

NEW DIRECTIONS IN NIELSEN-REIDEMEISTER THEORY

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ABSTRACT. The purpose of this expository paper is to present new directions in the classical Nielsen-Reidemeister fixed point theory. We describe twisted Burnside-Frobenius theorem, groups with R_∞ *property* and a connection between Nielsen fixed point theory and symplectic Floer homology.

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1. INTRODUCTION

1.1. Nielsen- Reidemeister fixed point theory. Let $f : X \rightarrow X$ be a map of a compact topological space X . Nielsen-Reidemeister fixed point theory suggests a way of counting the fixed points of the map f in the presence of the fundamental group of the space X . Let $p : \tilde{X} \rightarrow X$ be the universal covering of X and $\tilde{f} : \tilde{X} \rightarrow \tilde{X}$ a lifting of f , ie. $p \circ \tilde{f} = f \circ p$. Two liftings \tilde{f} and \tilde{f}' are called *conjugate* if there is a $\gamma \in \Gamma \cong \pi_1(X)$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. The subset $p(\text{Fix}(\tilde{f})) \subset \text{Fix}(f)$ is called *the fixed point class of f determined by the lifting class $[\tilde{f}]$* . A fixed point class is called *essential* if its index is nonzero. The number of lifting classes of f (and hence the number of fixed point classes, empty or not) is called the *Reidemeister Number* of f , denoted $R(f)$. This is a positive integer or infinity. The number of essential fixed point classes is called the *Nielsen number* of f , denoted by $N(f)$. The Nielsen number is always finite. $R(f)$ and $N(f)$ are homotopy type invariants. In the category of compact, connected polyhedra the Nielsen number of a map is equal to the least number of fixed points of maps with the same homotopy type as f . In Nielsen fixed point theory the main objects for investigation are the Nielsen and Reidemeister numbers and their modifications [58].

Our definition of a fixed point class is via the universal covering space. It essentially says: Two fixed point of f are in the same class iff there is a lifting \tilde{f} of f having fixed points above both of them. There is another way of saying this, which does not use covering space

explicitly, hence is very useful in identifying fixed point classes. Namely, two fixed points x_0 and x_1 of f belong to the same fixed point class iff there is a path c from x_0 to x_1 such that $c \cong f \circ c$ (homotopy relative endpoints). This can be considered as an equivalent definition of a non-empty fixed point class. Every map f has only finitely many non-empty fixed point classes, each a compact subset of X . Given a homotopy $H = \{h_t\} : f_0 \cong f_1$, we want to see its influence on fixed point classes of f_0 and f_1 . A homotopy $\tilde{H} = \{\tilde{h}_t\} : \tilde{X} \rightarrow \tilde{X}$ is called a lifting of the homotopy $H = \{h_t\}$, if \tilde{h}_t is a lifting of h_t for every $t \in I$. Given a homotopy H and a lifting \tilde{f}_0 of f_0 , there is a unique lifting \tilde{H} of H such that $\tilde{h}_0 = \tilde{f}_0$, hence by unique lifting property of covering spaces they determine a lifting \tilde{f}_1 of f_1 . Thus H gives rise to a one-one correspondence from liftings of f_0 to liftings of f_1 . This correspondence preserves the conjugacy relation. Thus there is a one-to-one correspondence between lifting classes and fixed point classes of f_0 and those of f_1 .

Given a selfmap $f : X \rightarrow X$ of a compact connected manifold X , the nonvanishing of the classical Lefschetz number $L(f)$ guarantees the existence of fixed points. Unfortunately, $L(f)$ yields no information about the size of the set of fixed points of f . However, the Nielsen number $N(f)$, a more subtle homotopy invariant, provides a lower bound on the size of this set. For $\dim X \geq 3$, a classical theorem of Wecken asserts that $N(f)$ is a sharp lower bound on the size of this set, that is, $N(f)$ is the minimal number of fixed points among all maps homotopic to f . Thus the computation of $N(f)$ is a central issue in fixed point theory.

Let G be a countable discrete group and $\phi : G \rightarrow G$ an endomorphism. Two elements $x, x' \in G$ are said to be ϕ -conjugate or *twisted conjugate*, iff there exists $g \in G$ with $x' = gx\phi(g^{-1})$. We shall write $\{x\}_\phi$ for the ϕ -conjugacy or *twisted conjugacy* class of the element $x \in G$. The number of ϕ -conjugacy classes is called the *Reidemeister number* of an endomorphism ϕ and is denoted by $R(\phi)$. If ϕ is

the identity map then the ϕ -conjugacy classes are the usual conjugacy classes in the group G .

Let $f : X \rightarrow X$ be given, and let a specific lifting $\tilde{f} : \tilde{X} \rightarrow \tilde{X}$ be chosen as reference. Let Γ be the group of covering translations of \tilde{X} over X . Then every lifting of f can be written uniquely as $\alpha \circ \tilde{f}$, with $\alpha \in \Gamma$. So elements of Γ serve as coordinates of liftings with respect to the reference \tilde{f} . Now for every $\alpha \in \Gamma$ the composition $\tilde{f} \circ \alpha$ is a lifting of f so there is a unique $\alpha' \in \Gamma$ such that $\alpha' \circ \tilde{f} = \tilde{f} \circ \alpha$. This correspondence $\alpha \rightarrow \alpha'$ is determined by the reference \tilde{f} , and is obviously a homomorphism. The endomorphism $\tilde{f}_* : \Gamma \rightarrow \Gamma$ determined by the lifting \tilde{f} of f is defined by $\tilde{f}_*(\alpha) \circ \tilde{f} = \tilde{f} \circ \alpha$. It is well known that $\Gamma \cong \pi_1(X)$. We shall identify $\pi = \pi_1(X, x_0)$ and Γ in the usual way.

We have seen that $\alpha \in \pi$ can be considered as the coordinate of the lifting $\alpha \circ \tilde{f}$. We can tell the conjugacy of two liftings from their coordinates: $[\alpha \circ \tilde{f}] = [\alpha' \circ \tilde{f}]$ iff there is $\gamma \in \pi$ such that $\alpha' = \gamma \alpha \tilde{f}_*(\gamma^{-1})$.

So we have the Reidemeister bijection: Lifting classes of f are in 1-1 correspondence with \tilde{f}_* -conjugacy classes in group π , the lifting class $[\alpha \circ \tilde{f}]$ corresponds to the \tilde{f}_* -conjugacy class of α .

By an abuse of language, we say that the fixed point class $p(\text{Fix}(\alpha \circ \tilde{f}))$, which is labeled with the lifting class $[\alpha \circ \tilde{f}]$, corresponds to the \tilde{f}_* -conjugacy class of α . Thus the \tilde{f}_* -conjugacy classes in π serve as coordinates for the fixed point classes of f , once a reference lifting \tilde{f} is chosen.

1.2. New directions. The interest in twisted conjugacy relation for group endomorphism has its origins not only in Nielsen-Reidemeister fixed point theory (see, e.g. [82, 58, 22]), but also in Selberg theory (see, eg. [90, 2]) and Algebraic Geometry (see, e.g. [50]).

A current important problem in this area is to obtain a twisted analogue of the celebrated Burnside-Frobenius theorem [25, 22, 32, 33, 95, 28, 31, 29, 54], that is, to show the equality of the Reidemeister number

of ϕ and the number of fixed points of the induced homeomorphism of an appropriate dual object.

If G is a finite group, then the classical Burnside-Frobenius theorem (see, e.g., [88], [63, p. 140]) says that the number of classes of irreducible representations is equal to the number of conjugacy classes of elements of G . Let \widehat{G} be the *unitary dual* of G , i.e. the set of equivalence classes of unitary irreducible representations of G .

If $\phi : G \rightarrow G$ is an automorphism, it induces a map $\widehat{\phi} : \widehat{G} \rightarrow \widehat{G}$, $\widehat{\phi}(\rho) = \rho \circ \phi$. Therefore, by the Burnside-Frobenius theorem, if ϕ is the identity automorphism of any finite group G , then we have $R(\phi) = \#\text{Fix}(\widehat{\phi})$.

In [25] it was discovered that this statement remains true for any automorphism ϕ of any finite group G . Indeed, if we consider an automorphism ϕ of a finite group G , then $R(\phi)$ is equal to the dimension of the space of twisted invariant functions on this group. Hence, by Peter-Weyl theorem (which asserts the existence of a two-side equivariant isomorphism $C^*(G) \cong \bigoplus_{\rho \in \widehat{G}} \text{End}(H_\rho)$), $R(\phi)$ is identified with the sum of dimensions d_ρ of twisted invariant elements of $\text{End}(H_\rho)$, where ρ runs over \widehat{G} , and the space of representation ρ is denoted by H_ρ . By the Schur lemma, $d_\rho = 1$, if ρ is a fixed point of $\widehat{\phi}$, and is zero otherwise. Hence, $R(\phi)$ coincides with the number of fixed points of $\widehat{\phi}$.

The attempts to generalize this theorem to the case of non-identical automorphism and of non-finite group (i.e., to identify the Reidemeister number of ϕ and the number of fixed points of $\widehat{\phi}$ on an appropriate dual object of G , provided that one of these numbers is finite) were inspired by the dynamical questions and were the subject of a series of papers [25, 22, 32, 33, 95, 30, 28, 31, 29, 54] In the paper [35] we studied the following property for a countable discrete group G and its automorphism ϕ : we say that the group is *ϕ -conjugacy separable* if its Reidemeister classes can be distinguished by homomorphisms onto finite groups, and we say that it is *twisted conjugacy separable* if it is ϕ -conjugacy separable for any automorphism ϕ with $R(\phi) < \infty$

(*strongly twisted conjugacy separable*, if we remove this finiteness restriction) (Definitions 2.3 and 2.5). This notion was used in [31] to prove the twisted Burnside-Frobenius theorem for polycyclic-by-finite groups with the finite-dimensional part of the unitary dual \widehat{G} as an appropriate dual object.

In chapter 2, after some preliminary considerations, we discuss following results:

- (1) Classes of twisted conjugacy separable groups: Polycyclic-by-finite groups are strongly twisted conjugacy separable groups [31, 35].
- (2) Twisted conjugacy separability respects some extensions: Suppose, there is an extension $H \rightarrow G \rightarrow G/H$, where the group H is a characteristic twisted conjugacy separable group; G/H is finitely generated FC-group (i.e., a group with finite conjugacy classes). Then G is a twisted conjugacy separable group [31, 35].
- (3) Examples of groups, which are not twisted conjugacy separable: HNN, Ivanov and Osin groups [31].
- (4) The affirmative answer to the twisted Dehn conjugacy problem for polycyclic-by-finite groups [35].
- (5) Twisted Burnside-Frobenius theorem for ϕ -conjugacy separable groups in the following formulation: Let G be an ϕ -conjugacy separable group. Then $R(\phi) = S_f(\phi)$ if one of these numbers is finite, where $S_f(\phi)$ is the number of fixed points of $\widehat{\phi} : \widehat{G}_f \rightarrow \widehat{G}_f$, $\widehat{\phi}(\rho) = \rho \circ \phi$, where \widehat{G}_f is the part of the unitary dual \widehat{G} , which is formed by the finite-dimensional representations [31, 35].

A number of examples of groups and automorphisms with finite Reidemeister numbers was obtained and studied in [22, 44, 26, 33, 30]. Using the same argument as in [32] one obtains from the twisted Burnside-Frobenius theorem the following dynamical and number-theoretical

consequence which, together with the twisted Burnside-Frobenius theorem itself, is very important for the realization problem of Reidemeister numbers in topological dynamics and the study of the Reidemeister zeta-function. Let $\mu(d)$, $d \in \mathbb{N}$, be the *Möbius function*, i.e. $\mu(d) = 0$ if d is divisible by a square different from one ; $\mu(d) = (-1)^k$ if d is not divisible by a square different from one , where k denotes the number of prime divisors of d ; $\mu(1) = 1$.

Congruences for Reidemeister numbers[31]: *Let $\phi : G \rightarrow G$ be an automorphism of a countable discrete twisted conjugacy separable group G such that all numbers $R(\phi^n)$ are finite. Then one has for all n ,*

$$\sum_{d|n} \mu(d) \cdot R(\phi^{n/d}) \equiv 0 \pmod{n}.$$

These congruences were proved previously in a number of special cases in [25, 22, 32, 33, 30] and are an analog of remarkable Dold congruences for the Lefschetz numbers of the iterations of a continuous map[16].

One step in the process to obtain twisted Burnside-Frobenius theorem is to describe the class of groups G for which $R(\phi) = \infty$ for any automorphism $\phi : G \rightarrow G$. We say that a group G has *R_∞ property* if all of its automorphisms ϕ satisfy $R(\phi) = \infty$.

The work of discovering which groups have *R_∞ property* was begun by Fel'shtyn and Hill in [25]. It was later shown by various authors that the following groups have *R_∞ property* : (1) non-elementary Gromov hyperbolic groups [23, 65], (2) Baumslag-Solitar groups $BS(m, n) = \langle a, b | ba^m b^{-1} = a^n \rangle$ except for $BS(1, 1)$ [24], (3) generalized Baumslag-Solitar groups, that is, finitely generated groups which act on a tree with all edge and vertex stabilizers infinite cyclic [64], (4) lamplighter groups $\mathbb{Z}_n \wr \mathbb{Z}$ if and only if $2|n$ or $3|n$ [46], (5) the solvable generalization Γ of $BS(1, n)$ given by the short exact sequence $1 \rightarrow \mathbb{Z}[\frac{1}{n}] \rightarrow \Gamma \rightarrow \mathbb{Z}^k \rightarrow 1$, as well as any group quasi-isometric to Γ [91]; such groups are quasi-isometric to $BS(1, n)$ [92] (note however that the class of

groups for which $R(\phi) = \infty$ for any automorphism ϕ is not closed under quasi-isometry) (6) saturated weakly branch groups (including the Grigorchuk group and the Gupta-Sidki group) [34], (7) The R. Thompson group F [7], (8) symplectic groups $Sp(2n, \mathbb{Z})$, the mapping class groups Mod_S of a compact surface S and the full braid groups $B_n(S)$ on n strings of a compact surface S in the cases where S is either the compact disk D , or the sphere S^2 [36].

The results of the present paper indicate that the further study of Reidemeister theory for these groups should go along the lines similar to those of the infinite case. On the other hand, this result reduces the class of groups for which the twisted Burnside-Frobenius conjecture [25, 32, 33, 95, 31, 29, 54] has yet to be verified.

One is interested in conditions which yield more specific information about the relationship between the Lefschetz number, the Nielsen number and the Reidemeister number. In the special case of selfmaps of a lens space, Franz [41] showed that all fixed point classes of such maps have the same fixed point index. From this it follows that one of two situations occurs, namely 1) $L(f) = 0$ implies $N(f) = 0$, in which case f is deformable to a fixed point free map, and 2) $L(f) \neq 0$ implies $N(f) = \#Coker(1 - f_{*1})$ where f_{*1} is the induced homomorphism on the first integral homology. When the Lefschetz number $L(f)$ is nonzero, the cardinality of $Coker(1 - f_{*1})$ is exactly equal to the Reidemeister number $R(f)$.

The evaluation subgroup $J(X)$, also known as the first Gottlieb group $G_1(X)$, of the fundamental group π is the image of the homomorphism $ev_* : \pi_1(X^X, 1_X) \rightarrow \pi_1(X, x_0)$ induced by the evaluation map at a point $x_0 \in X$. If $J(X)$ coincides with π , the analogous results are obtained as well. Namely, $L(f) = 0$ implies that $N(f) = 0$, and $L(f) \neq 0$ implies that $N(f) = R(f)$ for all maps f . This general type of result was first proved by B. Jiang (see [58] for more information on Nielsen fixed point theory) for a class of spaces now known as Jiang spaces.

Jiang-type results hold for all selfmaps of a large class of spaces including simply-connected spaces, generalized lens spaces, H -spaces, topological groups, orientable coset spaces of compact connected Lie groups, nilmanifolds, certain \mathcal{C} -nilpotent spaces where \mathcal{C} denotes the class of finite groups, certain solvmanifolds and infra-homogeneous spaces [96]. When $\dim X \geq 3$ and $N(f) = 0$, the selfmap f is deformable to be fixed point free by Wecken's theorem. For the equality $N(f) = R(f)$ to hold, one must first determine the finiteness of the Reidemeister number $R(f)$.

Unless $N(f) = 0$ for all homeomorphisms f , property R_∞ eliminates the possibility of a Jiang-type result, which would require the finiteness of the Reidemeister number. That is, if a group G has property R_∞ , a compact connected manifold with G as fundamental group will never satisfy a Jiang-type result.

In particular the fundamental group of surface of genus greater than 1 has property R_∞ . So, we need a new ideas and tools in this classical case. Recently a connection between symplectic Floer homology and Nielsen fixed point theory was discovered [37, 43]. The author came to the idea that Nielsen numbers are connected with Floer homology of surface diffeomorphisms at the Autumn 2000, after conversations with Joel Robbin and Dan Burghilea.

In the chapter 4 we discuss the connection between symplectic Floer homology group and Nielsen fixed point theory. We also describe symplectic zeta functions and an asymptotic invariant of monotone symplectomorphism. Generalised Arnold conjecture is formulated.

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2. TWISTED BURNSIDE-FROBENIUS THEOREM

2.1. Preliminary considerations. Following construction relates ϕ -conjugacy classes and some conjugacy classes of another group. Consider the action of \mathbb{Z} on G , i.e. a homomorphism $\mathbb{Z} \rightarrow \text{Aut}(G)$, $n \mapsto \phi^n$. Let Γ be a corresponding semi-direct product $\Gamma = G \rtimes \mathbb{Z}$:

$$(2.1) \quad \Gamma := \langle G, t \mid tgt^{-1} = \phi(g) \rangle$$

in terms of generators and relations, where t is a generator of \mathbb{Z} . The group G is a normal subgroup of Γ . As a set, Γ has the form

$$(2.2) \quad \Gamma = \sqcup_{n \in \mathbb{Z}} G \cdot t^n,$$

where $G \cdot t^n$ is the coset by G containing t^n .

Remark 2.1. Any usual conjugacy class of Γ is contained in some $G \cdot t^n$. Indeed, $gg't^n g^{-1} = gg'\phi^n(g^{-1})t^n$ and $tg't^n t^{-1} = \phi(g')t^n$.

Lemma 2.1. *Two elements x, y of G are ϕ -conjugate iff xt and yt are conjugate in the usual sense in Γ . Therefore $g \mapsto g \cdot t$ is a bijection from the set of ϕ -conjugacy classes of G onto the set of conjugacy classes of Γ contained in $G \cdot t$.*

Proof. If x and y are ϕ -conjugate then there is a $g \in G$ such that $gx = y\phi(g)$. This implies $gx = ytgt^{-1}$ and therefore $g(xt) = (yt)g$ so xt and yt are conjugate in the usual sense in Γ . Conversely, suppose xt and yt are conjugate in Γ . Then there is a $gt^n \in \Gamma$ with $gt^n xt = ytgt^n$. From the relation $txt^{-1} = \phi(x)$ we obtain $g\phi^n(x)t^{n+1} = y\phi(g)t^{n+1}$ and therefore $g\phi^n(x) = y\phi(g)$. Hence, y and $\phi^n(x)$ are ϕ -conjugate. Thus, y and x are ϕ -conjugate, because x and $\phi(x)$ are always ϕ -conjugate: $\phi(x) = x^{-1}x\phi(x)$. \square

2.2. Twisted conjugacy separability. We would like to give a generalization of the following well known notion.

Definition 2.2. A group G is *conjugacy separable* if any pair g, h of non-conjugate elements of G are non-conjugate in some finite quotient of G .

It was proved that polycyclic-by-finite groups are conjugacy separated ([83, 40], see also [85, Ch. 4]).

We can introduce the following notion, which coincides with the previous definition in the case $\phi = \text{Id}$.

Definition 2.3. [35] A group G is *ϕ -conjugacy separable* with respect to an automorphism $\phi : G \rightarrow G$ if any pair g, h of non- ϕ -conjugate elements of G are non- $\bar{\phi}$ -conjugate in some finite quotient of G respecting ϕ .

This notion is closely related to the notion $\text{RP}(\phi)$ introduced in [31].

Definition 2.4. [31] We say that a group G has the property RP if for any automorphism ϕ with $R(\phi) < \infty$ the characteristic functions f of Reidemeister classes (hence all ϕ -central functions) are periodic in the following sense.

There exists a finite group K , its automorphism ϕ_K , and epimorphism $F : G \rightarrow K$ such that

(1) The diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G \\ F \downarrow & & \downarrow F \\ K & \xrightarrow{\phi_K} & K \end{array}$$

commutes.

(2) $f = F^* f_K$, where f_K is a characteristic function of a subset of K .

If this property holds for a concrete automorphism ϕ , we will denote this by $\text{RP}(\phi)$.

One gets immediately the following statement.

Theorem 2.2. [31] *Suppose, $R(\phi) < \infty$. Then G is ϕ -conjugacy separable if and only if G is $\text{RP}(\phi)$.*

Proof. Indeed, let $F_{ij} : G \rightarrow K_{ij}$ distinguish i th and j th ϕ -conjugacy classes, where K_{ij} are finite groups, $i, j = 1, \dots, R(\phi)$. Let $F : G \rightarrow \bigoplus_{i,j} K_{ij}$, $F(g) = \sum_{i,j} F_{ij}(g)$, be the diagonal mapping and K its image. Then $F : G \rightarrow K$ gives $\text{RP}(\phi)$.

The opposite implication is evident. \square

Definition 2.5. [35] A group G is *twisted conjugacy separable* if it is ϕ -conjugacy separable for any ϕ with $R(\phi) < \infty$.

A group G is *strongly twisted conjugacy separable* if it is ϕ -conjugacy separable for any ϕ .

From Theorem 2.2 one immediately obtains

Corollary 2.3. [35] A group G is twisted conjugacy separable if and only if it is RP .

Theorem 2.4. [31] *Let $F : \Gamma \rightarrow K$ be a morphism onto a finite group K which separates two conjugacy classes of Γ in $G \cdot t$. Then the restriction $F_G := F|_G : G \rightarrow \text{Im}(F|_G)$ separates the corresponding (by the bijection from Lemma 2.1) ϕ -conjugacy classes in G .*

Proof. First of all let us remark that $\text{Ker}(F_G)$ is ϕ -invariant. Indeed, suppose $F_G(g) = F(g) = e$. Then

$$F_G(\phi(g)) = F(\phi(g)) = F(tgt^{-1}) = F(t)F(t)^{-1} = e$$

(the kernel of F is a normal subgroup).

Let gt and $\tilde{g}t$ be some representatives of the mentioned conjugacy classes. Then

$$F((ht^n)gt(ht^n)^{-1}) \neq F(\tilde{g}t), \quad \forall h \in G, n \in \mathbb{Z},$$

$$F(ht^n gt) \neq F(\tilde{g}t ht^n), \quad \forall h \in G, n \in \mathbb{Z},$$

$$F(h\phi^n(g)t^{n+1}) \neq F(\tilde{g}\phi(h)t^{n+1}), \quad \forall h \in G, n \in \mathbb{Z},$$

$$F(h\phi^n(g)) \neq F(\tilde{g}\phi(h)), \quad \forall h \in G, n \in \mathbb{Z},$$

in particular, $F(hg\phi(h^{-1})) \neq F(\tilde{g}) \forall h \in G$. □

Theorem 2.5. [35] *Let some class of conjugacy separable groups be closed under taking semidirect products by \mathbb{Z} . Then this class consists of strongly twisted conjugacy separable groups.*

Proof. This follows immediately from Theorem 2.4 and Theorem 2.2. □

2.2.1. *First examples: polycyclic-by-finite groups.* As an application we obtain another proof of the main theorem for polycyclic-by-finite groups.

Let $G' = [G, G]$ be the *commutator subgroup* or *derived group* of G , i.e. the subgroup generated by commutators. G' is invariant under any homomorphism, in particular it is normal. It is the smallest normal subgroup of G with an abelian factor group. Denoting $G^{(0)} := G$, $G^{(1)} := G'$, $G^{(n)} := (G^{(n-1)})'$, $n \geq 2$, one obtains *derived series* of G :

$$(2.3) \quad G = G^{(0)} \supset G' \supset G^{(2)} \supset \dots \supset G^{(n)} \supset \dots$$

If $G^{(n)} = e$ for some n , i.e. the series (2.3) stabilizes by trivial group, the group G is *solvable*;

Definition 2.6. A solvable group with derived series with cyclic factors is called *polycyclic group*.

Theorem 2.6. [31] *Any polycyclic-by-finite group is a strongly twisted conjugacy separable group.*

Proof. The class of polycyclic-by-finite groups is closed under taking semidirect products by \mathbb{Z} . Indeed, let G be an polycyclic-by-finite group. Then there exists a characteristic (polycyclic) subgroup P of finite index in G . Hence, $P \rtimes \mathbb{Z}$ is a polycyclic normal group of $G \rtimes \mathbb{Z}$ of the same finite index.

Polycyclic-by-finite groups are conjugacy separable ([83, 40], see also [85, Ch. 4]). It remains to apply Theorem 2.5. \square

2.3. Twisted conjugacy separability and extensions. It is known that conjugacy separability does not respect extensions. For twisted conjugacy separable groups the situation is much better under some finiteness conditions. More precisely one has the following statement.

Theorem 2.7. [31] *Suppose, there exists a commutative diagram*

$$(2.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H & \xrightarrow{i} & G & \xrightarrow{p} & G/H \longrightarrow 0 \\ & & \phi' \downarrow & & \downarrow \phi & & \downarrow \bar{\phi} \\ 0 & \longrightarrow & H & \xrightarrow{i} & G & \xrightarrow{p} & G/H \longrightarrow 0, \end{array}$$

where H is a normal subgroup of a finitely generated group G . Suppose, $R(\phi) < \infty$, G/H is a FC group, i.e., all conjugacy classes are finite, and H is a ϕ' -conjugacy separable group. Then G is a ϕ -conjugacy separable group.

Another variant of finiteness is $|G/H| < \infty$ (without the property $R(\phi) < \infty$).

2.4. Twisted Burnside-Frobenius theorem for ϕ -conjugacy separable groups.

Definition 2.7. Denote by \widehat{G}_f the subset of the unitary dual \widehat{G} related to finite-dimensional representations.

Theorem 2.8. [31] *Let G be an ϕ -conjugacy separable group. Then $R(\phi) = S_f(\phi)$ if one of these numbers is finite.*

Proof. The coefficients of finite-dimensional non-equivalent irreducible representations of G are linear independent by Frobenius-Schur theorem (see [12, (27.13)]). Moreover, the coefficients of non-equivalent unitary finite-dimensional irreducible representations are orthogonal to each other as functions on the universal compact group associated with

the initial group [14, 16.1.3] by the Peter-Weyl theorem. Hence, their linear combinations are orthogonal to each other as well.

It is sufficient to verify the following three statements:

1) If $R(\phi) < \infty$, then each ϕ -class function is a finite linear combination of twisted-invariant functionals being coefficients of points of $\text{Fix } \widehat{\phi}_f$.

2) If $\rho \in \text{Fix } \widehat{\phi}_f$, there exists one and only one (up to scaling) twisted invariant functional on $\rho(C^*(G))$ (this is a finite full matrix algebra).

3) For different ρ the corresponding ϕ -class functions are linearly independent. This follows from the remark at the beginning of the proof. Let us remark that the property RP implies in particular that ϕ -central functions (for ϕ with $R(\phi) < \infty$) are functionals on $C^*(G)$, not only $L^1(G)$, i.e. are in the Fourier-Stieltjes algebra $B(G)$.

The statement 1) follows from the RP property. Indeed, this ϕ -class function f is a linear combination of functionals coming from some finite collection $\{\rho_i\}$ of elements of \widehat{G}_f (these representations ρ_1, \dots, ρ_s are in fact representations of the form $\pi_i \circ F$, where π_i are irreducible representations of the finite group K and $F : G \rightarrow K$, as in the definition of RP). So,

$$f = \sum_{i=1}^s f_i \circ \rho_i, \quad \rho_i : G \rightarrow \text{End}(V_i), \quad f_i : \text{End}(V_i) \rightarrow \mathbb{C}, \quad \rho_i \neq \rho_j, (i \neq j).$$

For any $g, \tilde{g} \in G$ one has

$$\sum_{i=1}^s f_i(\rho_i(\tilde{g})) = f(\tilde{g}) = f(g\tilde{g}\phi(g^{-1})) = \sum_{i=1}^s f_i(\rho_i(g\tilde{g}\phi(g^{-1}))).$$

By the observation at the beginning of the proof concerning linear independence,

$$f_i(\rho_i(\tilde{g})) = f_i(\rho_i(g\tilde{g}\phi(g^{-1}))). \quad i = 1, \dots, s,$$

i.e. f_i are twisted-invariant. For any $\rho \in \widehat{G}_f$, $\rho : G \rightarrow \text{End}(V)$, any functional $\omega : \text{End}(V) \rightarrow \mathbb{C}$ has the form $a \mapsto \text{Tr}(ba)$ for some fixed $b \in \text{End}(V)$. Twisted invariance implies twisted invariance of b (evident details can be found in [32, Sect. 3]). Hence, b is intertwining between

ρ and $\rho \circ \phi$ and $\rho \in \text{Fix}(\widehat{\phi}_f)$. The uniqueness of intertwining operator (up to scaling) implies 2).

It remains to prove that $R(\phi) < \infty$ if $S_f(\phi) < \infty$. By the definition of a ϕ -conjugacy separable group the Reidemeister classes of ϕ can be separated by maps to finite groups. Hence, taking representations of these finite groups and applying the twisted Burnside-Frobenius theorem to these groups we obtain that for any pair of Reidemeister classes there exists a function being a coefficient of a finite-dimensional unitary representation, which distinguish these classes. Hence, if $R(\phi) = \infty$, then there are infinitely many linearly independent twisted invariant functions being coefficients of finite dimensional representations. But there are as many such functionals, as $S_f(\phi)$.

□

Corollary 2.9. [31] Let G be almost polycyclic group and ϕ its automorphism. Then $R(\phi) = S_f(\phi)$ if one of these numbers is finite.

2.5. Examples and counterexamples. Some of examples of groups, for which the twisted Burnside-Frobenius theorem in the above formulation is true, out of the class of polycyclic-by-finite groups were obtained by F. Indukaev [54]. Namely, it is proved that wreath products $A \wr \mathbb{Z}$ are RP groups, where A is a finitely generated abelian group (these groups are residually finite).

Now let us present some counterexamples to the twisted Burnside-Frobenius theorem in the above formulation for some discrete groups with extreme properties. Suppose, an infinite discrete group G has a finite number of conjugacy classes. Such examples can be found in [89] (HNN-group), [76, p. 471] (Ivanov group), and [77] (Osin group). Then evidently, the characteristic function of the unity element is not almost-periodic and the argument above is not valid. Moreover, let us show, that these groups give rise counterexamples to the above theorem.

In particular, they are not twisted conjugacy separable. Evidently, they are not conjugacy separable, because they are not residually finite.

Example 2.8. [31] For the Osin group Reidemeister number $R(\text{Id}) = 2$, while there is only trivial (1-dimensional) finite-dimensional representation. Indeed, Osin group is an infinite finitely generated group G with exactly two conjugacy classes. All nontrivial elements of this group G are conjugate. So, the group G is simple, i.e. G has no nontrivial normal subgroup. This implies that group G is not residually finite (by definition of residually finite group). Hence, it is not linear (by Mal'cev theorem [68], [84, 15.1.6]) and has no finite-dimensional irreducible unitary representations with trivial kernel. Hence, by simplicity of G , it has no finite-dimensional irreducible unitary representation with nontrivial kernel, except of the trivial one.

Let us remark that Osin group is non-amenable, contains the free group in two generators F_2 , and has exponential growth.

Example 2.9. [31] For large enough prime numbers p , the first examples of finitely generated infinite periodic groups with exactly p conjugacy classes were constructed by Ivanov as limits of hyperbolic groups (although hyperbolicity was not used explicitly) (see [76, Theorem 41.2]). Ivanov group G is infinite periodic 2-generator group, in contrast to the Osin group, which is torsion free. The Ivanov group G is also a simple group. The proof (kindly explained to us by M. Sapir) is the following. Denote by a and b the generators of G described in [76, Theorem 41.2]. In the proof of Theorem 41.2 on [76] it was shown that each of elements of G is conjugate in G to a power of generator a of order s . Let us consider any normal subgroup N of G . Suppose $\gamma \in N$. Then $\gamma = ga^s g^{-1}$ for some $g \in G$ and some s . Hence, $a^s = g^{-1} \gamma g \in N$ and from periodicity of a , it follows that also $a \in N$ as well as $a^k \in N$ for any k , because p is prime. Then any element h of G also belongs to N being of the form $h = \tilde{h} a^k (\tilde{h})^{-1}$, for some k , i.e., $N = G$. Thus, the group G is simple. The discussion can be completed in the same way as in the case of Osin group.

Example 2.10. In paper [51], Theorem III and its corollary, G. Higman, B. H. Neumann, and H. Neumann proved that any locally infinite countable group G can be embedded into a countable group G^* in which all elements except the unit element are conjugate to each other (see also [89]). The discussion above related Osin group remains valid for G^* groups.

2.6. Twisted Dehn conjugacy problem. The subject is closely related to some decision problem. Recall that M. Dehn in 1912 [13] (see [67, Ch. 1, §2; Ch. 2, §1]) has formulated in particular

Conjugacy problem: Does there exists an algorithm to determine whether an arbitrary pair of group words U, V in the generators of G define conjugate elements of G ?

Following question was posed by G. Makanin [71, Question 10.26(a)]:

Question: Does there exists an algorithm to determine whether for an arbitrary pair of group words U and V of a free group G and an arbitrary automorphism ϕ of G the equation $\phi(X)U = VX$ solvable in G ?

In [8] the affirmative answer to the Makanin's question is obtained.

In [3] the following problem, which generalizes the two above problems, was posed:

Twisted conjugacy problem: Does there exists an algorithm to determine whether for an arbitrary pair of group words U and V in the generators of G the equality $\phi(X)U = VX$ holds for some $W \in G$ and $\phi \in H$, where H is a fixed subset of $\text{Aut}(G)$?

We will discuss the twisted conjugacy problem for $H = \{\phi\}$.

Theorem 2.10. [35] *The twisted conjugacy problem has the affirmative answer for G being polycyclic-by-finite group and H be equal to a unique automorphism ϕ .*

Proof. It follows immediately from Theorem 2.6 by the same argument as in the paper of Mal'cev [69] (see also [75], where the property

of conjugacy separability was first formulated) for the (non-twisted) conjugacy problem. \square

In fact we have proved the following statement.

Theorem 2.11. [35] *If G is strongly twisted conjugacy separable then the twisted Dehn conjugacy problem is solvable for any automorphism of G .*

Corollary 2.12. The twisted conjugacy problem has the affirmative answer for G being wreath products $A \wr \mathbb{Z}$ and any automorphism of G , where A is a finitely generated abelian group.

Proof. It is proved in [54] that G is strongly twisted conjugacy separable. \square

Also one can study some more particular cases of this problem. In particular, one has

Theorem 2.13. [35] *Let G be a ϕ -conjugacy separable group. Then the twisted Dehn conjugacy problem is solvable for ϕ .*

From Corollary 3.4 and Proposition 3.5 in [72] one can obtain

Theorem 2.14. [35] *Suppose G is the fundamental group of a closed hyperbolic surface and $\phi : G \rightarrow G$ is virtually inner. Then the twisted Dehn conjugacy problem is solvable for ϕ .*

Recently, in [9] the twisted Dehn conjugacy problem was solved for virtually surface groups and an example of a finitely presented group with solvable conjugacy problem but unsolvable twisted conjugacy problem was given.

2.7. Some questions. [35] It is evident, that any conjugacy separable group is residually finite (because the unity element is an entire conjugacy class). This argument does not work for general Reidemeister classes. In this relation we have formulated in [35] several open questions:

Question 1: Does the ϕ -conjugacy separability imply residually finiteness?

Question 2: Does the ϕ -conjugacy separability imply residually finiteness, provided $R(\phi) < \infty$?

Question 3: Does the twisted conjugacy separability imply residually finiteness, provided the existence of ϕ with $R(\phi) < \infty$?

Question 4: Let G be a residually finite group and ϕ its automorphism with $R(\phi) < \infty$. Is G ϕ -conjugacy separable ?

The affirmative answer to the last question implies twisted Burnside-Frobenius theorem for ϕ .

3. GROUPS WITH *property* R_∞

Consider a group extension respecting homomorphism ϕ :

$$(3.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H & \xrightarrow{i} & G & \xrightarrow{p} & G/H \longrightarrow 0 \\ & & \phi \downarrow & & \downarrow \phi & & \downarrow \bar{\phi} \\ 0 & \longrightarrow & H & \xrightarrow{i} & G & \xrightarrow{p} & G/H \longrightarrow 0, \end{array}$$

where H is a normal subgroup of G . First, notice that the Reidemeister classes of ϕ in G are mapped epimorphically onto classes of $\bar{\phi}$ in G/H . Indeed,

$$(3.2) \quad p(\tilde{g})p(g)\bar{\phi}(p(\tilde{g}^{-1})) = p(\tilde{g}g\phi(\tilde{g}^{-1})).$$

Suppose that the Reidemeister number $R(\bar{\phi})$ is infinite, the previous remark then implies that the Reidemeister number $R(\phi)$ is infinite.

Let G be a group, and let φ be an automorphism of G of order m . G_φ be the group $G \rtimes_\varphi \mathbb{Z}_m = \langle G, t \mid \forall g \in G, tgt^{-1} = \varphi(g), t^m = 1 \rangle$.

The following lemma was proven by Delzant.

Lemma 3.1. [65, Lemma 3.4] *If K is a normal subgroup of a group Γ acting non-elementarily on a hyperbolic space, and if Γ/K is Abelian, then any coset of K contains infinitely many conjugacy classes.*

Theorem 3.2. [36] *If G_φ has a non-elementary action by isometries on a Gromov-hyperbolic length space, then G has infinitely many φ -twisted conjugacy classes.*

Proof. By elementary action, we mean an action consisting of elliptic elements, or with a global fixed point, or a global fixed pair, in the boundary of the hyperbolic space. The statement of the theorem follows immediately from Lemma 2.1 and Delzant Lemma 3.1 \square

Theorem 3.3. *The following groups have R_∞ property :*

- (1) *non-elementary Gromov hyperbolic groups* [23, 65]
- (2) *non-elementary relatively hyperbolic groups,*
- (3) *the mapping class groups Mod_S of a compact surface S (with a few exceptions)* [36],
- (4) *the full braid groups $B_n(S^2)$ on n strings of the sphere S^2* [36],

Proof. (1)-(2): Theorem 3.2 applies if G is a Gromov-hyperbolic group or relatively hyperbolic group and if φ has finite order in $Out(G)$. In fact, in this case, G_φ contains G as a subgroup of finite index, thus is quasi-isometric to G , and by quasi-isometry invariance, it is itself a Gromov-hyperbolic or relatively hyperbolic group. Now let assume that a automorphism of a hyperbolic or relatively hyperbolic group has infinite order in $Out(G)$. We describe main steps of the proof in this case (see [23, 65] for the details). By [79] and [5] Φ preserves some R -tree T with nontrivial minimal small action of G (recall that an action of G is small if all arcs stabilisers are virtually cyclic; the action of G on T is always irreducible(no global fixed point, no invariant line, no invariant end)). This means that there is an R -tree T equipped with an isometric action of G whose length function satisfies $l \cdot \Phi = \lambda l$ for some $\lambda \geq 1$.

Step 1. Suppose $\lambda = 1$. Then the Reidemeister number $R(\phi)$ is infinite

.

Step 2. Suppose $\lambda > 1$. Assume that arc stabilisers are finite, and there exists $N_0 \in N$ such that, for every $Q \in T$, the action of $StabQ$

on $\pi_o(T - Q)$ has at most N_0 orbits. Then the Reidemeister number $R(\phi)$ is infinite.

Step 3. If $\lambda > 1$, then T has finite arc stabilisers. If $\lambda > 1$ then from work of Bestwina-Feighn [4] it follows that there exists $N_0 \in \mathbb{N}$ such that, for every $Q \in T$, the action of $StabQ$ on $\pi_o(T - (Q))$ has at most N_0 orbits.

(3) Now let S be an oriented, compact surface of genus g and with p boundary components, where $3g + p - 4 > 0$. It is easy to see that the mapping Class Group Mod_S is a normal subgroup of the full mapping class group Mod_S^* , of index 2. The graph of curves of S , denoted $\mathcal{G}(S)$, is the graph whose vertices are the simple curves of S modulo isotopy. Two vertices (that is two isotopy classes of simple curves) are linked by an edge in this graph if they can be realized by disjoint curves. Both Mod_S and Mod_S^* act on $\mathcal{G}(S)$ in a non-elementary way. Now we use the non-elementary result of Masur and Minsky [70] (see also Bowditch [10]) that the complex of curves of an oriented surface (with genus g and p boundary components, and $3g + p - 4 > 0$) is Gromov-hyperbolic space.

Thus Theorem 3.2 is applicable for Mod_S and for φ_1 the automorphism induced by reversing the orientation of S , since in this case, $(Mod_S)_{\varphi_1} = Mod_S \rtimes_{\varphi_1} \mathbb{Z}_2 \simeq Mod_S^*$. For Mod_S and $\varphi_0 = Id$, we have $R(\varphi_0 = Id) = \infty$ because the group Mod_S has infinite an number of usual conjugacy classes. Finally, $Out(Mod_S) \simeq \{\overline{\varphi_0}, \overline{\varphi_1}\}$ (see [57]), which ensures that Mod_S has the R_∞ property if S is an orientable, compact surface of genus g with p boundary components, where $3g + p - 4 > 0$. The only cases not covered by this inequality are: i) S is the torus with at most one hole and ii) S is the sphere with at most 4 holes. The case of the torus with at most one hole follows from the section 3 in [36]. In the case of the sphere with at most 4 holes, this follows directly from the knowledge of $Out(ModS)$ and the cardinality of the mapping class group.

(4) Let $\phi : B_n(S^2) \rightarrow B_n(S^2)$ be an automorphism. Since the center of $B_n(S^2)$ is a characteristic subgroup, ϕ induces a homomorphism of the short exact sequence

$$1 \rightarrow \mathbb{Z}_2 \rightarrow B_n(S^2) \rightarrow \text{Mod}_{S_n^2} \rightarrow 1,$$

where $S_r^2 = S^2 - r$ open disks. This short exact sequence was obtained from the sequence in [6]. The result above implies that the group $\text{Mod}_{S_n^2}$ for $n > 3$, has the R_∞ property. Then the remark about extensions implies that the group $B_n(S^2)$ also has this property. For $n \leq 3$ the groups $B_n(S^2)$ are finite so they do not have the R_∞ property. \square

Relatively hyperbolic groups were introduced by Gromov[49] and since then various characterizations of relatively hyperbolic groups have been obtained. Examples of relatively hyperbolic groups are:

- the free products of finitely many finitely generated groups are hyperbolic relative to the factors.
- geometrically finite isometry groups of Hadamard manifolds of negatively pinched sectional curvature are hyperbolic relative to the maximal parabolic subgroups. This includes complete finite volume manifolds of negatively pinched sectional curvature.
- The amalgamation of relatively hyperbolic groups over parabolic subgroups is relatively hyperbolic, when the parabolic subgroup is maximal in at least one of the factors [Dah03a, Osi06a]
- CAT(0)-groups with isolated flats are hyperbolic relative to the flat stabilizers. Examples of CAT(0)-groups with isolated flats are listed in [HK05].
- Sela's limit groups are hyperbolic relative to non-cyclic maximal abelian subgroups [Dah03a]

3.1. Asymptotic expansions. Suppose we know that the number of the twisted conjugacy classes of a automorphism ϕ of a group G is infinite. The next natural step will be to write an asymptotic for the

number of twisted conjugacy classes with a norm smaller than x . First of all we need to define a norm of a twisted conjugacy class. In present section we realise this approach for pseudo-Anosov homeomorphism of a compact surface.

We assume X to be a compact surface of negative Euler characteristic and $f : X \rightarrow X$ is a pseudo-Anosov homeomorphism, i.e. there is a number $\lambda > 1$ and a pair of transverse measured foliations (F^s, μ^s) and (F^u, μ^u) such that $f(F^s, \mu^s) = (F^s, \frac{1}{\lambda}\mu^s)$ and $f(F^u, \mu^u) = (F^u, \lambda\mu^u)$. The mapping torus T_f of $f : X \rightarrow X$ is the space obtained from $X \times [0, 1]$ by identifying $(x, 1)$ with $(f(x), 0)$ for all $x \in X$. It is often more convenient to regard T_f as the space obtained from $X \times [0, \infty)$ by identifying $(x, s+1)$ with $(f(x), s)$ for all $x \in X, s \in [0, \infty)$. On T_f there is a natural semi-flow $\phi : T_f \times [0, \infty) \rightarrow T_f, \phi_t(x, s) = (x, s+t)$ for all $t \geq 0$. Then the map $f : X \rightarrow X$ is the return map of the semi-flow ϕ . A point $x \in X$ and a positive number $\tau > 0$ determine the orbit curve $\phi_{(x, \tau)} := \phi_t(x)_{0 \leq t \leq \tau}$ in T_f . The fixed points and periodic points of f then correspond to closed orbits of various periods. Take the base point x_0 of X as the base point of T_f . According to van Kampen's Theorem, the fundamental group $G := \pi_1(T_f, x_0)$ is obtained from π by adding a new generator z and adding the relations $z^g z^{-1} = \tilde{f}_*(g)$ for all $g \in \pi = \pi_1(X, x_0)$, where z is the generator of $\pi_1(S^1, x_0)$. This means that G is a semi-direct product $G = \pi \rtimes Z$ of π with Z .

There is a canonical projection $\tau : T_f \rightarrow R/Z$ given by $(x, s) \mapsto s$. This induces a map $\pi_1(\tau) : G = \pi_1(T_f, x_0) \rightarrow Z$.

The Reidemeister number $R(f)$ is equal to the number of homotopy classes of closed paths γ in T_f whose projections onto R/Z are homotopic to the path

$$\begin{aligned} \sigma : [0, 1] &\rightarrow R/Z \\ s &\mapsto s. \end{aligned}$$

Corresponding to this, there is a group-theoretical interpretation of $R(f)$ as the number of usual conjugacy classes of elements $\gamma \in \pi_1(T_f)$ satisfying $\pi_1(\tau)(\gamma) = z$.

Lemma 3.4. [93, 78] *The interior of the mapping torus $\text{Int}(T_f)$ admits a hyperbolic structure of finite volume if and only if f is isotopic to a pseudo-Anosov homeomorphism.*

So, if the surface X is closed and f is isotopic to a pseudo-Anosov homeomorphism, the mapping torus T_f can be realised as a hyperbolic 3-manifold, H^3/G , where H^3 is the Poincaré upper half space $\{(x, y, z) : z > 0, (x, y) \in \mathbb{R}^2\}$ with the metric $ds^2 = (dx^2 + dy^2 + dz^2)/z^2$. The closed geodesics on a hyperbolic manifold are in one-to-one correspondence with the free homotopy classes of loops. These classes of loops are in one-to-one correspondence with the conjugacy classes of loxodromic elements in the fundamental group of the hyperbolic manifold. This correspondence allowed Ch. Epstein (see [20], p.127) to study the asymptotics of such functions as $p_n(x) = \#\{\text{primitive closed geodesics of length less than } x \text{ represented by an element of the form } gz^n\}$ using the Selberg trace formula. A primitive closed geodesic is one which is not an iterate of another closed geodesic. Later, Phillips and Sarnak [80] generalised results of Epstein and obtained for n -dimensional hyperbolic manifold the asymptotic of the number of primitive closed geodesics of length at most x lying in fixed homology class. The proof of this result makes routine use of the Selberg trace formula. In the more general case of variable negative curvature, such asymptotics were obtained by Pollicott and Sharp [81]. They used a dynamical approach based on the geodesic flow. We will only need an asymptotic for $p_1(x)$. Note that closed geodesics represented by an element of the form gz are automatically primitive, because they wrap exactly once around the mapping torus (once around the generator z). We have following asymptotic expansion [20, 80, 81]

$$(3.3) \quad p_1(x) = \frac{e^{hx}}{x^{3/2}} \left(\sum_{n=0}^N \frac{C_n}{x^{n/2}} + o\left(\frac{1}{x^{N/2}}\right) \right),$$

for any $N > 0$, where $h = \dim T_f - 1 = 2$ is the topological entropy of the geodesic flow on the unit-tangent bundle ST_f , and the constant

$C_0 > 0$ depends on the volume of hyperbolic 3-manifold T_f . N. Anantharaman [1] has shown that the constants C_n vanish if n is odd. So, we have the following leading asymptotic behaviour:

$$(3.4) \quad p_1(x) \sim C_0 \frac{e^{hx}}{x^{3/2}}, \text{ as } x \rightarrow \infty$$

Notation. We write $f(x) \sim g(x)$ if $\frac{f(x)}{g(x)} \rightarrow 1$ as $x \rightarrow \infty$.

Now, using the one-to-one correspondences in Lemma 2.1 and Reidemeister bijection in Introduction we define the norm of the lifting class, or of the corresponding twisted conjugacy class $\{g\}_{\tilde{f}_*}$ in the fundamental group of the surface $\pi = \pi_1(X, x_0)$, as the length of the primitive closed geodesic γ on T_f , which is represented by an element of the form gz . So, for example, the norm function l^* on the set of twisted conjugacy classes equals $l^* = l \circ B$, where l is length function on geodesics ($l(\gamma)$ is the length of the primitive closed geodesic γ) and a norm map B is a bijection defined by formula $B(g) = gz$ between the fundamental group of the surface $\pi = \pi_1(X, x_0)$ and the first coset $\pi_1(X, x_0)z$ in the fundamental group $G := \pi_1(T_f, x_0) = \pi \rtimes Z$.

Lemma 3.5. 1) $B(\gamma g \tilde{f}_*((\gamma)^{-1})) = \gamma B(g) \gamma^{-1}$

2) Function $l^* = l \circ B$ is a non negative twisted class function on G

Proof. 1) $B(\gamma g \tilde{f}_*((\gamma)^{-1})) = \gamma g \tilde{f}_*(\gamma^{-1})z = \gamma g z \gamma^{-1} z^{-1} z = \gamma g z \gamma^{-1} = \gamma B(g) \gamma^{-1}$

2) $l^*(\gamma g \tilde{f}_*((\gamma)^{-1})) = l \circ B(\gamma g \tilde{f}_*((\gamma)^{-1})) = l(\gamma B(g) \gamma^{-1}) = l(B(g)) = l^*(g) \quad \square$

In [27] we introduced the following counting functions:

$\text{Tw}(x) = \#\{\text{twisted conjugacy classes for } \tilde{f}_* \text{ in the fundamental group of surface of norm less than } x\}$,

$L(x) = \#\{\text{lifting classes of } f \text{ of norm less than } x\}$. A norm function on the set of liftings of f equals $l^{**} = l^* \circ L$, where l^* is defined as above and $L(\tilde{f}' = \alpha \circ \tilde{f}) = \alpha$ is a function from the set of liftings of f to the group of their coordinates $\pi = \pi_1(X, x_0)$ (see the introduction). The

function l^{**} is a class function on the set of liftings, constant on the liftings classes.

Theorem 3.6. [27] *Let X be a closed surface of negative Euler characteristic and let $f : X \rightarrow X$ be a pseudo-Anosov homeomorphism. Then*

$$L(x) = Tw(x) = \frac{e^{2x}}{x^{3/2}} \left(\sum_{n=0}^N \frac{C_n}{x^{n/2}} + o\left(\frac{1}{x^{N/2}}\right) \right),$$

where the constant $C_0 > 0$ depends on the volume of the hyperbolic 3-manifold T_f , and the constants C_n vanish if n is odd.

Proof. The proof follows from Reidemeister bijection, Lemma 3.4, Lemma 2.1 and the asymptotic expansion (3.1).

Corollary 3.7. For pseudo-Anosov homeomorphisms of closed surfaces the Reidemeister number is infinite.

Question 3.1. How can one define the norm of a twisted conjugacy class in the general case?

4. NIELSEN FIXED POINT THEORY AND SYMPLECTIC FLOER HOMOLOGY

In the dimension two a diffeomorphism is symplectic if it preserves area. As a consequence, the symplectic geometry of surfaces lacks many of the interesting phenomena which are encountered in higher dimensions. For example, two symplectic automorphisms of a closed surface are symplectically isotopic iff they are homotopic, by a theorem of Moser[74]. On other hand symplectic fixed point theory is very nontrivial in dimension 2, as it is shown by the Poincare-Birkhoff theorem. It is known that symplectic Floer homology on surface is a simple model for the instanton Floer homology of the mapping torus of the surface diffeomorphism [86].

4.1. Symplectic Floer homology.

4.1.1. *Monotonicity.* In this section we discuss the notion of monotonicity as defined in [86, 43]. Monotonicity plays important role for Floer homology in two dimensions. Throughout this article, M denotes a closed connected and oriented 2-manifold of genus ≥ 2 . Pick an everywhere positive two-form ω on M .

Let $\phi \in \text{Symp}(M, \omega)$, the group of symplectic automorphisms of the two-dimensional symplectic manifold (M, ω) . The mapping torus of ϕ , $T_\phi = \mathbb{R} \times M / (t+1, x) \sim (t, \phi(x))$, is a 3-manifold fibered over $S^1 = \mathbb{R}/\mathbb{Z}$. There are two natural second cohomology classes on T_ϕ , denoted by $[\omega_\phi]$ and c_ϕ . The first one is represented by the closed two-form ω_ϕ which is induced from the pullback of ω to $\mathbb{R} \times M$. The second is the Euler class of the vector bundle $V_\phi = \mathbb{R} \times TM / (t+1, \xi_x) \sim (t, d\phi_x \xi_x)$, which is of rank 2 and inherits an orientation from TM .

$\phi \in \text{Symp}(M, \omega)$ is called **monotone**, if $[\omega_\phi] = (\text{area}_\omega(M)/\chi(M)) \cdot c_\phi$ in $H^2(T_\phi; \mathbb{R})$; throughout this article $\text{Symp}^m(M, \omega)$ denotes the set of monotone symplectomorphisms.

Now $H^2(T_\phi; \mathbb{R})$ fits into the following short exact sequence [86, 43]

$$(4.1) \quad 0 \longrightarrow \frac{H^1(M; \mathbb{R})}{\text{image}(\text{id} - \phi^*)} \xrightarrow{d} H^2(T_\phi; \mathbb{R}) \xrightarrow{r^*} H^2(M; \mathbb{R}), \longrightarrow 0.$$

where the map r^* is restriction to the fiber. The map d is defined as follows. Let $\rho : I \rightarrow \mathbb{R}$ be a smooth function which vanishes near 0 and 1 and satisfies $\int_0^1 \rho dt = 1$. If θ is a closed 1-form on M , then $\rho \cdot \theta \wedge dt$ defines a closed 2-form on T_ϕ ; indeed $d[\theta] = [\rho \cdot \theta \wedge dt]$. The map $r : M \hookrightarrow T_\phi$ assigns to each $x \in M$ the equivalence class of $(1/2, x)$. Note, that $r^* \omega_\phi = \omega$ and $r^* c_\phi$ is the Euler class of TM . Hence, by (4.1), there exists a unique class $m(\phi) \in H^1(M; \mathbb{R}) / \text{image}(\text{id} - \phi^*)$ satisfying $dm(\phi) = [\omega_\phi] - (\text{area}_\omega(M)/\chi(M)) \cdot c_\phi$, where χ denotes the Euler characteristic. Therefore, ϕ is monotone if and only if $m(\phi) = 0$.

We recall the fundamental properties of $\text{Symp}^m(M, \omega)$ from [86, 43]. Let $\text{Diff}^+(M)$ denotes the group of orientation preserving diffeomorphisms of M .

(Identity) $\text{id}_M \in \text{Symp}^m(M, \omega)$.

(Naturality) If $\phi \in \text{Symp}^m(M, \omega)$, $\psi \in \text{Diff}^+(M)$, then $\psi^{-1}\phi\psi \in \text{Symp}^m(M, \psi^*\omega)$.

(Isotopy) Let $(\psi_t)_{t \in I}$ be an isotopy in $\text{Symp}(M, \omega)$, i.e. a smooth path with $\psi_0 = \text{id}$. Then

$$m(\phi \circ \psi_1) = m(\phi) + [\text{Flux}(\psi_t)_{t \in I}]$$

in $H^1(M; \mathbb{R}) / \text{image}(\text{id} - \phi^*)$; see [86, Lemma 6]. For the definition of the flux homomorphism see [73].

(Inclusion) The inclusion $\text{Symp}^m(M, \omega) \hookrightarrow \text{Diff}^+(M)$ is a homotopy equivalence.

(Floer homology) To every $\phi \in \text{Symp}^m(M, \omega)$ symplectic Floer homology theory assigns a \mathbb{Z}_2 -graded vector space $HF_*(\phi)$ over \mathbb{Z}_2 , with an additional multiplicative structure, called the quantum cap product, $H^*(M; \mathbb{Z}_2) \otimes HF_*(\phi) \longrightarrow HF_*(\phi)$. For $\phi = \text{id}_M$ the symplectic Floer homology $HF_*(\text{id}_M)$ are canonically isomorphic to ordinary homology $H_*(M; \mathbb{Z}_2)$ and quantum cap product agrees with the ordinary cap product. Each $\psi \in \text{Diff}^+(M)$ induces an isomorphism $HF_*(\phi) \cong HF_*(\psi^{-1}\phi\psi)$ of $H^*(M; \mathbb{Z}_2)$ -modules.

(Invariance) If $\phi, \phi' \in \text{Symp}^m(M, \omega)$ are isotopic, then $HF_*(\phi)$ and $HF_*(\phi')$ are naturally isomorphic as $H^*(M; \mathbb{Z}_2)$ -modules.

This is proven in [86, Page 7]. Note that every Hamiltonian perturbation of ϕ (see [15]) is also in $\text{Symp}^m(M, \omega)$.

Now let g be a mapping class of M , i.e. an isotopy class of $\text{Diff}^+(M)$. Pick an area form ω and a representative $\phi \in \text{Symp}^m(M, \omega)$ of g . Then $HF_*(\phi)$ is an invariant of g , which is denoted by $HF_*(g)$. Note that $HF_*(g)$ is independent of the choice of an area form ω by Moser's isotopy theorem [74] and naturality of Floer homology.

4.1.2. *Floer homology.* Let $\phi \in \text{Symp}(M, \omega)$. There are two ways of constructing Floer homology detecting its fixed points, $\text{Fix}(\phi)$. Firstly, the graph of ϕ is a Lagrangian submanifold of $M \times M$, $(-\omega) \times \omega$ and its fixed points correspond to the intersection points of $\text{graph}(\phi)$ with the

diagonal $\Delta = \{(x, x) \in M \times M\}$. Thus we have the Floer homology of the Lagrangian intersection $HF_*(M \times M, \Delta, graph(\phi))$. This intersection is transversal if the fixed points of ϕ are nondegenerate, i.e. if 1 is not an eigenvalue of $d\phi(x)$, for $x \in \text{Fix}(\phi)$. The second approach was mentioned by Floer in [38] and presented with details by Dostoglou and Salamon in [15]. We follow here Seidel's approach [86] which, comparable with [15], uses a larger class of perturbations, but such that the perturbed action form is still cohomologous to the unperturbed. As a consequence, the usual invariance of Floer homology under Hamiltonian isotopies is extended to the stronger property stated above. Let now $\phi \in \text{Symp}^m(M, \omega)$, i.e. ϕ is monotone. Firstly, we give the definition of $HF_*(\phi)$ in the special case where all the fixed points of ϕ are non-degenerate, i.e. for all $y \in \text{Fix}(\phi)$, $\det(\text{id} - d\phi_y) \neq 0$, and then following Seidel's approach [86] we consider general case when ϕ has degenerate fixed points. Let $\Omega_\phi = \{y \in C^\infty(\mathbb{R}, M) \mid y(t) = \phi(y(t+1))\}$ be the twisted free loop space, which is also the space of sections of $T_\phi \rightarrow S^1$. The action form is the closed one-form α_ϕ on Ω_ϕ defined by $\alpha_\phi(y)Y = \int_0^1 \omega(dy/dt, Y(t)) dt$, where $y \in \Omega_\phi$ and $Y \in T_y\Omega_\phi$, i.e. $Y(t) \in T_{y(t)}M$ and $Y(t) = d\phi_{y(t+1)}Y(t+1)$ for all $t \in \mathbb{R}$.

The tangent bundle of any symplectic manifold admits an almost complex structure $J : TM \rightarrow TM$ which is compatible with ω in sense that $(v, w) = \omega(v, Jw)$ defines a Riemannian metric. Let $J = (J_t)_{t \in \mathbb{R}}$ be a smooth path of ω -compatible almost complex structures on M such that $J_{t+1} = \phi^* J_t$. If $Y, Y' \in T_y\Omega_\phi$, then $\int_0^1 \omega(Y'(t), J_t Y(t)) dt$ defines a metric on the loop space Ω_ϕ . So the critical points of α_ω are the constant paths in Ω_ϕ and hence the fixed points of ϕ . The negative gradient lines of α_ω with respect to the metric above are solutions of the partial differential equations with boundary conditions

$$(4.2) \quad \begin{cases} u(s, t) = \phi(u(s, t+1)), \\ \partial_s u + J_t(u) \partial_t u = 0, \\ \lim_{s \rightarrow \pm\infty} u(s, t) \in \text{Fix}(\phi) \end{cases}$$

These are exactly Gromov's pseudoholomorphic curves [48].

For $y^\pm \in \text{Fix}(\phi)$, let $\mathcal{M}(y^-, y^+; J, \phi)$ denote the space of smooth maps $u : \mathbb{R}^2 \rightarrow M$ which satisfy the equations (4.2). Now to every $u \in \mathcal{M}(y^-, y^+; J, \phi)$ we associate a Fredholm operator D_u which linearizes (4.2) in suitable Sobolev spaces. The index of this operator is given by the so called Maslov index $\mu(u)$, which satisfies $\mu(u) = \deg(y^+) - \deg(y^-) \bmod 2$, where $(-1)^{\deg y} = \text{sign}(\det(\text{id} - d\phi_y))$. We have no bubbling, since for surface $\pi_2(M) = 0$.

For a generic J , every $u \in \mathcal{M}(y^-, y^+; J, \phi)$ is regular, meaning that D_u is onto. Hence, by the implicit function theorem, $\mathcal{M}_k(y^-, y^+; J, \phi)$ is a smooth k -dimensional manifold, where $\mathcal{M}_k(y^-, y^+; J, \phi)$ denotes the subset of those $u \in \mathcal{M}(y^-, y^+; J, \phi)$ with $\mu(u) = k \in \mathbb{Z}$. Translation of the s -variable defines a free \mathbb{R} -action on 1-dimensional manifold $\mathcal{M}_1(y^-, y^+; J, \phi)$ and hence the quotient is a discrete set of points.

The energy of a map $u : \mathbb{R}^2 \rightarrow M$ is given by

$$E(u) = \int_{\mathbb{R}} \int_0^1 \omega(\partial_t u(s, t), J_t \partial_t u(s, t)) dt ds$$

for all $y \in \text{Fix}(\phi)$. P.Seidel has proved in [86] that if ϕ is monotone, then the energy is constant on each $\mathcal{M}_k(y^-, y^+; J, \phi)$. Since all fixed points of ϕ are nondegenerate the set $\text{Fix}(\phi)$ is a finite set and the \mathbb{Z}_2 -vector space $CF_*(\phi) := \mathbb{Z}_2^{\#\text{Fix}(\phi)}$ admits a \mathbb{Z}_2 -grading with $(-1)^{\deg y} = \text{sign}(\det(\text{id} - d\phi_y))$, for all $y \in \text{Fix}(\phi)$. The boundness of the energy $E(u)$ for monotone ϕ implies that the 0-dimensional quotients $\mathcal{M}_1(y_-, y_+, J, \phi)/\mathbb{R}$ are actually finite sets. Denoting by $n(y_-, y_+)$ the number of points mod 2 in each of them, one defines a differential $\partial_J : CF_*(\phi) \rightarrow CF_{*+1}(\phi)$ by $\partial_J y_- = \sum_{y_+} n(y_-, y_+) y_+$. Due to gluing theorem this Floer boundary operator satisfies $\partial_J \circ \partial_J = 0$. For gluing theorem to hold one needs again the boundness of the energy $E(u)$. It follows that $(CF_*(\phi), \partial_J)$ is a chain complex and its homology is by definition the Floer homology of ϕ denoted $HF_*(\phi)$. It is independent of J and is an invariant of ϕ .

If ϕ has degenerate fixed points one needs to perturb equations (4.2) in order to define the Floer homology. Equivalently, one could say that

the action form needs to be perturbed. The necessary analysis is given in [86], it is essentially the same as in the slightly different situations considered in [15]. But Seidel's approach also differs from the usual one in [15]. He uses a larger class of perturbations, but such that the perturbed action form is still cohomologous to the unperturbed.

4.2. Nielsen numbers and Floer homology.

4.2.1. Periodic diffeomorphisms.

Lemma 4.1. [59] *Let ϕ a non-trivial orientation preserving periodic diffeomorphism of a compact connected surface M of Euler characteristic $\chi(M) \leq 0$. Then each fixed point class of ϕ consists of a single point.*

There are two criteria for monotonicity which we use later on. Let ω be an area form on M and $\phi \in \text{Symp}(\Sigma, \omega)$.

Lemma 4.2. [43] *Assume that every class $\alpha \in \ker(\text{id} - \phi_*) \subset H_1(M; \mathbb{Z})$ is represented by a map $\gamma : S \rightarrow \text{Fix}(\phi)$, where S is a compact oriented 1-manifold. Then ϕ is monotone.*

Lemma 4.3. [43] *If ϕ^k is monotone for some $k > 0$, then ϕ is monotone. If ϕ is monotone, then ϕ^k is monotone for all $k > 0$.*

We shall say that $\phi : M \rightarrow M$ is a periodic map of period m , if ϕ^m is the identity map $\text{id}_M : M \rightarrow M$.

Theorem 4.4. [37] *If ϕ is a non-trivial orientation preserving periodic diffeomorphism of a compact connected surface M of Euler characteristic $\chi(M) < 0$, then ϕ is monotone with respect to some ϕ -invariant area form and*

$$HF_*(\phi) \cong \mathbb{Z}_2^{N(\phi)}, \quad \dim HF_*(\phi) = N(\phi),$$

where $N(\phi)$ denotes the Nielsen number of ϕ .

Proof. Let ϕ be a periodic diffeomorphism of least period l . First note that if $\tilde{\omega}$ is an area form on M , then area form $\omega := \sum_{i=1}^l (\phi^i)^* \tilde{\omega}$ is ϕ -invariant, i.e. $\phi \in \text{Symp}(M, \omega)$. By periodicity of ϕ , ϕ^l is the identity map $\text{id}_M : M \rightarrow M$. Then from Lemmas 4.2 and 4.3 it follows that ω can be chosen such that $\phi \in \text{Symp}^m(M, \omega)$.

Lemma 4.1 implies that every $y \in \text{Fix}(\phi)$ forms a different fixed point class of ϕ , so $\#\text{Fix}(\phi) = N(\phi)$. This has an immediate consequence for the Floer complex $(CF_*(\phi), \partial_J)$ with respect to a generic $J = (J_t)_{t \in \mathbb{R}}$. If $y^\pm \in \text{Fix}(\phi)$ are in different fixed point classes, then $\mathcal{M}(y^-, y^+; J, \phi) = \emptyset$. This follows from the first equation in (4.2). Then the boundary map in the Floer complex is zero $\partial_J = 0$ and \mathbb{Z}_2 -vector space $CF_*(\phi) := \mathbb{Z}_2^{\#\text{Fix}(\phi)} = \mathbb{Z}_2^{N(\phi)}$. This immediately implies $HF_*(\phi) \cong \mathbb{Z}_2^{N(\phi)}$ and $\dim HF_*(\phi) = N(\phi)$. □

4.2.2. Algebraically finite mapping classes. A mapping class of M is called algebraically finite if it does not have any pseudo-Anosov components in the sense of Thurston's theory of surface diffeomorphism. The term algebraically finite goes back to J. Nielsen.

In [43] the diffeomorphisms of finite type were defined. These are special representatives of algebraically finite mapping classes adopted to the symplectic geometry.

Definition 4.1. [43] We call $\phi \in \text{Diff}_+(M)$ of **finite type** if the following holds. There is a ϕ -invariant finite union $N \subset M$ of disjoint non-contractible annuli such that:

(1) $\phi|_{M \setminus N}$ is periodic, i.e. there exists $\ell > 0$ such that $\phi^\ell|_{M \setminus N} = \text{id}$.

(2) Let N' be a connected component of N and $\ell' > 0$ be the smallest integer such that $\phi^{\ell'}$ maps N' to itself. Then $\phi^{\ell'}|_{N'}$ is given by one of the following two models with respect to some coordinates $(q, p) \in I \times S^1$:

$$\text{(twist map)} \quad (q, p) \longmapsto (q, p - f(q))$$

(flip-twist map) $(q, p) \mapsto (1 - q, -p - f(q)),$

where $f : I \rightarrow \mathbb{R}$ is smooth and strictly monotone. A twist map is called positive or negative, if f is increasing or decreasing.

(3) Let N' and ℓ' be as in (2). If $\ell' = 1$ and $\phi|_{N'}$ is a twist map, then $\text{image}(f) \subset [0, 1]$, i.e. $\phi|_{\text{int}(N')}$ has no fixed points.

(4) If two connected components of N are homotopic, then the corresponding local models of ϕ are either both positive or both negative twists.

The term flip-twist map is taken from [61].

By M_{id} we denote the union of the components of $M \setminus \text{int}(N)$, where ϕ restricts to the identity.

The next lemma describes the set of fixed point classes of ϕ . It is a special case of a theorem by B. Jiang and J. Guo [61], which gives for any mapping class a representative that realizes its Nielsen number.

Lemma 4.5 (Fixed point classes). [61] *Each fixed point class of ϕ is either a connected component of M_{id} or consists of a single fixed point. A fixed point x of the second type satisfies $\det(\text{id} - d\phi_x) > 0$.*

The monotonicity of diffeomorphisms of finite type was investigated in details in [43]. Let ϕ be a diffeomorphism of finite type and ℓ be as in (1). Then ϕ^ℓ is the product of (multiple) Dehn twists along N . Moreover, two parallel Dehn twists have the same sign, by (4). We say that ϕ has uniform twists, if ϕ^ℓ is the product of only positive, or only negative Dehn twists.

Furthermore, we denote by ℓ the smallest positive integer such that ϕ^ℓ restricts to the identity on $M \setminus N$.

If ω' is an area form on M which is the standard form $dq \wedge dp$ with respect to the (q, p) -coordinates on N , then $\omega := \sum_{i=1}^{\ell} (\phi^i)^* \omega'$ is standard on N and ϕ -invariant, i.e. $\phi \in \text{Symp}(M, \omega)$. To prove that ω can be chosen such that $\phi \in \text{Symp}^m(M, \omega)$, Gaultschi distinguishes

two cases: uniform and non-uniform twists. In the first case he proves the following stronger statement.

Lemma 4.6. [43] *If ϕ has uniform twists and ω is a ϕ -invariant area form, then $\phi \in \text{Symp}^m(M, \omega)$.*

In the non-uniform case, monotonicity does not hold for arbitrary ϕ -invariant area forms.

Lemma 4.7. [43] *If ϕ does not have uniform twists, there exists a ϕ -invariant area form ω such that $\phi \in \text{Symp}^m(M, \omega)$. Moreover, ω can be chosen such that it is the standard form $dq \wedge dp$ on N .*

Theorem 4.8. [37] *If ϕ is a diffeomorphism of finite type of a compact connected surface M of Euler characteristic $\chi(M) < 0$ and if ϕ has only isolated fixed points, then ϕ is monotone with respect to some ϕ -invariant area form and*

$$HF_*(\phi) \cong \mathbb{Z}_2^{N(\phi)}, \quad \dim HF_*(\phi) = N(\phi),$$

where $N(\phi)$ denotes the Nielsen number of ϕ .

Proof. From Lemmas 4.6 and 4.7 it follows that ω can be chosen such that $\phi \in \text{Symp}^m(M, \omega)$. Lemma 4.5 implies that every $y \in \text{Fix}(\phi)$ forms a different fixed point class of ϕ , so $\#\text{Fix}(\phi) = N(\phi)$. This has an immediate consequence for the Floer complex $(CF_*(\phi), \partial_J)$ with respect to a generic $J = (J_t)_{t \in \mathbb{R}}$. If $y^\pm \in \text{Fix}(\phi)$ are in different fixed point classes, then $\mathcal{M}(y^-, y^+; J, \phi) = \emptyset$. This follows from the first equation in (4.2). Then the boundary map in the Floer complex is zero $\partial_J = 0$ and \mathbb{Z}_2 -vector space $CF_*(\phi) := \mathbb{Z}_2^{\#\text{Fix}(\phi)} = \mathbb{Z}_2^{N(\phi)}$. This immediately implies $HF_*(\phi) \cong \mathbb{Z}_2^{N(\phi)}$ and $\dim HF_*(\phi) = N(\phi)$. □

Theorem 4.9. [43] *Let ϕ be a diffeomorphism of finite type, then ϕ is monotone with respect to some ϕ -invariant area form and*

$$HF_*(\phi) = H_*(M_{\text{id}}, \partial_{M_{\text{id}}}; \mathbb{Z}_2) \oplus \mathbb{Z}_2^{L(\phi|M \setminus M_{\text{id}})}.$$

Here, L denotes the Lefschetz number.

Proof. The main idea of the proof is a separation mechanism for Floer connecting orbits. Together with the topological separation of fixed points discussed in theorem 4.8 , it allows us to compute the Floer homology of diffeomorphisms of finite type.

There exists a function $H : M \rightarrow \mathbb{R}$ such that $H|_{\text{int}(M_{id})}$ is a Morse function, meaning that all the critical points are non-degenerate and $H|M \setminus M_{id} = 0$. Let $(\psi_t)_{t \in \mathbb{R}}$ denote the Hamiltonian flow generated by H with respect to the fixed area form ω and set $\Phi := \phi \circ \psi_1$. Then $\text{Fix}(\Phi) = (\text{crit}(H) \cap M_{id}) \cup (\text{Fix}(\phi) \setminus M_{id})$. In particular, Φ only has non-degenerate fixed points. Let $N_0 \subset M_{id}$ be a collar neighborhood of ∂M_{id} . Let $x^-, x^+ \in \text{Fix}(\Phi) \cap M_{id}$ be in the same connected component of M_{id} . If $u \in \mathcal{M}(x^-, x^+; J, \Phi)$, then image $u \subset \Sigma_\delta$, where Σ_δ denotes the δ -neighborhood of $M_{id} \setminus N_0$ with respect to any of the metrics $\omega(\cdot, J_t)$ [86, 43]

Moreover, lemma 4.5 implies that every $y \in \text{Fix}(\phi) \setminus M_{id}$ forms a different fixed point class of Φ . This has an immediate consequence for the Floer complex $(CF_*(\Phi), \partial_J)$ with respect to a generic $J = (J_t)_{t \in \mathbb{R}}$. Namely, $(CF_*(\Phi), \partial_J)$ splits into the subcomplexes $(\mathcal{C}_1, \partial_1)$ and $(\mathcal{C}_2, \partial_2)$, where \mathcal{C}_1 is generated by $\text{crit}(H) \cap M_{id}$ and \mathcal{C}_2 by $\text{Fix}(\phi) \setminus M_{id}$. Moreover, \mathcal{C}_2 is graded by 0 and $\partial_2 = 0$ [43]. The homology of $(\mathcal{C}_1, \partial_1)$ is isomorphic to $H_*(M_{id}, \partial_+ M_{id}; \mathbb{Z}_2)$ [86, 43]. So $HF_*(\phi) \cong H_*(M_{id}, \partial_+ \Sigma_0; \mathbb{Z}_2) \oplus \mathbb{Z}_2^{\#\text{Fix}(\phi)|M \setminus M_{id}}$. Since every fixed point of $\phi|M \setminus M_{id}$ has fixed point index 1, the Lefschetz fixed point formula implies that $\#\text{Fix}(\phi) \setminus M_{id} = \Lambda(\phi|M \setminus M_{id})$.

□

Remark 4.2. In the theorem 4.8 the set M_{id} is empty and every fixed point of ϕ has fixed point index 1 [61]. The Lefschetz fixed point formula implies that $\#\text{Fix}(\phi) = N(\phi) = L(\phi)$. So, theorem 4.8 follows also from theorem 4.9.

4.3. Symplectic zeta functions and asymptotic invariant.

4.3.1. *Symplectic zeta functions.* Let $\Gamma = \pi_0(\text{Diff}^+(M))$ be the mapping class group of a closed connected oriented surface M of genus ≥ 2 . Pick an everywhere positive two-form ω on M . A isotopy theorem of Moser [74] says that each mapping class of $g \in \Gamma$, i.e. an isotopy class of $\text{Diff}^+(M)$, admits representatives which preserve ω . Due to Seidel[86] we can pick a monotone representative $\phi \in \text{Symp}^m(M, \omega)$ of g . Then $HF_*(\phi)$ is an invariant of g , which is denoted by $HF_*(g)$. Note that $HF_*(g)$ is independent of the choice of an area form ω by Moser's theorem and naturality of Floer homology. By lemma 4.3 symplectomorphisms ϕ^n are also monotone for all $n > 0$. Taking a dynamical point of view, we consider the iterates of monotone symplectomorphism ϕ and define the first symplectic zeta function of ϕ [37] as the following power series: $\chi_\phi(z) = \exp\left(\sum_{n=1}^{\infty} \frac{\chi(HF_*(\phi^n))}{n} z^n\right)$, where $\chi(HF_*(\phi^n))$ is the Euler characteristic of Floer homology group of ϕ^n . Then $\chi_\phi(z)$ is an invariant of g , which we denote by $\chi_g(z)$. Let us consider the Lefschetz zeta function $L_\phi(z) := \exp\left(\sum_{n=1}^{\infty} \frac{L(\phi^n)}{n} z^n\right)$, where $L(\phi^n) := \sum_{k=0}^{\dim X} (-1)^k \text{tr} \left[\phi_{*k}^n : H_k(M; \mathbf{Q}) \rightarrow H_k(M; \mathbf{Q}) \right]$ is the Lefschetz number of ϕ^n .

Theorem 4.10. [37] *Symplectic zeta function $\chi_\phi(z)$ is a rational function of z and*

$$\chi_\phi(z) = L_\phi(z) = \prod_{k=0}^{\dim X} \det(I - \phi_{*k} \cdot z)^{(-1)^{k+1}}.$$

Proof. If for every n all the fixed points of ϕ^n are non-degenerate, i.e. for all $x \in \text{Fix}(\phi^n)$, $\det(\text{id} - d\phi^n(x)) \neq 0$, then we have (see section 4): $\chi(HF_*(\phi^n)) = \sum_{x=\phi^n(x)} \text{sign}(\det(\text{id} - d\phi^n(x))) = L(\phi^n)$. If we have degenerate fixed points one needs to perturb equations (4.2) in order to define the Floer homology. The necessary analysis is given in [86] is essentially the same as in the slightly different situations considered in [15], where the above connection between the Euler characteristic and the Lefschetz number was firstly established. \square

In [37] we have defined the second symplectic zeta function for monotone symplectomorphism ϕ as the following power series: $F_\phi(z) = \exp\left(\sum_{n=1}^{\infty} \frac{\dim HF_*(\phi^n)}{n} z^n\right)$. Then $F_\phi(z)$ is an invariant of mapping class g , which we denote by $F_g(z)$.

Motivation for this definition was the theorem 4.4 and nice analytical properties of the Nielsen zeta function $N_\phi(z) = \exp\left(\sum_{n=1}^{\infty} \frac{N(\phi^n)}{n} z^n\right)$, see [22]. We denote the numbers $\dim HF_*(\phi^n)$ by N_n . Let $\mu(d), d \in \mathbb{N}$, be the Möbius function.

Theorem 4.11. [37] *Let ϕ be a non-trivial orientation preserving periodic diffeomorphism of least period m of a compact connected surface M of Euler characteristic $\chi(M) < 0$. Then the zeta function $F_\phi(z)$ is a radical of a rational function and $F_\phi(z) = \prod_{d|m} \sqrt[d]{(1-z^d)^{-P(d)}}$, where the product is taken over all divisors d of the period m , and $P(d)$ is the integer $P(d) = \sum_{d_1|d} \mu(d_1)N_{d/d_1}$.*

Remark 4.3. Given a symplectomorphism ϕ of surface M , one can form the symplectic mapping torus $M_\phi^4 = T_\phi^3 \times S^1$, where T_ϕ^3 is usual mapping torus (see section 3.1). Ionel and Parker [53] have computed the degree zero Gromov invariants (these are built from the invariants of Ruan and Tian) of M_ϕ^4 and of fiber sums of the M_ϕ^4 with other symplectic manifolds. This is done by expressing the Gromov invariants in terms of the Lefschetz zeta function $L_\phi(z)$. The result is a large set of interesting non-Kähler symplectic manifolds with computational ways of distinguishing them. In dimension four this gives a symplectic construction of the exotic elliptic surfaces of Fintushel and Stern. In higher dimensions it gives many examples of manifolds which are diffeomorphic but not equivalent as symplectic manifolds. Theorem 4.10 implies that the Gromov invariants of M_ϕ^4 are related to symplectic Floer homology of ϕ via zeta function $\chi_\phi(z) = L_\phi(z)$. We hope that the second symplectic zeta function $F_\phi(z)$ give rise to a new invariant of symplectic 4-manifolds.

4.3.2. *Topological entropy and the Nielsen numbers.* A basic relation between Nielsen numbers and topological entropy $h(f)$ was found by N. Ivanov [56]. We present here a very short proof of Jiang of the Ivanov's inequality.

Lemma 4.12. [56]

$$h(f) \geq \limsup_n \frac{1}{n} \cdot \log N(f^n)$$

Proof Let δ be such that every loop in X of diameter $< 2\delta$ is contractible. Let $\epsilon > 0$ be a smaller number such that $d(f(x), f(y)) < \delta$ whenever $d(x, y) < 2\epsilon$. Let $E_n \subset X$ be a set consisting of one point from each essential fixed point class of f^n . Thus $|E_n| = N(f^n)$. By the definition of $h(f)$, it suffices to show that E_n is (n, ϵ) -separated. Suppose it is not so. Then there would be two points $x \neq y \in E_n$ such that $d(f^i(x), f^i(y)) \leq \epsilon$ for $0 \leq i < n$ hence for all $i \geq 0$. Pick a path c_i from $f^i(x)$ to $f^i(y)$ of diameter $< 2\epsilon$ for $0 \leq i < n$ and let $c_n = c_0$. By the choice of δ and ϵ , $f \circ c_i \simeq c_{i+1}$ for all i , so $f^n \circ c_0 \simeq c_n = c_0$. This means x, y in the same fixed point class of f^n , contradicting the construction of E_n .

This inequality is remarkable in that it does not require smoothness of the map and provides a common lower bound for the topological entropy of all maps in a homotopy class.

We recall Thurston classification theorem for homeomorphisms of surface M of genus ≥ 2 .

Theorem 4.13. [94] *Every homeomorphism $\phi : M \rightarrow M$ is isotopic to a homeomorphism f such that either*

- (1) f is a periodic map; or
- (2) f is a pseudo-Anosov map, i.e. there is a number $\lambda > 1$ (stretching factor) and a pair of transverse measured foliations (F^s, μ^s) and (F^u, μ^u) such that $f(F^s, \mu^s) = (F^s, \frac{1}{\lambda}\mu^s)$ and $f(F^u, \mu^u) = (F^u, \lambda\mu^u)$; or
- (3) f is reducible map, i.e. there is a system of disjoint simple closed curves $\gamma = \{\gamma_1, \dots, \gamma_k\}$ in $\text{int}M$ such that γ is invariant by f (but γ_i

may be permuted) and γ has a f -invariant tubular neighborhood U such that each component of $M \setminus U$ has negative Euler characteristic and on each (not necessarily connected) f -component of $M \setminus U$, f satisfies (1) or (2).

The map f above is called the Thurston canonical form of f . In (3) it can be chosen so that some iterate f^m is a generalised Dehn twist on U . Such a f , as well as the f in (1) or (2), will be called standard. A key observation is that if f is standard, so are all iterates of f .

Lemma 4.14. [21] *Let f be a pseudo-Anosov homeomorphism with stretching factor $\lambda > 1$ of surface M of genus ≥ 2 . Then*

$$h(f) = \log(\lambda) = \limsup_n \frac{1}{n} \cdot \log N(f^n)$$

Lemma 4.15. [60] *Suppose f is a standard homeomorphism of surface M of genus ≥ 2 and λ is the largest stretching factor of the pseudo-Anosov pieces ($\lambda = 1$ if there is no pseudo-Anosov piece). Then*

$$h(f) = \log(\lambda) = \limsup_n \frac{1}{n} \cdot \log N(f^n)$$

4.3.3. *Asymptotic invariant.* The growth rate of a sequence a_n of complex numbers is defined by

$$\text{Growth}(a_n) := \max\{1, \limsup_{n \rightarrow \infty} |a_n|^{1/n}\}$$

which could be infinity. Note that $\text{Growth}(a_n) \geq 1$ even if all $a_n = 0$. When $\text{Growth}(a_n) > 1$, we say that the sequence a_n grows exponentially.

Definition 4.4. We define the asymptotic invariant $F^\infty(g)$ of mapping class $g \in \Gamma = \pi_0(\text{Diff}^+(M))$ to be the growth rate of the sequence $\{a_n = \dim HF_*(\phi^n)\}$ for a monotone representative $\phi \in \text{Symp}^m(M, \omega)$ of g :

$$F^\infty(g) := \text{Growth}(\dim HF_*(\phi^n))$$

Example 4.5. If ϕ is a non-trivial orientation preserving periodic diffeomorphism of a compact connected surface M of Euler characteristic $\chi(M) < 0$, then the periodicity of the sequence $\dim HF_*(\phi^n)$ implies that for the corresponding mapping class g the asymptotic invariant

$$F^\infty(g) := \text{Growth}(\dim HF_*(\phi^n)) = 1$$

Example 4.6. Let ϕ be a monotone diffeomorphism of finite type of a compact connected surface M of Euler characteristic $\chi(M) < 0$ and g a corresponding algebraically finite mapping class. Let U be the open regular neighborhood of the k reducing curves $\gamma_1, \dots, \gamma_k$ in the Thurston theorem, and M_j be the component of $M \setminus U$. Let F be a fixed point class of ϕ . Observe from [61] that if $F \subset M_j$, then $\text{ind}(F, \phi) = \text{ind}(F, \phi_j)$. So if F is counted in $N(\phi)$ but not counted in $\sum_j N(\phi_j)$, it must intersect U . But we see from [61] that a component of U can intersect at most 2 essential fixed point classes of ϕ . Hence we have $N(\phi) \leq \sum_j N(\phi_j)$. For the monotone diffeomorphism of finite type ϕ maps ϕ_j are periodic. Applying last inequality to ϕ^n and using remark 4.9 we have

$$\begin{aligned} 0 \leq \dim HF_*(\phi^n) &= \dim H_*(M_{\text{id}}^{(n)}, \partial M_{\text{id}}^{(n)}; \mathbb{Z}_2) + N(\phi^n | M \setminus M_{\text{id}}^{(n)}) \leq \\ &\leq \dim H_*(M_{\text{id}}^{(n)}, \partial M_{\text{id}}^{(n)}; \mathbb{Z}_2) + N(\phi^n) \\ &\leq \dim H_*(M_{\text{id}}^{(n)}, \partial M_{\text{id}}^{(n)}; \mathbb{Z}_2) + \sum_j N((\phi_j)^n) + 2k \leq \text{Const} \end{aligned}$$

by periodicity of ϕ_j . Taking the growth rate in n , we get that asymptotic invariant $F^\infty(g) = 1$.

4.4. Generalised Arnold conjecture. Concluding remarks . Let $\phi : M \rightarrow M$ be a Hamiltonian symplectomorphism of a compact symplectic manifold (M, ω) . In the nondegenerate case the Arnold conjecture asserts that

$$\# \text{Fix}(\phi) \geq \dim H_*(M, \mathbf{Q}) = \sum_{k=0}^{2n} b_k(M),$$

where $2n = \dim M$, $b_k(M) = \dim H_k(M, \mathbf{Q})$.

The Arnold conjecture was first proved by Eliashberg [19] for Riemann surfaces. For tori of arbitrary dimension the Arnold conjecture was proved in the celebrated paper by Conley and Zehnder[11]. The most important breakthrough was Floer's proof of the Arnold conjecture for monotone symplectic manifolds [39]. His proof was based on Floer homology. His method has been pushed through by Fukaya-Ono[42], Liu-Tian[66] and Hofer-Salamon[52] to establish the nondegenerate case of the Arnold conjecture for all symplectic manifolds.

The Hamiltonian symplectomorphism ϕ is isotopic to identity map id_M . In this case all fixed points ϕ are in the same Nielsen fixed point class. The Nielsen number of ϕ is 0 or 1 depending on Lefschetz number is 0 or not. So, the Nielsen number is very weak invariant to estimate the number of fixed points of ϕ for Hamiltonian symplectomorphism. From another side, as we saw in theorem 4.4, for the nontrivial periodic symplectomorphism ϕ of a surface, the Nielsen number of ϕ gives an exact estimation from below for the number of nondegenerate fixed points of ϕ . These considerations lead us to the following question

Question 4.7. How to estimate the number of nondegenerate fixed points of general(not necessary Hamiltonian) symplectomorphism?

4.4.1. *Algebraically finite mapping class.* If ψ is a diffeomorphism of finite type of surface M then $\psi \in \text{Symp}^m(M, \omega)$ for some ψ -invariant form ω . Suppose that symplectomorphism ϕ has only non-degenerate fixed points and ϕ is Hamiltonian isotopic to ψ . Then $\phi \in \text{Symp}^m(M, \omega)$ and $HF_*(\phi)$ is isomorphic to $HF_*(\psi)$. So, from theorem 4.9 it follows that

$$\begin{aligned} \#Fix(\phi) &\geq \dim HF_*(\phi) = \dim HF_*(\psi) = \\ &= \dim H_*(M_{\psi=id}, \partial M_{\psi=id}; \mathbb{Z}_2) + N(\psi|M \setminus M_{\psi=id}) = \\ &= \sum_{k=0}^2 b_k(M_{\psi=id}, \partial M_{\psi=id}; \mathbb{Z}_2) + N(\psi|M \setminus M_{\psi=id}) \end{aligned}$$

This estimation can be considered as a generalisation of Arnold conjecture because it implies Arnold conjecture for $\psi = id$. If ψ is nontrivial orientation preserving periodic diffeomorphism then theorem 4.4 implies an estimation

$$\#Fix(\phi) \geq \dim HF_*(\phi) = \dim HF_*(\psi) = N(\psi)$$

This estimation can be considered as a generalisation of Arnold conjecture for a nontrivial periodic mapping class.

4.4.2. *Pseudo-Anosov mapping class.* For pseudo-Anosov “diffeomorphism” in given pseudo-Anosov mapping class we also have, as in theorems 4.4, 4.8 and 4.9, a topological separation of fixed points [94, 61, 55], i.e the Nielsen number of pseudo-Anosov “diffeomorphism” equals to the number of fixed points and there are no connecting orbits between them. But we have the following difficulties. Firstly, a pseudo-Anosov “diffeomorphism” is a smooth and a symplectic automorphism only on the complement of his fixed points set. Nevertheless, M. Gerber and A. Katok [45] have found a smooth model for pseudo-Anosov “diffeomorphism” with the same dynamical properties. More precise they have constructed for every pseudo-Anosov “diffeomorphism” f a diffeomorphism f' which is topologically conjugate to f through a homeomorphism isotopic to identity. Diffeomorphism f' is a symplectomorphism, it has the same fixed points as f , which are also topologically separated, and it has the same Nielsen number as f . Secondly, in the case of a pseudo-Anosov “diffeomorphism” and its smooth model, we have to deal with fixed points of index $-p$ where $p > 1$. Such fixed points are degenerate from symplectic point of view and therefore need a local perturbation.

If ϕ is monotone symplectomorphism with nondegenerate fixed points in given pseudo-Anosov mapping class $\{\phi\} = g$ then

$$\#Fix(\phi) \geq \dim HF_*(\phi) = \dim HF_*(g)$$

To formulate the generalised Arnold conjecture in this case we need to know how to calculate in classical terms the Floer homology for monotone symplectomorphism with nondegenerate fixed points which represents given pseudo-Anosov mapping class g . For this we need, for example, to understand the contribution of degenerate fixed points of a smooth model of a pseudo-Anosov “diffeomorphism” to the Floer homology. In [18] Floer homology were calculated for certain class of pseudo-Anosov maps which are compositions of positive and negative Dehn twists along loops in M forming a tree-pattern.

4.4.3. *Reducible mapping class. Generalised Arnold conjecture.* Suppose now that symplectomorphism ϕ has only non-degenerate fixed points and that ϕ is Hamiltonian isotopic to a monotone symplectomorphism ψ in a reducible mapping class g which contains pseudo-Anosov components $g_{pA}^i, i = 1, \dots, s$ (see theorem 4.13).

Conjecture 4.16. The Floer complex $(CF_*(\psi), \partial_J)$ with respect to a generic $J = (J_t)_{t \in \mathbb{R}}$ splits into the subcomplexes $(\mathcal{C}_{fin}, \partial_{fin})$ and $(\mathcal{C}_{pA}^i, \partial_{pA}^i), i = 1, \dots, s$, where \mathcal{C}_{fin} corresponds to the finite type component g_{fin} of g and $\mathcal{C}_{pA}^i, i = 1, \dots, s$ correspond to pseudo-Anosov components $g_{pA}^i, i = 1, \dots, s$ of g . The homology of $(\mathcal{C}_{fin}, \partial_{fin})$ is isomorphic to $HF_*(g_{fin}) = H_*(M_{\psi=id}, \partial M_{\psi=id}; \mathbb{Z}_2) \oplus \mathbb{Z}_2^{N(\psi|M \setminus M_{\psi=id})}$ by theorem 4.8 and

$$\begin{aligned}
\#Fix(\phi) &\geq \dim HF_*(\phi) = \dim HF_*(\psi) = \dim HF_*(g) = \\
&= \dim HF_*(g_{fin}) + \sum_{i=1}^s \dim HF_*(g_{pA}^i) = \\
&= \dim H_*(M_{\psi=id}, \partial M_{\psi=id}; \mathbb{Z}_2) + N(\psi|M \setminus M_{\psi=id}) + \sum_{i=1}^s \dim HF_*(g_{pA}^i) = \\
&= \sum_{k=0}^2 b_k(M_{\psi=id}, \partial M_{\psi=id}; \mathbb{Z}_2) + N(\psi|M \setminus M_{\psi=id}) + \sum_{i=1}^s \dim HF_*(g_{pA}^i)
\end{aligned}$$

4.4.4. *Concluding remarks.* Due to P. Seidel [87] $\dim HF_*(\phi)$ is a new symplectic invariant of a four-dimensional symplectic manifold with nonzero first Betti number. This 4-manifold produced from symplectomorphism ϕ by a surgery construction which is a variation of earlier constructions due to McMullen-Taubes, Fintushel-Stern and J. Smith. We hope that our asymptotic invariant and symplectic zeta function $F_\phi(z)$ also give rise to a new invariants of contact 3- manifolds and symplectic 4-manifolds.

REFERENCES

1. N. Anantharaman, Precise counting results for closed orbits of Anosov flows. *Ann.Sci. Ecole Norm. Sup. (4)* 33(2000), no.1, 33-56.
2. J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Princeton University Press, Princeton, NJ, 1989. MR90m:22041
3. Valerij Bardakov, Leonid Bokut, and Andrei Vesnin, *Twisted conjugacy in free groups and Makanin's question*, Southeast Asian Bull. Math. **29** (2005), no. 2, 209–226, (E-print arxiv:math.GR/0401349).
4. M. Bestvina, M. Feighn, A combination theorem for negatively curved groups, *Jour. Diff. Geom.* 35(1992), 85-101.
5. I.Belegradek, A. Szczepanski, Endomorphisms of relatively hyperbolic groups. Arxiv: math/0501321, 2005.
6. J. S. Birman, Braids, links and mapping class groups, *Ann. Math. Stud.* **82**, Princeton University Press, 1974.
7. C. Bleak, A. Fel'shtyn, and D. L. Gonçalves, Twisted conjugacy classes in R. Thompson's group F, *E-print arXiv:math.GR/0704.3441*, 2007
8. O. Bogopolski, A. Martino, O. Maslakova, E. Ventura, Free-by-cyclic groups have solvable conjugacy problem, *Bulletin of the London Mathematical Society* 38(5)(2006), 787-794.
9. O. Bogopolski, A. Martino, O. Maslakova, E. Ventura, Orbit decidability and the conjugacy problem for some extensions of groups, *E-print arXiv: 0712.3104*, 2007
10. B. Bowditch, Intersection numbers and the hyperbolicity of the curve complex, *J. Reine Angew.Math.* 598(2006), 105-129.
11. C.Conley and E. Zehnder, The Birkhoff-Lewis fixed point theorem and a conjecture of V.I. Arnold. *Inventiones Mathematicae*, 73(1983), 33-49.

12. Charles W. Curtis and Irving Reiner, *Representation theory of finite groups and associative algebras*, Pure and Applied Mathematics, Vol. XI, Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.
- Dah03a. F. Dahmani, *Combination of convergence groups*, *Geom. Topol.* **7** (2003), 933–963 (electronic).
13. M. Dehn, *Über unendliche diskontinuierliche Gruppen*, *Math. Ann.* **71** (1911), no. 1, 116–144.
14. J. Dixmier, *C*-algebras*, North-Holland, Amsterdam, 1982.
15. S. Dostoglou and D. Salamon. Self dual instantons and holomorphic curves, *Annals of Math.*, 139 (1994), 581–640.
16. A. Dold, Fixed point indices of iterated maps, *Inventiones Math.* 74 (1983), 419-435.
17. C Druţu, Relatively hyperbolic groups: geometry and quasi-isometric invariance. E-print arXiv: math.GR/0605211.
18. E. Eftekhary, Floer cohomology of certain pseudo-Anosov maps, *J. Symplectic Geom.*2(2004), no.3, 357-375.
19. Ya. Eliashberg, Estimates on the number of fixed points of area preserving transformations. Syktyvkar University Preprint.
20. C. Epstein, The spectral theory of geometrically periodic hyperbolic 3-manifolds. *Memoirs of the AMS*, vol. 58, number 335, 1985.
21. A. Fathi, F. Laudenbach, and V. Poénaru, *Travaux de Thurston sur les surfaces*, Astérisque, vol. 66–67, Soc. Math. France, 1979.
22. A. Fel'shtyn, *Dynamical zeta functions*, *Nielsen theory and Reidemeister torsion*, *Mem. Amer. Math. Soc.* **147** (2000), no. 699, xii+146. MR2001a:37031
23. A. L. Fel'shtyn, The Reidemeister number of any automorphism of a Gromov hyperbolic group is infinite, *Zapiski Nauchnykh Seminarov POMI* **279** (2001), 229–241.
24. A. Fel'shtyn and D. Gonçalves, *Reidemeister numbers of any automorphism of Baumslag-Solitar group*, is infinite. *Geometry and Dynamics of Groups and Spaces*, *Progress in Mathematics*, v. 265(2008), 286-306, Birkhauser.
25. A. L. Fel'shtyn and R. Hill, The Reidemeister zeta function with applications to Nielsen theory and a connection with Reidemeister torsion, *K-theory* **8** no.4 (1994), 367–393.
26. A. Fel'shtyn, R. Hill, and P. Wong, *Reidemeister numbers of equivariant maps*, *Topology Appl.* **67** (1995), no. 2, 119–131.

27. Fel'shtyn A.L. Dynamical zeta functions and asymptotic expansions in Nielsen theory. Dynamical, Spectral, and Arithmetic Zeta Functions. Contemporary Mathematics, v. 290, 2002, 67-81.
28. A. Fel'shtyn, E. Troitsky, Theorie de Burnside-Frobenius tordue pour les groupes polycycliques-par-fini. C. R. Acad. Sci. Paris, Ser. I 340(250)(accepted).
29. A. Fel'shtyn and E. Troitsky, *Geometry of Reidemeister classes and twisted Burnside theorem*, Journal of K-theory 1(2008), 1-40.
30. A. Fel'shtyn, F. Indukaev, and E. Troitsky, *Twisted Burnside theorem for two-step torsion-free nilpotent groups, C^* -algebras and Elliptic Theory II*. Trends in Mathematics 2008, 15p., Birkhauser.
31. A. Fel'shtyn and E. Troitsky, Twisted Burnside - Frobenius theory for discrete groups, *J. Reine Angew. Math. (Crelle's Journal)* 613 (2007)(math.GR/0606179).
32. A. L. Fel'shtyn and E. Troitsky, Twisted Burnside theorem for countable groups and Reidemeister numbers, Noncommutative Geometry and Number Theory, Vieweg, Braunschweig, 2006, 141–154 (math.RT/0606155).
33. A. Fel'shtyn, E. Troitsky, and A. Vershik, Twisted Burnside theorem for type II_1 groups: an example, Mathematical Research Letters 13(2006), no.5, 719–728 (math.RT/0606161).
34. A. Fel'shtyn, Y. Leonov, E. Troitsky, Reidemeister numbers of saturated weakly branch groups. E-print arXiv: math.GR/0606725. Preprint MPIM 2006-79. (to appear in *Geometria Dedicata*).
35. Alexander Fel'shtyn and Evgenij Troitsky, Twisted conjugacy separable groups, E-print arXiv: math.GR/0606764. Preprint MPIM 2006-81.
36. Alexander Fel'shtyn and Daciberg L. Gonçaves, Twisted conjugacy classes in Symplectic groups, Mapping class groups and Braid groups(including an Appendix written with Francois Dahmani).E-print arxiv: math.GR/0708.2628.
37. Fel'shtyn A.L. Floer homology, Nielsen theory and symplectic zeta functions. Proceedings of the Steklov Institute of Mathematics, Moscow, vol. 246, 2004, pp. 270-282.
38. A. Floer. Morse theory for Lagrangian intersections. J. Differential Geom., 28(1988), 513-547.
39. A. Floer. Symplectic fixed points and holomorphic spheres. Comm. Math. Phys., 120(2),575–611, 1989.
40. E. Formanek, *Conjugacy separability in polycyclic groups*, J. Algebra 42 (1976), 1–10.

41. W. Franz, Abbildungsklassen und fixpunktklassen dreidimensionaler Linsenräume. (German) *J. Reine Angew. Math.* 185 (1943), 65–77.
42. K. Fukaya, K. Ono, Arnold conjecture and Gromov-Witten invariants for general symplectic manifolds, *Topology* 38(1999), 933-1048.
43. R. Gaultschi. Floer homology of algebraically finite mapping classes J. Symplectic Geom. 1(2003), no.4, 715-765.
44. D. L. Gonçalves and P. Wong, Twisted conjugacy classes in exponential growth groups, *Bulletin of the London Mathematical Society* **35** (2003), 261–268.
45. M. Gerber, A. Katok . Smooth models of Thurston’s pseudo-Anosov maps. *Ann. scient. Ec. Norm. Sup.* t.15(1982), 173-204.
46. D. L. Gonçalves and P. Wong, Twisted conjugacy classes in wreath products, *Internat. J. Alg. Comput.* **16** No. 5 (2006), 875–886.
47. M. Gromov, Groups of polynomial growth and expanding maps. *Publicationes Mathematicae*, 53, 1981, 53–78.
48. M. Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*82(1985), 307-347.
49. M. Gromov, Hyperbolic groups, in: S. Gersten, ed., “Essays in Group Theory” *MSRI Publications* **8** Springer-Verlag, Berlin, New York, Heidelberg, 1987, 75–263.
50. A. Grothendieck, *Formules de Nielsen-Wecken et de Lefschetz en géométrie algébrique*, Séminaire de Géométrie Algébrique du Bois-Marie 1965-66. SGA 5, Lecture Notes in Math., vol. 569, Springer-Verlag, Berlin, 1977, pp. 407–441.
51. Graham Higman, B. H. Neumann, and Hanna Neumann, *Embedding theorems for groups*, J. London Math. Soc. **24** (1949), 247–254.
52. H. Hofer and D. Salamon, Floer homology and Novikov’s ring, in Floer Memorial Volume, 1995, 483-524, Birkhauser.
- HK05. G. C. Hruska and B. Kleiner, *Hadamard spaces with isolated flats*, *Geom. Topol.* **9** (2005), 1501–1538 (electronic), With an appendix by the authors and M. Hindawi.
53. E.-N. Ionel, Th. Parker, Gromov invariants and symplectic maps, *Math. Ann.* 314(1999), 127-158.
54. Fedor Indukaev, *Twisted Burnside theory for discrete Heisenberg groups and some wreath products*. *Vestnik Mosk. Univ., ser.1, Math.Mech.*, n.6, 2007, 9-17.
55. N. V. Ivanov, Nielsen numbers of maps of surfaces. *Journal Sov. Math.*, 26, (1984).

56. N. V. Ivanov, Entropy and the Nielsen Numbers. Dokl. Akad. Nauk SSSR 265 (2) (1982), 284-287 (in Russian); English transl.: Soviet Math. Dokl. 26 (1982), 63-66.
57. N. V. Ivanov, Automorphisms of Teichmüller modular groups, *Lecture Notes in Mathematics* **1346** Springer, Berlin, 1988.
58. B. Jiang, *Lectures on Nielsen fixed point theory*, Contemp. Math., vol. 14, Amer. Math. Soc., Providence, RI, 1983.
59. B. Jiang. Fixed point classes from a differentiable viewpoint. In Fixed point theory, Lecture Notes in Math., vol. 886, 163–170. Springer, 1981.
60. B. Jiang, Estimation of the number of periodic orbits. Pacific Jour. Math., 172(1996), 151-185.
61. B. Jiang and J. Guo. Fixed points of surface diffeomorphisms. Pac. J. Math., 160(1):67–89, 1993.
62. M. Kapovich, “Hyperbolic Manifolds and Discrete Groups,” Birkhäuser, Boston, Basel, Berlin, 2000.
63. A. A. Kirillov, *Elements of the theory of representations*, Springer-Verlag, Berlin Heidelberg New York, 1976.
64. G. Levitt, On the automorphism group of generalised Baumslag-Solitar groups. Geom. Topol. 11(2007), 473-515.
65. G. Levitt and M. Lustig, Most automorphisms of a hyperbolic group have very simple dynamics, *Ann. Scient. Éc. Norm. Sup.* **33** (2000), 507–517.
66. G. Liu and G. Tian, Floer homology and Arnold conjecture, J.Diff.Geom., 49(1998), 1-74.
67. Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory*, Springer-Verlag, Berlin, 1977, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.
68. A. I. Mal’cev, *On the faithful representations of infinite groups by matrices*, Mat. Sb. (NS) **8(50)** (1940), 405–422, (in Russian. English translation: Amer. Math. Soc. Transl. (2), **45** (1965), 1–18).
69. A.I. Malcev, *On homomorphisms onto finite groups*, Uchen. Zapiski Ivanovsk. ped. instituta **18** (1958), no. 5, 49–60, (=Selected Papers, Vol. 1, 1976, 450–461).
70. H. A. Masur, Y. N. Minsky, Geometry of the complex of curves I: hyperbolicity, Invent. Math. 138 (1999),
71. V. D. Mazurov and E. I. Khukhro (eds.), *The Kourovka notebook*, augmented ed., Rossiiskaya Akademiya Nauk Sibirskoe Otdelenie, Institut Matematiki im. S. L. Soboleva, Novosibirsk, 2002, Unsolved problems in group theory.

72. Armando Martino, *A proof that all Seifert 3-manifold groups and all virtual surface groups are conjugacy separable*, J. Algebra 313(2007), no.2,773-781.
73. D. McDuff and D. A. Salamon. Introduction to symplectic topology. Oxford Mathematical Monographs. Oxford Science Publications, 1998.
74. J. Moser. On the volume elements on a manifold. Trans. Amer. Math. Soc., 120:286–294, 1965.
75. A. W. Mostowski, *On the decidability of some problems in special classes of groups*, Fund. Math. **59** (1966), 123–135.
76. A. Yu. Ol' shanski, *Geometry of defining relations in groups*, Mathematics and its Applications (Soviet Series), vol. 70, Kluwer Academic Publishers Group, Dordrecht, 1991, Translated from the 1989 Russian original by Yu. A. Bakhturin.
77. D. Osin, *Small cancellations over relatively hyperbolic groups and embedding theorems*, Arxiv e-print math.GR/0411039, 2004.
- Osi06a. D. V. Osin, *Relative Dehn functions of amalgamated products and HNN-extensions*, Topological and asymptotic aspects of group theory, Contemp. Math., vol. 394, Amer. Math. Soc., 2006, pp. 209–220.
78. J.-P. Otal, Le theoreme d'hyperbolisation pour les varietes fibrees de dimension 3. Asterisque vol. 235, 1996.
79. F. Paulin, Sur les automorphismes exterieurs des groupes hyperboliques. Ann. Sc. ENS 30(1997), 147-167.
80. R. Phillips, P. Sarnak, Geodesics in homology classes. Duke math. journal, v.55, n.2, 1987, p. 287-297.
81. M. Pollicott, R. Sharp, Asymptotic expansions for closed orbits in homology classes. Geom. Dedicata 87(2001), no.1-3, 123-160.
82. K. Reidemeister, Automorphismen von Homotopiekettenringen, *Math. Ann.* **112** (1936), 586–593.
83. V. N. Remeslennikov, *Conjugacy in polycyclic groups. (Russian)*, Algebra i Logika **8** (1969), 712–725.
84. Derek J. S. Robinson, *A course in the theory of groups*, second ed., Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1996.
85. Daniel Segal, *Polycyclic groups*, Cambridge Tracts in Mathematics, no. 82, Cambridge University Press, Cambridge, 1983.
86. P. Seidel. Symplectic Floer homology and the mapping class group. Pacific J. Math. 206(2002), no. 1, 219-229.
87. P. Seidel. Braids and symplectic four-manifolds with abelian fundamental group. Turkish J. Math. 26(2002), no.1, 93-100.

88. Jean-Pierre Serre, *Linear representations of finite groups*, Springer-Verlag, New York, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.
89. ———, *Trees*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.
(1970), 605–617.
90. Salahoddin Shokranian, *The Selberg-Arthur trace formula*, Lecture Notes in Mathematics, vol. 1503, Springer-Verlag, Berlin, 1992, Based on lectures by James Arthur. MR1176101 (93j:11029)
91. Jennifer Taback and Peter Wong, Twisted conjugacy and quasi-isometry invariance for generalized solvable Baumslag-Solitar groups, E-print arxiv:math.GR/0601271, 2006.
92. Jennifer Taback and Peter Wong, A note on twisted conjugacy and generalized Baumslag-Solitar groups, E-print arXiv:math.GR/0606284, 2006.
93. W. Thurston, *The geometry and topology of 3-manifolds*. Princeton University, 1978.
94. W. P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc.*, 19(2):417–431, 1988.
95. E. Troitsky, *Noncommutative Riesz theorem and weak Burnside type theorem on twisted conjugacy*, *Funct. Anal. Pril.* **40** (2006), no. 2, 44–54, In Russian, English translation: *Funct. Anal. Appl.* **40** (2006), No. 2, 117–125 (Preprint 86 (2004), Max-Planck-Institut für Mathematik, math.OA/0606191).
96. P. Wong, Fixed point theory for homogeneous spaces—a brief survey. *Handbook of topological fixed point theory*, 265–283, Springer, Dordrecht, 2005.

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