_2\right) \xrightarrow{\nu_u \circ h_* \beta_u} h_* u_* \left(u^* g_1^* A_1 \otimes u^* g_2^* A_2\right) \xrightarrow{h_* u_* \left(\zeta_{g_1} \otimes \zeta_{g_2}\right)} h_* u_* \left(g_1'^* u^* A_1 \otimes g_2'^* u^* A_2\right)$$

with respect to $h_* u_*$. Making use of the definition of κ_σ in (4.9), this adjoint works out to be

$$u^* h^* \left(\bigotimes_i f_i A_i\right) \xrightarrow{\nu_u \circ \left(\bigotimes_i u^* \zeta^{-1}_{f_i, g_i}\right) \circ u^* \nu_h} \bigotimes_i u^* g_i^* f_i^* f_{i*} A_i \xrightarrow{\bigotimes_i u^* g_i^* \gamma_{f_i}} \bigotimes_i u^* g_i^* A_i \xrightarrow{\bigotimes_i \zeta_{g_i}} \bigotimes_i g_i'^* u^* A_i$$

Composing with $h'^* u^* \left(f_{1*}A_1 \otimes f_{2*}A_2\right) \xrightarrow{\eta_h^{-1}} u^* h^* \left(f_{1*}A_1 \otimes f_{2*}A_2\right)$ and simplifying we see that the left adjoint of (4.15) with respect to $u_* h'_*$ is

$$h'^* u^* \left(\bigotimes_i f_{i*} A_i\right) \xrightarrow{\left(\bigotimes_i \zeta^{-1}_{f'_i, g'_i}\right) \circ \nu_{h'} \circ h'^* \nu_u} \bigotimes_i g_i'^* f_i'^* u^* f_{i*} A_i \xrightarrow{\bigotimes_i g_i'^* \zeta^{-1}_{f_i}} \bigotimes_i g_i'^* u^* f_i^* f_{i*} A_i \xrightarrow{\bigotimes_i g_i'^* u^* \gamma_{f_i}} \bigotimes_i g_i'^* u^* A_i.$$

Similarly, the left adjoint of the lower-left half

$$f_{1*}A_1 \otimes f_{2*}A_2 \xrightarrow{\beta_u} u_* u^* \left(f_{1*}A_1 \otimes f_{2*}A_2\right) \xrightarrow{u_* \left(\bigotimes_i \mu_{f_i}\right) \circ u_* \nu_u} u_* \left(f'_{1*} u^* A_1 \otimes f'_{2*} u^* A_2\right) \xrightarrow{u_* \kappa_{\sigma'}} u_* h'_* \left(g_1'^* u^* A_1 \otimes g_2'^* u^* A_2\right)$$

of (4.15) with respect to $u_* h'_*$ works out as

$$h'^* u^* \left(\bigotimes_i f_{i*} A_i\right) \xrightarrow{\left(\bigotimes_i \zeta^{-1}_{f'_i, g'_i}\right) \circ \nu_{h'} \circ h'^* \nu_u} \bigotimes_i g_i'^* f_i'^* u^* f_{i*} A_i \xrightarrow{\bigotimes_i g_i'^* f_i'^* \mu_{f_i}} \bigotimes_i g_i'^* f_i'^* f'_{i*} u^* A_i \xrightarrow{\otimes_i g_i'^* \gamma_{f'_i}} \bigotimes_i g_i'^* u^* A_i$$

It therefore suffices to show that the following diagram commutes for $i = 1, 2$ and for all $A_i \in D(Y_i)$

$$\begin{tikzcd}
f_i'^* u^* f_{i*} A_i \arrow[r, "g_i'^* \zeta^{-1}_{f_i}"] \arrow[d, "f_i'^* \mu_{f_i}"'] & u^* f_i^* f_{i*} A_i \arrow[d, "u^* \gamma_{f_i}"] \\
f_i'^* f'_{i*} u^* A_i \arrow[r, "\gamma_{f'_i}"'] & u^* A_i.
\end{tikzcd} \tag{4.16}$$

By definition of μ_{f_i} in (2.34) the right adjoint with respect to $f_i'^*$ of $f_i'^* u^* f_{i*} \xrightarrow{\zeta^{-1}_{f_i}} u^* f_i^* f_{i*} \xrightarrow{u^* \gamma_{f_i}} u^*$ is precisely $u^* f_{i*} \xrightarrow{\mu_{f_i}} f'_{i*} u^*$. So the right adjoint with respect to $f_i'^*$ of (4.16) is the diagram

$$\begin{tikzcd}
u^* f_{i*} A_i \arrow[d, "\mu_{f_i}"'] \arrow[dr, "\mu_{f_i}"] & \\
f'_{i*} u^* A_i \arrow[r, "\mathrm{Id}"'] & f'_{i*} u^* A_i
\end{tikzcd}$$

which clearly commutes. $\square$

4.4. **The adjunction counits for the pushforward Fourier-Mukai kernels.** We can now apply the generalities of the previous two sections to obtain an alternative decomposition to that in Theorem 3.1 of the morphism of Fourier-Mukai kernels which induces the canonical adjunction morphism $\Phi_E^{ladj} \Phi_E \to \mathrm{Id}$ in case where E is a pushforward of an object on some $Z \hookrightarrow X_1 \times X_2$.

Let X_1 and X_2 be a pair of separable schemes of finite type over k. Let $Z \xrightarrow{\iota_Z} X_1 \times X_2$ be a closed immersion proper over both X_1 and X_2. Denote by π_{Z1} the composition $Z \xrightarrow{\iota_Z} X_1 \times X_2 \xrightarrow{\pi_1} X_1$. Consider the following fiber squares:

$$\sigma_{12}\colon \begin{tikzcd}
Z \times X_1 \arrow[r, "\iota_{Z12}"] \arrow[d, "\pi_{Z12}"'] & X_1 \times X_2 \times X_1 \arrow[d, "\pi_{12}"] \\
Z \arrow[r, "\iota_Z"'] & X_1 \times X_2
\end{tikzcd} \quad \text{and} \quad \sigma_{23}\colon \begin{tikzcd}
X_1 \times Z \arrow[r, "\iota_{Z23}"] \arrow[d, "\pi_{Z23}"'] & X_1 \times X_2 \times X_1 \arrow[d, "\pi_{23}"] \\
Z \arrow[r, "\iota_Z"'] & X_1 \times X_2.
\end{tikzcd}$$

Then $Z' = (Z \times X_1) \cap (X_1 \times Z) \xrightarrow{\iota_{Z'}} X_1 \times X_2 \times X_1$ fits into the fiber square

$$\sigma\colon \quad \begin{tikzcd} Z' \arrow[r, "\iota'_{23}"] \arrow[d, "\iota'_{12}"'] \arrow[dr, "\iota_{Z'}"] & X_1 \times Z \arrow[d, "\iota_{Z23}"] \\ Z \times X_1 \arrow[r, "\iota_{Z12}"'] & X_1 \times X_2 \times X_1. \end{tikzcd} \qquad . \tag{4.17}$$

Let σ_Δ denote the square obtained from (4.17) by base change $X_1 \times X_2 \xrightarrow{\Delta} X_1 \times X_2 \times X_1$:

$$\begin{tikzcd} Z \arrow[rrrr, "\mathrm{Id}"] \arrow[dr, "\Delta"] \arrow[ddd, "\mathrm{Id}"'] & & & & Z \arrow[dl, "\Delta"'] \arrow[ddd, "\iota_Z"] \\ & Z' \arrow[rr, "\iota'_{23}"] \arrow[d, "\iota'_{12}"'] \arrow[drr, "\iota_{Z'}"] & & X_1 \times Z \arrow[d, "\iota_{Z23}"] \arrow[dr, phantom, "\sigma_{Z23}"] & \\ & Z \times X_1 \arrow[rr, "\iota_{Z12}"'] \arrow[drr, phantom, "\sigma_{Z12}"] & & X_1 \times X_2 \times X_1 & \\ Z \arrow[ur, "\Delta"] \arrow[rrrr, "\iota_Z"'] & & & & X_1 \times X_2 \arrow[ul, "\Delta"'] \end{tikzcd} \tag{4.18}$$

Observe that:

- Composition $Z \xrightarrow{\Delta} Z' \xrightarrow{\iota_{Z'}} X_1 \times X_1 \times X_1$ equals $Z \xrightarrow{\pi_{Z1}} X_1 \xrightarrow{\Delta} X_1 \times X_1$.
- Compositions $Z \xrightarrow{\Delta} Z' \xrightarrow{\iota'_{12}} Z \times X_1 \xrightarrow{\pi_{Z12}} Z$ and $Z \xrightarrow{\Delta} Z \times X_1 \xrightarrow{\pi_{Z12}} Z$ are the identity map.
- Compositions $Z \xrightarrow{\Delta} Z' \xrightarrow{\iota'_{23}} X_1 \times Z \xrightarrow{\pi_{Z23}} Z$ and $Z \xrightarrow{\Delta} X_1 \times Z \xrightarrow{\pi_{Z23}} Z$ are the identity map.

Theorem 4.1. *Let $E_Z \in D(Z)$ be such that $E = \iota_{Z*}(E_Z)$ is perfect in $D(X_1 \times X_2)$. Let Φ_E be the Fourier-Mukai transform $D(X_1) \to D(X_2)$ with kernel E. The adjunction counit $\Phi_E^{ladj}\Phi_E \to \mathrm{Id}_{X_1}$ is isomorphic to the morphism of Fourier-Mukai transforms induced by the composition:*

$$\begin{tikzcd} Q_Z = \pi_{13*}\left(\iota_{Z12*}\pi^*_{Z12}E_Z \otimes \iota_{Z23*}\pi^*_{Z23}\,\mathbf{R}\mathcal{H}om\left(E_Z, \pi^!_{Z1}(\mathcal{O}_{X_1})\right)\right) \arrow[d, "\pi_{13*}\kappa_\sigma"] \\ \pi_{13*}\iota_{Z'*}\left(\iota'^*_{12}\pi^*_{Z12}E_Z \otimes \iota'^*_{23}\pi^*_{Z23}\,\mathbf{R}\mathcal{H}om\left(E_Z, \pi^!_{Z1}(\mathcal{O}_{X_1})\right)\right) \arrow[d, "\pi_{13*}\iota_{Z'*}\beta_\Delta"] \\ \pi_{13*}\iota_{Z'*}\Delta_*\Delta^*\left(\iota'^*_{12}\pi^*_{Z12}E_Z \otimes \iota'^*_{23}\pi^*_{Z23}\,\mathbf{R}\mathcal{H}om\left(E_Z, \pi^!_{Z1}(\mathcal{O}_{X_1})\right)\right) \arrow[d, "\Delta_*\pi_{Z1*}\left(\left(\zeta_{\pi_{Z12},\iota'_{12},\Delta}\otimes\zeta_{\pi_{Z23},\iota'_{23},\Delta}\right)\circ\nu_\Delta\right)\circ\eta_{\Delta,\pi_{Z1}}\circ\eta^{-1}_{\pi_{13},\iota_{Z'},\Delta}", "\simeq"'] \\ \Delta_*\pi_{Z1*}\left(E_Z \otimes \mathbf{R}\mathcal{H}om\left(E_Z, \pi^!_{Z1}(\mathcal{O}_{X_1})\right)\right) \arrow[d, "\Delta_*\pi_{Z1*}\left(\mathrm{ev}_{E_Z}\right)"] \\ \Delta_*\pi_{Z1*}\pi^!_{Z1}(\mathcal{O}_{X_1}) \arrow[d, "\Delta_*\epsilon_{\pi_{Z1}}"] \\ \Delta_*\mathcal{O}_{X_1} \end{tikzcd} \tag{4.19}$$

Proof. Assume first that X_2 is proper. By Theorem 3.1 the adjunction counit $\Phi_E^{ladj}\Phi_E \to \mathrm{Id}_{X_1}$ is induced by the morphism of Fourier-Mukai kernels which we reproduce here for the convenience of our readers:

$$\pi_{13*}\left(\pi_{12}^* E \otimes \pi_{23}^*\left(E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})\right)\right) \xrightarrow{\beta_\Delta} \pi_{13*}\Delta_*\Delta^*\left(\pi_{12}^* E \otimes \pi_{23}^*\left(E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})\right)\right) \tag{4.20}$$

$$\pi_{13*}\Delta_*\Delta^*\left(\pi_{12}^* E \otimes \pi_{23}^*\left(E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})\right)\right) \simeq \Delta_*\pi_{1*}\left(E \otimes E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})\right) \tag{4.21}$$

$$\Delta_*\pi_{1*}\left(E \otimes E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{\Delta_*\pi_{1*}(\mathrm{ev}_E \otimes \mathrm{Id})} \Delta_*\pi_{1*}\left(\pi_1^!(\mathcal{O}_{X_1})\right) \tag{4.22}$$

$$\Delta_*\pi_{1*}\left(\pi_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{\Delta_*\epsilon_{\pi_1}} \Delta_*\mathcal{O}_{X_1}. \tag{4.23}$$

where the connecting isomorphism (4.21) is $\Delta_*\pi_{1*}\left(\left(\zeta_{\pi_{12},\Delta} \otimes \zeta_{\pi_{23},\Delta}\right) \circ \nu_\Delta\right) \circ \eta_{\Delta,\pi_1} \circ \eta^{-1}_{\pi_{13},\Delta}$.

We have $E = \iota_{Z*}E_Z$ and

$$E^\vee \otimes \pi_1^!\mathcal{O}_{X_1} = (\iota_{Z*}E_Z)^\vee \otimes \pi_1^!\mathcal{O}_{X_1} \xrightarrow{(2.10)} \mathbf{R}\mathcal{H}om\left(\iota_{Z*}E_Z, \pi_1^!\mathcal{O}_{X_1}\right) \xrightarrow{\delta^{-1}_{\iota_Z}} \iota_{Z*}\mathbf{R}\mathcal{H}om\left(E_Z, \pi_{Z1}^!\mathcal{O}_{X_1}\right). \tag{4.24}$$

Using the isomorphisms $\pi_{12}^*\iota_{Z*} \xrightarrow{\mu_{\sigma^T_{12}}} \iota_{Z12*}\pi_{Z12}^*$ and $\pi_{23}^*\iota_{Z*} \xrightarrow{\mu_{\sigma^T_{23}}} \iota_{Z23*}\pi_{Z23}^*$ and functoriality of β_Δ, we see that (4.20) is isomorphic to

$$\begin{array}{c} \pi_{13*}\left(\iota_{Z12*}\pi_{Z12}^*E_Z \otimes \iota_{Z23*}\pi_{Z23}^*\mathbf{R}\mathcal{H}om\left(E_Z, \pi_{Z1}^!\mathcal{O}_{X_1}\right)\right) \\ \Big\downarrow{\scriptstyle \pi_{13*}\beta_\Delta} \\ \pi_{13*}\Delta_*\Delta^*\left(\iota_{Z12*}\pi_{Z12}^*E_Z \otimes \iota_{Z23*}\pi_{Z23}^*\mathbf{R}\mathcal{H}om\left(E_Z, \pi_{Z1}^!\mathcal{O}_{X_1}\right)\right). \end{array} \tag{4.25}$$

By Prop. 4.1 it also follows that (4.22)-(4.23) is isomorphic to the composition

$$\begin{array}{c} \Delta_*\pi_{1*}\left(\iota_{Z*}E_Z \otimes \iota_{Z*}\mathbf{R}\mathcal{H}om\left(E_Z, \pi_{Z1}^!\mathcal{O}_{X_1}\right)\right) \\ \Big\downarrow{\scriptstyle \Delta_*\pi_{1*}\kappa_{\sigma_\Delta}} \\ \Delta_*\pi_{1*}\iota_{Z*}\left(E_Z \otimes \mathbf{R}\mathcal{H}om\left(E_Z, \pi_{Z1}^!\mathcal{O}_{X_1}\right)\right) \\ \Big\downarrow{\scriptstyle \Delta_*\pi_{1*}\iota_{Z*}\mathrm{ev}_{E_Z}} \\ \Delta_*\pi_{1*}\iota_{Z*}\iota_Z^!\pi_1^!(\mathcal{O}_{X_1}) \\ \Big\downarrow{\scriptstyle \Delta_*\pi_{1*}\epsilon_{\iota_Z}} \\ \Delta_*\pi_{1*}\pi_1^!(\mathcal{O}_{X_1}) \\ \Big\downarrow{\scriptstyle \Delta_*\epsilon_{\pi_1}} \\ \Delta_*(\mathcal{O}_{X_1}). \end{array} \tag{4.26}$$

The connecting isomorphism from (4.25) to (4.26) works out to be

$$\begin{array}{c} \pi_{13*}\Delta_*\Delta^*\left(\iota_{Z12*}\pi_{Z12}^*E_Z \otimes \iota_{Z23*}\pi_{Z23}^*\mathbf{R}\mathcal{H}om\left(E_Z, \pi_{Z1}^!\mathcal{O}_{X_1}\right)\right) \\ \simeq\Big\downarrow{\scriptstyle (\mu_{\sigma_{Z12}} \otimes \mu_{\sigma_{Z23}}) \circ \nu_\Delta} \\ \pi_{13*}\Delta_*\left(\iota_{Z*}\Delta^*\pi_{Z12}^*E_Z \otimes \iota_{Z*}\Delta^*\pi_{Z23}^*\mathbf{R}\mathcal{H}om\left(E_Z, \pi_{Z1}^!\mathcal{O}_{X_1}\right)\right) \\ \simeq\Big\downarrow{\scriptstyle \Delta_*\pi_{1*}(\iota_{Z*}\zeta_{\pi_{Z12},\Delta} \otimes \iota_{Z*}\zeta_{\pi_{Z23},\Delta}) \circ \eta_{\Delta,\pi_1} \circ \eta^{-1}_{\pi_{13},\Delta}} \\ \Delta_*\pi_{1*}\left(\iota_{Z*}E_Z \otimes \iota_{Z*}\mathbf{R}\mathcal{H}om\left(E_Z, \pi_{Z1}^!\mathcal{O}_{X_1}\right)\right) \end{array} \tag{4.27}$$

By functoriality the bottom isomorphism of (4.27) commutes with the top morphism of (4.26), so we conclude that (4.20)-(4.23) is isomorphic to the composition of

$$
\begin{array}{c}
\pi_{13*}\left(\iota_{Z12*}\pi^*_{Z12}E_Z \otimes \iota_{Z23*}\pi^*_{Z23}\,\mathbf{R}\,\mathcal{H}om\left(E_Z, \pi^!_{Z1}\mathcal{O}_{X_1}\right)\right) \\
\Big\downarrow {\scriptstyle \pi_{13*}\beta_\Delta} \\
\pi_{13*}\Delta_*\Delta^*\left(\iota_{Z12*}\pi^*_{Z12}E_Z \otimes \iota_{Z23*}\pi^*_{Z23}\,\mathbf{R}\,\mathcal{H}om\left(E_Z, \pi^!_{Z1}\mathcal{O}_{X_1}\right)\right) \\
\simeq \Big\downarrow {\scriptstyle \pi_{13*}\Delta_*((\mu_{Z12}\otimes\mu_{Z23})\circ\nu_\Delta)} \\
\pi_{13*}\Delta_*\left(\iota_{Z*}\Delta^*\pi^*_{Z12}E_Z \otimes \iota_{Z*}\Delta^*\pi^*_{Z23}\,\mathbf{R}\,\mathcal{H}om\left(E_Z, \pi^!_{Z1}\mathcal{O}_{X_1}\right)\right) \\
\Big\downarrow {\scriptstyle \pi_{13*}\Delta_*\kappa_{\sigma_\Delta}} \\
\pi_{13*}\Delta_*\iota_{Z*}\left(\Delta^*\pi^*_{Z12}E_Z \otimes \Delta^*\pi^*_{Z23}\,\mathbf{R}\,\mathcal{H}om\left(E_Z, \pi^!_{Z1}\mathcal{O}_{X_1}\right)\right)
\end{array}
\tag{4.28}
$$

with

$$
\begin{array}{c}
\pi_{13*}\Delta_*\iota_{Z*}\left(\Delta^*\pi^*_{Z12}E_Z \otimes \Delta^*\pi^*_{Z23}\,\mathbf{R}\,\mathcal{H}om\left(E_Z, \pi^!_{Z1}\mathcal{O}_{X_1}\right)\right) \\
\simeq \Big\downarrow {\scriptstyle \Delta_*\pi_{1*}\left(\zeta_{\pi_{Z12},\Delta}\otimes\zeta_{\pi_{Z23},\Delta}\right)\circ\eta_{\Delta,\pi_1}\circ\eta^{-1}_{\pi_{13},\Delta}} \\
\Delta_*\pi_{1*}\iota_{Z*}\left(E_Z \otimes \mathbf{R}\,\mathcal{H}om\left(E_Z, \pi^!_{Z1}\mathcal{O}_{X_1}\right)\right) \\
\Big\downarrow {\scriptstyle \Delta_*\pi_{1*}\iota_{Z*}\,\mathrm{ev}_{E_Z}} \\
\Delta_*\pi_{1*}\iota_{Z*}\iota^!_Z\pi^!_1(\mathcal{O}_{X_1}) \\
\Big\downarrow {\scriptstyle \Delta_*\pi_{1*}\epsilon_{\iota_Z}} \\
\Delta_*\pi_{1*}\pi^!_1(\mathcal{O}_{X_1}). \\
\Big\downarrow {\scriptstyle \Delta_*\epsilon_{\pi_1}} \\
\Delta_*\mathcal{O}_{X_1}
\end{array}
\tag{4.29}
$$

The claim of the theorem follows by applying the base change for Künneth maps of Prop. 4.4(2) to (4.28) and noting that as $\pi_{Z1} = \pi_1 \circ \iota_Z$ so by compatibility of the $(f_*, f^\times)$ adjunction with pseudo-functoriality, counits $\pi_{1*}\epsilon_{\iota_Z}$ and ϵ_{π_1} at the bottom of (4.29) compose to give $\epsilon_{\pi_{Z1}}$.

Suppose now X_2 is not proper. Then, following Section 3.2, we compactify X_2 by choosing an open immersion $j\colon X_2 \to \bar{X}_2$ with $\bar{X}_2$ proper. Similar to the conventions in Section 3.2, we use j to also denote all the compactification maps induced by $j\colon X_2 \to \bar{X}_2$ and we put a bar over various objects and morphisms to denote their compactified versions. E.g. we denote the inclusion $Z \xrightarrow{\iota_Z} X_1 \times X_2 \xrightarrow{j} X_1 \times \bar{X}_2$ by $\bar{\iota}_Z$. By the argument above the compactified version of the composition (4.19) gives a morphism $\bar{Q}_Z \to \Delta_{\mathcal{O}_{X_1}}$ which induces the compactified adjunction counit $\Phi^{ladj}_{\bar{E}}\Phi_{\bar{E}} \to \mathrm{Id}_{X_1}$. By the results of Section 3.2 the compactified and the uncompactified adjunction counits are naturally isomorphic, therefore to prove the claim of the theorem it suffices to exhibit an isomorphism $\bar{Q}_Z \xrightarrow{\sim} Q_Z$ which composed with the uncompactified (4.19) gives the compactified (4.19).

All the morphisms in (4.19) except for the first one are independent of the ambient space X_2. To be more precise, we have $\bar{\pi}_{13}\circ\bar{\iota}_{Z'} = \bar{\pi}_{13}\circ j\circ\iota_{Z'} = \pi_{13}\circ\iota_{Z'}$, and hence the compactified versions of last four morphisms in (4.19) are isomorphic to the uncompactified ones via pseudofunctoriality isomorphisms. It therefore suffices to find an isomorphism $\bar{Q}_Z \xrightarrow{\sim} Q_Z$ that would make the following diagram commute:

$$\begin{tikzcd}
Q \arrow[r, "\pi_{13*}\kappa_\sigma"] & \pi_{13*}\iota_{Z'*}\left(\iota'^*_{12}\pi^*_{Z12}E_Z \otimes \iota'^*_{23}\pi^*_{Z23}\,\mathbf{R}\mathcal{H}om\left(E_Z, \pi^!_{Z1}\mathcal{O}_{X_1}\right)\right) \\
\bar{Q} \arrow[u, "\sim"] \arrow[r, "\bar{\pi}_{13*}\kappa_{\bar{\sigma}}"'] & \bar{\pi}_{13*}\bar{\iota}_{Z'*}\left(\iota'^*_{12}\pi^*_{Z12}E_Z \otimes \iota'^*_{23}\pi^*_{Z23}\,\mathbf{R}\mathcal{H}om\left(E_Z, \pi^!_{Z1}\mathcal{O}_{X_1}\right)\right). \arrow[u, "\simeq"', "\eta_{\pi_{13},\iota_{Z'}}\circ\eta^{-1}_{\bar{\pi}_{13},\bar{\iota}_{Z'}}"]
\end{tikzcd} \tag{4.30}$$

But $\pi_{13*} \simeq \bar{\pi}_{13*}j_*$ and square σ is obtained from square $\bar{\sigma}$ by the base change $j\colon X_1 \times X_2 \times X_1 \to X_1 \times \bar{X}_2 \times X_1$. So the desired statement is precisely the base change for Künneth maps of Prps. 4.4. $\square$

We have similarly:

Theorem 4.2. *Under the assumptions of Theorem 4.1 let $\Psi_E\colon D(X_2) \to D(X_1)$ be the Fourier-Mukai transform with kernel E. The adjunction counit $\Psi_E\Psi_E^{radj} \to \mathrm{Id}$ is isomorphic to the morphism of Fourier-Mukai transforms induced by the composition:*

$$\begin{tikzcd}
Q'_Z = \pi_{13*}\left(\iota_{Z12*}\pi^*_{Z12}\,\mathbf{R}\mathcal{H}om\left(E_Z, \pi^!_{Z1}\mathcal{O}_{X_1}\right) \otimes \iota_{Z23*}\pi^*_{Z23}E_Z\right) \arrow[d, "\pi_{13*}\kappa_\sigma"] \\
\pi_{13*}\iota_{Z'*}\left(\pi^*_{Z12}\,\mathbf{R}\mathcal{H}om\left(E_Z, \pi^!_{Z1}\mathcal{O}_{X_1}\right) \otimes \pi^*_{Z23}E_Z\right) \arrow[d, "\pi_{13*}\iota_{Z'*}\beta_\Delta"] \\
\pi_{13*}\iota_{Z'*}\Delta_*\Delta^*\left(\pi^*_{Z12}\,\mathbf{R}\mathcal{H}om\left(E_Z, \pi^!_{Z1}\mathcal{O}_{X_1}\right) \otimes \pi^*_{Z23}E^\vee_Z\right) \arrow[d, "\simeq"', "\Delta_*\pi_{Z1*}\left(\left(\zeta_{\pi_{Z12},\iota'_{12},\Delta}\otimes\zeta_{\pi_{Z23},\iota'_{23},\Delta}\right)\circ\nu_\Delta\right)\circ\eta_{\Delta,\pi_{Z1}}\circ\eta^{-1}_{\pi_{13},\iota_{Z'},\Delta}"] \\
\Delta_*\pi_{Z1*}\left(\mathbf{R}\mathcal{H}om\left(E_Z, \pi^!_{Z1}\mathcal{O}_{X_1}\right) \otimes E_Z\right) \arrow[d, "\Delta_*\pi_{Z1*}\,\mathrm{ev}_{E_Z}"] \\
\Delta_*\pi_{Z1*}\pi^!_{Z1}(\mathcal{O}_{X_1}) \arrow[d, "\Delta_*\epsilon_{\pi_{Z1}}"] \\
\Delta_*\mathcal{O}_{X_1}.
\end{tikzcd} \tag{4.31}$$

One of the main advantages of the alternative decompositions offered by Theorems 4.1 and 4.2 is that most of the morphisms in them can become isomorphisms under fairly reasonable assumptions on Z, X_1 and X_2. We can then write down twists of Φ_E and Ψ_E fairly easily, for example:

Corollary 4.5. *Let X_1 and X_2 be separable schemes of finite type over a field k. Let $Z \xrightarrow{\iota_Z} X_1 \times X_2$ be a regular closed immersion proper over X_1 and X_2. Suppose $\pi_{Z1*}\mathcal{O}_Z = \mathcal{O}_{X_1}$ where π_{Z1} is the composition $Z \xrightarrow{\iota_Z} X_1 \times X_2 \xrightarrow{\pi_1} X_1$. Suppose also that $Z \times X_1$ and $X_1 \times Z$ are* Tor*-independent inside $X_1 \times X_2 \times X_1$ and denote by Z' their intersection. Denote by $\iota_{Z'}$ the inclusion $Z' \hookrightarrow X_1 \times X_2 \times X_1$.*

Then the Fourier-Mukai kernel of the dual co-twist of $\Phi_{\mathcal{O}_Z}\colon D(X_1) \to D(X_2)$ is $\pi_{13*}\iota_{Z'*}\left(\mathcal{L} \otimes \mathcal{I}_{\Delta'}[1]\right)$ where $\mathcal{I}_{\Delta'}$ is the ideal sheaf of the diagonal Z in Z' and $\mathcal{L}$ is the pullback of $\pi^!_{Z1}(\mathcal{O}_{X_1})$ via $X_1 \times Z$ to Z'.

Proof. The Fourier-Mukai kernel of the dual co-twist of Φ_E is the cone of the morphism of kernels underlying $\Phi_E^{ladj}\Phi_E \to \mathrm{Id}$. Applying Theorem 4.1, we note that under the assumptions of this corollary, all the morphisms in (4.19) become isomorphisms with the exception of

$$\begin{tikzcd}
\pi_{13*}\iota_{Z'*}\left(\iota'^*_{12}\pi^*_{Z12}E_Z \otimes \iota'^*_{23}\pi^*_{Z23}\,\mathbf{R}\mathcal{H}om\left(E_Z, \pi^!_{Z1}\mathcal{O}_{X_1}\right)\right) \arrow[d, "\pi_{13*}\iota_{Z'*}\beta_\Delta"] \\
\pi_{13*}\iota_{Z'*}\Delta_*\Delta^*\left(\iota'^*_{12}\pi^*_{Z12}E_Z \otimes \iota'^*_{23}\pi^*_{Z23}\,\mathbf{R}\mathcal{H}om\left(E_Z, \pi^!_{Z1}\mathcal{O}_{X_1}\right)\right).
\end{tikzcd}$$

Since $E_Z = \mathcal{O}_Z$ the above simplifies to the direct image under $\pi_{13*}\iota_{Z'*}$ of

$$\iota'^{*}_{23}\pi^{*}_{Z23}\pi^{!}_{Z1}(\mathcal{O}_{X_1}) \xrightarrow{\beta_\Delta} \Delta_*\Delta^*\left(\iota'^{*}_{23}\pi^{*}_{Z23}\pi^{!}_{Z1}(\mathcal{O}_{X_1})\right).$$

Write $\mathcal{L}$ for $\iota'^{*}_{23}\pi^{*}_{Z23}\pi^{!}_{Z1}(\mathcal{O}_{X_1})$. By Lemma 2.1 (with $f = \mathrm{Id}$) the morphism $\mathcal{L} \xrightarrow{\beta_\Delta} \Delta_*\Delta^*\mathcal{L}$ is isomorphic to $\mathcal{L} \otimes (\mathcal{O}_{Z'} \to \Delta_*\Delta^*\mathcal{O}_{Z'})$. Since $\mathcal{O}_{Z'} \xrightarrow{\beta_\Delta} \Delta_*\Delta^*\mathcal{O}_{Z'}$ is just the sheaf restriction $\mathcal{O}_{Z'} \to \Delta_*\mathcal{O}_Z$, its cone is $\mathcal{I}_{\Delta'}[1]$ and the claim follows. $\square$

5. An example

Let us give a concrete example of using the results of section 4. For this example we choose the naive derived category transform induced by the Mukai flop. This transform is not an equivalence - it was proved by Namikawa in [Nam03] by direct comparison of Hom spaces. Below we use Cor. 4.5 to compute the kernel which defines its dual co-twist as the Fourier-Mukai transform. We stress that the value of this section lies not in the answer itself, but in demonstrating how the methods of the paper apply to obtain it. However, the reader may observe that the kernel we obtain is a line bundle supported on the zero-section of the product. We shall demonstrate in [AL] that this is the reason for the braiding which occurs between natural spherical twists in the derived categories of the cotangent bundles of complete flag varieties (see [KT07], §4).

Let V be a 3-dimensional vector space and let X_1 be the scheme $T^*\mathbb{P}(V)$, that is - the total space of the cotangent bundle of $\mathbb{P}(V)$. Similarly, let X_2 be the scheme $T^*\mathbb{P}(V^\vee)$. These schemes admit the following description:

$$X_1 = \left\{ \begin{tikzcd} 0 & \subset & U_1 \arrow[ll, bend right, "\alpha"'] & \subset & V \arrow[ll, bend right, "\alpha"'] \end{tikzcd} \right\} := \left\{\ U_1 \subset V,\ \alpha \in \mathrm{End}(V) \ \middle|\ \dim U_1 = 1, \alpha(V) \subseteq U_1,\ \alpha(U_1) = 0\ \right\}$$

$$X_2 = \left\{ \begin{tikzcd} 0 & \subset & U_2 \arrow[ll, bend right, "\alpha"'] & \subset & V \arrow[ll, bend right, "\alpha"'] \end{tikzcd} \right\} := \left\{\ U_2 \subset V,\ \alpha \in \mathrm{End}(V) \ \middle|\ \dim U_2 = 2, \alpha(V) \subseteq U_2,\ \alpha(U_2) = 0\ \right\}$$

We also have a variety

$$Z = \left\{ \begin{tikzcd} 0 & \subset & U_1 & \subset & U_2 \arrow[llll, bend right, "\alpha"'] & \subset & V \arrow[llll, bend right, "\alpha"'] \end{tikzcd} \right\}$$

with natural "forgetful" maps $\phi_k \colon Z \to X_k$ which forget one of the subspaces. Each map ϕ_k is isomorphic to the blow-up of the zero section carved out by $\alpha = 0$ in X_k. Both blowups have the same exceptional divisor $F \subset Z$ which is carved out by $\alpha = 0$:

$$F = \{0 \subset U_1 \subset U_2 \subset V\}.$$

The resulting birational transformation $X_1 \dashrightarrow X_2$ which transforms the zero-section $\mathbb{P}(V) \hookrightarrow X_1$ into the zero-section $\mathbb{P}(V^\vee) \hookrightarrow X_2$ is a local model of a four-dimensional Mukai flop. Note that maps ϕ_k are proper and, since each map ϕ_k is a blowup of X_k, we have $\phi_{k*}\mathcal{O}_Z = \mathcal{O}_{X_k}$.

Let Φ be the functor $\phi_{2*}\phi_1^*$ from $D(X_1)$ to $D(X_2)$ and let us compute its dual co-twist. The functor Φ is a Fourier-Mukai transform with the kernel $\iota_{Z*}\mathcal{O}_Z$, where $\iota_Z = \phi_1 \times \phi_2 : Z \to X_1 \times X_2$. We have:

$$X_1 \times X_2 \times X_1 = \left\{ \begin{tikzcd} 0 & \subset & U_1, U_2, U_1' \arrow[ll, bend right, "{\alpha_1,\alpha_2,\alpha_1'}"'] & \subset & V \arrow[ll, bend right, "{\alpha_1,\alpha_2,\alpha_1'}"'] \end{tikzcd} \right\}$$

$$Z \times X_1 = \left\{ \begin{tikzcd} 0 & \subset & U_1 & \subset & U_2 \arrow[llll, bend right, "\alpha_1=\alpha_2"'] & \subset & V \arrow[llll, bend right, "\alpha_1=\alpha_2"'] \end{tikzcd},\ \begin{tikzcd} 0 & \subset & U_1' \arrow[ll, bend right, "\alpha_1'"'] & \subset & V \arrow[ll, bend right, "\alpha_1'"'] \end{tikzcd} \right\}$$

$$X_1 \times Z = \left\{ \begin{tikzcd}[column sep=small] 0 & \subset & U_1 \arrow[ll, bend right, "\alpha_1"'] & \subset & V \arrow[ll, bend right, "\alpha_1"'] \end{tikzcd} , \quad \begin{tikzcd}[column sep=small] 0 & \subset & U_1' & \subset & U_2 \arrow[llll, bend right, "\alpha_2=\alpha_1'"'] & \subset & V \arrow[llll, bend right, "\alpha_2=\alpha_1'"'] \end{tikzcd} \right\}.$$

It follows that $Z' = (Z \times X_1) \cap (X_1 \times Z) \subset X_1 \times X_2 \times X_1$ can be described as

$$Z' = \left\{ \begin{tikzcd}[column sep=small] 0 & \subset & U_1, U_1' & \subset & U_2 \arrow[llll, bend right, "\alpha"'] & \subset & V \arrow[llll, bend right, "\alpha"'] \end{tikzcd} \;\middle|\; \alpha(V) \subseteq U_1 \cap U_1' \right\}.$$

Observe that for any point of Z' we have $U_1 = U_1'$ or $\alpha = 0$ (or both). Therefore Z' consists of two irreducible components: the diagonal ΔZ and the zero section

$$P = \{0 \subset U_1, U_1' \subset U_2 \subset V\}.$$

The intersection $\Delta Z \cap P$ considered as a subvariety of ΔZ is the exceptional divisor F of the blowups $Z \xrightarrow{\phi_i} X_i$ described above. On the other hand, let $P \xrightarrow{\phi_{13}} \mathbb{P}(V) \times \mathbb{P}(V)$ be the map which forgets the subspace U_2. It is the blowup of the diagonal of $\mathbb{P}(V) \times \mathbb{P}(V)$ and its exceptional divisor in P is carved out by $U_1 = U_1'$, i.e. it is $F = \Delta Z \cap P$ again.

By Cor. 4.5 the dual co-twist of Φ is the Fourier-Mukai transform $X_1 \to X_1$ with kernel

$$K = \pi_{13*} \iota_{Z'*} \left(\mathcal{L} \otimes \mathcal{I}_\Delta[1] \right) \in D(X_1 \times X_1).$$

Here $\iota_{Z'}$ is the inclusion $Z' \hookrightarrow X_1 \times X_2 \times X_1$, $\mathcal{I}_\Delta$ is the ideal sheaf of ΔZ in Z' and $\mathcal{L}$ is the pullback of $\phi_1^!(\mathcal{O}_{X_1})$ to Z' via $X_1 \times Z$.

Since $Z \xrightarrow{\phi_1} X_1$ is the blow-up of the zero-section $\mathbb{P}(V) \hookrightarrow X_1$ whose codimension is 2, we know that $\phi_1^!(\mathcal{O}_{X_1})$ is the line bundle $\mathcal{O}_Z(F)$ where F is the exceptional divisor of the blow-up. On the other hand, pulling back along the projection

$$Z \to \mathbb{P}(V) \times \mathbb{P}(V^\vee)$$

induces an isomorphism

$$\operatorname{Pic} Z \simeq \operatorname{Pic} \mathbb{P}(V) \times \operatorname{Pic} \mathbb{P}(V^\vee).$$

A simple calculation shows that $\mathcal{O}_Z(F)$ is the pullback of $\mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(V^\vee)}(-1,-1)$. Similarly

$$\operatorname{Pic} Z' \simeq \operatorname{Pic} \mathbb{P}(V) \times \operatorname{Pic} \mathbb{P}(V^\vee) \times \operatorname{Pic} \mathbb{P}(V)$$

and $\mathcal{L}$, being the pullback to Z' of $\phi_1^!(\mathcal{O}_{X_1})$ via $X_1 \times Z$, is then the pullback of $\mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(V^\vee)\times\mathbb{P}(V)}(0,-1,-1)$.

Since Z' has two irreducible components ΔZ and P, we have $\mathcal{I}_\Delta \simeq \iota_{P*}\mathcal{O}_P(-\Delta Z \cap P)$ where ι_P is the inclusion $P \hookrightarrow Z'$. We therefore have $K \simeq \pi_{13*}\iota_{Z'*}\iota_{P*}\left(\iota_P^*\mathcal{L} \otimes \mathcal{O}_P(-F)[1]\right)$. A simple computation shows that $\mathcal{O}_P(-F)$ is the pullback of $\mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(V^\vee)\times\mathbb{P}(V)}(-1,1,-1)$ and therefore $\iota_P^*\mathcal{L} \otimes \mathcal{O}_P(-F)$ is the pullback of $\mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(V^\vee)\times\mathbb{P}(V)}(-1,0,-2)$. We conclude that $K \simeq \pi_{13*}\iota_{Z'*}\iota_{P*}\phi_{13}^*\left(\mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(V)}(-1,-2)[1]\right)$.

Now observe that the following diagram commutes

$$\begin{tikzcd} P \arrow[r, "\iota_P"] \arrow[d, "\phi_{13}"'] & Z' \arrow[r, "\iota_{Z'}"] & X_1 \times X_2 \times X_1 \arrow[d, "\pi_{13}"] \\ \mathbb{P}(V) \times \mathbb{P}(V) \arrow[rr, "\iota_0"'] & & X_1 \times X_1 \end{tikzcd}$$

where ι_0 is the zero-section inclusion of $\mathbb{P}(V) \times \mathbb{P}(V)$ into $X_1 \times X_1$. We conclude that

$$K \simeq \iota_{0*}\phi_{13*}\phi_{13}^*\left(\mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(V)}(-1,-2)[1]\right) \simeq \iota_{0*}\left(\mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(V)}(-1,-2)[1]\right).$$

Appendix A. The unabridged proof of Theorem 3.1

Here we give a complete version of the proof of Theorem 3.1. It contains explicit computations of all the connecting isomorphisms which we left out of the version in the main body of the paper so as to emphasise the meaningful part of the proof. The version below is for referees and others who relish seeing how the monoidal structure of the inverse image functor commutes with pseudofunctoriality and with the associativity of tensor product. Lasciate ogne speranza, voi ch'intrate.

Proof. Set

$$Q' = \pi_{23}^* \left(\pi_1^! \mathcal{O}_{X_1} \otimes E^\vee\right) \otimes \pi_{12}^* E$$

so that $Q = \pi_{13*} Q'$. Since $\pi_{12} \circ \Delta = \pi_{23} \circ \Delta = \mathrm{Id}$ we have a natural isomorphism

$$\Delta^* Q' \xrightarrow{\nu_\Delta} \Delta^* \pi_{23}^* \left(\pi_1^! \mathcal{O}_{X_1} \otimes E^\vee\right) \otimes \Delta^* \pi_{12}^* E \xrightarrow{\zeta_{\pi_{23},\Delta} \otimes \zeta_{\pi_{12},\Delta}} \left(\pi_1^! \mathcal{O}_{X_1} \otimes E^\vee\right) \otimes E. \tag{A.1}$$

We therefore define a morphism

$$\Delta^* Q' \xrightarrow{\text{(A.1)}} \left(\pi_1^! \mathcal{O}_{X_1} \otimes E^\vee\right) \otimes E \xrightarrow{E \otimes \left(E^\vee \otimes (-)\right) \to \mathrm{Id}} \pi_1^! \mathcal{O}_{X_1}. \tag{A.2}$$

In these terms, the morphism of FM-transforms $D(X) \to D(X)$ induced by $Q \xrightarrow{(3.3)-(3.6)} \Delta \mathcal{O}_X$ is:

$$\tilde{\pi}_{2*} \left(\pi_{13*} Q' \otimes \tilde{\pi}_1^*(-)\right) \xrightarrow{\mathrm{Id} \to \Delta_* \Delta^*} \tilde{\pi}_{2*} \left(\pi_{13*} \Delta_* \Delta^* Q' \otimes \tilde{\pi}_1^*(-)\right) \tag{A.3}$$

$$\tilde{\pi}_{2*} \left(\pi_{13*} \Delta_* \Delta^* Q' \otimes \tilde{\pi}_1^*(-)\right) \xrightarrow{\eta_{\Delta,\pi_1} \circ \eta^{-1}_{\pi_{13},\Delta}} \tilde{\pi}_{2*} \left(\Delta_* \pi_{1*} \Delta^* Q' \otimes \tilde{\pi}_1^*(-)\right) \tag{A.4}$$

$$\tilde{\pi}_{2*} \left(\Delta_* \pi_{1*} \Delta^* Q' \otimes \tilde{\pi}_1^*(-)\right) \xrightarrow{\text{(A.2)}} \tilde{\pi}_{2*} \left(\Delta_* \pi_{1*} \pi_1^! \mathcal{O}_{X_1} \otimes \tilde{\pi}_1^*(-)\right) \tag{A.5}$$

$$\tilde{\pi}_{2*} \left(\Delta_* \pi_{1*} \pi_1^! \mathcal{O}_{X_1} \otimes \tilde{\pi}_1^*(-)\right) \xrightarrow{\pi_{1*}\pi_1^! \to \mathrm{Id}} \tilde{\pi}_{2*} \left(\Delta_* \mathcal{O}_{X_1} \otimes \tilde{\pi}_1^*(-)\right) \tag{A.6}$$

On the other hand, Φ_E is the composition of functors π_1^*, $E \otimes (-)$ and π_{2*}. Each of these functors has a left adjoint, these adjoints are $\pi_{1*}(\pi_1^! \mathcal{O}_{X_1} \otimes -)$, $E^\vee \otimes (-)$ and π_2^*, respectively. Therefore, the adjunction counit $\Phi_E^{\mathrm{ladj}} \Phi_E \to \mathrm{Id}$ is the composition of the three corresponding adjunction counits:

$$\pi_{1*} \left(\pi_1^! \mathcal{O}_{X_1} \otimes \left(E^\vee \otimes \pi_2^* \pi_{2*} \left(E \otimes \pi_1^* \left(-\right)\right)\right)\right) \xrightarrow{\pi_2^* \pi_{2*} \to \mathrm{Id}} \pi_{1*} \left(\pi_1^! \mathcal{O}_{X_1} \otimes \left(E^\vee \otimes \left(E \otimes \pi_1^* \left(-\right)\right)\right)\right) \tag{A.7}$$

$$\pi_{1*} \left(\pi_1^! \mathcal{O}_{X_1} \otimes \left(E^\vee \otimes \left(E \otimes \pi_1^* \left(-\right)\right)\right)\right) \xrightarrow{E^\vee \otimes (E \otimes (-)) \to \mathrm{Id}} \pi_{1*} \left(\pi_1^! \mathcal{O}_{X_1} \otimes \pi_1^* \left(-\right)\right) \tag{A.8}$$

$$\pi_{1*} \left(\pi_1^! \mathcal{O}_{X_1} \otimes \pi_1^* \left(-\right)\right) \to \mathrm{Id} \tag{A.9}$$

The claim of the theorem is that the composition (A.7)-(A.9) is isomorphic to the composition (A.3)-(A.6).

Let us clarify some terminology. We say that two morphisms of functors $f \to g$ and $f' \to g'$ are isomorphic if there exist connecting isomorphisms $f \xrightarrow{\sim} f'$ and $g \xrightarrow{\sim} g'$ such that the diagram

$$\begin{tikzcd} f \arrow[r] \arrow[d, "\sim"'] & g \arrow[d, "\sim"] \\ f' \arrow[r] & g' \end{tikzcd} \tag{A.10}$$

commutes. Clearly it is an equivalence relation on the set of all morphisms between all functors between two given categories. In particular, it is transitive.

If we further have a morphism of functors $g \to h$ which is isomorphic to a morphism of functors $g'' \to h''$ then $f \to g \to h$ is isomorphic to $f' \to g' \xrightarrow{\sim} g'' \to h''$, where the connecting isomorphism $g' \xrightarrow{\sim} g''$ is the composition of the inverse of the connecting isomorphism $g \xrightarrow{\sim} g'$ with the connecting isomorphism $g \xrightarrow{\sim} g''$.

Our strategy therefore is to consecutively replace the morphisms which compose (3.19)-(3.21) by isomorphic ones until we obtain (3.15)-(3.18).

Observe that the following diagram, whose vertical arrows are all isomorphisms, commutes:

$$
\begin{tikzcd}
\pi_{1*}\left(\pi_1^! \mathcal{O}_{X_1} \otimes \left(E^\vee \otimes \pi_2^* \pi_{2*}\left(E \otimes \pi_1^*(-)\right)\right)\right) \arrow[r, "(A.7)"] \arrow[d, "\rho^{-1}"'] & \pi_{1*}\left(\pi_1^! \mathcal{O}_{X_1} \otimes \left(E^\vee \otimes \left(E \otimes \pi_1^*(-)\right)\right)\right) \arrow[d, "\rho^{-1}"] \\
\pi_{1*}\left(\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \pi_2^* \pi_{2*}\left(E \otimes \pi_1^*(-)\right)\right) \arrow[r, "\operatorname{Id} \otimes \left(\pi_2^*\pi_{2*} \to \operatorname{Id}\right)"] \arrow[d, "\mu"'] & \pi_{1*}\left(\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \left(E \otimes \pi_1^*(-)\right)\right) \arrow[d, "\operatorname{Id} \otimes \left(\eta_{\pi_{23},\Delta} \circ \zeta^{-1}_{\pi_{12},\Delta}\right)"] \\
\pi_{1*}\left(\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \pi_{23*}\pi_{12}^*\left(E \otimes \pi_1^*(-)\right)\right) \arrow[r, "\beta_\Delta"] \arrow[d, "\alpha_{\pi_{23}}"'] & \pi_{1*}\left(\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \pi_{23*}\Delta_*\Delta^*\pi_{12}^*\left(E \otimes \pi_1^*(-)\right)\right) \arrow[d, "\nu_\Delta^{-1} \circ \alpha_\Delta \circ \alpha_{\pi_{23}}"] \\
\pi_{1*}\pi_{23*}\left(\pi_{23}^*\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \pi_{12}^*\left(E \otimes \pi_1^*(-)\right)\right) \arrow[r, "\beta_\Delta"] \arrow[d, "\rho^{-1} \circ \left(\operatorname{Id} \otimes \nu_{\pi_{12}}\right)"'] & \pi_{1*}\pi_{23*}\Delta_*\Delta^*\left(\pi_{23}^*\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \pi_{12}^*\left(E \otimes \pi_1^*(-)\right)\right) \arrow[d, "\rho^{-1} \circ \left(\operatorname{Id} \otimes \nu_{\pi_{12}}\right)"] \\
\pi_{1*}\pi_{23*}\left(\left(\pi_{23}^*\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \pi_{12}^* E\right) \otimes \pi_{12}^*\pi_1^*(-)\right) \arrow[r, "\beta_\Delta"] \arrow[d, "\operatorname{Id}"'] & \pi_{1*}\pi_{23*}\Delta_*\Delta^*\left(\left(\pi_{23}^*\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \pi_{12}^* E\right) \otimes \pi_{12}^*\pi_1^*(-)\right) \arrow[d, "\nu_\Delta"] \\
\pi_{1*}\pi_{23*}\left(\left(\pi_{23}^*\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \pi_{12}^* E\right) \otimes \pi_{12}^*\pi_1^*(-)\right) \arrow[r, "\nu_\Delta \circ \beta_\Delta"] & \pi_{1*}\pi_{23*}\Delta_*\left(\Delta^*\left(\pi_{23}^*\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \pi_{12}^* E\right) \otimes \Delta^*\pi_{12}^*\pi_1^*(-)\right)
\end{tikzcd}
\tag{A.11}
$$

The first square in it commutes by functoriality of ρ^{-1}, the second commutes by Lemma 3.2, the third commutes by Lemma 2.1, the fourth commutes by functoriality of β_Δ and the fifth commutes tautologically.

We now want to simplify the connecting isomorphism in the right column of (A.11). By compatibility of the projection formula with pseudofunctoriality (see diagram (2.29)) we have an equality

$$\alpha_\Delta \circ \alpha_{\pi_{23}} \circ (\operatorname{Id} \otimes \eta_{\pi_{23},\Delta}) = \left(\zeta^{-1}_{\pi_{23},\Delta} \otimes \operatorname{Id}\right) \circ \eta_{\pi_{23},\Delta} \circ (\alpha_{\pi_{23}\circ\Delta})$$

of two morphisms

$$\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \Delta^*\pi_{12}^*\left(E \otimes \pi_1^*(-)\right) \longrightarrow \pi_{23*}\Delta_*\left(\Delta^*\pi_{23}^*\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \Delta^*\pi_{12}^*\left(E \otimes \pi_1^*(-)\right)\right).$$

Since $\pi_{23} \circ \Delta = \operatorname{Id}$, we have $\alpha_{\pi_{23}\circ\Delta} = \operatorname{Id}$. It follows that the right-hand column of (A.11) equals to

$$
\begin{tikzcd}
\pi_{1*}\left(\pi_1^! \mathcal{O}_{X_1} \otimes \left(E^\vee \otimes \left(E \otimes \pi_1^*(-)\right)\right)\right) \arrow[d, "\rho^{-1}"] \\
\pi_{1*}\left(\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \left(E \otimes \pi_1^*(-)\right)\right) \arrow[d, "\left(\zeta^{-1}_{\pi_{23},\Delta} \otimes \zeta^{-1}_{\pi_{12},\Delta}\right) \circ \eta_{\pi_{23},\Delta}"] \\
\pi_{1*}\pi_{23*}\Delta_*\left(\Delta^*\pi_{23}^*\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \Delta^*\pi_{12}^*\left(E \otimes \pi_1^*(-)\right)\right) \arrow[d, "\nu_\Delta \circ \rho^{-1} \circ \left(\operatorname{Id} \otimes \nu_{\pi_{12}}\right) \circ \nu_\Delta^{-1}"] \\
\pi_{1*}\pi_{23*}\Delta_*\left(\Delta^*\left(\pi_{23}^*\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \pi_{12}^* E\right) \otimes \Delta^*\pi_{12}^*\pi_1^*(-)\right)
\end{tikzcd}
\tag{A.12}
$$

Note that ν_Δ^{-1} and $\operatorname{Id} \otimes \nu_{\pi_{12}}$ commute by functoriality. Note further, that by the compatibility of the map ν_Δ with the associativity of the tensor product (see diagram (2.25)) we have an equality

$$\nu_\Delta \circ \rho^{-1} \circ \nu_\Delta^{-1} = \left(\nu_\Delta^{-1} \otimes \operatorname{Id}\right) \circ \rho^{-1} \circ (\operatorname{Id} \otimes \nu_\Delta)$$

of two morphisms

$$\Delta^*\pi_{23}^*\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \Delta^*\left(\pi_{12}^* E \otimes \pi_{12}^*\pi_1^*(-)\right) \longrightarrow \Delta^*\left(\pi_{23}^*\left(E^\vee \otimes \pi_1^! \mathcal{O}_{X_1}\right) \otimes \pi_{12}^* E\right) \otimes \Delta^*\pi_{12}^*\pi_1^*(-).$$

It follows that composition (A.12) equals to

$$\begin{tikzcd}
\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes\left(E^\vee\otimes\left(E\otimes\pi_1^*\left(-\right)\right)\right)\right) \arrow[d, "\rho^{-1}"] \\
\pi_{1*}\left(\left(E^\vee\otimes\pi_1^!\mathcal{O}_{X_1}\right)\otimes\left(E\otimes\pi_1^*\left(-\right)\right)\right) \arrow[d, "\left(\zeta^{-1}_{\pi_{23},\Delta}\otimes\left(\nu_\Delta\circ\nu_{\pi_{12}}\circ\zeta^{-1}_{\pi_{12},\Delta}\right)\right)\circ\eta_{\pi_{23},\Delta}"] \\
\pi_{1*}\pi_{23_*}\Delta_*\left(\Delta^*\pi_{23}^*\left(E^\vee\otimes\pi_1^!\mathcal{O}_{X_1}\right)\otimes\left(\Delta^*\pi_{12}^*E\otimes\Delta^*\pi_{12}^*\pi_1^*\left(-\right)\right)\right) \arrow[d, "\left(\nu_\Delta^{-1}\otimes\mathrm{Id}\right)\circ\rho^{-1}"] \\
\pi_{1*}\pi_{23_*}\Delta_*\left(\Delta^*\left(\pi_{23}^*\left(E^\vee\otimes\pi_1^!\mathcal{O}_{X_1}\right)\otimes\pi_{12}^*E\right)\otimes\Delta^*\pi_{12}^*\pi_1^*\left(-\right)\right)
\end{tikzcd} \tag{A.13}$$

By compatibility of ν with pseudofunctoriality (see diagram (2.26)) we have an equality

$$\nu_\Delta\circ\nu_{\pi_{12}}\circ\zeta^{-1}_{\pi_{12},\Delta}=\nu_{\pi_{12}\circ\Delta}\circ\left(\zeta^{-1}_{\pi_{12},\Delta}\otimes\zeta^{-1}_{\pi_{12},\Delta}\right)$$

of morphisms

$$E\otimes\pi_1^*(-)\longrightarrow\Delta^*\pi_{12}^*E\otimes\Delta^*\pi_{12}^*\pi_1^*(-).$$

Since $\pi_{12}\circ\Delta=\mathrm{Id}$ we further have $\nu_{\pi_{12}\circ\Delta}=\mathrm{Id}$. Therefore

$$\left(\zeta^{-1}_{\pi_{23},\Delta}\otimes\left(\nu_\Delta\circ\nu_{\pi_{12}}\circ\zeta^{-1}_{\pi_{12},\Delta}\right)\right)\circ\eta_{\pi_{23},\Delta}=\left(\zeta^{-1}_{\pi_{23},\Delta}\otimes\left(\zeta^{-1}_{\pi_{12},\Delta}\otimes\zeta^{-1}_{\pi_{12},\Delta}\right)\right)\circ\eta_{\pi_{23},\Delta}$$

in (A.13). Finally, by functoriality of ρ and of $\eta_{\pi_{23},\Delta}$ we have

$$\rho^{-1}\circ\left(\zeta^{-1}_{\pi_{23},\Delta}\otimes\left(\zeta^{-1}_{\pi_{12},\Delta}\otimes\zeta^{-1}_{\pi_{12},\Delta}\right)\right)\circ\eta_{\pi_{23},\Delta}=\left(\left(\zeta^{-1}_{\pi_{23},\Delta}\otimes\zeta^{-1}_{\pi_{12},\Delta}\right)\otimes\zeta^{-1}_{\pi_{12},\Delta}\right)\circ\eta_{\pi_{23},\Delta}\circ\rho^{-1}.$$

We conclude that (A.13) equals to

$$\begin{tikzcd}
\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes\left(E^\vee\otimes\left(E\otimes\pi_1^*\left(-\right)\right)\right)\right) \arrow[d, "\rho^{-1}\circ\rho^{-1}"] \\
\pi_{1*}\left(\left(\left(E^\vee\otimes\pi_1^!\mathcal{O}_{X_1}\right)\otimes E\right)\otimes\pi_1^*\left(-\right)\right) \arrow[d, "\left(\left(\nu_\Delta^{-1}\circ\left(\zeta^{-1}_{\pi_{23},\Delta}\otimes\zeta^{-1}_{\pi_{12},\Delta}\right)\right)\otimes\zeta^{-1}_{\pi_{12},\Delta}\right)\circ\eta_{\pi_{23},\Delta}"] \\
\pi_{1*}\pi_{23_*}\Delta_*\left(\Delta^*\left(\pi_{23}^*\left(E^\vee\otimes\pi_1^!\mathcal{O}_{X_1}\right)\otimes\pi_{12}^*E\right)\otimes\Delta^*\pi_{12}^*\pi_1^*\left(-\right)\right)
\end{tikzcd} \tag{A.14}$$

Recall now that we write Q' for $\pi_{23}^*\left(E^\vee\otimes\pi_1^!\mathcal{O}_{X_1}\right)\otimes\pi_{12}^*E$ and note that $\nu_\Delta^{-1}\circ\left(\zeta^{-1}_{\pi_{23},\Delta}\otimes\zeta^{-1}_{\pi_{12},\Delta}\right)$ in (A.14) is precisely the inverse of isomorphism (A.1). So what we have shown above is that (A.7) is isomorphic to

$$\pi_{1*}\pi_{23*}\left(Q'\otimes\pi_{12}^*\pi_1^*(-)\right)\xrightarrow{\nu_\Delta\circ\beta_\Delta}\pi_{1*}\pi_{23*}\Delta_*\left(\Delta^*Q'\otimes\Delta^*\pi_{12}^*\pi_1^*(-)\right) \tag{A.15}$$

with the connecting isomorphism on the RHS being

$$\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes\left(E^\vee\otimes\left(E\otimes\pi_1^*\left(-\right)\right)\right)\right)\xrightarrow{\left((\mathrm{A.1})^{-1}\otimes\zeta^{-1}_{\pi_{12},\Delta}\right)\circ\eta_{\pi_{23},\Delta}\circ\rho^{-1}\circ\rho^{-1}}\pi_{1*}\pi_{23*}\Delta_*\left(\Delta^*Q'\otimes\Delta^*\pi_{12}^*\pi_1^*(-)\right).$$

As $\pi_1\circ\pi_{23}=\tilde{\pi}_2\circ\pi_{13}$ and $\pi_1\circ\pi_{12}=\tilde{\pi}_1\circ\pi_{13}$ (see diagram (3.1)) we have the following commutative square

$$\begin{tikzcd}
\pi_{1*}\pi_{23*}\left(Q'\otimes\pi_{12}^*\pi_1^*(-)\right) \arrow[r, "(\mathrm{A.15})"] \arrow[d, "\left(\mathrm{Id}\otimes\left(\zeta^{-1}_{\tilde{\pi}_1,\pi_{13}}\circ\zeta_{\pi_1,\pi_{12}}\right)\right)\circ\eta_{\tilde{\pi}_2,\pi_{13}}\circ\eta^{-1}_{\pi_1,\pi_{23}}"'] & \pi_{1*}\pi_{23*}\Delta_*\left(\Delta^*Q'\otimes\Delta^*\pi_{12}^*\pi_1^*(-)\right) \arrow[d, "\left(\mathrm{Id}\otimes\left(\zeta^{-1}_{\tilde{\pi}_1,\pi_{13}}\circ\zeta_{\pi_1,\pi_{12}}\right)\right)\circ\eta_{\tilde{\pi}_2,\pi_{13}}\circ\eta^{-1}_{\pi_1,\pi_{23}}"] \\
\tilde{\pi}_{2*}\pi_{13*}\left(Q'\otimes\pi_{13}^*\tilde{\pi}_1^*(-)\right) \arrow[r, "\nu_\Delta\circ\beta_\Delta"'] & \tilde{\pi}_{2*}\pi_{13*}\Delta_*\left(\Delta^*Q'\otimes\Delta^*\pi_{13}^*\tilde{\pi}_1^*(-)\right)
\end{tikzcd}.$$

We finally conclude that (A.7) is isomorphic to

$$\tilde{\pi}_{2*}\pi_{13*}\left(Q' \otimes \pi_{13}^{*}\tilde{\pi}_{1}^{*}(-)\right) \xrightarrow{\nu_{\Delta}\circ\beta_{\Delta}} \tilde{\pi}_{2*}\pi_{13*}\Delta_{*}\left(\Delta^{*}Q' \otimes \Delta^{*}\pi_{13}^{*}\tilde{\pi}_{1}^{*}(-)\right) \tag{A.16}$$

with the connecting isomorphism on the RHS being

$$\pi_{1*}\left(\pi_{1}^{!}\mathcal{O}_{X_1} \otimes \left(E^{\vee} \otimes \left(E \otimes \pi_{1}^{*}(-)\right)\right)\right) \xrightarrow{\left((\mathrm{A.1})^{-1}\otimes\zeta^{-1}_{\tilde{\pi}_1,\pi_{13},\Delta}\right)\circ\eta_{\tilde{\pi}_2,\pi_{13},\Delta}\circ\rho^{-1}\circ\rho^{-1}} \tilde{\pi}_{2*}\pi_{13*}\Delta_{*}\left(\Delta^{*}Q' \otimes \Delta^{*}\pi_{13}^{*}\tilde{\pi}_{1}^{*}(-)\right). \tag{A.17}$$

Here we have used the fact that by pseudofunctoriality relations (2.19) and (2.20) we have

$$\eta_{\tilde{\pi}_2,\pi_{13}} \circ \eta^{-1}_{\pi_1,\pi_{23}} \circ \eta_{\pi_{23},\Delta} = \eta_{\tilde{\pi}_2,\pi_{13}} \circ \eta_{\pi_1\circ\pi_{23},\Delta} = \eta_{\tilde{\pi}_2,\pi_{13}} \circ \eta_{\tilde{\pi}_2\circ\pi_{13},\Delta} = \eta_{\tilde{\pi}_2,\pi_{13},\Delta}$$

and similarly $\zeta^{-1}_{\tilde{\pi}_1,\pi_{13}} \circ \zeta_{\pi_1,\pi_{12}} \circ \zeta^{-1}_{\pi_{12},\Delta} = \zeta^{-1}_{\tilde{\pi}_1,\pi_{13},\Delta}$.

Next, we note that the following diagram commutes:

$$\begin{tikzcd}
\pi_{1*}\left(\pi_{1}^{!}\mathcal{O}_{X_1} \otimes \left(E^{\vee} \otimes \left(E \otimes \pi_{1}^{*}(-)\right)\right)\right) \arrow[r, "(\mathrm{A.8})"] \arrow[d, "\rho^{-1}\circ\rho^{-1}"'] & \pi_{1*}\left(\pi_{1}^{!}\mathcal{O}_{X_1} \otimes \pi_{1}^{*}(-)\right) \arrow[d, "\mathrm{Id}"] \\
\pi_{1*}\left(\left(\left(\pi_{1}^{!}\mathcal{O}_{X_1} \otimes E^{\vee}\right) \otimes E\right) \otimes \pi_{1}^{*}(-)\right) \arrow[r, "\left((-)\otimes E^{\vee}\right)\otimes E\to\mathrm{Id}"] \arrow[d, "(\mathrm{A.1})^{-1}\otimes\mathrm{Id}"'] & \pi_{1*}\left(\pi_{1}^{!}\mathcal{O}_{X_1} \otimes \pi_{1}^{*}(-)\right) \arrow[d, "\mathrm{Id}"] \\
\pi_{1*}\left(\Delta^{*}Q' \otimes \pi_{1}^{*}(-)\right) \arrow[r, "(\mathrm{A.2})"] \arrow[d, "\left(\mathrm{Id}\otimes\zeta^{-1}_{\tilde{\pi}_1,\Delta}\right)\circ\eta_{\tilde{\pi}_2,\Delta}"'] & \pi_{1*}\left(\pi_{1}^{!}\mathcal{O}_{X_1} \otimes \pi_{1}^{*}(-)\right) \arrow[d, "\left(\mathrm{Id}\otimes\zeta^{-1}_{\tilde{\pi}_1,\Delta}\right)\circ\eta_{\tilde{\pi}_2,\Delta}"] \\
\tilde{\pi}_{2*}\Delta_{*}\pi_{1*}\left(\Delta^{*}Q' \otimes \pi_{1}^{*}\Delta^{*}\tilde{\pi}_{1}^{*}(-)\right) \arrow[r, "(\mathrm{A.2})"] & \tilde{\pi}_{2*}\Delta_{*}\pi_{1*}\left(\pi_{1}^{!}\mathcal{O}_{X_1} \otimes \pi_{1}^{*}\Delta^{*}\tilde{\pi}_{1}^{*}(-)\right)
\end{tikzcd} \tag{A.18}$$

Here the top square commutes by Lemma 2.3, the second square commutes by the definition of map (A.2) and the third square commutes by the functoriality. Therefore (A.8) is isomorphic to

$$\tilde{\pi}_{2*}\Delta_{*}\pi_{1*}\left(\Delta^{*}Q' \otimes \pi_{1}^{*}\Delta^{*}\tilde{\pi}_{1}^{*}(-)\right) \xrightarrow{(\mathrm{A.2})} \tilde{\pi}_{2*}\Delta_{*}\pi_{1*}\left(\pi_{1}^{!}\mathcal{O}_{X_1} \otimes \pi_{1}^{*}\Delta^{*}\tilde{\pi}_{1}^{*}(-)\right). \tag{A.19}$$

And finally, the following square

$$\begin{tikzcd}
\pi_{1*}\left(\pi_{1}^{!}\mathcal{O}_{X_1} \otimes \pi_{1}^{*}(-)\right) \arrow[r, "(\mathrm{A.9})"] \arrow[d, "\left(\mathrm{Id}\otimes\zeta^{-1}_{\tilde{\pi}_1,\Delta}\right)\circ\eta_{\tilde{\pi}_2,\Delta}"'] & \mathrm{Id} \arrow[d, "\zeta^{-1}_{\tilde{\pi}_1,\Delta}\circ\eta_{\tilde{\pi}_2,\Delta}"] \\
\tilde{\pi}_{2*}\Delta_{*}\pi_{1*}\left(\pi_{1}^{!}\mathcal{O}_{X_1} \otimes \pi_{1}^{*}\Delta_{*}\tilde{\pi}_{1}^{*}(-)\right) \arrow[r, "\pi_{1*}\left(\pi_1^{!}\mathcal{O}_{X_1}\otimes\pi_1^{*}(-)\right)\to\mathrm{Id}"] & \tilde{\pi}_{2*}\Delta_{*}\Delta^{*}\tilde{\pi}_{1}^{*}(-)
\end{tikzcd} \tag{A.20}$$

commutes by functoriality. Therefore (A.9) is isomorphic to

$$\tilde{\pi}_{2*}\Delta_{*}\pi_{1*}\left(\pi_{1}^{!}\mathcal{O}_{X_1} \otimes \pi_{1}^{*}\Delta_{*}\tilde{\pi}_{1}^{*}(-)\right) \xrightarrow{\pi_{1*}\left(\pi_1^{!}\mathcal{O}_{X_1}\otimes\pi_1^{*}(-)\right)\to\mathrm{Id}} \tilde{\pi}_{2*}\Delta_{*}\Delta^{*}\tilde{\pi}_{1}^{*}(-). \tag{A.21}$$

We now compute the connecting isomorphisms. Composing the inverse of (A.17), the isomorphism in the right column of (A.11), with the isomorphism in the left column of (A.18) we obtain

$$\tilde{\pi}_{2*}\pi_{13*}\Delta_{*}\left(\Delta^{*}Q' \otimes \Delta^{*}\pi_{13}^{*}\tilde{\pi}_{1}^{*}(-)\right) \xrightarrow{\left(\mathrm{Id}\otimes\left(\zeta^{-1}_{\tilde{\pi}_1,\Delta}\circ\zeta_{\tilde{\pi}_1,\pi_{13},\Delta}\right)\right)\circ\eta_{\tilde{\pi}_2,\Delta}\eta^{-1}_{\tilde{\pi}_2,\pi_{13},\Delta}} \tilde{\pi}_{2*}\Delta_{*}\pi_{1*}\left(\Delta^{*}Q' \otimes \pi_{1}^{*}\Delta^{*}\tilde{\pi}_{1}^{*}(-)\right)$$

and by pseudofunctoriality relations (2.19) and (2.20) this is equal to

$$\tilde{\pi}_{2*}\pi_{13*}\Delta_{*}\left(\Delta^{*}Q' \otimes \Delta^{*}\pi_{13}^{*}\tilde{\pi}_{1}^{*}(-)\right) \xrightarrow{\left(\mathrm{Id}\otimes\left(\zeta^{-1}_{\Delta,\pi_1}\circ\zeta_{\pi_{13},\Delta}\right)\right)\circ\eta_{\Delta,\pi_1}\circ\eta^{-1}_{\pi_{13},\Delta}} \tilde{\pi}_{2*}\Delta_{*}\pi_{1*}\left(\Delta^{*}Q' \otimes \pi_{1}^{*}\Delta^{*}\tilde{\pi}_{1}^{*}(-)\right). \tag{A.22}$$

On the other hand, the composition of the inverse of the isomorphism in the right column of (A.18) with the isomorphism in the left column of (A.20) is clearly Id.

We can now conclude that the adjunction counit $\Phi_E^{\mathrm{ladj}}\Phi_E \to \mathrm{Id}$, being the composition of (A.7), (A.8) and (A.9), is isomorphic to the composition of (A.16), (A.22), (A.19) and (A.21). The claim of the theorem then follows from the fact that the following diagram commutes:

$$\begin{tikzcd}
\tilde\pi_{2*}\left(\pi_{13*}Q' \otimes \tilde\pi_1^*(-)\right) \arrow[r, "\sim"] \arrow[d, "(A.3)"'] & \tilde\pi_{2*}\pi_{13*}\left(Q' \otimes \pi_{13}^*\tilde\pi_1^*(-)\right) \arrow[d, "(A.16)"] \\
\tilde\pi_{2*}\left(\pi_{13*}\Delta_*\Delta^*Q' \otimes \tilde\pi_1^*(-)\right) \arrow[r, "\sim"] \arrow[d, "(A.4)"'] & \tilde\pi_{2*}\pi_{13*}\Delta_*\left(\Delta^*Q' \otimes \Delta^*\pi_{13}^*\tilde\pi_1^*(-)\right) \arrow[d, "(A.22)"] \\
\tilde\pi_{2*}\left(\Delta_*\pi_{1*}\Delta^*Q' \otimes \tilde\pi_1^*(-)\right) \arrow[r, "\sim"] \arrow[d, "(A.5)"'] & \tilde\pi_{2*}\Delta_*\pi_{1*}\left(\Delta^*Q' \otimes \pi_1^*\Delta^*\tilde\pi_1^*(-)\right) \arrow[d, "(A.19)"] \\
\tilde\pi_{2*}\left(\Delta_*\pi_{1*}\pi_1^!\mathcal{O}_{X_1} \otimes \tilde\pi_1^*(-)\right) \arrow[r, "\sim"] \arrow[d, "(A.6)"'] & \tilde\pi_{2*}\Delta_*\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1} \otimes \pi_1^*\Delta^*\tilde\pi_1^*(-)\right) \arrow[d, "(A.21)"] \\
\tilde\pi_{2*}\left(\Delta_*\mathcal{O}_{X_1} \otimes \tilde\pi_1^*(-)\right) \arrow[r, "\sim"] & \tilde\pi_{2*}\Delta_*\Delta^*\tilde\pi_1^*(-)
\end{tikzcd} \tag{A.23}$$

where the horizontal isomorphisms are all due to the projection formula. To see that diagram (A.23) indeed commutes, observe that its topmost square commutes by Lemma 2.1, the middle two commute by functoriality and the lowermost square commutes by Lemma 2.2. □

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