

**REPRESENTATION THEORETIC PATTERNS IN THREE
DIMENSIONAL CRYO-ELECTRON MICROSCOPY I - THE
INTRINSIC RECONSTITUTION ALGORITHM**

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ABSTRACT. In this paper, we describe and study a mathematical framework for cryo-electron microscopy. The main result, is a proof of the admissibility (correctness) and the numerical stability of the intrinsic reconstitution algorithm which was introduced by Singer and Shkolnisky in [7]. In addition, we explain how the various numerical observations reported in that work, follow from basic representation theoretic principles.

0. INTRODUCTION

”Three dimensional electron microscopy” [1] is the name commonly given to methods in which the three dimensional structure of a macromolecular complex is obtained from the set of images taken by an electron microscope. The most general and widespread of this methods is single-particle reconstruction. In this method the three dimensional structure is determined from images of randomly oriented and positioned identical macromolecular complexes, also referred to as *molecular particles*. A variant of this method is called cryo-electron microscopy (or cryo-EM for short) where multitude of molecular particles are rapidly immobilized in a thin layer of ice and maintained at liquid nitrogen temperature throughout the imaging process. Single particle reconstruction from cryo-electron microscopy images is of particular interest, since it promises to be an entirely general technique which does not require crystallization or other special preparation stages and is beginning to reach sufficient resolution to allow the trace of polypeptide chains and the identification of residues in protein molecules [3, 5, 11].

Over the years, several methods were proposed for single particle reconstruction from cryo-EM images. Present methods are based on the ”Angular Reconstitution” algorithm of Van Heel [9], which was also developed independently by Vainshtein and Goncharov [2]. However, these methods fail with particles that are too small, cryo-EM images that are too noisy or at resolutions where the signal-to-noise ratio becomes too small.

0.1. Main results. In [7], a novel algorithm, referred to in this paper as *the intrinsic reconstitution algorithm*, for single particle reconstruction from cryo-EM images was presented. The appealing property of this new algorithm is that it exhibits remarkable numerical stability to noise. The admissibility (correctness) and the numerical stability of this algorithm were verified in an overwhelming number of numerical simulations, albeit, a formal justification was still missing. In this paper,

we prove the admissibility and the numerical stability of the intrinsic reconstitution algorithm. The proof relies on the study of a certain operator C , of geometric origin, referred to as the *common lines operator*. Specifically,

- Admissibility, depends, among other things, on the fact that the maximal eigenspace of C is three dimensional.
- Numerical stability, depends on the existence of a spectral gap which separates the maximal eigenvalue λ_{\max} from the rest of the spectrum.

In this regard, the main technical result of this paper is a complete description of the spectral properties of the common lines operator. In the course, we describe a formal mathematical framework for cryo-EM which explains how the various numerical observations reported in [7] follow from basic representation theoretic principles, thus putting that work on firm mathematical grounds.

The remainder of the introduction is devoted to a detailed description of the intrinsic reconstitution algorithm and to the explanation of the main ideas and results of this paper.

0.2. Mathematical model. When modeling the mathematics of cryo-EM its more convenient to think of a fixed macromolecular complex which is observed from different directions by the electron microscope. In more details, the macromolecular complex is modeled by a real valued function ϕ on a three dimensional Euclidian vector space $V \simeq \mathbb{R}^3$, which describes the electric potential due to the charge density in the complex. A viewing direction of the electron microscope is modeled by a point x on the unit sphere $X = S(V)$. The interaction of the beam from the electron microscope with the complex is modeled by a real valued function R_x on the orthogonal plane $P_x = x^\perp$, given by the Radon transform of ϕ along the direction x , that is

$$R_x(v) = \int_{L_x} \phi(v+l) dl,$$

for every $v \in P_x$, where L_x is the line passing thorough x and dl is the Euclidian measure on L_x .

The data collected from the experiment is a collection of Radon images $R_x : P_x \rightarrow \mathbb{R}$, $x \in X_N$; where $X_N \subset X$ consists of N points. The main empirical assumption is

Empirical assumption: The points $x \in X_N$ are distributed independently and uniformly at random.

We emphasize that, in practice, the embeddings $i_x : P_x \hookrightarrow V$, $x \in X_N$ are not known. What one is given is the collection of Radon images $\{R_x : P_x \rightarrow \mathbb{R} : x \in X_N\}$, where each plane P_x , $x \in X_N$ should be considered as an abstract Euclidian plane.

The main problem of cryo-EM is to reconstruct the orthogonal embeddings $\{i_x : P_x \hookrightarrow V : x \in X_N\}$ from the Radon images $\{R_x : P_x \rightarrow \mathbb{R} : x \in X_N\}$. We will refer to this problem as the *cryo-EM reconstruction problem* and note that, granting its solution, the function ϕ can be computed (approximately) using the inverse Radon transform.

0.3. The Fourier slicing property and the common lines datum. The first step of the reconstruction is to extract from the Radon images a certain linear algebra datum, referred to as *the common lines datum*, which captures a basic relation in three dimensional Euclidian geometry.

First one notes that for every $x \in X_N$, the following relation holds

$$(0.1) \quad \widehat{R}_x = \widehat{\phi}|_{P_x},$$

where the operation $\widehat{(-)}$ on the left hand side denotes the Euclidian Fourier transform on the plane P_x and the operation $\widehat{(-)}$ on the right hand side denotes the Euclidian Fourier transform on V . This relation follows easily from the standard properties of the Fourier transform and is sometimes referred to as the *Fourier slicing property* (see [6]). The key observation, first made by Klug (see [4]), is that (0.1) implies, for every pair of different points $x, y \in X_N$, that the functions \widehat{R}_x and \widehat{R}_y must agree on the line of intersection (common line), that is

$$\widehat{R}_x|_{P_x \cap P_y} = \widehat{R}_y|_{P_x \cap P_y}.$$

Hence, if the function ϕ is generic enough then one can compute from each pair of images \widehat{R}_x and \widehat{R}_y the operator $C_N(x, y) : P_y \rightarrow P_x$, which identifies the line of intersection between the two planes. Formally, this operator is given by the composition $C_{x,y} \circ C_{y,x}^T$, where $C_{x,y}$ and $C_{y,x}$ are the tautological embeddings

$$\begin{aligned} C_{x,y} &: P_x \cap P_y \hookrightarrow P_x, \\ C_{y,x} &: P_x \cap P_y \hookrightarrow P_y. \end{aligned}$$

0.4. The intrinsic reconstitution algorithm. The intrinsic reconstitution algorithm reconstructs the orthogonal embeddings $\{i_x : P_x \rightarrow V : x \in X_N\}$ from the common lines datum $\{C_N(x, y) : (x, y) \in X_N \times X_N\}$. The crucial step is to construct an **intrinsic** model of the three dimensional Euclidian vector space V which is expressed solely in terms of the common lines datum. The construction proceeds in four steps:

Ambient vector space: We define the $2N$ dimensional Euclidian vector space

$$\mathcal{H}_N = \bigoplus_{x \in X_N} P_x.$$

Common lines operator: We define the symmetric operator $C_N : \mathcal{H}_N \rightarrow \mathcal{H}_N$ by

$$C_N(s)(x) = \frac{1}{|X_N|} \sum_{y \in X_N} C_N(x, y)(s(y)).$$

Intrinsic model: We define the Euclidian subspace¹

$$\mathbb{V}_N = \bigoplus_{\lambda > 1/3} \mathcal{H}_N(\lambda),$$

where $\mathcal{H}_N(\lambda)$ denote the eigenspace of C_N associated with the eigenvalue λ .

Intrinsic maps: For every $x \in X_N$, we define the map

$$\varphi_x = \sqrt{2/3} \cdot (\text{Pr}_x)^t : P_x \rightarrow \mathbb{V}_N,$$

where $\text{Pr}_x : \mathbb{V}_N \rightarrow P_x$ is the orthogonal projection on the component P_x .

The fact that the vector space \mathbb{V}_N is of the right dimension is granted by the following theorem:

¹The condition $\lambda > 1/3$ in the definition of \mathbb{V}_N will be clarified when we will discuss the spectral gap property in the next subsection.

Theorem 1. *For sufficiently large N , we have*

$$\dim \mathbb{V}_N = 3.$$

The fact that the collection of maps $\{\varphi_x : P_x \rightarrow \mathbb{V}_N : x \in X_N\}$ solves the cryo-EM reconstruction problem is the content of the following theorem:

Theorem 2. *There exists an (approximated) isometry $\tau_N : V \approx \mathbb{V}_N$ which satisfies the following property:*

$$\tau_N \circ i_x = \varphi_x,$$

for every $x \in X_N$.

Remark 1. *Theorem 2 implies that the vector space V equipped with the tautological embeddings $\{i_x : x \in X_N\}$ is isomorphic to the intrinsic vector space \mathbb{V}_N equipped with the mappings $\{\varphi_x : x \in X_N\}$. Hence, for all practical purposes they are indistinguishable. This is, to our judgement, an elegant formal example, realizing the general phylosophy about the appearance of structure form data.*

0.5. Analytic set-up. The proofs of Theorems 1 and 2 are based on an approximation argument of the discrete from the continuous, which we are going to explain next.

Let $\mathfrak{H} \rightarrow X$ be the vector bundle on the unit sphere whose fiber at a point $x \in X$ is the plane $P_x = x^\perp$ and let \mathcal{H} denote the vector space of smooth global sections $\Gamma(X, \mathfrak{H})$. The vector space \mathcal{H} is equipped with an inner product, given by

$$(s_1, s_2) = \int_{x \in X} B(s_1(x), s_2(x)) dx,$$

where dx is the Haar measure on the unit sphere. The common lines datum can be used to form a kernel of a symmetric integral operator $C : \mathcal{H} \rightarrow \mathcal{H}$ which is given by

$$C(s)(x) = \int_{y \in X} C(x, y)(s(y)) dy,$$

The main difference from the discrete scenario is that, here, the space \mathcal{H} supports a representation of the orthogonal group $O(V)$ and the main observation is that the operator C commutes with the group action. This enables to understand the operator C in terms of the representation theory of the orthogonal group and, consequently, to compute its spectrum and to describe the associated eigenspaces. In this regard, the main technical result of this paper is

Theorem 3. *The operator C admits a discrete spectrum $\lambda_1, \lambda_2, \dots, \lambda_n, \dots, \lambda_\infty = 0$ such that*

$$\lambda_n = \frac{(-1)^{n-1}}{n(n+1)}.$$

Moreover, $\dim \mathcal{H}(\lambda_n) = 2n + 1$.

An immediate consequence of Theorem 3 is that the maximal eigenvalue of C is $\lambda_{\max} = 1/2$, its multiplicity is equal 3 and there exists a **spectral gap** of $\lambda_1 - \lambda_3 = 5/12$, which separates it from the rest of the spectrum. Now, consider the vector space $\mathbb{V} = \mathcal{H}(\lambda_{\max})$ and define the maps

$$\varphi_x = \sqrt{2/3} \cdot (ev_x)^t : P_x \rightarrow \mathbb{V}.$$

The second main result of this paper asserts that the vector space V equipped with the collection of tautological embeddings $\{i_x : x \in X\}$ is isomorphic to the vector space \mathbb{V} equipped with the mappings $\{\varphi_x : x \in X\}$ and, in addition, this isomorphism is proportional to the canonical morphism $\alpha_{can} : V \rightarrow \mathcal{H}$ which sends a vector $v \in V$ to the section $\alpha_{can}(v) \in \mathcal{H}$ whose value at the point x is the orthogonal projection of v on the plane P_x . All of this is summarized in the following theorem:

Theorem 4. *The morphism $\tau = \sqrt{3/2} \cdot \alpha_{can}$ maps the vector space V isometrically onto $\mathbb{V} \subset \mathcal{H}$. Moreover,*

$$\tau \circ i_x = \varphi_x,$$

for every $x \in X$.

0.5.1. *Proof of Theorems 1 and 2.* The proof is based on the following three assertions:

Assertion 1: The vector space \mathcal{H}_N approximates the vector space \mathcal{H} .

Assertion 2: The operator C_N approximates the integral operator C .

Assertion 3: The vector space \mathbb{V}_N approximates the vector space \mathbb{V} .

The validity of the first two assertions depends on our principal assumption that the points in X_N are chosen independently and uniformly at random, which implies that the Haar measure on X is approximated by the (normalized) counting measure on X_N . The validity of the third assertion depends also on the existence of a spectral gap for the operator C which implies that the maximal eigenspace can be computed in a numerically stable manner. Consequently, Theorems 1 and 2 follow from Theorem 4.

Remark 2. *The reconstruction of the orthogonal maps $\{i_x : x \in X_N\}$ is a non-linear problem because of the orthogonality constraint. One of the appealing properties of the intrinsic reconstitution algorithm is that it reduces this problem to a problem in linear algebra - the computation of the maximal eigenspace of a linear operator. Another appealing property is its remarkable stability to noise which, using a bit of random matrix theory arguments (see [7]), follows from the spectral gap property. Other existing reconstruction methods, like the angular reconstitution method (see [9] and [2]), do not enjoy this important stability property.*

0.6. **Structure of the paper.** The paper consists of three sections except of the introduction.

- In Section 1, we begin by introducing the basic analytic set-up which underlies cryo-EM. Then, we proceed to formulate the main results of this paper, which are: A complete description of the spectral properties of the common lines operator C (Theorem 5) and the admissibility of the intrinsic reconstitution algorithm (Theorem 6).
- In Section 2, we prove Theorem 5, in particular, we develop all the representation theoretic machinery which is needed for the proof.
- Finally, in Appendix A, we give the proofs of all technical statements which appeared in the previous sections.

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1. PRELIMINARIES AND MAIN RESULTS

1.1. Set up. Let (V, B) be a three dimensional Euclidian vector space over \mathbb{R} . The reader can take $V = \mathbb{R}^3$ equipped with the standard inner product $B_{std} : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$. Let $O(V) = O(V, B)$ denote the group of orthogonal transformations with respect to the inner product B ; let $SO(V) \subset O(V)$ denote the subgroup of orthogonal transformation which have determinant one; let $\theta \in O(V)$ denote the element $-Id$. Let $S(V)$ denote the unit sphere in V , that is, $S(V) = \{v \in V : B(v, v) = 1\}$.

1.2. The vector bundle of planes. Let $\mathfrak{H} \rightarrow S(V)$ be the real vector bundle with fibers $\mathfrak{H}|_x = x^\perp$ and let $\mathcal{H} = \Gamma(S(V), \mathfrak{H})$ denote the space of smooth global sections. The vector bundle \mathfrak{H} admits a fiberwise Euclidian structure induced from the one on V , which in turns yields a (pre) Euclidian structure on \mathcal{H} (here, the word pre just means that \mathcal{H} is not complete). In general, in this paper we will not distinguish between an Euclidian/Hermitian vector space and its completion and the correct choice between the two will be clear from the context.

In addition, \mathfrak{H} admits a natural $O(V)$ equivariant structure which induces an orthogonal action of $O(V)$ on the space of global sections \mathcal{H} which sends a section $s \in \mathcal{H}$ to a section $g \cdot s$, given by

$$(g \cdot s)(x) = gs(g^{-1}x),$$

for every $x \in S(V)$. This makes \mathcal{H} into an Euclidian representation of $O(V)$. We will also consider the complexified vector bundle $\mathbb{C}\mathfrak{H}$ and its space of global sections $\mathbb{C}\mathcal{H} = \Gamma(S(V), \mathbb{C}\mathfrak{H})$. The vector bundle $\mathbb{C}\mathfrak{H}$ is equipped with an Hermitian inner product induced from the Hermitian product $\langle -, - \rangle$ on $\mathbb{C}V$ which is given by

$$\langle u, v \rangle = B(\bar{u}, v),$$

where $\overline{(-)} : \mathbb{C}V \rightarrow \mathbb{C}V$ is the Galois conjugation. Consequently, $\mathbb{C}\mathcal{H}$ is a (pre) Hermitian vector space supporting a (real) unitary representation of the group $O(V)$.

1.3. The operator of common lines. We define an integral operator $C : \mathcal{H} \rightarrow \mathcal{H}$ capturing a basic relation in three dimensional Euclidian geometry.

The operator C is defined as follows: For every pair of points $x, y \in S(V)$ such that $x \neq \pm y$, consider the intersection $x^\perp \cap y^\perp$ of the corresponding orthogonal planes, which is a line in V . There are two tautological embeddings

$$\begin{aligned} C_{x,y} & : x^\perp \cap y^\perp \hookrightarrow x^\perp, \\ C_{y,x} & : x^\perp \cap y^\perp \hookrightarrow y^\perp. \end{aligned}$$

Using these embeddings we can define a rank one operator $C(x, y) : \mathfrak{H}|_y \rightarrow \mathfrak{H}|_x$, given by the composition $C_{x,y} \circ C_{y,x}^t$. The collection $\{C(x, y) : x \neq \pm y\}$ yields a well defined smooth section of $\mathfrak{H} \boxtimes \mathfrak{H}^*$ on the complement of the union of the diagonal and the anti-diagonal. It is not difficult to verify that this section extends to a

distribution section $C \in \Gamma'(S(V) \times S(V), \mathfrak{H} \boxtimes \mathfrak{H}^*)$ which, in turns, establishes a kernel for an integral operator $C : \mathcal{H} \rightarrow \mathcal{H}$ given by

$$C(s)(x) = \int_{y \in S(V)} C(x, y) (s(y)) dy,$$

for every $s \in \mathcal{H}$, where we take dy to be the normalized Haar measure on the sphere. Since $C(x, y) = C(y, x)^t$, this implies that C is a symmetric operator. In addition, it is evident that C commutes with the $O(V)$ action, namely $C(g \cdot s) = g \cdot C(s)$ for every $s \in \mathcal{H}$ and $g \in O(V)$.

The operator C is referred to as the *operator of common lines* and the main technical part of this paper will be devoted to the investigation of this operator.

1.4. Main results. Our goal is to describe an **intrinsic** model of the Euclidian vector space V which is expressed in the terms of the common lines operator and the Euclidian structure of \mathcal{H} alone.

The main technical result of this paper is

Theorem 5. *The operator C admits a discrete spectrum $\lambda_1, \lambda_2, \dots, \lambda_n, \dots, \lambda_\infty = 0$, such that*

$$\lambda_n = \frac{(-1)^{n-1}}{n(n+1)}.$$

Moreover, $\dim \mathcal{H}(\lambda_n) = 2n + 1$.

For a proof, see Section 2.

Intrinsic model: Take $\mathbb{V} = \mathcal{H}(\lambda_{\max})$ to be the maximal eigenspace of C .

There are two immediate implications of Theorem 5 that we will consider:

- The vector space \mathbb{V} is three dimensional.
- There exists a spectral gap of $\lambda_1 - \lambda_3 = 5/12$ which separates λ_{\max} from the rest of the spectrum.

Fix $r = \sqrt{3/2}$. Let $ev_x : \mathcal{H} \rightarrow x^\perp$ denote the evaluation morphism at the point $x \in S(V)$. For every $x \in S(V)$, define the morphism

$$\varphi_x = r^{-1} \cdot (ev_{x|\mathbb{V}})^t : x^\perp \rightarrow \mathbb{V}.$$

Let $\alpha_{can} : V \rightarrow \mathcal{H}$ be the canonical morphism which sends a vector $v \in V$ to the section $\alpha_{can}(v) \in \mathcal{H}$, defined by

$$\alpha_{can}(v)(x) = \text{Pr}_x(v),$$

where Pr_x is the orthogonal projection on the plane x^\perp . The morphism α_{can} is a morphism of representations of the orthogonal group $O(V)$.

Finally, define

$$\tau = r \cdot \alpha_{can} : V \rightarrow \mathcal{H}.$$

The morphism τ identifies the Euclidian vector space V together with the tautological embeddings $\{i_x : x^\perp \rightarrow V\}$ with the Euclidian vector space \mathbb{V} together with the maps $\{\varphi_x : x^\perp \rightarrow \mathbb{V}\}$. All of this is summarized in the following theorem:

Theorem 6. *The map τ maps V isometrically onto \mathbb{V} . Moreover,*

$$\tau \circ i_x = \varphi_x,$$

for every $x \in S(V)$.

For a proof, see Appendix A (the proof uses the results and terminology of Section 2).

2. SPECTRAL ANALYSIS OF THE OPERATOR OF COMMON LINES

2.1. Set-up. It will be convenient to extend the set-up a bit.

2.1.1. Auxiliary vector bundles. We introduce the following auxiliary vector bundles on $S(V)$. Let $\mathfrak{N} \rightarrow S(V)$ be the vector bundle of normal lines with fibers $\mathfrak{N}|_x = \mathbb{R}x$ and let $\mathcal{N} = \Gamma(S(V), \mathfrak{N})$ denote the corresponding space of global sections. Let $V_{S(V)}$ denote the trivial vector bundle with fiber at each point equal V and let $\mathcal{V} = \Gamma(S(V), V_{S(V)}) = \mathcal{F} \otimes V$ where $\mathcal{F} = C^\infty(S(V), \mathbb{R})$, we have

$$\mathcal{V} = \mathcal{H} \oplus \mathcal{N}.$$

All the vector bundles are equipped with a fiberwise Euclidian structure which is induced from the one on V and consequently the spaces of global sections are Euclidian. In addition, all the vector bundles are equipped with a natural $O(V)$ equivariant structure which is compatible with the Euclidian structure and consequently the spaces of global sections form Euclidian representations of the group $O(V)$. We will consider these spaces as representations of the subgroup $SO(V)$ and remember also the action of the special element $\theta \in O(V)$ which commutes with the action of $SO(V)$.

2.1.2. The operator of orthographic lines. We define an integral operator $O : \mathcal{H} \rightarrow \mathcal{H}$ which we refer to as the *operator of orthographic lines*. This operator captures another basic relation in three dimensional Euclidian geometry which, in some sense, stands in duality with the common lines relation.

The operator O is defined by the following kernel: For every pair of points $x, y \in S(V)$ such that $x \neq \pm y$, consider the pair of unit vectors

$$\begin{aligned} o_{x,y} &= \frac{\text{Pr}_x(y)}{\|\text{Pr}_x(y)\|} \in x^\perp, \\ o_{y,x} &= \frac{\text{Pr}_y(x)}{\|\text{Pr}_y(x)\|} \in y^\perp. \end{aligned}$$

In words, the vector $o_{x,y}$ is the normalized projection of the vector y on the plane x^\perp and similarly the vector $o_{y,x}$ is the normalized projection of the vector x on the plane y^\perp . Define a rank one operator $O(x, y) : y^\perp \rightarrow x^\perp$ by

$$O(x, y)(v) = B(o_{y,x}, v) o_{x,y},$$

for every $v \in y^\perp$. The collection $\{O(x, y) : x \neq \pm y\}$ yields a well defined smooth section of $\mathfrak{H} \boxtimes \mathfrak{H}^*$ on the complement of the union of the diagonal and the anti-diagonal which extends to a distribution section $O \in \Gamma'(S(V) \times S(V), \mathfrak{H} \boxtimes \mathfrak{H}^*)$ which, in turns, yields a symmetric integral operator $O : \mathcal{H} \rightarrow \mathcal{H}$ which commutes with the $O(V)$ action.

Given a pair of unit vectors $x \neq \pm y$, the following observations are in order.

- The orthographic lines $\mathbb{R}o_{x,y} \in x^\perp$ and $\mathbb{R}o_{y,x} \in y^\perp$ are orthogonal to the common line $x^\perp \cap y^\perp$.
- The kernel $O(x, y)$ satisfy $O(x, -y) = O(-x, y) = -1 \cdot O(x, y)$ which means that $O(x, y)$ depends on the choice of the unit vectors x, y and not only on the planes x^\perp, y^\perp . This should be contrasted with the analogue

property of the kernel $C(x, y)$ which satisfies $C(x, -y) = C(-x, y) = C(x, y)$.

2.1.3. The operator of parallel translation. We define the integral operator $T = C - O : \mathcal{H} \rightarrow \mathcal{H}$. For every $x, y \in S(V)$, such that $x \neq \pm y$, the kernel $T(x, y) : y^\perp \rightarrow x^\perp$ is a full rank operator and it is not difficult to verify that $T(x, y)$ is the operator of parallel translation along the unique geodesic (large circle) connecting the point y with x . Consequently, we will refer to T as the operator of *parallel translations*.

The strategy that we are going to follow is to study the spectral properties of the operator T , from which, as it turns out, the spectral properties of the operators C and O can be derived.

2.2. Isotypic decompositions. The spaces $\mathcal{H}, \mathcal{V}, \mathcal{N}$ and \mathcal{F} form Euclidian representations of the group $SO(V)$ and as such decompose into isotypic components²

$$\begin{aligned}\mathcal{H} &= \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \\ \mathcal{N} &= \bigoplus_{n=0}^{\infty} \mathcal{N}_n, \\ \mathcal{V} &= \bigoplus_{n=0}^{\infty} \mathcal{V}_n, \\ \mathcal{F} &= \bigoplus_{n=0}^{\infty} \mathcal{F}_n,\end{aligned}$$

where we use the subscript n to denote the isotypic component which consists of the unique irreducible representation of $SO(V)$ of dimension $2n + 1$. In addition, the element $\theta \in O(V)$ acts on all these spaces, thus decompose them into a direct sum of two components

$$\begin{aligned}\mathcal{H} &= \mathcal{H}^+ \oplus \mathcal{H}^-, \\ \mathcal{N} &= \mathcal{N}^+ \oplus \mathcal{N}^-, \\ \mathcal{V} &= \mathcal{V}^+ \oplus \mathcal{V}^-, \\ \mathcal{F} &= \mathcal{F}^+ \oplus \mathcal{F}^-, \end{aligned}$$

where we use the superscript $+$ to denote the component on which θ acts as Id and the superscript $-$ to denote the component on which θ acts as $-Id$. We will refer to the $+$ component as the symmetric component and to the $-$ component as the anti-symmetric component.

The following theorem summarizes the properties of these decompositions which we will require in the sequel.

Theorem 7. *The following properties hold*

- (1) *Each isotypic component \mathcal{F}_n is an irreducible representation. Moreover, $\mathcal{F}_n = \mathcal{F}_n^+$ when n is even and $\mathcal{F}_n = \mathcal{F}_n^-$ when n is odd.*

²We remind the reader that an isotypic component is a representation which is a direct sum of copies a single irreducible representation.

- (2) Each isotypic component \mathcal{N}_n is an irreducible representation. Moreover, $\mathcal{N}_n = \mathcal{N}_n^+$ when n is even and $\mathcal{N}_n = \mathcal{N}_n^-$ when n is odd.
- (3) The isotypic component $\mathcal{H}_0 = 0$ and each isotypic component \mathcal{H}_n , $n \geq 1$ decomposes under θ into a direct sum of two irreducible representations $\mathcal{H}_n^+ \oplus \mathcal{H}_n^-$.
- (4) The isotypic component \mathcal{V}_0 is equal to the symmetric trivial representation $\mathbf{1}^+$ and each isotypic component \mathcal{V}_n , $n \geq 1$ decomposes under θ into a direct sum of three irreducible representations $\mathcal{H}_n^+ \oplus \mathcal{H}_n^- \oplus \mathcal{N}_n^?$ where

$$? = \begin{cases} + & n \text{ even} \\ - & n \text{ odd} \end{cases} .$$

For a proof, see Appendix A.

Since the operators C, O and T commute with the $O(V)$ action, they preserve all the above decompositions.

Proposition 1. *The following properties hold:*

- The operator T acts as scalar operator on \mathcal{H}_n , moreover, $T|_{\mathcal{H}_n} = \lambda_n Id$ where $\lambda_n \neq 0$.
- The isotypic component $\mathcal{H}_n^+ \subset \ker C$, moreover, $C|_{\mathcal{H}_n^+} = \lambda_n Id$.
- The isotypic component $\mathcal{H}_n^- \subset \ker O$, moreover, $O|_{\mathcal{H}_n^-} = -\lambda_n Id$.

For a proof, see Appendix A.

The rest of this section is devoted to the computation of the eigenvalues λ_n .

2.3. Computation of the eigenvalues. Fix a point $x \in S(V)$. Let $T_x = \{g \in SO(V) : gx = x\}$ be the subgroup of rotations around x . Let $(H, E, F) \in \mathbb{C}Lie(SO(V))$ be an sl_2 triple associated with T_x .

2.3.1. Spherical decomposition. For each $n \geq 1$, the complexified (Hilbert) space $\mathbb{C}\mathcal{H}_n$ admits an isotypic decomposition with respect to the action of T_x

$$\mathbb{C}\mathcal{H}_n = \bigoplus_{m=-n}^n \mathcal{H}_n^m,$$

where H acts on \mathcal{H}_n^m by $2mId$. Since $\mathbb{C}\mathcal{H}_n = \mathbb{C}\mathcal{H}_n^+ \oplus \mathbb{C}\mathcal{H}_n^-$, each \mathcal{H}_n^m is two dimensional.

2.3.2. Strategy. Given a section $u_n \in \mathbb{C}\mathcal{H}_n$, by Proposition 1, $Tu_n = \lambda_n u_n$. If, in addition, $u_n \in \mathcal{H}_n^+$ then, as will be shown, u_n can be chosen such that $u_n(x) \neq 0$. Under this choice, the eigenvalue λ_n can be computed from the equation

$$\lambda_n \langle u_n(x), u_n(x) \rangle = \langle u_n(x), Tu_n(x) \rangle.$$

The following proposition gives an explicit formula for $\langle u_n(x), Tu_n(x) \rangle$. But first, we need to introduce one additional terminology.

Fix a unit vector $l_0 \in S(x^\perp) \subset V$ and let $A_{l_0} \in Lie(T_{l_0})$ be a vector such that the morphism $\exp : [0, 2\pi) \rightarrow T_{l_0}$ given by $\exp(\theta) = e^{\theta A_{l_0}}$ is an isomorphism.

Proposition 2. *The following equation holds:*

$$(2.1) \quad \lambda_n \langle u_n(x), u_n(x) \rangle = \int_0^\pi \mu(\theta) \langle u_n(x), e^{-\theta A_{l_0}} u_n(e^{\theta A_{l_0}} x) \rangle d\theta,$$

where $\mu(\theta) = \sin(\theta)/2$.

For a proof, see Appendix A.

Our strategy is to construct "good" section $u_n \in \mathcal{H}_n^1$ and then to use Equation (2.1).

2.3.3. Construction of a "good" section. For every $n \geq 0$, choose a highest weight vector $\psi_n \in \mathbb{C}\mathcal{F}_n$ with respect to (H, E, F) ($H\psi_n = 2n\psi_n$). In addition, choose a highest weight vector $v_1 \in \mathbb{C}V$ ($Hv_1 = 2v_1$).

For every $n \geq 1$, the section $\psi_{n-1} \otimes v_1$ is a highest weight vector in $\mathbb{C}\mathcal{V}_n$. In order to get a section in \mathcal{H}_n^1 , first step, apply the lowering operator F and consider the section $\tilde{u}_n = F^{n-1}(\psi_{n-1} \otimes v_1) \in \mathcal{V}_n^1$.

Let us denote by P_{n-1} the weight zero spherical function $F^{n-1}\psi_{n-1} \in \mathcal{F}_{n-1}^0$ and note that, under appropriate choice of coordinates, P_{n-1} is the classical spherical function on the sphere $S(V)$. The following proposition gives an explicit expression of \tilde{u}_n in terms of the function P_{n-1} and the vector v_1 .

Proposition 3. *The section \tilde{u}_n can be written as*

$$(2.2) \quad \tilde{u}_n = P_{n-1} \otimes v_1 + \frac{1}{n}EP_{n-1} \otimes Fv_1 + \frac{1}{2n(n+1)}E^2P_{n-1} \otimes F^2v_1.$$

For a proof, see Appendix A.

The second step in order to get a section in \mathcal{H}_n^1 is to define

$$u_n(y) = P_y \tilde{u}_n(y),$$

for every $y \in S(V)$, where P_y is the orthogonal projector on y^\perp .

Proposition 4. *The section u_n is symmetric or anti-symmetric depending on the parity of n as follows*

$$\begin{aligned} u_n &\in \mathcal{H}_n^{+,1} \text{ when } n \text{ is even,} \\ u_n &\in \mathcal{H}_n^{-,1} \text{ when } n \text{ is odd.} \end{aligned}$$

For a proof, see Appendix A.

Now, we are ready to finish the computation. Using Equation (2.1), we can write

$$\lambda_n \langle u_n(x), u_n(x) \rangle = \int_0^\pi \mu(\theta) \langle e^{\theta A_{i_0}} u_n(x), \tilde{u}_n(e^{\theta A_{i_0}} x) \rangle d\theta.$$

Considering formula (2.2), it is evident that the functions EP_{n-1} and E^2P_{n-1} must vanish at $x \in S(V)$ since these are function of weight different then zero with respect to the action of T_x . Hence

$$u_n(x) = \tilde{u}_n(x) = P_{n-1}(x) v_1 \in \mathbb{C}x^{\perp,1}.$$

For $k = 0, 1, 2$, define the integrals

$$\begin{aligned} I_n^k &= \frac{1}{\langle u_n(x), u_n(x) \rangle} \int_0^\pi \mu(\theta) \langle e^{\theta A_{i_0}} u_n(x), E^k P_{n-1}(e^{\theta A_{i_0}} x) F^k v_1 \rangle d\theta \\ &= \frac{1}{P_{n-1}(x) \|v_1\|^2} \int_0^\pi \mu(\theta) E^k P_{n-1}(e^{\theta A_{i_0}} x) \langle e^{\theta A_{i_0}} v_1, F^k v_1 \rangle d\theta, \end{aligned}$$

and note that the eigen value λ_n can be expressed as

$$(2.3) \quad \lambda_n = I_n^0 + \frac{1}{n}I_n^1 + \frac{1}{2n(n+1)}I_n^2.$$

Theorem 8 (Main technical statement). *For $k = 0, 1, 2$, the integrals I_n^k are equal to*

$$\begin{aligned} I_n^0 &= \begin{cases} 1 & n = 1 \\ 1/6 & n = 2 \\ 0 & n \geq 3 \end{cases}, \\ I_n^1 &= \begin{cases} 0 & n = 1 \\ -2/3 & n = 2 \\ 0 & n \geq 3 \end{cases}, \\ I_n^2 &= \begin{cases} 0 & n = 1 \\ 0 & n = 2 \\ 2(-1)^{n-1} & n \geq 3 \end{cases}. \end{aligned}$$

For a proof, see Subsection 2.4.

Consequently, using Theorem 8 and Equation (2.3) we obtain the desired formula

$$\lambda_n = \frac{(-1)^{n-1}}{n(n+1)},$$

which proves Theorem 5.

2.4. Proof of the main technical statement. Let (e_1, e_2, e_3) be an orthonormal basis of V . Silently, the reader should think of the basis vector e_3 as standing in place of the fixed unit vector $x \in S(V)$ and of the basis vector e_2 as standing in place of the vector $l_0 \in S(x^\perp)$. It is possible to choose vectors $A_{e_i} \in Lie(T_{e_i})$ which satisfy the relations

$$\begin{aligned} [A_{e_3}, A_{e_1}] &= A_{e_2}, \\ [A_{e_3}, A_{e_2}] &= -A_{e_1}, \\ [A_{e_1}, A_{e_2}] &= A_{e_3}, \end{aligned}$$

and, in addition, satisfy $[A_{e_i}, A_{e_j}] = A_{e_i}e_j$ for every $1 \leq i, j \leq 3$. We can define the following sl_2 triple (H, E, F) which is associated with T_{e_3}

$$\begin{aligned} H &= -2iA_{e_3}, \\ E &= iA_{e_2} - A_{e_1}, \\ F &= A_{e_1} + iA_{e_2}. \end{aligned}$$

2.4.1. Spherical coordinates. We introduce spherical coordinates $f : (0, 2\pi) \times (0, \pi) \rightarrow S(V)$ given by $f(\varphi, \theta) = g_\varphi \cdot (\cos(\theta)e_3 + \sin(\theta)e_1)$ where

$$g_\varphi = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In the coordinates (φ, θ) the operators H, E, F are given by the following formulas

$$\begin{aligned} H &= 2i\partial_\varphi, \\ E &= -e^{-i\varphi} (i\partial_\theta + \cot(\theta) \partial_\varphi), \\ F &= -e^{i\varphi} (i\partial_\theta - \cot(\theta) \partial_\varphi). \end{aligned}$$

2.4.2. *Highest weight vector in V .* The vector $v_1 = -e_1 + ie_2$ is a highest weight vector in V and we note that $\|v_1\|^2 = 2$. For $k = 0, 1, 2$, let us denote by $j^k(\theta)$ the function $\langle e^{\theta A_{e_2}} v_1, F^k v_1 \rangle$. Explicit calculation reveals that

$$\begin{aligned} j^0(\theta) &= \cos(\theta) + 1, \\ j^1(\theta) &= 2i \sin(\theta), \\ j^2(\theta) &= 2 \cos(\theta) - 2. \end{aligned}$$

2.4.3. *Spherical function in \mathcal{F}_n .* For $n \geq 0$, let $P_n \in \mathcal{F}_n^0$ denote the unique weight zero spherical function which satisfies the normalization condition $P_n(e_3) = 1$. Define the generating function

$$G(\varphi, \theta, t) = \sum_{n=0}^{\infty} P_n(\varphi, \theta) t^n.$$

The generating function G admits an explicit formula.

Theorem 9 ([8]).

$$(2.4) \quad G(\varphi, \theta, t) = (1 - 2t \cos(\theta) + t^2)^{-1/2}.$$

Remark 3. Note that $G(0, 0, t) = (1 - t)^{-1} = \sum_{n=0}^{\infty} t^n$ which is compatible with the normalization condition $P_n(0, 0) = 1$.

Applying the raising operator E we obtain the generating functions

$$\begin{aligned} EG(\varphi, \theta, t) &= \sum_{n=0}^{\infty} EP_n(\varphi, \theta) t^n, \\ E^2G(\varphi, \theta, t) &= \sum_{n=0}^{\infty} E^2P_n(\varphi, \theta) t^n. \end{aligned}$$

Granting formula (2.4), explicit calculation reveals that

$$\begin{aligned} EG(\varphi, \theta, t) &= ie^{-i\varphi} t \sin(\theta) (1 - 2t \cos(\theta) + t^2)^{-3/2}, \\ E^2G(\varphi, \theta, t) &= -3e^{-2i\varphi} t^2 \sin^2(\theta) (1 - 2t \cos(\theta) + t^2)^{-5/2}. \end{aligned}$$

2.4.4. *Putting everything together.* In terms of our choice of the highest weight vector v_1 and the spherical functions P_n 's, the integrals I_n^k , $k = 0, 1, 2$, are given in the spherical coordinates (φ, θ) by

$$\begin{aligned} I_n^k &= \frac{1}{P_{n-1}(x) \|v_1\|^2} \int_0^\pi \mu(\theta) E^k P_{n-1}(0, \theta) j^k(\theta) d\theta \\ &= \frac{1}{2} \int_0^\pi \mu(\theta) E^k P_{n-1}(0, \theta) j^k(\theta) d\theta. \end{aligned}$$

For $k = 0, 1, 2$, define the generating functions

$$I^k(t) = \sum_{n=0}^{\infty} I_{n+1}^k t^n.$$

Each $I^k(t)$ can be expressed as the integral

$$I^k(t) = \frac{1}{2} \int_0^{\pi} \mu(\theta) E^k G(0, \theta, t) j^k(\theta) d\theta.$$

Explicit calculation of the integrals $I^k(t)$ reveals that

$$\begin{aligned} I^0(t) &= \frac{1}{2} \left(1 + \frac{1}{3}t \right), \\ I^1(t) &= \frac{1}{2} \left(-\frac{4}{3}t \right), \\ I^2(t) &= \frac{1}{2} \left(4(1+t)^{-1} - 4t - 4 \right) = 2 \sum_{n=2}^{\infty} (-1)^n t^n. \end{aligned}$$

From the above formulas, using Equation (2.3), we get

$$\lambda_n = \frac{(-1)^{n-1}}{n(n+1)},$$

for every $n \geq 1$. This finish the proof of Theorem 8.

APPENDIX A. PROOFS

A.1. Proof of Theorem 6. Since τ is a morphism of $O(V)$ representations, it maps V isometrically onto \mathcal{H}_1^- - the unique anti-symmetric copy (θ acts by -1) of the three dimensional representation of $SO(V)$, which, by Proposition 1, coincides with $\mathbb{V} = \mathcal{H}(\lambda_{\max})$.

Evidently, τ is an isometry, up to a scalar. Hence, it is enough to show that $Tr(\tau \circ \tau^T) = 3$, which we verify as follows:

$$\begin{aligned} Tr(\tau \circ \tau^T) &= r^2 \cdot Tr(\alpha_{can} \circ \alpha_{can}^T) = \frac{3}{2} \int_{x \in S(V)} Tr(i_x^T \circ i_x) dx \\ &= \frac{3}{2} \int_{x \in S(V)} 2dx = 3. \end{aligned}$$

Finally, the relation $\tau \circ i_x = \varphi_x$ follows from

$$e_{x|_{\mathbb{V}}} \circ \alpha_{can} = Pr_x.$$

This concludes the proof of the theorem.

A.2. Proof of Theorem 7. Property 1 is the classical result of spherical harmonics on the two dimensional sphere, which can be found for example in [8]. We just note that the representation \mathcal{F}_n consists of the restriction to $S(V)$ of harmonic polynomials of degree n , which implies that $\mathcal{F}_n = \mathcal{F}_n^+$ when n is even and $\mathcal{F}_n = \mathcal{F}_n^-$ when n is odd.

Property 2 follows from property 1 since \mathcal{N} can be trivialized using the $O(V)$ invariant section $s \in \mathcal{N}$ where $s(y) = y$, for every $y \in S(V)$.

We now prove Properties 3 and 4 simultaneously.

Since $\mathcal{V} = \mathcal{F} \otimes V$ as a representation of $O(V)$, we can compute the isotypic components of \mathcal{V} in terms of the isotypic components of \mathcal{F} . The computation proceeds as follows:

- For $n = 0$, $\mathcal{F}_0 \otimes V = V^-$.
- For $n \geq 1$, $\mathcal{F}_n \otimes V = (\mathcal{F}_n \otimes V)_{n-1}^? \oplus (\mathcal{F}_n \otimes V)_n^? \oplus (\mathcal{F}_n \otimes V)_{n+1}^?$ where

$$? = \begin{cases} + & n \text{ odd} \\ - & n \text{ even} \end{cases}.$$

The decomposition of $\mathcal{F}_n \otimes V$ as a representation of $SO(V)$ is computed using the branching rules of a tensor product and the action of $\theta \in O(V)$ is derived from the facts that $V = V^-$ and Property 1.

This implies that the isotypic components of \mathcal{V} are

- For $n = 0$, $\mathcal{V}_n = \mathbf{1}^+$.
- For odd $n \geq 1$, $\mathcal{V}_n = (\mathcal{F}_{n-1} \otimes V)_n^- \oplus (\mathcal{F}_n \otimes V)_n^+ \oplus (\mathcal{F}_{n+1} \otimes V)_n^-$.
- For even $n \geq 1$, $\mathcal{V}_n = (\mathcal{F}_{n-1} \otimes V)_n^+ \oplus (\mathcal{F}_n \otimes V)_n^- \oplus (\mathcal{F}_{n+1} \otimes V)_n^+$.

Combined with Property 2 and the fact that $\mathcal{V} = \mathcal{H} \oplus \mathcal{N}$ yields Properties 3,4. This concludes the proof of the Theorem.

A.3. Proof of Proposition 1. Fix $n \geq 1$. Denote $\mathcal{H} = \mathcal{H}_n$ and $\mathcal{H}^\pm = \mathcal{H}_n^\pm$. The statement that $\mathcal{H}^+ \subset \ker C$ follows from the facts that $C(x, -y) = C(-x, y) = C(x, y)$ and that a section $s \in \mathcal{H}^+$ satisfies that $s(-x) = -\theta(s)(x) = -s(x)$. Similarly, the statement that $\mathcal{H}^- \subset \ker O$ follows from the facts that $O(x, -y) = O(-x, y) = -O(x, y)$ and that a section $s \in \mathcal{H}^-$ satisfies that $s(-x) = -\theta(s)(x) = s(x)$.

Since, by definition, $T = C - O$, this implies that $C|_{\mathcal{H}^-} = T|_{\mathcal{H}^-}$ and $-O|_{\mathcal{H}^+} = T|_{\mathcal{H}^+}$. Since T commutes with the action of $SO(V)$ and \mathcal{H}^\pm are irreducible representations

$$T|_{\mathcal{H}^\pm} = \lambda^\pm Id.$$

We are left to show that $\lambda^+ = \lambda^-$. The argument proceeds as follows:

Let us denote by $ev_x : \mathbb{C}\mathcal{H} \rightarrow \mathbb{C}x^\perp$ the evaluation map at the point x . Since x is fixed by the group T_x , this implies that ev_x is a morphism of representations of T_x . Moreover, ev_x induces an isomorphism of weight spaces

$$ev_x : \mathcal{H}^{\pm,1} \xrightarrow{\cong} x^{\perp,1}.$$

Fix sections $u^\pm \in \mathcal{H}^{\pm,1}$ normalized to be of norm 1. Applying Formula (2.1), we can write

$$\begin{aligned} \text{(A.1)} \quad \lambda^\pm \langle ev_x u^\pm, ev_x u^\pm \rangle &= \int_0^\pi \mu(\theta) \langle ev_x u^\pm, ev_x (e^{\theta A_{i_0}} u^\pm) \rangle d\theta \\ &= \langle ev_x u^\pm, ev_x (\pi^\pm(\bar{\mu}) u_n) \rangle, \end{aligned}$$

where $\pi^\pm : T_x \rightarrow U(\mathbb{C}\mathcal{H}^\pm)$ are the group actions restricted to the subgroup T_x and $\bar{\mu}$ is the function on T_x corresponding to μ via the isomorphism $e^{\theta A_{i_0}}$.

Equation (A.1), implies that

$$\lambda^\pm = \langle u^\pm, \pi^\pm(\bar{\mu}) u^\pm \rangle_{\mathbb{C}\mathcal{H}^\pm}.$$

This implies that λ^\pm are characterized solely in terms of the irreducible representation $\pi^\pm : SO(V) \rightarrow U(\mathbb{C}\mathcal{H}^\pm)$, which, in turns, implies that $\lambda^+ = \lambda^-$.

This concludes the proof of the proposition.

A.4. Proof of Proposition 2. Let $f : T_x \times (0, \pi) \rightarrow S(V)$ be the spherical coordinates on $S(V)$ given by $f(g, \theta) = ge^{\theta A_{i_0}}x$. In these coordinates, the normalized Haar measure on $S(V)$ is given by $dg \boxtimes \mu(\theta) d\theta$ where dg is the normalized Haar measure on T_x and $\mu(\theta) = \sin(\theta)/2$.

The section $u_n \in \mathcal{H}_n^1$ is a character vector with respect to the group T_x , let us denote this character by $\chi : T_x \rightarrow S^1$ and note that we have $g \cdot u_n = \chi(g) u_n$, for every $g \in T_x$. Now, compute

$$\begin{aligned}
\lambda_n \langle u_n(x), u_n(x) \rangle &= \langle u_n(x), T u_n(x) \rangle \\
&= \int_{y \in S(V)} \langle u_n(x), T(x, y) u_n(y) \rangle dy \\
&= \int_{T_x} dg \int_0^\pi \mu(\theta) \langle u_n(x), T(x, ge^{\theta A_{i_0}}x) u_n(ge^{\theta A_{i_0}}x) \rangle d\theta \\
&= \int_{T_x} dg \int_0^\pi \mu(\theta) \langle u_n(x), gT(x, e^{\theta A_{i_0}}x) g^{-1} u_n(ge^{\theta A_{i_0}}x) \rangle d\theta \\
&= \int_{T_x} dg \int_0^\pi \mu(\theta) \langle g^{-1} u_n(x), T(x, e^{\theta A_{i_0}}x) g^{-1} u_n(ge^{\theta A_{i_0}}x) \rangle d\theta \\
&= \int_{T_x} dg \int_0^\pi \mu(\theta) \langle u_n(x), T(x, e^{\theta A_{i_0}}x) u_n(e^{\theta A_{i_0}}x) \rangle d\theta \\
&= \int_0^\pi \mu(\theta) \langle u_n(x), e^{-\theta A_{i_0}} u_n(e^{\theta A_{i_0}}x) \rangle d\theta,
\end{aligned}$$

where, step 4 follows from the fact that T commutes with the action of $SO(V)$ which is equivalent to the property that $T(gx, gy) = gT(x, y)g^{-1}$, for every $x, y \in S(V)$ and $g \in SO(V)$ which, in particular, implies that $T(x, ge^{\theta A_{i_0}}x) = T(gx, ge^{\theta A_{i_0}}x) = gT(x, e^{\theta A_{i_0}}x)g^{-1}$ and step 7 follows from the fact that $T(x, e^{\theta A_{i_0}}x)$ is the operator of parallel translation along the big circle connecting the point $e^{\theta A_{i_0}}x$ with the point x .

This concludes the proof of the proposition.

A.5. Proof of Proposition 3. First we note the following simple fact: The operator $EF : \mathbb{C}\mathcal{V}_n \rightarrow \mathbb{C}\mathcal{V}_n$ preserve the weight spaces \mathcal{V}_n^l , and, moreover, since $\mathbb{C}\mathcal{V}_n$ is a representation of highest weight $2n$ with respect to the sl_2 triple (H, E, F) we have

$$(A.2) \quad EF|_{\mathcal{V}_n^l} = (n+l)(n-l+1)Id,$$

for $l = -n, \dots, n$.

Now, calculate

$$\tilde{u}_n = F^{n-1}(\psi_{n-1} \otimes v_1) = \sum_{i=0}^{n-1} \binom{n-1}{i} F^{n-1-i} \otimes F^i (\psi_{n-1} \otimes v_1).$$

Since $\mathbb{C}V$ is a representation of highest weight 2 with respect to the sl_2 triple (H, E, F) , all tensors of the form $(-)\otimes F^k v_1$, for $k \geq 3$, vanish. This implies that the above sum is equal to

$$F^{n-1}\psi_{n-1} \otimes v_1 + (n-1)F^{n-2}\psi_{n-1} \otimes Fv_1 + \frac{(n-1)(n-2)}{2}F^{n-3}\psi_{n-1} \otimes F^2v_1.$$

Recall that $P_{n-1} = F^{n-1}\psi_{n-1}$. Explicit calculation, using formula (A.2), reveals that

$$\begin{aligned} F^{n-1}\psi_{n-1} &= \frac{1}{n(n-1)}EP_{n-1}, \\ F^{n-3}\psi_{n-1} &= \frac{1}{(n-2)(n-1)n(n+1)}E^2P_{n-1}. \end{aligned}$$

Combining all the above yields the desired formula for \tilde{u}_n .

This concludes the proof of the proposition.

A.6. Proof of Proposition 4. The statement follows directly from the facts that $V = V^-$ which implies that $\theta(F^k v_1) = -F^k v_1$ and that $P_{n-1} \in \mathcal{F}_{n-1}^?$ where

$$? = \begin{cases} + & n \text{ odd} \\ - & n \text{ even} \end{cases}.$$

This concludes the proof of the proposition.

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