GENERALIZATIONS OF CLAUSEN'S FORMULA AND ALGEBRAIC TRANSFORMATIONS OF CALABI–YAU DIFFERENTIAL EQUATIONS

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Abstract. We provide certain unusual generalizations of Clausen's and Orr's theorems for solutions of generalized hypergeometric equations of order 4 and 5. As application, we present several examples of algebraic transformations of Calabi–Yau differential equations.

Introduction

In our study of Picard–Fuchs differential equations of Calabi–Yau type [2], [3] we discovered some curious relations between hypergeometric series

$$ {}_mF_{m-1}\left(\begin{matrix} a_1, a_2, \dots, a_m \\ b_2, \dots, b_m \end{matrix} \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_m)_n}{(b_2)_n \cdots (b_m)_n} \frac{z^n}{n!} \tag{1} $$

and their natural generalizations. Although our original motivation is in the differential equations themselves, we are intrigued by seeing that many of our identities can be extended to a more general form, which does not use all the properties of their Calabi–Yau prototypes. Besides the very classical examples of such identities, like Orr-type theorems in [14, Section 2.5], we have already indicated an example related to a Calabi–Yau equation in [2, Proposition 6]. The main aim of the present article is to systemize our findings and present algebraic transformations of certain hypergeometric and related series in a general form. Specializations to Calabi–Yau examples [2], [3] are discussed in some detail.

The paper is organized as follows. In Section 1 we review the notion of a Calabi–Yau differential equation, while in Section 2 we recall some 'standard' relations between Calabi–Yau differential equations of order 2 and 3, and of order 4 and 5; these two sections may be regarded as an expanded introductory part. Section 3 is devoted to algebraic transformations of 2nd and 3rd order differential equations; in Section 4 we discuss transformations of higher order equations with applications to Calabi–Yau examples. In Section 5 we indicate a natural formal invariant of 4th

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order Calabi–Yau differential equations that can be used to verify whether two such equations are related by an algebraic transformation. In Section 6 we discuss our strategies to find and prove algebraic transformations for differential equations.

1. Calabi–Yau differential equations

Certain differential equations look better than others, at least arithmetically. To illustrate this principle, consider the differential equation

$$\big(\theta^2 - z(11\theta^2 + 11\theta + 3) - z^2(\theta+1)^2\big)y = 0, \qquad \text{where} \quad \theta = z\frac{\mathrm{d}}{\mathrm{d}z}\,. \tag{2}$$

What is special about it? First of all, it has a unique analytic solution $y_0(z) = f(z)$ with $f(0) = 1$; another solution may be given in the form $y_1(z) = f(z)\log z + g(z)$ with $g(0) = 0$. Secondly, the coefficients in the Taylor expansion $f(z) = \sum_{n=0}^{\infty} A_n z^n$ are integral, $f(z) \in 1 + z\mathbb{Z}[[z]]$, which can be hardly seen from the defining recurrence

$$(n+1)^2 A_{n+1} - (11n^2 + 11n + 3)A_n - n^2 A_{n-1} = 0 \quad \text{for } n = 0, 1, \dots, \quad A_0 = 1 \tag{3}$$

(cf. (1)), but follows from the explicit expression

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}, \qquad n = 0, 1, \dots, \tag{4}$$

due to R. Apéry [5]; note that these numbers appear in Apéry's proof of the irrationality of $\zeta(2)$. Thirdly, the expansion $q(z) = \exp(y_1(z)/y_0(z)) = z\exp(g(z)/f(z))$ also has integral coefficients, $q(z) \in z\mathbb{Z}[[z]]$. This follows from the fact that the functional inverse $z(q)$,

$$z(q) = q\prod_{n=1}^{\infty} (1 - q^n)^{5\left(\frac{n}{5}\right)}, \tag{5}$$

where $\left(\frac{n}{5}\right)$ denotes the Legendre symbol, lies in $q\mathbb{Z}[[q]]$. The formula in (5), due to F. Beukers [7], shows that $z(q)$ is a modular function with respect to the congruence subgroup $\Gamma_1(5)$ of $SL_2(\mathbb{Z})$.

If the reader is not so much surprised by these integrality properties, then try to find more such cases, replacing the differential operator in (2) by the more general one

$$\theta^2 - z(a\theta^2 + a\theta + b) + cz^2(\theta+1)^2. \tag{6}$$

To ensure the required integrality one easily gets $a, b, c \in \mathbb{Z}$, but for a generic choice of the parameters already the second feature ($y_0(z) = f(z) \in 1 + z\mathbb{Z}[[z]]$) fails 'almost always'. In fact, this problem was studied by F. Beukers [9] and D. Zagier [22]. The exhaustive experimental search in [22] resulted in 14 (non-degenerate) examples of the triplets $(a, b, c) \in \mathbb{Z}^3$ when both this and the third property (the integrality of the corresponding expansion $z(q)$) happen; the latter follows from modular interpretations of $z(q)$.

A natural extension of the above problem to 3rd order linear differential equations is prompted by the other Apéry's sequence used in his proof [5] of the irrationality of $\zeta(3)$. One takes the family of differential operators

$$\theta^3 - z(2\theta+1)(\hat{a}\theta^2+\hat{a}\theta+\hat{b}) + \hat{c}z^2(\theta+1)^3 \tag{7}$$

and looks for the cases when the two solutions $f(z) \in 1+z\mathbb{C}[[z]]$ and $f(z)\log z+g(z)$ with $g(0)=0$ of the corresponding differential equation satisfy $f(z)\in\mathbb{Z}[[z]]$ and $\exp(g(z)/f(z))\in\mathbb{Z}[[z]]$. Apart from some degenerate cases, we have found in [2] again 14 triplets $(\hat{a},\hat{b},\hat{c})\in\mathbb{Z}^3$ meeting the integrality conditions; the second one holds in all these cases as a modular bonus. Apéry's example corresponds to the case $(\hat{a},\hat{b},\hat{c})=(17,5,1)$.

How can one generalize the above problem of finding 'arithmetically nice' linear differential equations (operators)? An approach we followed in [2], [3], at least up to order 5, was not specifying the form of the operator, like in (6) and (7), but posing the following:

(i) the differential equation is of Fuchsian type, that is, all its singular points are regular; in addition, the local exponents at $z=0$ are zero;
(ii) the unique analytic solution $y_0(z)=f(z)$ with $f(0)=1$ at the origin have integral coefficients, $f(z)\in 1+z\mathbb{Z}[[z]]$; and
(iii) the solution $y_1(z)=f(z)\log z+g(z)$ with $g(0)=0$ gives rise to the integral expansion $\exp(y_1(z)/y_0(z))\in z\mathbb{Z}[[z]]$.

Requirement (i), known as the condition of *maximally unipotent monodromy* (MUM), means that the corresponding differential operator written as a polynomial in variable z with coefficients from $\mathbb{C}[[\theta]]$ has constant term θ^m, where m is the order — degree in θ; the local monodromy around 0 consists of a single Jordan block of maximal size. Note that (i) guarantees the uniqueness of the above $y_0(z)$ and $y_1(z)$. Condition (ii) can be usually relaxed to $f(Cz)\in 1+z\mathbb{Z}[[z]]$ for some positive integer C (without the scaling $z\mapsto Cz$, many of the resulting formulas look 'more natural').

In fact, in [2], [3] we posed on 4th (and 5th) order differential equations some extra conditions as well: about the structure of the projective monodromy group (see the 'Calabi–Yau' or 'self-duality' condition (13) below) and about the integrality of a related sequence of numbers, known as instanton numbers in the physics literature. These arise as coefficients in the Lambert expansion of the so-called Yukawa coupling, which we review in Section 5. However, it seems that in all examples these additional conditions are satisfied automatically when (i)–(iii) hold.

Our experimental search [2], [3] resulted in more than 350 examples of such operators which we called differential operators of *Calabi–Yau type*, since some of these examples can be identified with Picard–Fuchs differential equations for the periods of 1-parameter families of Calabi–Yau manifolds. For an entry in our table from [3], checking (i) is trivial, (ii) usually follows from an explicit form of the coefficients of $f(z)$ (when it is available), while (iii) can be verified in certain cases using some of Dwork's p-adic techniques. Substantial progress in this direction was obtained recently by C. Krattenthaler and T. Rivoal [16]. According to standard conjectures (see, e.g., [4]) all our operators should be of geometric origin, meaning that they correspond (as subquotients of the local systems) to factors of Picard–Fuchs equations satisfied by period integrals for some family of varieties over the projective line.

Basic examples of Calabi–Yau differential equations are given by the general hypergeometric differential equation

$$\Big(\theta\prod_{j=2}^{m}(\theta+b_j-1)-z\prod_{j=1}^{m}(\theta+a_j)\Big)y=0 \tag{8}$$

of order m satisfied by the hypergeometric series (1). The equation (8) has (smallest possible) degree 1 in z and condition (i) forces $b_2=\dots=b_m=1$ to hold. The latter is the main reason for our identities below to involve the hypergeometric series with this special form of the lower parameters.

2. Symmetric and antisymmetric squares

Given a 2nd order linear homogeneous differential equation

$$y''+Py'+Qy=0, \qquad \text{where} \quad {}'=\frac{\mathrm{d}}{\mathrm{d}z}, \tag{9}$$

and a pair of its two linearly independent solutions $y_0=y_0(z)$ and $y_1=y_1(z)$, one can easily construct the 3rd order differential equation whose solutions are y_0^2, y_0y_1, and y_1^2:

$$y'''+3Py''+(2P^2+P'+4Q)y'+(4PQ+2Q')y=0, \tag{10}$$

called the *symmetric square* of equation (9) (see [20, Chap. 14, Exercise 10]). Clearly, equation (10) is independent of a choice of solutions y_0, y_1 of the equation (9). A hypergeometric example of the relationship between solutions of (9) and (10) is *Clausen's formula*

$${}_2F_1\left(\left.\begin{matrix} a,\, b \\ a+b+\frac12 \end{matrix}\,\right|\, z\right)^2 = {}_3F_2\left(\left.\begin{matrix} 2a,\, 2b,\, a+b \\ a+b+\frac12,\, 2a+2b \end{matrix}\,\right|\, z\right). \tag{11}$$

The situation changes drastically when one goes to linear homogeneous differential equations of order higher than 2. In principle, there is no difficulty in writing formulas similar to (10) for the symmetric cubes, biquadratics, etc, but unfortunately, as far as we know, this never results in some nontrivial identities for the hypergeometric series (1).

If the coefficients of a 4th order linear differential equation

$$y^{(4)}+Py'''+Qy''+Ry'+Sy=0 \tag{12}$$

satisfy the relation

$$R=\frac12PQ-\frac18P^3+Q'-\frac34PP'-\frac12P'' \tag{13}$$

then the operator is said to satisfy the *Calabi–Yau condition* [2]; it expresses the self-duality of (12). If y_0,y_1,y_2,y_3 are linearly independent solutions, then condition (13) implies that the six functions

$$w_{jk} = W(y_j, y_k) = \det\begin{pmatrix} y_j & y_k \\ y_j' & y_k' \end{pmatrix}, \qquad 0 \le j < k \le 3, \tag{14}$$

are linearly dependent over $\mathbb{C}$. These functions satisfy a 5th order linear differential equation

$$y^{(5)} + \widetilde{P}y^{(4)} + \widetilde{Q}y''' + \widetilde{R}y'' + \widetilde{S}y' + \widetilde{T}y = 0, \tag{15}$$

independent of choice of solutions y_0, y_1, y_2, y_3 of (13) and called the anti-symmetric square of (13) (see, for example, [2, Proposition 1]).

Proposition 1 ([1], [21]). *Suppose that a* 5*th order equation* (15) *is the antisymmetric square of a* 4*th order linear differential equation. Let* $U = U(z)$ *be an arbitrary function. Then, for any pair* w_0, w_1 *of solutions of* (15)*, the function*

$$y = W(w_0, w_1)^{1/2} \cdot U \tag{16}$$

satisfies a 4*th order equation* (13) *whose coefficients* P*,* Q*,* R*, and* S *are differential polynomials in* $\widetilde{P}$*,* $\widetilde{Q}$*,* $\widetilde{R}$*,* $\widetilde{S}$*,* $\widetilde{T}$*, and* U*.* (The explicit expressions are given in [1].)

Following [1] we call the resulting 4th order equation (13) with the choice

$$U = z^{5/2} \cdot \exp\biggl\{-\frac{1}{5}\int^z \widetilde{P}(z)\,\mathrm{d}z\biggr\} \tag{17}$$

the *Yifan Yang pullback* of equation (15), or the *YY-pullback* for short.

As an example, the YY-pullback of the equation

$$\bigl(\theta^5 - z(\theta+\tfrac12)(\theta+\alpha)(\theta+1-\alpha)(\theta+\beta)(\theta+1-\beta)\bigr)y = 0, \tag{18}$$

satisfied by the hypergeometric function

$${}_5F_4\biggl(\begin{matrix} \frac12, \alpha, 1-\alpha, \beta, 1-\beta \\ 1, \quad 1, \quad 1, \quad 1 \end{matrix}\biggm| z\biggr), \tag{19}$$

is given [1] by

$$\begin{aligned} \Bigl(\theta^4 &- z\bigl(2(\theta+\tfrac12)^4 + \tfrac12(\theta+\tfrac12)^2(\alpha(1-\alpha)+\beta(1-\beta)+3) \\ &\qquad - \tfrac14\alpha(1-\alpha)\beta(1-\beta) + \tfrac18\alpha(1-\alpha) + \tfrac18\beta(1-\beta)\bigr) \\ &+ z^2\bigl(\theta+\tfrac12+\tfrac12(\alpha+\beta)\bigr)\bigl(\theta+\tfrac12+\tfrac12(\alpha+1-\beta)\bigr)\bigl(\theta+\tfrac12+\tfrac12(1-\alpha+\beta)\bigr) \\ &\quad\times\bigl(\theta+\tfrac12+\tfrac12(1-\alpha+1-\beta)\bigr)\Bigr)y = 0. \end{aligned} \tag{20}$$

As we will see in Theorems 2–7 below, many Calabi–Yau differential equations related by algebraic transformations are Hadamard products of 2nd and 3rd order Picard–Fuchs differential equations, with 0 a MUM point. Recall that the *Hadamard product* of two series $f(z) = \sum_{n=0}^\infty A_n z^n$ and $\hat f(z) = \sum_{n=0}^\infty \widehat{A}_n z^n$ is defined by the formula $f(z) * \hat f(z) = \sum_{n=0}^\infty A_n \widehat{A}_n z^n$. If $f(z)$ and $\hat f(z)$ are the analytic solutions of two differential equations $Dy = 0$ and $\widehat{D}y = 0$, respectively, then their Hadamard product $f(z) * \hat f(z)$ satisfies a differential equation $\widetilde{D}y = 0$ (we pick the one of minimal order), which we then call the *Hadamard product* of the two equations. The differential operator $\widetilde{D}$ in this case is the Hadamard product of the corresponding operators D and $\widehat{D}$. The Hadamard product is the analytic representation of the multiplicative convolution. In particular, the singular points of the operator $\widetilde{D}$ consist of the products of singular points of D and of $\widehat{D}$. The Hadamard product of operators of geometric origin is again of geometric origin (see, e.g., [4]).

We will say that two (Calabi–Yau) differential equations or operators are *equivalent* if they are related by an algebraic transformation. Note that algebraic transformations preserve the order of differential equations with rational or algebraic coefficients.

3. Apéry-like differential operators

To illustrate the above theorems and also to present some further algebraic transformations, we will list 2nd and 3rd order Calabi–Yau equations keeping their names used in [2], [3], and [15].

In writing down the series for analytic solutions of the Calabi–Yau differential equations, one usually re-normalize the variable $z \mapsto Cz$ in order to make the series expansions lying in $1 + z\mathbb{Z}[[z]]$ (see condition (ii) in Section 1). Basic examples are ${}_2F_1$-hypergeometric series satisfying 2nd order differential equations, and there are exactly four such series having MUM at the origin (that is, satisfying condition (i)):

$$ {}_2F_1\left(\begin{matrix}\alpha,\, 1-\alpha\\ 1\end{matrix}\,\middle|\, C_\alpha z\right) = \sum_{n=0}^{\infty} A_n z^n \in 1 + z\mathbb{Z}[[z]], \tag{21}$$

where

$$\begin{aligned} &\text{(A)}\quad \alpha = \tfrac{1}{2},\quad C_{1/2} = 16 = 2^4, &&\text{(B)}\quad \alpha = \tfrac{1}{3},\quad C_{1/3} = 27 = 3^3,\\ &\text{(C)}\quad \alpha = \tfrac{1}{4},\quad C_{1/4} = 64 = 2^6, &&\text{(D)}\quad \alpha = \tfrac{1}{6},\quad C_{1/6} = 432 = 2^4\cdot 3^3. \end{aligned} \tag{22}$$

The corresponding differential operators are

$$\theta^2 - C_\alpha z(\theta+\alpha)(\theta+1-\alpha), \tag{23}$$

and the integrality of the expansions in (21) follows from the explicit formulas

$$\begin{aligned} &\text{(A)}\quad A_n = \binom{2n}{n}^2, &&\text{(B)}\quad A_n = \frac{(3n)!}{n!^3},\\ &\text{(C)}\quad A_n = \frac{(4n)!}{n!^2(2n)!}, &&\text{(D)}\quad A_n = \frac{(6n)!}{n!(2n)!(3n)!}. \end{aligned} \tag{24}$$

These four hypergeometric instances (A)–(D) are particular examples of the 2nd order differential operators having the form (6); they correspond to the choice $c = 0$. Besides the above hypergeometric cases, D. Zagier found in [22] four *Legendrian* and six *sporadic* equations.

The Legendrian examples are obtained from the hypergeometric ones by a simple rational transformation that interchanges $1/C_\alpha$ and ∞):

$$\frac{1}{1-C_\alpha z}\cdot {}_2F_1\left(\begin{matrix}\alpha,\, 1-\alpha\\ 1\end{matrix}\,\middle|\, \frac{-C_\alpha z}{1-C_\alpha z}\right) \in 1 + z\mathbb{Z}[[z]], \tag{25}$$

where

$$\text{(e)}\quad \alpha=\tfrac12,\qquad \text{(h)}\quad \alpha=\tfrac13,\qquad \text{(i)}\quad \alpha=\tfrac14,\qquad \text{(j)}\quad \alpha=\tfrac16; \tag{26}$$

the corresponding differential operators

$$\theta^2 - C_\alpha z\big(\theta^2+(\theta+1)^2-\alpha(1-\alpha)\big)+C_\alpha^2 z^2(\theta+1)^2 \tag{27}$$

have the form (6) with $c=a^2/4$.

These four hypergeometric operators and their Legendrian companions have a nice geometric origin: they are Picard–Fuchs operators of the extremal rational elliptic surfaces with three singular fibres [17], [12].

The sporadic examples of (6) (when $c\ne0$ and $c\ne a^2/4$ as in the hypergeometric and Legendrian cases, respectively) with the corresponding analytic solutions $\sum_{n=0}^\infty A_nz^n\in 1+z\mathbb{Z}[[z]]$ are as follows:

$$\begin{aligned}
&\text{(a)}\quad a=7,\ \ b=2,\ c=-8, && A_n=\sum_k\binom{n}{k}^3;\\
&\text{(b)}\quad a=11,\, b=3,\ c=-1, && A_n=\sum_k\binom{n}{k}^2\binom{n+k}{n};\\
&\text{(c)}\quad a=10,\, b=3,\ c=9, && A_n=\sum_k\binom{n}{k}^2\binom{2k}{k};\\
&\text{(d)}\quad a=12,\, b=4,\ c=32, && A_n=\sum_k\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k};\\
&\text{(f)}\quad a=9,\ \ b=3,\ c=27, && A_n=\sum_k(-1)^k3^{n-3k}\binom{n}{3k}\frac{(3k)!}{k!^3};\\
&\text{(g)}\quad a=17,\, b=6,\ c=72, && A_n=\sum_{k,l}(-1)^k8^{n-k}\binom{n}{k}\binom{k}{l}^3.
\end{aligned} \tag{28}$$

These six sporadic operators also have a geometric origin; they arise as Picard–Fuchs equations of the six families of elliptic curves with four reduced singular fibres [6], [17], although the connection between the operators and rational elliptic surfaces is not one-to-one (cf. [22]).

The story for the 3rd order differential operators of the form (7) looks very similar to the one for order 2. We also have four hypergeometric examples

$${}_3F_2\biggl(\begin{matrix}\frac12,\ \alpha,\ 1-\alpha\\ 1,\ 1\end{matrix}\biggm| 4C_\alpha z\biggr)\in 1+z\mathbb{Z}[[z]],$$

four operators of Legendre type and six sporadic operators.

The 'Legendrian' 3rd order examples originate from the series

$$\begin{aligned}\sum_{n=0}^{\infty} A_n z^n &= \sum_{n=0}^{\infty} (C_\alpha z)^n \sum_{k=0}^{n} \left(\frac{(\alpha)_k (1-\alpha)_{n-k}}{k!\,(n-k)!} \right)^2 \\ &= \frac{1}{1-C_\alpha z}\, {}_3F_2\left(\left. \begin{matrix} \frac12, \alpha, 1-\alpha \\ 1, \quad 1 \end{matrix} \right| \frac{-4C_\alpha z}{(1-C_\alpha z)^2} \right) \\ &= \frac{1}{1-C_\alpha z}\, {}_2F_1\left(\left. \begin{matrix} \alpha, 1-\alpha \\ 1 \end{matrix} \right| \frac{-C_\alpha z}{1-C_\alpha z} \right)^2 \end{aligned} \tag{29}$$

(we use [14, Section 2.5, Theorem IX] and the Euler transformation [14, p. 31, Eq. (1.7.1.3)]), where

$$(\beta) \quad \alpha = \tfrac12, \qquad (\iota) \quad \alpha = \tfrac13, \qquad (\vartheta) \quad \alpha = \tfrac14, \qquad (\kappa) \quad \alpha = \tfrac16; \tag{30}$$

the corresponding differential operators are

$$\theta^3 - C_\alpha z (2\theta+1)\big(\theta(\theta+1) + \alpha^2 + (1-\alpha)^2\big) + C_\alpha^2 z^2 (\theta+1)^3. \tag{31}$$

The sporadic 3rd order examples of (7) with analytic solutions $\sum_{n=0}^{\infty} A_n z^n \in 1 + z\mathbb{Z}[[z]]$ are given in the following list:

(32)

$$\begin{array}{lll} (\delta) & \hat a = 7,\ \hat b = 3,\ \hat c = 81, & A_n = \displaystyle\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}; \\ (\eta) & \hat a = 11, \hat b = 5,\ \hat c = 125, & A_n = \displaystyle\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right); \\ (\alpha) & \hat a = 10, \hat b = 4,\ \hat c = 64, & A_n = \displaystyle\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k}; \\ (\epsilon) & \hat a = 12, \hat b = 4,\ \hat c = 16, & A_n = \displaystyle\sum_k \binom{n}{k}^2 \binom{2k}{n}^2; \\ (\zeta) & \hat a = 9,\ \hat b = 3,\ \hat c = -27, & A_n = \displaystyle\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}; \\ (\gamma) & \hat a = 17, \hat b = 5,\ \hat c = 1, & A_n = \displaystyle\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2. \end{array}$$

The following theorem gives a natural bijection between the differential operators (6) and (7), in particular, between the above 14 pairs of arithmetic operators.

Theorem 1. *Let the triplets (a,b,c) and $(\hat a, \hat b, \hat c)$ be related by the formulas*

$$\hat a = a, \quad \hat b = a - 2b \quad \text{and} \quad \hat c = a^2 - 4c. \tag{33}$$

For the differential operators D and $\widehat{D}$ given in (6) and (7), denote by $f(z)$ and $\hat f(z)$ the analytic solutions of $Dy = 0$ and $\widehat{D}y = 0$, respectively, with $f(0) = \hat f(0) = 1$.

Then

$$f(z)^2 = \frac{1}{1-az+cz^2}\,\hat f\left(\frac{-z}{1-az+cz^2}\right). \tag{34}$$

Proof. Writing down the general 3rd order differential equations for the functions on the left- and right-hand sides of (34), respectively, is a routine exercise in `Maple` to show that the transformation is the right one. □

In fact, there is a natural geometric construction that explains this bijection, which we will sketch now. The 2nd order operators are Picard–Fuchs operators for special families of elliptic curves E_t, where $t \in Y = \mathbb{P}^1$. In each of the cases the rational curve Y covers the modular curve $X_0(N)$ for some N, so that each elliptic curve E_t comes with a cyclic subgroup of order N. The quotient of E_t by this cyclic subgroup turns out to be $E_{\iota(t)}$, where $\iota\colon Y \to Y$ is an involution corresponding to the Atkin–Lehner involution that acts as $\tau \mapsto -1/(N\tau)$ on the elliptic modular parameter. The product of the elliptic curves $A_t := E_t \times E_{\iota(t)}$ can now be considered as parametrized by the rational curve $Z := Y/\iota$. The Picard–Fuchs equation for the holomorphic 2-form for this family is of order 3 and thus can be seen as a 'twisted' square of the corresponding 2nd order operators. We refer to [19], [18] and [11] for details about this construction.

The specific form of the transformation can be understood by noting that the quotient map $Y \to Z$ is described by a degree 2 rational map $f\colon \mathbb{P}^1 \to \mathbb{P}^1$. If the pre-image of 0 consists of the points 0 and ∞, and the pre-image of ∞ of the two other singular points of the 2nd order operator (6) (that is, of the roots of $1-az+cz^2$), then one is led to a map of the form $z \mapsto ez/(1-az+cz^2)$. The singular points of the 3rd order operator (7) consist of 0, ∞ and the roots of the equation $1 - 2\hat a z + \hat c z^2 = 0$, and these have to coincide with the image of the two critical points of the map, which one computes to be the roots of $e^2 + 2aez + z^2(a^2 - 4c)$. Hence one can take $e = -1$, $\hat a = a$, and $\hat c = a^2 - 4c$. The factor in front of $\hat f$ in (34) is needed to get the local exponents agreed, but the value of $\hat b$ remains undetermined by these considerations.

4. Hadamard products and algebraic transformations

Recall that the YY-pullback of the differential equation (18) is (20). The latter 4th order equation has the unique analytic solution of the form

$$\widetilde{F}(z) \in 1 + z\mathbb{C}[[z]] \tag{35}$$

at the origin, since all exponents at $z = 0$ are zero (in other words, equation (20) has MUM at the origin). Roughly speaking, we may call the function $\widetilde{F}(z)$ an antisymmetric square root of (19). The following theorem may be viewed as a generalization of Clausen's formula (11) in the special case $a + b = \frac{1}{2}$.

Theorem 2. *Let*

$$f_\alpha(z) = \frac{1}{1-z}\,{}_2F_1\left(\left.\begin{matrix}\alpha,\,1-\alpha\\ 1\end{matrix}\,\right|\,\frac{-z}{1-z}\right) = \sum_{n=0}^\infty a_n z^n \tag{36}$$

and

$$f_\beta(z) = \frac{1}{1-z}\,{}_2F_1\biggl(\begin{matrix}\beta, 1-\beta\\ 1\end{matrix}\biggm| \frac{-z}{1-z}\biggr) = \sum_{n=0}^\infty b_n z^n, \tag{37}$$

and let $F(z)$ be the Hadamard product of the series $f_\alpha(z)$ and $f_\beta(z)$,

$$F(z) = \sum_{n=0}^\infty a_n b_n z^n. \tag{38}$$

Then for the analytic solution (35) of the YY-pullback (20) of (18) we have

$$F(z) = \frac{1+z/4}{(1-z/4)^2}\widetilde{F}\biggl(\frac{-z/4}{(1-z/4)^2}\biggr), \tag{39}$$

equivalently,

$$\widetilde{F}(z) = \frac{2}{1-z+\sqrt{1-z}}F\biggl(\frac{-z/2}{1-z/2+\sqrt{1-z}}\biggr). \tag{40}$$

Remark 1. The coefficients a_n in the expansion (36) may be given by the formulas

$$a_n = \sum_{k=0}^n (-1)^k \binom{n}{k}\frac{(\alpha)_k(1-\alpha)_k}{k!^2} = \sum_{k=0}^n \frac{(\alpha)_k}{k!}\,\frac{(1-\alpha)_{n-k}^2}{(n-k)!^2}; \tag{41}$$

similar formulas, but with the replacement α by β are available for the coefficients b_n. The statement of Theorem 2 is a version of the experimental observation in [1, Section 3.2]. Recalling the relationship between solutions of (18) and (20) we can write our final formula (40) as follows (reminding a little of Clausen's original formula (11)):

$$\begin{aligned}
&\sum_{n=0}^\infty z^n \sum_{k=0}^n \frac{(\frac12)_k(\frac12)_{n-k}(\alpha)_k(\alpha)_{n-k}(1-\alpha)_k(1-\alpha)_{n-k}(\beta)_k(\beta)_{n-k}(1-\beta)_k(1-\beta)_{n-k}}{k!^5(n-k)!^5}\\
&\qquad\times\biggl(1+(2k-n)\sum_{j=0}^{k-1}\biggl(\frac{1}{\frac12+j}+\frac{1}{\alpha+j}+\frac{1}{1-\alpha+j}+\frac{1}{\beta+j}+\frac{1}{1-\beta+j}\biggr)\biggr)\\
&\quad=\frac{4(1-z)}{(1-z+\sqrt{1-z})^2}\biggl(\sum_{n=0}^\infty\biggl(\frac{-z/2}{1-z/2+\sqrt{1-z}}\biggr)^n\\
&\qquad\times\sum_{j=0}^n(-1)^j\binom{n}{j}\frac{(\alpha)_j(1-\alpha)_j}{j!^2}\cdot\sum_{k=0}^n(-1)^k\binom{n}{k}\frac{(\beta)_k(1-\beta)_k}{k!^2}\biggr)^2.
\end{aligned}$$

Proof of Theorem 2. The sequence (41) satisfies the recursion

$$(n+1)^2a_{n+1} - \bigl(n^2+(n+1)^2-\alpha(1-\alpha)\bigr)a_n + n^2a_{n-1} = 0, \tag{42}$$

and a similar formula is valid for the sequence b_n. Taking the Hadamard product a_nb_n in (38) as described in [2, Section 7] gives a 4th order recursion (which is too plain to be stated here). The corresponding differential operator $\mathscr{L}$ annihilating the series (38) is of order 6 and is factorable, $\mathscr{L} = \mathscr{L}_1\mathscr{L}_2$, where $\mathscr{L}_1$ is of order 2.

Computing $\mathscr{L}_1$ and performing `leftdivision`$(\mathscr{L}, \mathscr{L}_1, [\mathrm{d}/\mathrm{d}z, z])$ in `Maple`, we find $\mathscr{L}_2$ which can be written in the form

$$\begin{aligned}\mathscr{L}_2 = \theta^4 &- z(2\theta^4 + 8\theta^3 - 2(s-4)\theta^2 - 2(s-2)\theta + p - s + 1)\\ &- z^2(\theta^4 - 12\theta^3 - 26\theta^2 + 4(s-5)\theta + 4p - s^2 + 4s - 7)\\ &+ z^3(4\theta^4 + 8\theta^3 - 4(s+3)\theta^2 - 4(s+4)\theta - 6p + 2s^2 + 2s - 8)\\ &- z^4(\theta^4 + 16\theta^3 + 16\theta^2 - 4(s-2)\theta + 4p - s^2)\\ &- z^5(2\theta^4 - 2(s+2)\theta^2 - 2(s+2)\theta + p - s - 1) + z^6(\theta+1)^4,\end{aligned}$$

where $s = \alpha(1-\alpha) + \beta(1-\beta)$ and $p = \alpha(1-\alpha)\beta(1-\beta)$. Then we finish the proof by performing the transformation

$$z = \frac{-4Z}{(1-Z)^2}, \qquad y(z) = \frac{(1-Z)^2}{2(1+Z)} Y(Z),$$

which transforms the equation for $\widetilde{F}(z)$ to $\mathscr{L}_2 Y = 0$. (Some precaution is necessary, since `Maple` does not cancel common factors in the coefficients of the resulting differential equation.) This equation has the unique analytic solution at the origin and both expansions in (40) lie in $1 + z\mathbb{C}[[z]]$. □

In the cases $\alpha, \beta \in \{\frac12, \frac13, \frac14, \frac16\}$, Theorem 2 provides the equivalences for the YY-pullbacks of 5th order Calabi–Yau hypergeometric differential equations and the 4th order Hadamard products of Legendrian cases (25)–(27).

Our next family of transformations concerns with the series

$$g_\alpha(z) = {}_2F_1\left(\begin{matrix}\alpha, \alpha\\ 1\end{matrix}\,\middle|\, z\right) \cdot {}_2F_1\left(\begin{matrix}1-\alpha, 1-\alpha\\ 1\end{matrix}\,\middle|\, z\right) = \frac{1}{1-z}\,{}_3F_2\left(\begin{matrix}\frac12, \alpha, 1-\alpha\\ 1, \ 1\end{matrix}\,\middle|\, \frac{-4z}{(1-z)^2}\right); \tag{43}$$

the particular cases correspond to the Legendrian 3rd order examples from Section 3. Writing $g_\alpha(z) = \sum_{n=0}^\infty a_n z^n$ and using the first representation in (43) one finds that

$$a_n = \sum_{k=0}^n \left(\frac{(\alpha)_k (1-\alpha)_{n-k}}{k!\,(n-k)!}\right)^2, \tag{44}$$

and this sequence satisfies the recursion

$$(n+1)^3 a_{n+1} - (2n+1)\big(n(n+1) + \alpha^2 + (1-\alpha)^2\big)a_n + n^3 a_{n-1} = 0. \tag{45}$$

Applying twice the Euler transformation [14, p. 31, Eq. (1.7.1.3)] to the first expression in (43) gives one a way to express $g_\alpha(z)$ as the square:

$$g_\alpha(z) = \frac{1}{1-z}\,{}_2F_1\left(\begin{matrix}\alpha, 1-\alpha\\ 1\end{matrix}\,\middle|\, \frac{-z}{1-z}\right)^2. \tag{46}$$

Theorem 3. *Let $F(z)$ be the Hadamard product of* ${}_2F_1\left(\begin{matrix}\alpha,\,1-\alpha\\ 1\end{matrix}\,\middle|\, z\right)$ *and* $f_\beta(z)$ *in* (37), *and let* $G(z)$ *be the Hadamard product of* ${}_2F_1\left(\begin{matrix}\beta,\,1-\beta\\ 1\end{matrix}\,\middle|\, z\right)$ *and* $g_\alpha(z)$ *in* (43), (46). *Let* $\widetilde{G}(z)\in 1+z\mathbb{C}[[z]]$ *be the analytic solution of the YY-pullback*

$$
\begin{aligned}
&\Bigl(\theta^4 - z\bigl(4(\theta+\tfrac12)^4 + (4-2\alpha(1-\alpha)+\beta(1-\beta)(\theta+\tfrac12)^2+\tfrac18\\
&\qquad\qquad + (\alpha(1-\alpha)-\tfrac14)(\beta(1-\beta)-\tfrac12)\bigr)\\
&\quad + z^2\bigl(6(\theta+1)^4 + \bigl(\tfrac{15}{2}-4\alpha(1-\alpha)+3\beta(1-\beta)\bigr)(\theta+1)^2+\tfrac34\\
&\qquad\qquad + \alpha(1-\alpha)\beta(1-\beta)+\alpha^2(1-\alpha)^2-\alpha(1-\alpha)\bigr)\\
&\quad - z^3(\theta+\tfrac32)^2\bigl(4(\theta+\tfrac32)^2+3-2\alpha(1-\alpha)-3\beta(1-\beta)\bigr)\\
&\quad + z^4(\theta+\tfrac32)(\theta+\tfrac52)(\theta+\beta+\tfrac32)(\theta-\beta+\tfrac52)\Bigr)y=0
\end{aligned}
\tag{47}
$$

of the 5th order linear differential equation satisfied by $G(z)$. *Then*

$$
F(z)=\widetilde{G}\left(\frac{-z}{1-z}\right) \quad\text{and}\quad \widetilde{G}(z)=F\left(\frac{-z}{1-z}\right).
\tag{48}
$$

Proof. A routine in the spirit of the proof of Theorem 2. ☐

In [23] we consider a quadratic transformation of a ${}_5F_4$-series with a particular instance

$$
{}_5F_4\left(\begin{matrix}\frac12,\,\frac12,\,\frac12,\,\frac12,\,\frac12\\ 1,\,1,\,1,\,1\end{matrix}\,\middle|\, z\right)=\frac{1}{(1-z)^{1/2}}\sum_{n=0}^{\infty}\left(\frac{-4z}{(1-z)^2}\right)^n\frac{(\frac14)_n(\frac34)_n}{n!^2}a_n,
\tag{49}
$$

where

$$
a_n=\sum_{k=0}^{n}\frac{(\frac12)_k^3}{k!^3}\,\frac{(\frac12)_{n-k}}{(n-k)!}=\sum_{k=0}^{n}\left(\frac{(\frac14)_k(\frac34)_{n-k}}{k!(n-k)!}\right)^2
\tag{50}
$$

are coefficients in the power expansion of $g_{1/4}(z)$ in (43).

The series on the left-hand side in (49) is the special case $\alpha=\beta=1/2$ of (19), and Theorem 2 gives an example of quadratic transformation of the corresponding YY-pullback (20). Our next theorem gives another quadratic transformation for the series $F(z)$ from (38) in this case.

Theorem 4. *Let* $f(z)=f_{1/2}(z)$ *and* $\widehat{f}(z)=f_{1/4}(z)$, *where* $f_\alpha(z)$ *is defined in* (36), *and let* $F(z)$ *be the Hadamard square of the series* $f(z)$,

$$
F(z)=\sum_{n=0}^{\infty}z^n\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\frac12)_k^2}{k!^2}\right)^2,
\tag{51}
$$

and $\widehat{F}(z)$ *the Hadamard product of* ${}_2F_1\left(\begin{matrix}\frac14,\,\frac34\\ 1\end{matrix}\,\middle|\, z\right)$ *and* $\widehat{f}(z)$,

$$
\widehat{F}(z)=\sum_{n=0}^{\infty}z^n\,\frac{(\frac14)_n(\frac34)_n}{n!^2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\frac14)_k(\frac34)_k}{k!^2}\,.
\tag{52}
$$

Then

$$F(z)=\frac{1}{\sqrt{1-6z+z^2}}\widehat{F}\biggl(\frac{-16z(1-z)^2}{(1-6z+z^2)^2}\biggr). \tag{53}$$

Proof. A routine; cf. Section 6 below. □

Our next result refers to a generic set of the (complex) parameters α, a, b, and c, while the three additional parameters $\hat a$, $\hat b$ and $\hat c$ are defined in accordance with (33). The Hadamard product of the differential operators $\theta^2-z(\theta+\alpha)(\theta+1-\alpha)$ (which is the un-normalized version of (23)) and (7) is

$$\begin{aligned}\theta^5&-z(2\theta+1)(\theta+\alpha)(\theta+1-\alpha)(\hat a\theta^2+\hat a\theta+\hat b)\\&+\hat cz^2(\theta+1)(\theta+\alpha)(\theta+1-\alpha)(\theta+1+\alpha)(\theta+2-\alpha);\end{aligned} \tag{54}$$

its 4th order YY-pullback reads

$$\begin{aligned}D=\theta^4&-z\bigl(4\hat a\bigl(\theta+\tfrac12\bigr)^4+\bigl((p+4)\hat a-2\hat b\bigr)\bigl(\theta+\tfrac12\bigr)^2+\tfrac14(1-p)\hat a-\tfrac12(1-2p)\hat b\bigr)\\&+z^2\bigl((6\hat a^2-8\hat c)(\theta+1)^4+\bigl(\tfrac32(5+2p)\hat a^2-4\hat a\hat b-2(13+2p)\hat c\bigr)(\theta+1)^2\\&\qquad+\tfrac34\hat a^2-(1-p)\hat a\hat b+\hat b^2-(2+2p-p^2)\hat c\bigr)\\&-(\hat a^2-4\hat c)z^3\bigl(\theta+\tfrac32\bigr)^2\bigl(4\hat a\bigl(\theta+\tfrac32\bigr)^2+3(1+p)\hat a-2\hat b\bigr)\\&+(\hat a^2-4\hat c)^2z^4\bigl(\theta+\tfrac32\bigr)\bigl(\theta+\tfrac52\bigr)\bigl(\theta+\tfrac32+\alpha\bigr)\bigl(\theta+\tfrac52-\alpha\bigr),\end{aligned} \tag{55}$$

where $p=\alpha(1-\alpha)$.

Theorem 5. *Let $\widehat{F}(z)\in 1+z\mathbb{C}[[z]]$ be the analytic solution of the differential equation $Dy=0$ with D defined in (55), and let $F(z)\in 1+z\mathbb{C}[[z]]$ be the Hadamard product of*

$$f_\alpha(z)=\frac{1}{1-z}\cdot {}_2F_1\biggl(\begin{matrix}\alpha,\ 1-\alpha\\ 1\end{matrix}\biggm|\frac{-z}{1-z}\biggr)$$

and the analytic solution of the differential equation with differential operator (6). Then

$$F(z)=\frac{1-cz^2}{(1-az+cz^2)^{3/2}}\widehat{F}\biggl(\frac{-z}{1-az+cz^2}\biggr). \tag{56}$$

Proof. As before, the proof is just an extensive check using `Maple`; already the differential equation for the Hadamard product $F(z)$ is too spacious to be given here. □

We remark that Theorem 1 in Section 3 may be regarded as a limiting case $\alpha\to 0$ of Theorem 5.

Note that Theorems 2 and 3 are special cases of Theorem 5, but in the former cases we can explicitly write down the Hadamard products involved, through hypergeometric series. Theorem 5 provides us with equivalences relating the sporadic cases (28) and (32), namely, it gives us the following table of 24 equivalences:

	(A)	(B)	(C)	(D)
(δ)	(e)$*$(a)	(h)$*$(a)	(i)$*$(a)	(j)$*$(a)
(η)	(e)$*$(b)	(h)$*$(b)	(i)$*$(b)	(j)$*$(b)
(α)	(e)$*$(c)	(h)$*$(c)	(i)$*$(c)	(j)$*$(c)
(ϵ)	(e)$*$(d)	(h)$*$(d)	(i)$*$(d)	(j)$*$(d)
(ζ)	(e)$*$(f)	(h)$*$(f)	(i)$*$(f)	(j)$*$(f)
(γ)	(e)$*$(g)	(h)$*$(g)	(i)$*$(g)	(j)$*$(g)

Theorem 2 gives in a nice way the equivalence of the YY-pullbacks of the 5th order hypergeometric on the one side and Hadamard products of two 2nd order Legendrian cases on the other side, while Theorem 3 provides the equivalence of (X)$*$(x) and the YY-pullback of (X)$*$(ξ), where (X) is one of the hypergeometric cases (21), (22), (x) is one of the 2nd order Legendrian equations (25), (26) and (ξ) is the corresponding 3rd order Legendrian equation (30), (31).

We already established the algebraic connection for the YY-pullback of the left-hand side in (49) and between (e)$*$(e) (Theorem 2); in (49) we have the equivalence implying, in particular, the equivalence of the YY-pullback of (C)$*$(ϑ) and of (e)$*$(e). Finally, the equivalence of the YY-pullback of (C)$*$(ϑ) and of (C)$*$(i) follows from Theorem 3; this implies the equivalence of (e)$*$(e) and (C)$*$(i) which is also the subject of Theorem 4.

We now illustrate Theorem 5 by an explicit example of an algebraic transformation relating two Calabi–Yau equations.

Example 1. Let us write the algebraic transformation for the equivalence of (the YY-pullback of) (C)$*$(γ) and (i)$*$(g).

We have the following solutions of the 5th order equation for (C)$*$(γ):

$$\begin{aligned} w_0(z) &= \sum_{n=0}^{\infty} z^n \frac{(4n)!}{n!^2(2n)!} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}^2, \\ w_1(z) &= w_0(z)\log z + \sum_{n=1}^{\infty} z^n \frac{(4n)!}{n!^2(2n)!} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}^2 \\ &\qquad \times (4H_{4n} - 2H_{2n} - 2H_n - 2H_{n-k} + 2H_{n+k}), \end{aligned}$$

with the YY-pullback

$$\widehat{F}(z) = (1 - 2176z + 4096z^2)^{-1/2}\big(w_0(z)\cdot\theta w_1(z) - \theta w_0(z)\cdot w_1(z)\big)^{1/2}, \tag{57}$$

where we used the data $C_\alpha = 64$ and $\hat{a} = 17$, $\hat{c} = 1$ for cases (C) and (γ).

For (i) and (g) we have $C_\alpha = 64$ and $a = 17$, $c = 72$, and the analytic solution is

$$F(z) = \sum_{n=0}^{\infty} z^n \sum_{0\le j\le i\le n} (-1)^i 8^{n-i} \binom{n}{i}\binom{i}{j}^3 \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(4k)!}{k!^2(2k)!}. \tag{58}$$

Then Theorem 5 gives us the transformation

$$F(z) = \frac{1 - 294912z^2}{(1 - 1088z + 294912z^2)^{3/2}} \cdot \widehat{F}\bigg(\frac{-z}{1 - 1088z + 294912z^2}\bigg), \tag{59}$$

which we can write in a way looking like Clausen's formula:

(60)
$$\begin{aligned}&\biggl(\sum_{n=0}^{\infty} z^n \sum_{0\le j\le i\le n}(-1)^i 8^{n-i}\binom{n}{i}\binom{i}{j}^3\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(4k)!}{k!^2(2k)!}\biggr)^2\\ &\quad=\frac{1}{1-1088z+294912z^2}\sum_{n=0}^{\infty}\biggl(\frac{-z}{1-1088z+294912z^2}\biggr)^n\\ &\qquad\times\sum_{k=0}^{n}\frac{(4k)!}{k!^2(2k)!}\,\frac{(4n-4k)!}{(n-k)!^2(2n-2k)!}\sum_{j=0}^{n-k}\binom{n-k}{j}^2\binom{n-k+j}{n-k}^2\\ &\qquad\times\sum_{l=0}^{k}\binom{k}{l}^2\binom{k+l}{k}^2\bigl(1+(2k-n)(4H_{4k}-2H_{2k}-2H_k-2H_{k-l}+2H_{k+l})\bigr).\end{aligned}$$

Theorem 6. *Let $F(z)$ be the Hadamard square of the series $f_\alpha(z)$ given in (36), while $\widetilde{F}(z)$ the Hadamard product of $(1-4z)^{-1/2}=\sum_{n=0}^{\infty}\binom{2n}{n}z^n$ and $g_\alpha(z)$ in (43), (46). Then*

$$F(z)=\frac{1}{1+z}\widetilde{F}\biggl(\frac{z}{(1+z)^2}\biggr). \tag{61}$$

As before, Calabi–Yau applications of Theorem 6 correspond to the choices $\alpha\in\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}\}$.

Finally, we present more algebraic transformations of 4th order Calabi–Yau differential equations which are not covered by the above theorems but look quite nice (to our taste).

Theorem 7. *The following identities are valid:*

(62)
$$\begin{aligned}&\sum_{n=0}^{\infty}\binom{2n}{n}\sum_{k,l}(-1)^{k+l}\binom{n}{k}\binom{n}{l}\binom{k+l}{k}^3 z^n\\ &\quad=\frac{1}{\sqrt{1-4z}}\sum_{n=0}^{\infty}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{n}\binom{3k}{n}\biggl(\frac{z}{1-4z}\biggr)^n,\end{aligned}$$

(63)
$$\begin{aligned}&\sum_{n=0}^{\infty}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}z^n\\ &\quad=\frac{1}{\sqrt{1-32z}}\sum_{n=0}^{\infty}\binom{2n}{n}\sum_{k,l}(-1)^{n-k}2^{3(n-k)}\binom{n}{k}\binom{k}{l}^2\binom{2l}{l}\binom{2k-2l}{k-l}\biggl(\frac{z}{1-32z}\biggr)^n,\end{aligned}$$

(64)
$$\sum_{n=0}^\infty \binom{2n}{n} \sum_{k,l} \binom{n}{k}\binom{n}{l}\binom{k}{l}\binom{k+l}{k}\binom{2l}{l}\binom{2k}{k-l} z^n$$
$$= \frac{1}{\sqrt{1-4z}} \sum_{n=0}^\infty \binom{2n}{n}^2 \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \biggl(\frac{z}{1-4z}\biggr)^n,$$

(65)
$$\sum_{n=0}^\infty \biggl(\sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{n}\biggr)^2 z^n$$
$$= \frac{1}{1+z} \sum_{n=0}^\infty \binom{2n}{n} \sum_{k,l} \binom{n}{k}\binom{n}{l}\binom{k+l}{k}\binom{2l}{l}\binom{l}{k-l} \biggl(\frac{z}{(1+z)^2}\biggr)^n,$$

(66)
$$\sum_{n=0}^\infty \biggl(\sum_{k=0}^n \binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}\biggr)^2 z^n$$
$$= \frac{1}{1-32z} \sum_{n=0}^\infty \binom{2n}{n} \sum_{k=0}^{[n/2]} 2^{n-2k} \binom{n}{k}\binom{n-k}{k}\binom{2k}{k}\binom{2n-2k}{n-k} \biggl(\frac{z}{(1-32z)^2}\biggr)^n,$$

(67)
$$\sum_{n=0}^\infty \biggl(\sum_{k=0}^{[n/3]} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}\biggr)^2 z^n$$
$$= \frac{1}{1-27z} \sum_{n=0}^\infty \sum_{k=0}^{[n/3]} (-1)^{n-k} \biggl(\binom{2n-3k-1}{n} + \binom{2n-3k}{n}\biggr)$$
$$\times \frac{(3k)!}{k!^3}\,\frac{(3n-3k)!}{(n-k)!^3} \biggl(\frac{z}{(1-27z)^2}\biggr)^n.$$

5. The invariance of Yukawa couplings

As we have already seen, algebraic transformations transform Calabi–Yau equations into similar ones, but looking sometimes quite different. Such transformations, however preserve (in a certain precise sense, which we describe below) the *Yukawa coupling* of the corresponding differential equations. Recall that the *Yukawa coupling* K can be defined, up to a normalization constant factor, through the quotient $t(z) = y_1(z)/y_0(z) \in \log z + z\mathbb{Q}[[z]]$ of the two solutions $y_0(z) \in 1 + z\mathbb{Z}[[z]]$ and $y_1(z)$ of a Calabi–Yau equation (12) (see Section 1) as

$$K = \frac{1}{y_0^2 \cdot (\mathrm{d}t/\mathrm{d}z)^3} \exp\biggl(-\frac{1}{2}\int^z P(z)\,\mathrm{d}z\biggr), \tag{68}$$

and this function is often viewed as a function of $q = e^t$, since its q-expansion in the case of a degenerating family of Calabi–Yau threefolds is supposed to encode the counting of rational curves of various degrees on a mirror manifold.

On the other hand, we did find several examples of Calabi–Yau equations whose Yukawa couplings coincide, although it is not obvious to see that the equations themselves are indeed equivalent in the sense that they are related by an algebraic transformation. At this moment of writing, we have discovered and proved algebraic transformations for *all* pairs of Calabi–Yau equations with equal Yukawa couplings tabulated in [3] (see also the diploma thesis of M. Bogner [10]). Most of these transformations (at least those that follow a general pattern) were given in Section 4 and some of them are immediate consequences of already proved Theorems 2–7, while Section 6 describes some of our strategies for finding the transformations.

It is a routine to write down the 4th order linear differential equation

$$\frac{\mathrm{d}^4Y}{\mathrm{d}x^4}+\widehat{P}\frac{\mathrm{d}^3Y}{\mathrm{d}x^3}+\widehat{Q}\frac{\mathrm{d}^2Y}{\mathrm{d}x^2}+\widehat{R}\frac{\mathrm{d}Y}{\mathrm{d}x}+\widehat{S}Y=0 \tag{69}$$

for the function $Y(x)=v(x)\cdot y\big(z(x)\big)$. For example, we have

$$\widehat{P}=-6\frac{z''}{z'}+z'P+4\frac{v'}{v}, \tag{70}$$

where the prime stands for the x-derivative.

Clearly, the new equation (69) does not necessarily have rational coefficients but it does after posing certain conditions on $v(x)$ and $z(x)$ (for instance, assuming their rationality). Continuing the computation in (70) we obtain

Proposition 2 (cf. [13])**.** *Denote*

$$U_z(P,Q)=Q-\frac{3}{2}\frac{\mathrm{d}P}{\mathrm{d}z}-\frac{3}{8}P^2. \tag{71}$$

Then

$$U_x(\widehat{P},\widehat{Q})-(z')^2U_z(P,Q)=5\{z,x\}, \tag{72}$$

where

$$\{z,x\}=\frac{z'''}{z'}-\frac{3}{2}\left(\frac{z''}{z'}\right)^2 \tag{73}$$

is the Schwarzian derivative.

Our next statement shows the invariance of the Yukawa coupling.

Proposition 3. *Let $Y(x)=v(x)\cdot y\big(z(x)\big)$, where $z(x)=x+O(x^2)$ and $v(x)=1+O(x)$. Then the Yukawa couplings defined in accordance with* (68) *coincide:*

$$K_{Y(x)}=K_{y(z)}. \tag{74}$$

Proof. Clearly, it is enough to treat the case $v(x)=1$. We have the formula (68) implying

$$K=\frac{y_0^4}{\det\begin{pmatrix}y_0 & y_1\\ \mathrm{d}y_0/\mathrm{d}z & \mathrm{d}y_1/\mathrm{d}z\end{pmatrix}^3}\exp\biggl(-\frac{1}{2}\int^z P(z)\,\mathrm{d}z\biggr).$$

Furthermore,

$$\frac{\mathrm{d}Y_0}{\mathrm{d}x} = z'\frac{\mathrm{d}y_0}{\mathrm{d}z}, \quad \frac{\mathrm{d}Y_1}{\mathrm{d}x} = z'\frac{\mathrm{d}y_1}{\mathrm{d}z}, \quad \text{and} \quad \widetilde{P} = -6\frac{z''}{z'} + z'P,$$

hence

$$\begin{aligned} K_{Y(x)} &= \frac{Y_0^4}{\det\begin{pmatrix} Y_0 & Y_1 \\ \mathrm{d}Y_0/\mathrm{d}x & \mathrm{d}Y_1/\mathrm{d}x \end{pmatrix}^3} \exp\biggl(-\frac{1}{2}\int^x \widetilde{P}(x)\,\mathrm{d}x\biggr) \\ &= \frac{y_0(z)^4}{\det\begin{pmatrix} y_0(z) & y_1(z) \\ z'\mathrm{d}y_0/\mathrm{d}z & z'\mathrm{d}y_1/\mathrm{d}z \end{pmatrix}^3} \exp\biggl(-\frac{1}{2}\int^x \Bigl(-6\frac{z''}{z'} + z'P\bigl(z(x)\bigr)\Bigr)\,\mathrm{d}x\biggr) \\ &= \frac{y_0(z)^4}{\det\begin{pmatrix} y_0(z) & y_1(z) \\ \mathrm{d}y_0/\mathrm{d}z & \mathrm{d}y_1/\mathrm{d}z \end{pmatrix}^3} \exp\biggl(-\frac{1}{2}\int^z P(z)\,\mathrm{d}z\biggr) \\ &= K_{y(z)}. \end{aligned}$$

Here we used $z'(0) = 1$ when we integrated

$$3\int_0^x \frac{z''}{z'}\,\mathrm{d}x = 3\log z'(x) - 3\log z'(0).$$

□

From the transformation formulas for passing from (12) to (69) through the map $Y(x) = v(x)\cdot y\bigl(z(x)\bigr)$, we find that the Calabi–Yau condition (13) is preserved. This is very hard to see by direct computation, since one gets an enormous 4th order non-linear differential equation for $z(x)$.

Conjecture. *If Yukawa couplings coincide, then there exists an algebraic transformation between corresponding Calabi–Yau differential equations.*

In fact, Proposition 3 states that the Yukawa coupling defined by (68) is preserved by *any formal* coordinate transformation $z(x) = x + \cdots$. However, the requirement for the transformed equation to be of Calabi–Yau type (in particular, to have rational functions as coefficients) should lead to the *algebraicity* of such a transformation.

6. Proof of Theorem 4: Guessing algebraic transformations

This section does not only provide a proof of Theorem 4, — we illustrate as well our strategies to guess algebraic transformations for Calabi–Yau differential equations on the example of Theorem 4; more precisely, we show how to 'discover' the equivalence of (e)∗(e) and (C)∗(i). We distinguish three methods, two analytic and one algebraic. They also provide proofs of the discovered algebraic transformation as soon as the fact of its existence is established.

First of all we indicate a way to recognize in `Maple` whether a function $R(z)$, given by its Taylor expansion at the origin, is rational or algebraic and, if it is, to find a closed expression. For this, one applies `seriestodiffeq`$(R, \widehat{R}(z))$ (with `gfun`) and then `dsolve` to $\widehat{R}(z)$.

6.1. **Analytic guessing.** Compute the z-expansions (30 terms, say) of the mirror maps $\widetilde{q}$ and q, then write

$$\widetilde{q}(z) = \pm q(\pm z + a_2 z^2 + \cdots + a_{30} z^{30} + \cdots)$$

(the signs belong together); expand the latter equality up to z^{31} to get a system of linear equations for unknowns $a_2, \dots, a_{30}$. It takes `Maple` a few minutes to solve the system; this finds the inner transformation. Then compute the power series expansion of the outer transformation multiple and use `gfun`.

6.2. **Schwarzian relation.** In passing from (e)$*$(e) to (C)$*$(i), we can use the 'magic Schwarzian relation' (Proposition 2) for $z(x)$ in the form $z(x) = -x + \cdots$. Then we find recursively the expansion for $\widetilde{z}(x) = -256z(x/256)$:

$$\widetilde{z}(x) = x + 10x^2 + 83x^3 + 628x^4 + 4501x^5 + 31134x^6 + 210023x^7 + \cdots,$$

which `Maple` easily recognizes as expansion of a rational function:

$$\widetilde{z}(x) = \frac{x(1-x)^2}{(1-6x+x^2)^2}.$$

Furthermore, denoting by $Y(z), \widehat{Y}(z) \in 1 + z\mathbb{Z}[[z]]$ the analytic solutions of (e)$*$(e) and (C)$*$(i), respectively, it remains to let `Maple` identify the quotient of

$$Y(z) \quad \text{and} \quad \widehat{Y}\left(\frac{-z(1-256z)^2}{(1-6\cdot 256z + 256^2z^2)^2}\right)$$

with the algebraic function $(1-6\cdot 256z + 256^2z^2)^{-1/2}$.

6.3. **Local monodromy considerations.** The operator for the Hadamard product (e)$*$(e) is

$$D = \theta^4 - 16z(16\theta^4 + 128\theta^3 + 112\theta^2 + 48\theta + 9) + \cdots - 2^{40}z^5(\theta+1)^4$$

The discriminant is $(1+256z)^2(1-256z)^3$ but the point $z = -1/256$ turns out to be an apparent singularity. The point $z = 1/256$ is also a MUM point and a calculation shows that

$$K(q)_{z=1/256} = K(q^2)_{z=0} = K(q^2)_{z=\infty}.$$

Thus, the operator D has three MUM points and no other singularities.

The operator corresponding to the Hadamard product (C)$*$(i) is

$$\theta^4 - 16z(4\theta+1)(4\theta+3)(32\theta^2+32\theta+13) + 2^{16}z^2(4\theta+1)(4\theta+3)(4\theta+5)(4\theta+7),$$

an operator with discriminant $(1-4096z)^2$. The exponents at $z = 1/4096$ are $0, 1/4, 3/4, 1$; at $z = \infty$ they are $1/4, 3/4, 5/4, 7/4$, hence these points have local monodromy of order 4. All these facts are easily checked with `Maple` using `formal sol` (within `DEtools`). If we try to think of (e)$*$(e) as a pullback of (C)$*$(i) via a rational map $R(z) = P(z)/Q(z)$, we see that it requires to have the following properties: $R^{-1}(0) = \{0, 1/256, \infty\}$ and the ramification over $z = 1/256$ is of order 2. Therefore, the degree of $R(z)$ is four. We also require $R^{-1}(1/4096) = \{-1/256\}$ as we want the ramification index at $z = 1/256$ to be 4. But then we cannot require the same ramification index over $z = \infty$, hence we assume that $R^{-1}(\infty)$ consists of two points, each with ramification index 2. The pullback then has exponents 1/2 at these points, and we have to divide by the square root of the polynomial defining these points. Combining the information we see that

$$R(z) = c\frac{z(1-256z)^2}{q(z)^2}, \qquad q(z) = 1 + az + bz^2,$$

where a, b and c are certain constants. We now determine the constants by requiring that $R(-1/256) = 1/4096$ and

$$R\biggl(x - \frac{1}{256}\biggr) = \frac{1}{4096} + 0\cdot x + 0\cdot x^2 + 0\cdot x^3 + \cdots.$$

We find

$$a = -1536, \quad b = 65536, \quad c = -1.$$

If $\widehat{Y}(z)$ is the solution of (C)∗(i), then the function $\widehat{Y}(R(z))$ has the square-root behavior at the pre-image of $z = \infty$, that is, at the roots of $q(z) = 1 - 1536z + 65536z^2$, hence

$$Y(z) = \frac{1}{\sqrt{q(z)}}\,\widehat{Y}\biggl(\frac{-z(1-256z)^2}{q(z)^2}\biggr)$$

has the same local properties as the solution of (e)∗(e) and, in fact, coincides with it.

The same method can be used to find the transformation in the other cases. For example, from the fact that the Hadamard product multiplies the singular points of the operators, it follows without further calculation that the operators (54) and (55) have 0, ∞ and the roots of $1 - 2\hat{a}z + \hat{c}z^2$ as singularities. The Hadamard product of $f_\alpha(z)$ and operator (6) has its singularities at 0, ∞ and the roots of $1 - az + cz^2$. A possible transformation of degree two will have to map these singular points in exactly the same way as in Theorem 1, hence we are again led to consider $z \mapsto -z/(1 - az + cz^2)$. In this case, the prefactor can also be determined by looking at the local exponents.

7. Concluding remarks

It is definitely not our goal here to stress on consequences of our theorems, since we feel that the transformations, and even their existence, are beautiful by themselves. Our results provide (albeit rather indirect) geometric interpretations of several (YY-)pullbacks from the table in [3]; before, such pullbacks were of geometric origin only conjecturally, and no relation to Calabi–Yau geometry was known. Another application of our transformation theorems, having a more arithmetic flavor, is the integrality of the analytic solutions of the pullbacks (condition (ii) of Section 1), as well as the integrality of the corresponding mirror maps (condition (iii)) when the results in [16] are applicable. There are many aspects that can be discussed elsewhere.

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