Tightness and computing distances in the curve complex.

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Abstract: We give explicit bounds on the intersection number between any
curve on a tight multigeodesic and the two ending curves. We use this to con-
struct all tight multigeodesics and so conclude that distances are computable.
The algorithm applies to all surfaces. We recover the finiteness result of Masur-
Minsky for tight geodesics. The central argument makes no use of the geometric
limit arguments seen in the recent work of Bowditch (2003) and Masur-Minsky
(2000), and is enough to deduce a computable version of the acylindricity the-
orem of Bowditch.

Keywords: Curve complex, multigeodesic, train track.

§0. Introduction. Let $\Sigma$ be a closed connected oriented surface and let
$\Pi \subseteq \Sigma$ be a finite subset. In [Harv], Harvey associates to the pair $(\Sigma, \Pi)$ a
simplicial complex $C(\Sigma, \Pi)$ called the curve complex. This is defined as follows.
We shall say that an embedded loop in $\Sigma - \Pi$ is trivial if it bounds a disc and
peripheral if it bounds a once punctured disc. Let $X = X(\Sigma, \Pi)$ be the set of
all free homotopy classes of non-trivial and non-peripheral embedded loops in
$\Sigma - \Pi$. The elements of $X$ will be referred to as curves. We take $X$ to be the
set of vertices and deem a family of distinct curves $\{\gamma_0, \gamma_1, \ldots, \gamma_k\}$ to span a
$k$-simplex if any two curves can be disjointly realised in $\Sigma - \Pi$. The mapping
class group has cocompact simplicial action on $C$. This has been exploited by
various authors, see for example [BF], [Hare] and [Iva].

With the exception of only a few cases, namely $\Sigma$ is a 2-sphere and $|\Pi| \leq
4$ and $\Sigma$ is a torus and $|\Pi| \leq 1$, $X$ is non-empty and the curve complex is
connected. For these non-exceptional cases, it can be verified that the simplicial
dimension $C$ of $C$ is equal to $3(\text{genus}(\Sigma) - 1) + |\Pi| - 1$. We see that $(\Sigma, \Pi)$ is
in fact non-exceptional if and only if $C(\Sigma, \Pi) > 0$. In each subsequent section,
it is to be assumed that $(\Sigma, \Pi)$ is such that $C(\Sigma, \Pi) > 0$.

When $C(\Sigma, \Pi) > 0$ the curve complex can be endowed with a path-metric
by declaring all edge lengths to be equal to 1. All that is important here is the
1-skeleton $G$ of $C$, with which $C$ is quasi-isometric. The induced metric on $G$
is known to be unbounded and hyperbolic in the sense of Gromov [MaMi1],
[Bow1]. The boundary of $G$ is homeomorphic to the space of minimal geodesic
laminations filling $\Sigma - \Pi$, given any hyperbolic metric on $\Sigma - \Pi$, endowed with the
“measure forgetting” topology [Kla], [Ham]. The curve complex plays a key
role in Minsky et al’s approach to Thurston’s ending lamination conjecture. A
second approach has been proposed by Rees [R].

All this at first sight suggests that we may apply the methods of hyperbolic
groups and spaces to study various groups acting on $G$, in particular the mapping
class group and its subgroups. The curve graph, though, is not locally finite
or even fine: As early as the 2-ball around any vertex of $G$ these problems are
manifest. Tight multigeodesics, introduced in [MaMi2] and further studied in
[Bow2] and [Bow3], address this problem. Their introduction has been fruitful: Masur and Minsky used these to study the conjugacy problem in the mapping class group and Bowditch used these to describe the action of the mapping class group on the curve complex.

Masur and Minsky [MaMi2] showed that there are only finitely many tight multigeodesics between any two vertices of the curve graph and Bowditch [Bow2] improved on this, showing that there are only uniformly boundedly many curves in any given slice. We go some way to re-establishing these results, though our bounds depend on the intersection number of the two ending vertices. Since the arguments given here do not rely on passing to geometric limits, our results can be viewed as addressing the local finiteness problems as well as offering computability. We see how to construct geodesics, all tight multigeodesics and compute the distance between any two vertices. These notions of tightness are perhaps stronger than we need.

We introduce the key idea of chords and pulses to measure the interleaving in the surface of curves lying on a geodesic in $G$. Pulse is preserved by the action of the mapping class group. We establish bounds on pulse that apply to all geodesics and, combining these with an appropriate tightness criterion, we establish the finiteness of tight multigeodesics. These methods are readily applicable to related complexes.

Lastly, in §7 we use our results to compute stable lengths of all mapping classes.

§1. Tightness in the curve graph and the main results. Let us remind ourselves of a few definitions. Associated to any two curves $\alpha$ and $\beta$ is their geometric intersection number $\iota(\alpha, \beta)$, namely the minimal cardinality of the set $a \cap b$ among all $a \in \alpha$ and $b \in \beta$. Note $\iota(\alpha, \alpha) = 0$ for all $\alpha$, and $d(\alpha, \beta) \leq 1$ if and only if $\iota(\alpha, \beta) = 0$. For any two curves $\alpha$ and $\beta$, we have have $d(\alpha, \beta) \leq \iota(\alpha, \beta)+1$ (see [Bow1] for a logarithmic bound).

For us, paths in the curve graph shall be sequences of vertices $\gamma_0, \gamma_1, \ldots, \gamma_n$ such that $\gamma_i \neq \gamma_{i+1}$ and $\iota(\gamma_i, \gamma_{i+1}) = 0$, that is $\gamma_i$ and $\gamma_{i+1}$ are adjacent, for each $i$. A geodesic in $G$ is a distance realising path.

We shall recall the notion of tight multigeodesic due to Bowditch [Bow2], but that of Masur and Minsky [MaMi2] works equally well here. Recall that a multicurve is a collection of pairwise distinct curves of pairwise zero intersection number. Intersection number on multicurves is defined additively. Recall that a multipath is a sequence of multicurves $(v_i)_{i=0}^n$ such that $d(\gamma_i, \gamma_j) = |i - j|$ for all $\gamma_i \in v_i, \gamma_j \in v_j$ and $i < j$. We say that a multipath $(v_i)_{i=0}^n$ is tight at $v_j$ ($1 \leq j \leq n-1$) if for all curves $\delta$, whenever $\iota(\delta, v_j) > 0$ we have $\iota(\delta, v_{j-1}) + \iota(\delta, v_{j+1}) > 0$. We say that $(v_i)_{i=0}^n$ is tight if tight at each $v_j$ ($1 \leq j \leq n-1$). A tight multipath $(v_i)_{i=0}^n$ is a tight multigeodesic if $d(\gamma_0, \gamma_n) = n$ for some (hence any) $\gamma_0 \in v_0$ and $\gamma_n \in v_n$. The existence of tight multigeodesics was established in [MaMi2]. Whether we can always connect two vertices of $G$ by a tight geodesic, rather than having to use multicurves, remains open.

The main result may be stated as follows.

Lemma 1 There is an explicit increasing function $F : \mathbb{N} \rightarrow \mathbb{N}$ such that the following holds. Let $(v_i)_{i=0}^n$ be any multigeodesic tight at $v_1$. Then, $\iota(v_0, v_n) \leq F(\iota(v_0, v_n))$. 

2
Note that $F(n)$ grows superexponentially with $n$. In particular, the loss of the uniformity of the bounds of [Bow2] appears to be the price of computability. Even so, these bounds are enough to deduce the visual connectivity of $\partial G$ by bi-infinite geodesics and other familiar facts. Consequences of Lemma 1 include the following.

**Theorem 2** There exists an explicit algorithm which takes as input $\Sigma, \Pi$ and any two curves $\alpha$ and $\beta$ in $\Sigma - \Pi$ and returns all tight multigeodesics connecting $\alpha$ to $\beta$.

From this we conclude that distances are computable.

**Theorem 3** There exists an explicit algorithm which takes as input $\Sigma, \Pi$ and any two curves $\alpha$ and $\beta$ in $\Sigma - \Pi$ and returns the distance between $\alpha$ and $\beta$ in $G(\Sigma, \Pi)$.

In his unpublished thesis, J. Leasure [Lea] gives a version of Theorem 3 for closed surfaces of genus at least two. We are grateful to Richard P. Kent IV for alerting us to this.

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§2. **An overview of the proof to Lemma 1 and the first few cases.** Neighbouring multicurves on a tight multigeodesic tend to drag one another round the surface and shield each other from other curves. Consider any multigeodesic $(v_i)_i^0$ and any simple realisation $c_i$ for $v_i$, each $i$, such that $c_i \cap c_{i+1} = \emptyset$ for each $i$, $|c_i \cap c_n| = i(v_i, v_n)$ each $i \leq n - 2$ and $c_i \cap c_j \cap c_n = \emptyset$ for each $i < j \leq n - 2$. Suppose that two components $J_1$ and $J_2$ of $c_n - c_0$ are connected by three subarcs of $c_1$, denoted $g_1, g_2$ and $g_3$, that are otherwise disjoint from $c_n$ and are homotopic relative to $c_n - c_0$. Tightness at $v_1$ implies that the ends of at least two of these subarcs, say $g_1$ and $g_2$, are separated by a point from $c_2 \cap c_n$. Now $c_1$ and $c_2$ are disjoint so we conclude that there must be a subarc $h$ of $c_2$ connecting $J_1$ and $J_2$ and sandwiched between $g_1$ and $g_2$. This subarc of $c_2$ is shielded from $c_0$ by $c_1$. Indeed, if $g_1$ and $g_2$ continue from $J_2$ and both return to $J_1$ while remaining homotopic relative to $c_n - c_0$ then $h$ continues to be trapped in between $g_1$ and $g_2$ all the way back to $J_1$. If the ends of $h$ are not separated on $J_1$ by a point from $c_3 \cap c_n$ then we may surger $h$ along $c_n$ to find a new simple loop $c_2'$ disjoint from both $c_0$ and $c_3$. In particular, when $\Pi$ is empty $c_2'$ represents a curve, denoted $\gamma_2'$, and we have succeeded in finding a new multipath $v_0, \gamma_2', v_3$, contradicting $d(v_0, v_3) = 3$. We conclude that the ends of $h$ must be separated by $c_3$.

This analysis continues along $(v_i)_i^0$ to higher indices. Suppose that a long subarc of $c_{n-2}$ is fellow travelled by two long subarcs of $c_1$, one on either side, and kept apart from $c_1$ by long subarcs from $c_2, c_3, \ldots, c_{n-3}$ in turn. Then we may surger $c_{n-2}$ along $c_n - c_0$ and arrive at a new curve $\gamma_2'_{n-2}$ having zero intersection with $v_0$. Furthermore, $\iota(v_{n-1}, v_n) = 0$ and so this time we are guaranteed
\( \iota(\gamma_{n-2}, v_{n-1}) = 0 \). We have succeeded in finding a multipath connecting \( v_0 \) and \( v_{n-1} \) of length less than \( n-1 \) and hence we have a contradiction.

We shall see that if we start with a multigeodesic \((v_i)_0^n\) which is tight at \( v_1 \) and is such that \( \iota(v_1, v_3) \) is large relative to \( \iota(v_0, v_n) \) (we shall quantify this in terms of \( F \) in §5) then this is exactly the situation we find ourselves in. Furthermore, the argument only requires tightness at \( v_1 \).

We now deal with the cases \( n = 2 \) and \( n = 3 \) separately, and here after assume \( n \geq 4 \).

**Proposition 4** For each multigeodesic \( v_0, v_1, v_2 \) we have \( \iota(v_1, v_2) = 0 \). For each multigeodesic \( v_0, v_1, v_2, v_3 \) we have \( \iota(v_1, v_3) \leq 2\iota(v_0, v_3) \).

**Proof** We note that for any multigeodesic \( u_0, u_1, u_2 \) tight at \( u_1 \) and any multicurve \( z \), we have \( \iota(z, u_1) \leq 2(\iota(z, u_0) + \iota(z, u_2)) \). In particular, when \( z = v_3 \) we have \( \iota(v_1, v_3) \leq 2(\iota(v_0, v_3) + \iota(v_2, v_3)) = 2(\iota(v_0, v_3) + 0) = 2\iota(v_0, v_3) \). \( \square \)

The same argument fails for \( n \geq 4 \) since \( v_2 \) and \( v_n \) are no longer adjacent.

§3. The idea of pulse. We introduce a measure for the interleaving in the surface \( \Sigma - \Pi \) of curves lying on a tight multigeodesic in \( G(\Sigma, \Pi) \). The same ideas can be applied to geodesics and multigeodesics tight at a given vertex.

For any given positive integer \( n \), let \( F_n \) denote the free monoid of rank \( n \) generated by the set \( \{e_1, e_2, \ldots, e_n\} \). Let \( R \) denote the relation set \( \{e_i e_j = e_j e_i : i < j - 1\} \) and let \( N \) denote the congruence on \( F_n \) generated by \( R \). We form the quotient \( F_n/N \) and refer to the elements of this monoid as chords. For any word \( w \in F_n \), denote by \( w(i) \) the \( i \)th letter appearing in \( w \). For each \( i, j \) define \( |e_i - e_j| = |i - j| \).

Chords naturally arise in the context of paths and multipaths in the curve graph. Consider a path or multipath \((v_i)_0^n\) and choose simple representatives \( c_i \) for \( v_i \) once more, so that \( c_i \cap c_{i+1} = \emptyset \) each \( i \), \( |c_i \cap c_n| = \iota(v_i, v_n) \) for each \( i \leq n - 2 \) and \( c_i \cap c_j \cap c_n = \emptyset \) for each \( i < j \leq n - 2 \). Let \( J \) be any component of \( c_n - c_0 \). Orientate \( J \) and use this orientation to enumerate the points of \( J \cap \bigcup_{j=0}^{n-2} c_j \). This enumeration spells out an element \( w \) of \( F_n \) by identifying a point from \( c_i \) with the \( i \)th generator \( e_i \) of \( F_n \), each \( i \). Tightness at \( v_i \) implies that \( w \) cannot be of the form \( w = w_1 e_1^3 w_2 \), for some \( w_1, w_2 \in F_n \) and each \( i \). We will consider various subsets of \( J \cap \bigcup_{j=0}^{n-2} c_j \), and enumerate their elements with the orientation on \( J \) to determine \( m \)-pulse.

Now suppose that \( |w(i) - w(i + 1)| > 1 \). Then \( \gamma_{w(i)} \) and \( \gamma_{w(i+1)} \) have non-zero intersection number and so we may homotope both \( c_{w(i)} \) and \( c_{w(i+1)} \) near \( J \) so as to transpose the two points of intersection. If we re-enumerate, we arrive at a second word \( w' \in F_n \) with \( \overline{w'} = \overline{w} \). In this way, paths may be viewed as defining chords and tight multigeodesics pinched chords. Note also that each word in a chord induced by a path or multipath can be induced by the same path or multipath, just by considering transpositions.

Let us set about defining the \( m \)-pulse of a given word and then for a given chord, for each \( 2 \leq m \leq n \). For each word \( w \in F_n \) we consider subwords \( u \) satisfying the following three conditions. Firstly, both the initial and the final letters in \( u \) are equal to \( e_1 \). Secondly, for each \( i \) we have \( |u(i) - u(i + 1)| \leq 1 \). Thirdly, between any two successive \( e_1 \)’s in \( u \) there is exactly one \( e_m \). We define the \( m \)-pulse of a such a subword \( u \) to be equal to the number of times \( e_m \) appears
in $u$. We define the $m$-pulse of $w$ to be the maximal $m$-pulse arising among all such subwords $u$ of $w$ and denote it by $p_m(w)$. Even when $u$ satisfies these criteria and is maximal with respect to inclusion among all such subwords, it need not realise the $m$-pulse of $w$.

**Lemma 5** Suppose that $v, w \in F_n$ represent the same chord, that is $\overline{v} = \overline{w}$. Then, there is a natural one-to-one correspondence between subwords of $v$ and subwords of $w$. In particular, this restricts to a correspondence between subwords of $v$ satisfying our three criteria and subwords of $w$ satisfying our three criteria and preserves their $m$-pulse, each $2 \leq m \leq n$.

**Proof** Any two elements of a chord are related by a finite sequence of transpositions. The result follows by an induction on the length of such sequences. Note that each transposition fixes every subword satisfying our three criteria and so preserves $m$-pulse, each $m$. ♦

For $2 \leq m \leq n$, we define the $m$-pulse of a given chord to be equal to the $m$-pulse of one (hence any) representative word, and denote this by $p_m(\overline{w})$. We have just seen that this is well-defined.

Let us complete this section with a few examples and remarks. Chords may be represented and are determined by words from $\{e_1, e_2,\ldots, e_n\}$. For instance, $e_1e_2e_3e_4e_5$ and $e_1e_2e_4e_2e_3e_4$ represent the same chord since the first $e_4$ and the second $e_2$ may be transposed. The words $e_1e_2e_1e_2e_1$ and $e_1e_1\ldots e_1e_2e_1e_1\ldots e_1e_2$ represent different chords although their 2-pulses both equal 2. The words $e_1e_1e_2e_1e_1e_1e_1e_2e_1$ and $e_1e_2e_2e_1e_2e_2e_1e_1$ define different chords although both their 2-pulses and their lengths are equal. When chords are induced by a given geodesic, the tightness property prevents repetition. If we bound the $m$-pulse on a chord, each $m$, then we bound its length.

Lastly, note that pulse is symmetric and almost additive: For each $m$, the $m$-pulse of the concatenation of two words is either the sum of each $m$-pulse or one more than this sum. The 1-pulse of a word or chord should always be regarded as zero and, for each $2 \leq m \leq n - 1$, the $m + 1$-pulse of a chord is at most the $m$-pulse. Summing the 2-pulses over each component of $c_n - c_0$ closely approximates $\iota(v_1, v_n)$.

**Lemma 6** Suppose that $(v_i)_0^n$ is a multigeodesic tight at $v_1$. Then $\sum p_2(\overline{w}) \leq \iota(v_1, v_n) \leq 2 \sum p_2(\overline{w}) + \iota(v_0, v_n)$, where the summations are taken over the components of $c_n - c_0$.

**§4. Train tracks relative to the ends of a geodesic in the curve graph.** Let us fix a smooth structure on $\Sigma$. A train track $\tau$ in $\Sigma - \Pi$ is a smooth branched 1-submanifold such that the Euler characteristic of the smooth double of each component of $\Sigma - (\Pi \cup \tau)$ is negative. This rules out discs, once-punctured discs and discs with one or two boundary singularities as complementary regions. Train tracks were introduced by Thurston to study geodesic laminations.

It is standard to refer to the branch points of a train track as switches and the edges between switches as branches. We say that $\tau$ is generic if each switch has valence three. By sliding branches along branches, if need be, we can take a train track and return a generic train track. This is a convenient option since it greatly simplifies our counting arguments. A train subpath $p : I \rightarrow \tau$ is a
continuous map on a closed interval \( I \subseteq \mathbb{R} \) such that \( p(n) \) is a switch for each \( n \in \mathbb{Z} \setminus I, p^{-1}(v) \in \mathbb{Z} \) for each switch \( v \) and \( \partial I \subseteq \mathbb{Z} \cup \{ \pm \infty \} \).

A smooth simple closed loop \( c \) is carried by \( \tau \) if there exists a smooth map \( \phi : \Sigma - \Pi \to \Sigma - \Pi \) homotopic to the identity map on \( \Sigma - \Pi \) such that \( \phi|_c \) is an immersion and \( \phi(c) \subseteq \tau \). We refer to \( \phi \) as a carrying map or supporting map. If \( \tau \) carries \( c \) then we have a measure on the branch set of \( \tau \) by counting the number of times the subpath \( \phi(c) \) traverses any given branch. This measure satisfies a switch condition: At each switch, the total inward measure is equal to the total outward measure.

We recall a useful combinatorial lemma relating the number of switches and the number of branches of a train track to the Euler characteristic of \( \Sigma - \Pi \). This is Corollary 1.1.3 from [PenH].

**Lemma 7** Let \( \tau \) be any train track in \( \Sigma - \Pi \), let \( s \) denote the number of switches and \( e \) the number of branches. Then:

i). \( s \leq -6\chi(\Sigma - \Pi) - 2|\Pi| \);

ii). \( e \leq -9\chi(\Sigma - \Pi) - 3|\Pi| \).

Let \( (v_i^n) \) be any multipath in \( G(\Sigma, \Pi) \) and choose smooth and simple realisations \( c_i \) for \( v_i \), each \( i \), such that \( c_i \cap c_{i+1} = \emptyset \) for each \( i \), \( c_i \cap c_n = \emptyset \) for each \( i \leq n - 2 \) and \( c_i \cap c_j \cap c_n = \emptyset \) for each \( i < j \leq n - 2 \). We construct a train track \( \tau \) which will carry all of \( c_i \) and all those subarcs of each \( c_i \) \( (2 \leq i \leq n - 2) \) which end on \( c_n \) and which are trapped between subarcs of \( c_1 \) over large distances.

There exists a smooth surjection \( \phi : \Sigma - \Pi \to \Sigma - \Pi \) homotopic to the identity map such that the restriction of \( \phi \) to \( c_1 \) is an immersion onto a smooth branched 1-submanifold \( \tau \) of \( \Sigma - \Pi \) with the characterising property that any two components of \( c_1 - c_n \) homotopic relative to \( c_n - c_0 \) are carried into the same edge of \( \tau \) and \( \tau \) is to be disjoint from \( c_0 \). Each branch point necessarily belongs to one component of \( c_n - c_0 \) and each component of \( c_n - c_0 \) contains at most one branch point. We now check that \( \tau \) defines a train track, with each branch point viewed as a switch and each edge thought of as a branch, and that \( \tau \) is unique up to isotopy.

**Lemma 8** \( \tau \) is a train track.

**Proof** Note that no region complementary to \( \tau \) can be diffeomorphic to a disc with smooth boundary or a monogon (disc with one outward pointing singularity) by the minimality of \(|c_1 \cap c_n|\).

Suppose for contradiction that \( E \) is a bigon component, that is a disc with two outward pointing singularities, of \( \Sigma - (\Pi \cup \tau) \). The two subarcs of \( \partial E \) connecting the two singularities of \( \partial E \) are homotopic to one another relative to \( c_n - c_1 \). Hence \( E \) must intersect \( c_0 \), for otherwise these two subarcs of \( \partial E \) would have been collapsed into a single branch of \( \tau \). Since \( \tau \) and \( c_0 \) are disjoint, so \( \partial E \) and \( c_0 \) are disjoint. Hence \( E \) contains a component of \( c_0 \) which is therefore homotopically trivial. This is absurd, and we conclude that \( \tau \) is a train track.

\( \Box \)

**Lemma 9** Suppose that \( c_0^i, c_1^i, \ldots, c_n^i \ (i = 1, 2) \) are two such realisations for \( v_0, v_1, \ldots, v_n \) and that \( \tau_1 \) and \( \tau_2 \) are the resulting train tracks, respectively. Then \( \tau_1 \) and \( \tau_2 \) are isotopic.
Proof This follows since $c_0^1 \cup c_1^1 \cup c_n^1$ and $c_0^2 \cup c_1^2 \cup c_n^2$ are isotopic. ◊

It is worth pointing out that the same construction for $c_i$ ($i = 2, 3, \ldots, n-2$) will not necessarily yield a train track but instead a bigon train track, that is we allow the complementary regions to be bigons. Each complementary bigon will contain at least one point from $c_0 \cap c_n$ and there would be at most $t(v_0, v_n)$ bigons.

To each switch $v$ of $\tau$ we can associate the set $\phi^{-1}(v) \cup \bigcup_{i=0}^{n-2} c_i$, which we henceforth denote by $D(v)$. Let us assume that $c_0, c_1, \ldots, c_n$ are such that $|D(v)|$ is minimal for each switch $v$. Orientate $c_n$ and use this orientation to enumerate the points of $D(v)$. This gives us a word $w$ in $F_{n-2}$. Thus, for each integer $2 \leq m \leq n - 2$, we may associate to the switch $v$ the $m$-pulse of the chord $w$.

We may use pulse on switches to define measures on the branch set of $\tau$. Suppose that $v_1$ and $v_2$ are adjacent switches of $\tau$ connected by a branch $b$. In what follows, the topological closure of $b$ in $\Sigma - \Pi$ is denoted $c(b)$.

Lemma 10 For each integer $2 \leq m \leq n - 2$, the $m$-pulse of $\phi^{-1}(c(b)) \cap D(v_1)$ is equal to the $m$-pulse of $\phi^{-1}(c(b)) \cap D(v_2)$.

Proof For each $x \in \phi^{-1}(c(b)) \cap D(v_1)$, define $q(x)$ to be the end on $\phi^{-1}(c(b)) \cap D(v_2)$ of the subarc of the $c_i$ containing $x$, beginning at $x$ and collapsing to $b$. We denote this arc by $[x, q(x)]$. The arc $[x, q(x)]$ crosses a second arc $[y, q(y)]$ only if the corresponding multicurves $c_i$ and $c_j$ satisfy $|i - j| \geq 2$. ◊

We define the $m$-pulse of the branch $b$ to be equal to the $m$-pulse associated to $\phi^{-1}(c(b)) \cap D(v_1)$, or indeed the $m$-pulse associated to $\phi^{-1}(c(b)) \cap D(v_2)$, and denote it by $p_m(b)$. This measure on the branch set of $\tau$ satisfies a coarse switch condition. For any switch $v$ we may choose one of two directions at $v$ and partition the branches incident on $v$ as outgoing and incoming. Denote the corresponding branches by $b_1, b_2, \ldots, b_s$ and $b_{s+1}, b_{s+2}, \ldots, b_{s+t}$, respectively.

Lemma 11 “Coarse switch condition.” For each integer $2 \leq m \leq n - 2$, we have:

i). $|\sum_1^s p_m(b_i) - \sum_1^{s+t} p_m(b_j)| \leq s + t - 2$;
ii). $p_m(v) - s + 1 \leq \sum_1^s p_m(b_i) \leq p_m(v)$;
ii'). $p_m(v) - t + 1 \leq \sum_1^{s+t} p_m(b_j) \leq p_m(v)$.

In particular, if $\tau$ is generic we have the following.

Corollary 12 Suppose that $\tau$ is generic and that $b_1$ and $b_2$ are outgoing. For each $m$ we have:

i). $0 \leq p_m(b_3) - p_m(b_1) - p_m(b_2) \leq 1$;
ii). $p_m(v) - 1 \leq p_m(b_1) + p_m(b_2) \leq p_m(v)$;
iii). $p_m(b_3) = p_m(v)$.

We shall only require Corollary 12i). and we prove it directly.

Proof Suppose that $v$ is a trivalent switch with outgoing branches $b_1$ and $b_2$ and incoming branch $b_3$. Each component of $c_1 \cap \phi^{-1}(b_3)$ either goes on to
be supported by $b_1$ or by $b_2$. All those components of $c_m \cap \phi^{-1}(b_2)$ that are trapped between components of $\phi^{-1}(b_1 \cup \{v\} \cup b_1)$ go on to be supported by $b_1$ ($i = 2, 3$). However, where these components of $c_1$ diverge subarcs of $c_m$ may escape. This reduces the total outgoing $m$-pulse by at most one, if at all. That is, we have part 1) of Corollary 11. ◆

Each arc $g$ supported by $\tau$ and whose ends each lie on components of $b - a$ containing a switch of $\tau$ defines a train subpath $g_\phi$ of $\tau$. For a train path $q$ in $\tau$ we define the support $\text{Supp}(q)$ of $q$ to be the set of all subarcs $g$ of each $c_i$ ($i = 1, 2, \ldots, n - 2$) with $g = g_\phi$. Suppose that $q : I \rightarrow \tau$ is a train path with $0 \in I \subseteq [0, \infty)$. We call the branch of $\tau$ containing $q(i+1/2)$ the $i$th branch of $q$. For each integer $m \geq 1$ and each $i \geq 0$ we define the $m$-pulse of the $i$th branch of $q$ to be equal to the $m$-pulse of either end of $\phi^{-1}(q(0, 1)) \cap \bigcup \text{Supp}(q|_{[0,i+1]})$. We denote this by $p_{m,q}(i)$. Note Lemma 10 tells us that this is well defined. Note also that $p_{m,q}(0)$ is precisely the $m$-pulse of the 0th branch traversed by $q$, each $m$.

**Lemma 13** “Trains run out of fuel.” Let $q : [0, \infty) \rightarrow \tau$ be any train path in $\tau$. Then:

1. $p_{m,q}(i + 1) \leq p_{m,q}(i)$ for all $m$ and all $i$;
2. $p_{m,q}(i) \rightarrow 0$ as $i \rightarrow \infty$.

**Proof** Each time $q$ arrives at a switch of $\tau$, subarcs of $c_1$ that have so far induced $q$ may diverge and there can be no gain in pulse. Hence i) holds. If $p_{m,q}(i) \geq 1$ for all $i$ then, since $\Sigma$ is orientable, we conclude that $c_1$ has two freely homotopic components and this is absurd. Hence ii) holds. ◆

The sum of all 2-pulses over each branch of $\tau$ resembles a reduced intersection number for $c_1$ and $c_n$ relative to $c_0$. Suppose that $\iota(v_0, v_n)$ is large. Then most of the regions complementary to $c_0 \cup c_n$ are squares. Whenever $c_1$ meets an edge of one of these squares, necessarily from $c_n$, it must go on to meet the edge opposite. Consider a sequence $S_1, S_2, \ldots, S_k$ of closed squares whose interiors are complementary to $c_0 \cup c_n$ and such that $S_i \cap S_{i+1} \subseteq c_n$, each $i$. Whenever $c_1$ meets one of the outer edges of $\bigcup^k_i S_i$ then $c_1$ remains trapped in $\bigcup^k_i S_i$ and goes on to meet each edge $S_i \cap S_{i+1}$ before exiting at the other outer edge. However, the components of $c_1 \cap \bigcup^k_i S_i$ all collapse into a single branch of $\tau$. Now suppose that $S_1$ is met by $c_1$. If any two such components of $c_1$ are separated by a subarc of $c_2$ then the 2-pulse on this branch is precisely one less than the number of components of $c_1 \cap \bigcup^k_i S_i$.

§5. **Proof of Lemma 1.** Let $(v_i)_0^n$ be any multigeodesic in $G(\Sigma, \Pi)$ tight at $v_1$ with $n \geq 4$. Let $c_i$ be any realisation for $v_i$, each $i$, such that $c_i \cap c_{i+1} = \emptyset$ for each $i \leq n - 1$, $|c_i \cap c_n| = \iota(v_i, v_n)$ each $i \leq n - 2$ and $c_i \cap c_j \cap c_n = \emptyset$ each $i < j \leq n - 2$. Recall the construction of the train track $\tau$ relative to $v_0$ and $v_n$ and carrying all of $c_1$, as described in §4. We can endow each branch of $\tau$ with a family of measures and each of these satisfies a coarse switch condition. For any train subpath $q$ of $\tau$ we defined a time measure associated to $q$ by considering those subarcs of each $c_i$ that induce $q$ via $\phi$.

We define the function $K_n : \{1, 2, \ldots, n-1\} \rightarrow \mathbb{N}$ by the recurrence relation

$$K_n(j) = 2^{-(9(3\iota(\Sigma, \Pi) - 3\iota(\Pi)))(1 + K_n(j+1))},$$

8
for all \( j = 2, 3, \ldots, n - 2 \), with the boundary condition \( K_n(n - 1) = 1 \) and where \( \chi(\Sigma - \Pi) \) denotes the Euler characteristic of \( \Sigma - \Pi \).

**Lemma 14** Let \( m \geq 2 \). Suppose that all the branches of \( \tau \) have \( m + 1 \)-pulse at most \( K_n(m + 1) \) and that at least one branch of \( \tau \) has \( m \)-pulse at least \( K_n(m) \). Then there exists a subarc \( h \) of \( c_m \) disjoint from \( c_0 \) and whose ends lie on, but is otherwise disjoint from, one component of \( c_n - (c_0 \cup c_{m+1}) \).

**Proof** We remark that \( K_n(j) \) was chosen to guarantee the existence of a circuit \( q : [0, k] \to \tau \) such that \( p_{m,q}(k - 1) \geq 1 \). Hence \( \text{Supp}(q) \) contains at least two subarcs \( g_1, g_2 \) of \( c_1 \) beginning and ending on the same component of \( c_n - (c_0 \cup c_{m+1}) \) and homotopic relative to \( c_n - c_0 \) that trap a subarc \( h \) of \( c_m \) whose ends lie on the same component of \( c_n - c_0 \). \( \diamondsuit \)

**Corollary 15** The \( m \)-pulse on each branch of \( \tau \) is at most \( K_n(m) \), each \( m \geq 2 \).

**Proof** Suppose, for contradiction, that there exists \( m \) and a branch of \( \tau \) whose \( m \)-pulse is at least \( K_n(m) \). We take \( m \) to be maximal subject to this property. Now by Lemma 14 we deduce that there exists a subarc \( h \) of \( c_m \) beginning and ending on the same component of \( c_n \) and disjoint from \( c_0 \). When \( \Pi \) is empty we know that the union of \( h \) and the subinterval of \( c_n - c_0 \) connecting its ends defines a curve, denoted \( \delta \), and we know that this curve has zero intersection with both \( v_0 \) and \( v_{m+1} \). We have found a multipath \( v_0, \delta, v_{m+1} \) of length two. Since \( d(v_0, v_{m+1}) = m + 1 \geq 2 + 1 = 3 \) we have a contradiction.

When \( \Pi \) is non-empty we only have to be slightly more careful since \( \delta \) may be peripheral. Instead, if we define \( K'_n \) by the recurrence relation

\[
K'_n(j) = 2^{-2(\chi(\Sigma - \Pi) + 3|\Pi|)(1 + K'_n(j + 1))}
\]

with the same boundary condition \( K'_n(n - 1) = 1 \), then we can ask for the second return to the same component of \( c_n - (c_0 \cup c_{m+1}) \). By considering the boundary components of a regular neighbourhood of the union of \( h \) and the subinterval of \( c_n - c_0 \) connecting the ends of \( h \), we again find a curve \( \delta \) which again has zero intersection with both \( v_0 \) and \( v_{m+1} \). \( \diamondsuit \)

**Corollary 16** Suppose that \( (v_1)''_0 \) is a multigeodesic tight at \( v_1 \). Then \( \iota(v_1, v_2) \leq 2(1 + K_n(2))\iota(v_0, v_n) \) if \( \Pi \neq \emptyset \) and \( \iota(v_1, v_n) \leq 2(1 + K_n(2))\iota(v_0, v_n) \) if \( \Pi = \emptyset \).

**Proof** Let us consider \( \Pi \neq \emptyset \). We have seen that the 2-pulse on each branch of \( \tau \) is at most \( K_n(2) \). Since \( (v_1)''_0 \) is tight at \( v_1 \) we have \( \iota(v_1, v_n) \leq 2(1 + \max\{p_2(b) : b \text{ is a branch of } \tau\})\iota(\alpha, \beta) \leq (1 + K_n(2))\iota(\alpha, \beta) \). \( \diamondsuit \)

We conclude the proof of Lemma 1.

**Proof** (of Lemma 1). Since \( d(v_0, v_n) \leq \iota(v_0, v_n) + 1 \) and \( d(v_0, v_n) = n \) we have \( K'_n(2) \leq K_{n(\alpha, \beta) + 1} \). Hence \( F(s) = (1 + K'_{s+1}(2))s \), each \( s \in \mathbb{N} \), suffices. \( \diamondsuit \)

§6. The proof of Theorem 2 and Theorem 3. In this section, we prove the main implications of Lemma 1: We establish a finiteness result for tight multigeodesics and we establish the computability of distances in the curve graph.
Lemma 17 There exists an explicit increasing function \( F_1 : \mathbb{N} \rightarrow \mathbb{N} \) such that the following holds. Let \((v_i)_0^n\) be any tight multigeodesic. Then, \(\iota(v_j, v_n) \leq F_1(\iota(v_0, v_n))\) for all \(j\).

Proof This follows by an inductive argument using Lemma 1, noting that \(\iota(v_j, v_n) \leq F^j(\iota(v_0, v_n))\) for \(1 \leq j \leq n - 2\). Now \(F_1\) defined by \(F_1(k) = F^k(k)\) will suffice. \(\diamondsuit\)

By considering \((v_{n-i})_0^n\) instead, we deduce the following.

Corollary 18 There exists an explicit increasing function \( F_2 : \mathbb{N} \rightarrow \mathbb{N} \) such that the following holds. Let \((v_i)_0^n\) be any tight multigeodesic. Then, \(\iota(v_j, v_n) \leq F_2(\iota(v_0, v_n))\) and \(\iota(v_0, v_j) \leq F_2(\iota(v_0, v_n))\) for all \(j\).

Let \(\alpha\) and \(\beta\) be any two vertices of the curve graph. The set of multipaths of length at most \(\iota(\alpha, \beta) + 1\) connecting \(\alpha\) to \(\beta\) for which each multicurve verifies the bounds in Corollary 18 has size uniformly and explicitly bounded in terms of \(\iota(\alpha, \beta)\), genus(\(\Sigma\)) and \(|\Pi|\). In particular, this set contains all the tight multigeodesics connecting \(\alpha\) to \(\beta\) and we are left to look for these in a bounded search space. This concludes the proof of Theorem 2. Since tight multigeodesics are distance realising multipaths, we deduce Theorem 3.

§7. The computability of stable lengths. Given a metric space \(X\) and an isometry \(h : X \rightarrow X\) we define the stable length, \(||h||\), of \(h\) to be equal to \(\lim_{n \rightarrow \infty} d(x, h^n x)/n\). See [BGS] for more details. It is easily verified that that \(||h||\) does not depend on the choice of \(x\). We say that a mapping class \(h\) is pseudo-Anosov if for any two curves \(\alpha\) and \(\beta\) we have \(\iota(\alpha, h^n(\beta)) \rightarrow \infty\) as \(n \rightarrow \infty\) (see [FLP]). We consider \(\mathcal{G}\) endowed with usual path-metric, and prove the computability of the stable length of any given pseudo-Anosov mapping class.

Let us first recall a few results. In [Bow1], not only is the hyperbolicity of the curve complex re-established but it is also shown that one can compute hyperbolicity constants. In [Bow2], it is established that there exists a positive integer \(N = N(\text{genus}(\Sigma), |\Pi|)\) such that for each pseudo-Anosov mapping class \(h, h^N\) has a geodesic axis in \(\mathcal{G}\). This implies the stable lengths of pseudo-Anosov mapping classes are both positive and uniformly rational. It is not known how to compute \(N\). Lastly, in a \(k\)-hyperbolic geodesic metric space each geodesic rectangle is \(8k\)-narrow (so that any point on any one side of the rectangle is within \(8k\) of the union of the other three). We now state the result in full:

Theorem 19 There is an algorithm which takes as input \(\Sigma, \Pi, N(\Sigma, \Pi)\) and a pseudo-Anosov mapping class \(h\) and returns \(||h||\).

Proof Fix a choice of \(k\) such that \(\mathcal{G}\) is \(k\)-hyperbolic. Choose an integer \(M \geq 18k\). Let us suppose that \(h\) is the \(N\)th power of a pseudo-Anosov. Then, \(h\) is again pseudo-Anosov and has a geodesic axis denoted \(L\). Choose any curve \(\alpha\) and construct a geodesic \(\alpha, h^M \alpha\) from \(\alpha\) to \(h^M \alpha\) in \(\mathcal{G}\). A central vertex \(\beta\) of \(\alpha, h^M \alpha\) must lie within \(8k\) of \(L\). Now construct a geodesic \(\beta, h^M \beta\) from \(\beta\) to \(h^M \beta\). We have \(M||h|| = ||h^M|| \leq d(\beta, h^M \beta) + 16k \leq Md(\beta, h\beta) + 16k\). Hence \(||h|| \leq d(\beta, h\beta) + 16k/M < d(\beta, h\beta) + 1\). As \(||h||\) is an integer, so \(||h|| \leq d(\beta, h\beta)\).
Further, \( d(\beta, h^M \beta) \leq ||h^M|| + 16k = M||h|| + 16k \) and so \( d(\beta, h^M \beta)/M \leq ||h|| + 16k/M < ||h|| + 1. \)

Combining the two inequalities, we have \( ||h|| \leq d(\beta, h^M \beta)/M < ||h|| + 1. \) Hence \( \lfloor d(\beta, h^M \beta)/M \rfloor = ||h||. \)

Notice that in the above we do not find an axis \( L. \) It would interesting to find a way of doing so.

References.


