OPEN STRING THEORY AND PLANAR ALGEBRAS

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Abstract. In this note we show that abstract planar algebras are algebras over the topological operad of moduli spaces of stable maps with Lagrangian boundary conditions, which in the case of the projective line are described in terms of real rational functions. These moduli spaces appear naturally in the formulation of open string theory on the projective line. We also show two geometric ways to obtain planar algebras from real algebraic geometry, one based on string topology and one on Gromov-Witten theory. In particular, through the well known relation between planar algebras and subfactors, these results establish a connection between open string theory, real algebraic geometry, and subfactors of von Neumann algebras.

1. Introduction

The purpose of this paper is to show that planar algebras arise as algebras over an operad of moduli spaces of stable maps to $\mathbb{P}^1$ with Lagrangian boundary conditions, which can be described in terms of real algebraic curves. In particular, the results presented in this paper can be interpreted as a connection between open string theory on $\mathbb{P}^1$ and Jones' theory of subfactors of von Neumann algebras, using as intermediate steps the relation between open string theory on $\mathbb{P}^1$ and certain moduli spaces $R_g(\mathbb{P}^1, d)$ of maps to $\mathbb{P}^1$ with Lagrangian boundary conditions, combined with the main result we prove here, which relates the latter to the theory of planar algebras developed in [6]. The connection to subfactors can then be seen by invoking the result of [15] and its reformulation in terms of planar algebras of [5] and [9].

The main point involved in our description of planar algebras in terms of real algebraic geometry is an identification of (weighted) planar tangles as the combinatorial datum that encodes the components of the moduli spaces $R_g(\mathbb{P}^1, d)$ of stable M-maps to $\mathbb{P}^1$ with Lagrangian boundary conditions. The terminology M-maps here refers to the fact that they are realized by maximal real algebraic curves. We also describe the compositions of planar tangles and the trace map in this geometric setting. More precisely, we prove the following statement.

Theorem 1.1. *Abstract planar algebras are topological operads of* $R_g(\mathbb{P}^1, d)$.

In particular, by restricting to the case with $g = 0$, we obtain the case of the Temperley-Lieb algebras.

The first two sections of this paper respectively review the relation between open string theory on $\mathbb{P}^1$ and the moduli spaces $R_g(\mathbb{P}^1, d)$, as well as the well known relation between subfactors and planar algebras. The main original contribution of this paper is in §4 and §5. In §4 we connect the two previous topics by proving Theorem 1.1 and in §5 we then describe two different methods, both based on real algebraic geometry, for constructing planar algebras as representations of the planar operad, the first based on algebraic loop spaces and string topology and the other based on a real version of Gromov–Witten theory.

2. Real algebraic geometry and open string theory

2.1. Complex and real curves, and bordered Riemann surfaces. Recall that a bordered Riemann surface of type (g, h) is a Riemann surface with boundary, with g handles and h boundary components, oriented according to the orientation induced by the complex structure on the Riemann surface. We also denote, as in [1], [7] the complex double of a bordered Riemann surface Σ by $\Sigma_{\mathbb{C}}$, with σ the antiholomorphic involution on $\Sigma_{\mathbb{C}}$.

2.1.1. *Maximal real curves.* A maximal real curve, denoted M-curve, is a real algebraic curve $(\Sigma_{\mathbb{C}}, \sigma)$ with the maximal number of connected components of the real part $\Sigma_{\mathbb{R}}$ of σ. By Harnack's bound this maximal number is $g + 1$ for a curve of genus g.

The real part $\Sigma_{\mathbb{R}}$ of a real structure σ divides $\Sigma_{\mathbb{C}}$ into two 2-dimensional discs, respectively denoted by Σ^+ and Σ^-, minus a set of interior (open) discs $D_1, \ldots, D_g$, having $\Sigma_{\mathbb{R}}$ as their common boundary in $\Sigma_{\mathbb{C}}$. The real structure σ interchanges $\Sigma^{\pm}$, and the complex orientations of $\Sigma^{\pm}$ induce two opposite orientations on $\Sigma_{\mathbb{R}}$, called its complex orientations. The quotient $\overline{\Sigma} := \Sigma_{\mathbb{C}}/\sigma$ is isomorphic to Σ^+. Here $\Sigma^+ \simeq \overline{\Sigma}$ is a bordered Riemann surface and the real algebraic curve $(\Sigma_{\mathbb{C}}, \sigma)$ is its complex double, *cf.* [16].

2.2. Open strings on $\mathbb{P}^1$ and the moduli spaces of real maps. We present here briefly the setting of open string theory on $\mathbb{P}^1_{\mathbb{C}}$ with a Lagrangian submanifold $\mathbb{P}^1_{\mathbb{R}}$. We especially focus on the role of the moduli space of stable maps with Lagrangian boundary conditions.

2.3. M-maps and their moduli spaces. We define stable M-maps following a setting similar to that of [7].

Definition 2.1. *An M-map (of degree d) is a pair (Σ^+, f^+) consisting of the following data.*

- *A bordered Riemann surface Σ^+ whose complex double $(\Sigma_{\mathbb{C}}, \sigma)$ is an M-curve of genus g, and with boundary components $\partial\Sigma = \partial D_* \cup \partial D_1 \cup \cdots \cup \partial D_g$.*
- *A map f^+ of degree d which is the restriction of a real stable morphism $f : \Sigma_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$, such that the singularities of f are all of degree two.*

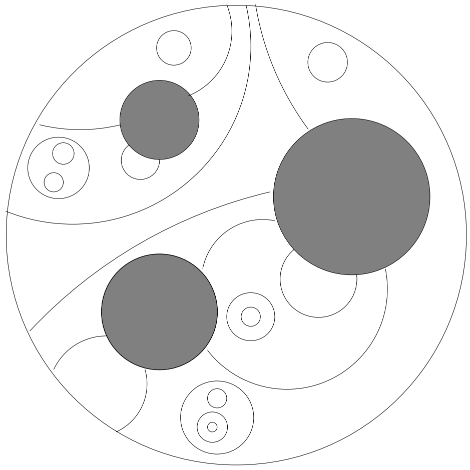

FIGURE 1. Planar tangle

- *A set consisting of a critical point of f for each non-empty component of $\Sigma_{\mathbb{R}}$.*

An M-map is stable if it does not admit any infinitesimal automorphisms.

The moduli space $R_g(\mathbb{P}^1, d)$ is the space of isomorphism classes of stable M-maps. The geometric and topological properties of the space $R_g(\mathbb{P}^1, d)$ has been extensively studied in [11]. The construction in [11] is in fact more general; it considers an arbitrary symplectic manifold X and its Lagrangian manifold L as the target of stable maps.

3. Planar algebras and subfactors

3.1. **Planar algebras.** We recall here briefly the basic definitions and facts of the theory of planar algebras developed in [6]. We follow closely the short survey given in [2].

3.1.1. *Planar tangles.* A planar k-tangle T consists of the unit disc $D \subset \mathbb{C}$ together with a finite collection of discs $D_1, D_2, \ldots, D_g$ inside D. The boundaries of D and of each interior disc D_i are decorated by an even number of marked points, $2k$ points on ∂D and $2k_i$ points on each ∂D_i. The interior of $D \smallsetminus \bigcup_i D_i$ also contains a collection of non-intersecting strings, which are either closed strings or have as boundary the marked points on the $\partial D \cup \partial D_i$. Each marked point lies on the boundary of one of these strings.

The complementary region $D \smallsetminus (\{\text{strings}\} \bigcup_i D_i)$ admits a checkerboard black and white coloring as well as a choice of a white region at each D_i.

3.1.2. *Planar operad.* Two planar tangles T and S can be composed whenever the number of marked points on the boundary of S matches the number of marked points on the boundary of one of the interior discs D_j of T. The

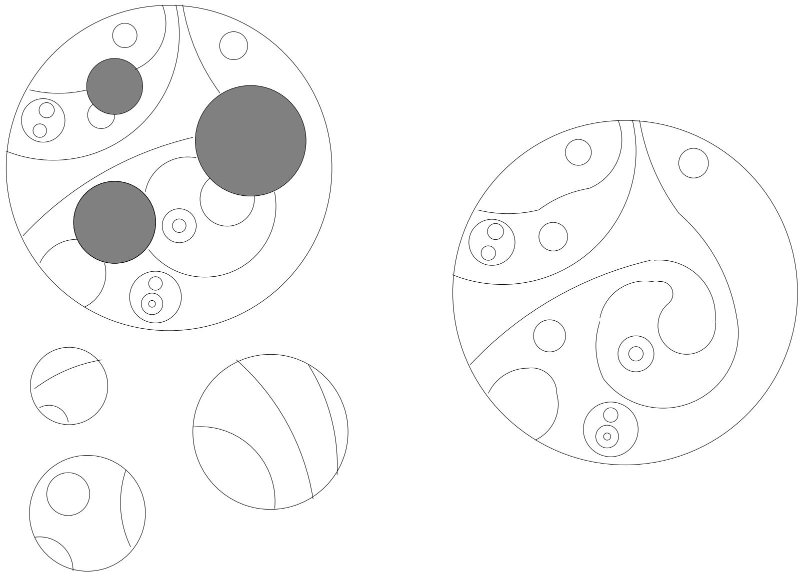

FIGURE 2. Planar operad: compositions of tangles

composition $T \circ_j S$ is then given by gluing a rescaled copy of S in place of the interior of the disc D_j, so as to match the shadings and the marked white regions. This operation is well-defined, with the coloring and choice of white region eliminating any possible rotational ambiguity. Also the result only depends on the isotopy classes.

The planar operad $\mathcal{P}$ consists of all orientation-preserving diffeomorphism classes of planar k-tangles that fix the boundary ∂D. The structure of operad is given by the compositions of tangles, defined as above. An example of compositions is given in Figure 2. In the figures 1 and 2 the black/white coloring and the marked white regions are not shown for simplicity.

3.1.3. *Planar algebras.* One then defines planar algebras ([6], [2]) as algebras over the planar operad $\mathcal{P}$, in the sense described in [12].

This means (see *e.g.* [2]) that a planar algebra $\mathcal{P}$ is a family of vector spaces $\{V_k\}_{k>0}$ together with a morphism Z from the planar operad $\mathcal{P}$ to the (colored) operad Hom of multilinear maps between vector spaces.

3.1.4. *Partition function and trace.* The morphism $Z : \mathcal{P} \to \mathrm{Hom}$ from the planar operad to the operad of multilinear maps of vector spaces has the following properties.

Given a tangle T_ℓ, where 2ℓ is the number of points on ∂D, one obtains a multilinear map

$$Z(T_\ell) : \otimes_{i=1}^{n} V_{\ell_i} \to V_\ell, \tag{3.1}$$

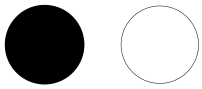

FIGURE 3. The planar tangles $T_{0,\emptyset,b/w}$

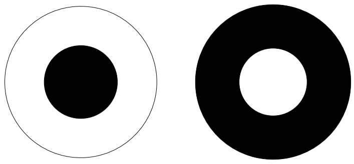

FIGURE 4. The planar tangles $T_{0,\emptyset,\mathcal{L},b/w}$

satisfying the composition property

$$Z(T \circ_j S) = Z(T) \circ_j Z(S). \tag{3.2}$$

Moreover, one has the following properties, which give Z an interpretation as a "partition function" associated to a planar algebra.

• Normalization: let $T_{0,\emptyset,b/w}$ be the colored discs of Figure 3. Then

$$Z(T_{0,\emptyset,b/w}) = 1. \tag{3.3}$$

• Parameters: Consider the case of planar tangles as in Figure 4, given by a simple curve with no boundary inside D and the two possible choices of coloring. Then the corresponding linear maps under Z are of the form

$$Z(T_{0,\emptyset,\mathcal{L},b}) = \delta_1, \quad Z(T_{0,\emptyset,\mathcal{L},w}) = \delta_2, \tag{3.4}$$

for two parameters δ_i.

These two rules give a procedure to *eliminate ovals* from a planar tangle, replacing them in the image under Z by a multiplicative factor of the form $\delta_1^m \delta_2^n$, depending on the parameters δ_i. This is illustrated in one example in Figure 5, where one encodes a configuration of ovals inside a planar tangle via a rooted tree and computes the corresponding multiplicative factor accordingly.

3.2. **Subfactors and planar algebras.** Our main focus in this paper is on a geometric framework for planar algebras based on real algebraic geometry and open string theory. However, we mention briefly the well known connection between planar algebras and the theory of subfactors of von Neumann algebras ([2],[5], [6],[8], [9], [15]), since, in light of our interpretation of planar algebras it leads naturally to possible constructions of subfactors from data of moduli spaces in real algebraic geometry.

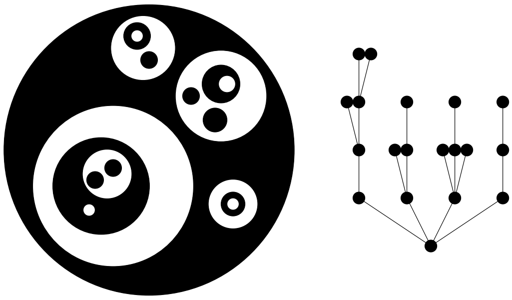

FIGURE 5. Ovals in planar tangles encoded by rooted trees

In [15], Popa showed that one can associate to a subfactor a *standard invariant*, which is a planar algebra. Not all planar algebras arise as the standard invariant of a subfactor, but one can characterize those that have this property, as in §4 of [6].

A subfactor planar algebra is a planar algebra for which all the vector spaces V_k are finite dimensional, with $\dim V_{0,b/w} = 1$, and with $\delta_1 = \delta_2 \neq 0$, which has an involution on each V_k induced by the reflection of tangles with the property that the partition function Z is a sesquilinear form with respect to this involution. The reconstruction theorem, in the form given in [5], shows how to associate to a subfactor planar algebra with $\delta > 1$ a subfactor $M_0 \subset M_1$ obtained from a tower of type II_1 factors $M_k = Gr_k(V) = \oplus_{n\geq k} V_k$, with a faithful tracial state tr_k, so that the standard invariant of the subfactor, constructed as in [15] by considering the relative commutants $M_0' \cap M_k$, gives canonical identifications $M_0' \cap M_k \simeq V_k$ which induce a morphism of involutive planar algebras.

4. Planar algebras and open string theory

4.1. The moduli space $R_g(\mathbb{P}^1, d)$ and planar tangles. We first define weighted planar tangles, which are planar tangles with additional decorations, and we use them to distinguish the connected components of the moduli space $R_g(\mathbb{P}^1, d)$.

Definition 4.1. *Let T be a planar tangle as in §3.1.1. Then, T is called weighted if each interior string and each boundary segment is decorated with a non-negative integer weight. A weighted tangle is of weight d if the sum of weights for all strings and boundary segments is d.*

The following statement is a slight generalization of a similar result in [13] for genus zero covers of $\mathbb{P}^1$.

Proposition 4.2. *There is a 1-1 correspondence between the set of connected components of the moduli space $R_g(\mathbb{P}^1, d)$ and the set of weighted planar tangles.*

Proof. We associate a tangle to each stable M-map $(\Sigma^+, f^+) \in R_g(\mathbb{P}^1, d)$ by simply pulling back the real line $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \infty$ by the map f^+,

$$(\Sigma^+, f^+) \mapsto T := (f^+)^{-1}(\mathbb{P}^1_{\mathbb{R}}). \tag{4.1}$$

Let $(\Sigma_{\mathbb{C}}, f)$ be the complex double of (Σ^+, f^+). Let $Crit(f)$ be the set of critical values of f, and let $Crit_{\mathbb{R}}(f)$ and $Crit_{\mathbb{C}}(f)$ denote respectively the set of real critical values and its complement. To determine weights, consider an arbitrary point in $x \in \mathbb{P}^1_{\mathbb{R}} \backslash Crit_{\mathbb{R}}(f)$. For each string (or boundary segment), we define the function $w(x)$ as the number of preimages of x lying on this string (or boundary segment). Then the weight of this string (or boundary segment) is the minimum of $w(x)$. Note that the weight of a closed string in Σ^+ coincides with the multiplicity of f^+ restricted to this string.

We first check that this indeed gives a weighted planar tangle. Since Σ^+ is half of a maximal real curve of genus g, it is topologically a disc minus a collection of g discs. The preimage $(f^+)^{-1}(\mathbb{P}^1_{\mathbb{R}})$ gives the strings inside Σ^+ or segments in the boundary of Σ^+.

We now check the condition that these tangles are expected to satisfy. These strings intersect only in the boundary $\partial\Sigma^+$ of Σ^+. We have the following possibilities.

Strings inside Σ^+. If the strings had intersections in $\Sigma^+ \setminus \partial\Sigma^+$, then such an intersection point z would be a critical point with real critical value. Hence, the same is also true for the conjugate $\sigma(z) \in \Sigma_{\mathbb{C}}$ in the complex double $\Sigma_{\mathbb{C}}$ of Σ^+. However, this contradicts the genericity condition of $f : \Sigma \to \mathbb{P}^1_{\mathbb{C}}$ (see Defn. 2.1).

Strings and the boundary $\partial\Sigma^+$. All critical points of f which have real critical values must be real. If z is a critical point with a real critical value, then $\bar{z}$ is a critical point with the same critical value. Therefore, $z = \bar{z}$, since f is generic. For each such critical point, $f^{-1}(\mathbb{P}^1_{\mathbb{R}})$ contains exactly four arcs in $\Sigma_{\mathbb{C}}$ incident to it. Two of these arcs are the arcs lying in $\partial D_i \subset \mathbb{R}\Sigma$, while the other two interchange under the involution σ. In particular, the other endpoints of these two arcs coincide.

Next, we need to show that the number of critical points at each boundary component ∂D_i of Σ^+ is even, as required for the data to define a planar tangle. Note that the set of all critical points of the real map $f : \Sigma \to \mathbb{P}^1_{\mathbb{C}}$ consists of an even number of points. The order of this set is in fact $2(d + g - 1)$ due to the Riemann–Hurwitz formula. On the other hand, the real maps $(\Sigma_{\mathbb{C}}, f)$ degenerate when their critical points collide, in particular when a complex conjugate pair of critical points degenerates to a real point of Σ. Under such a degeneration, it can only split into a pair of real ramification points lying in the same real component ∂D_i. Since there exist stable maps $f : \Sigma_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ without any real ramification points, we show that the number of critical points is even for each boundary component ∂D_i via a simple induction on these types of degenerations.

We need to show that, for the tangle we associated to elements of a connected component of $R_g(\mathbb{P}^1, d)$, the the isotopy class does not change. It is clear that the isotopy class of these tangles will not change under small perturbations and it can only change through intersections of strings. As we have already observed above, any intersection of strings violates the genericity condition. Namely, changes of isotopy class of tangles can happen when one passes through the discriminant locus to another component of $R_g(\mathbb{P}^1, d)$.

Finally, to show that we have a bijection, we then need to show that any arbitary tangle of weight d can be obtained in this way. For a given such tangle, an element of $R_g(P^1, d)$ can be constructed by gluing a set of weighted pairs of pants which is in fact $\Sigma^+ \setminus \{\text{strings}\}$. The gluing prescription of weighted pants is given in Thm 1 in [13] based on [14]. $\square$

In the following, we denote by C_T the connected component of $R_g(\mathbb{P}^1, d)$ corresponding to a given weighted tangle T.

4.2. **Sewing stable M-maps.** Let (Σ_1^+, f_1^+) and (Σ_2^+, f_2^+) be a pair of stable M-maps such that the restriction of f_1^+ to the boundary component $\partial D_{i_1} \subset \Sigma_1^+$ agrees with the restriction of f_2^+ onto $\partial D_{i_2} \subset \Sigma_2^+$. Assume that the boundary components ∂D_{i_1} and ∂D_{i_2} carry opposite orientations. Then we can sew the M-maps (Σ_k^+, f_k^+) along their boundaries ∂D_{i_k} for $k = 1, 2$.

Let $R_{g_1}(\mathbb{P}^1, d_1) \times_{ij} R_{g_2}(\mathbb{P}^1, d_2)$ denote the space of pairs described above. This space is in fact a fiber product in the following way.

Recall that the space of *algebraic loops* is the space of morphisms $f : \mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^1_{\mathbb{R}}$. The moduli spaces $R_{g_k}(\mathbb{P}^1, d_k), k = 1, 2$ admit evaluation maps to $L^{alg}(\mathbb{P}^1_{\mathbb{R}})$,

$$ev_{i_k} : R_{g_k}(\mathbb{P}^1, d_k) \to L^{alg}(\mathbb{P}^1_{\mathbb{R}}), \quad (\Sigma_k^+, f_k^+) \mapsto f_k^+|_{\partial D_{i_k}}$$

Then the space described above is the fibered product

$$R_{g_1}(\mathbb{P}^1, d_1) \times_{L^{alg}(\mathbb{P}^1_{\mathbb{R}})} R_{g_2}(\mathbb{P}^1, d_2).$$

The sewing operation described above provides us with a morphism

$$R_{g_1}(\mathbb{P}^1, d_1) \times_{ij} R_{g_2}(\mathbb{P}^1, d_2) \to R_{g_1+g_2-1}(\mathbb{P}^1, d_1 + d_2). \tag{4.2}$$

The induced map on the sets of connected components of these spaces defines a corresponding map at the level of weighted tangles, which agrees with the composition of tangles.

4.2.1. *Sewing boundary segments.* There is a similar sewing along the boundary segments which fits better with the example that we discuss in §4.6 below.

Let (Σ_1^+, f_1^+) and (Σ_2^+, f_2^+) be a pair of stable M-maps such that the restriction of f_1^+ to the boundary segment $I_{j_1} \subset \partial D_{i_1} \subset \Sigma_1^+$ agrees with the restriction of f_2^+ onto $I_{j_2} \subset \partial D_{i_2} \subset \Sigma_2^+$. If ∂D_{i_1} and ∂D_{i_2} carry opposite orientations, then we can sew the M-maps (Σ_k^+, f_k^+) along their boundary segments ∂I_{j_k} for $k = 1, 2$ and obtain a new M-map of degree $d_1 + d_2$.

The space of such pairs is also a fiber product. The appropriate subspaces of the moduli spaces $R_{g_k}(\mathbb{P}^1, d_k), k = 1, 2$ admit evaluation maps to the space of algebraic paths $P^{alg}(\mathbb{R})$, which is the space of morphisms $f : \mathbb{R} \to \mathbb{R}$. Note that this map is not defined for all components C_T of $R_{g_k}(\mathbb{P}^1, d_k)$ since the tangles T in general need not have boundary segments (*e.g.* if there is no critical point on a boundary component of Σ^+, then it can only map into the algebraic loop space as above). Then, by using the appropriate subspaces of $R_{g_k}(\mathbb{P}^1, d_k)$, we obtain the space of such pairs as a fiber product similar to the above construction (4.2).

4.3. **The trace.** Let (Σ^+, f^+) be in $C_T \subset R_g(\mathbb{P}^1, d)$ and let S be the closed string which is the common boundary of a pair of weighted pants P_1, P_2 in Σ^+. Let $\{\partial D_1, \ldots, \partial D_l, S\}$ and $\{\partial D_{l+1}, \ldots, \partial D_k, S\}$ be the sets of boundaries of P_1, and P_2 respectively. Let the w_i be the weights of ∂D_i and w_s be the weight of S.

We first note that the deformations of such a pair of weighted pants and their maps to $\mathbb{P}^1$ are given by the fiber product

$$R_l(\mathbb{P}^1, w_1 + \cdots + w_l) \times_s R_{k-l}(\mathbb{P}^1, w_{l+1} + \cdots + w_k)$$

which is determined by the evaluation map $(P_i, f_i) \mapsto f|_S$. As we have already seen above, there is a morphism of this product into the moduli space $R_{k-1}(P^1, w_1 + \cdots + w_k)$. This map in fact removes the string S in Σ^+ *i.e.,* it provides the trace operator in geometric terms in the above setting. This map is given by the gluing of a pair of weighted pants P_1 and P_2 as in [13].

4.4. **The involution.** We can define an involution $*$ by replacing the complex structure J of Σ^+ by $-J$. This is equivalent to replacing Σ^+ with the other half $\sigma(\Sigma^+)$ of $\Sigma_{\mathbb{C}}$. This operation reverses the orientations of the strings and checkerboard shadings of the corresponding planar tangle.

4.5. **Planar algebras and the moduli space of real maps $R_g(\mathbb{P}^1, d)$.** The collection of connected components of $R_g(\mathbb{P}^1, d)$ provides a topological operad $\mathcal{P}_{\mathbb{P}^1}$. The following statement then follows directly from the discussion of the previous subsections.

Theorem 4.3. *The operad of (weighted) planar tangles is the topological operad $\mathcal{P}_{\mathbb{P}^1}$.*

4.6. **An example: Temperley-Lieb algebras.** Consider the moduli space $R_0(\mathbb{P}^1, d)$ of M-maps of genus zero. Such M-maps have $2d - 2$ critical points. Then, the tangles that distinguish the components C_T of $R_0(\mathbb{P}^1, d)$ are the planar tangles on a disc with $2k$ points on their boundaries where $0 \leq 2k \leq 2d - 2$. In other words, these tangles connect the first k points starting at the marked critical point in Σ^+ to the second k points without having any crossings. These are in fact the tangles that generate the

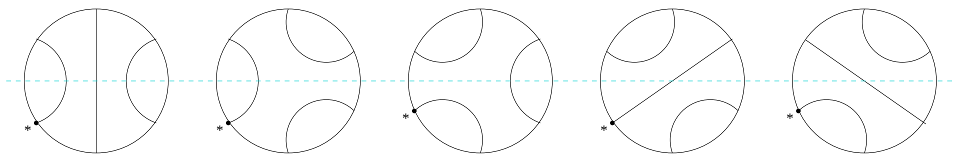

FIGURE 6. The bases of TL_3

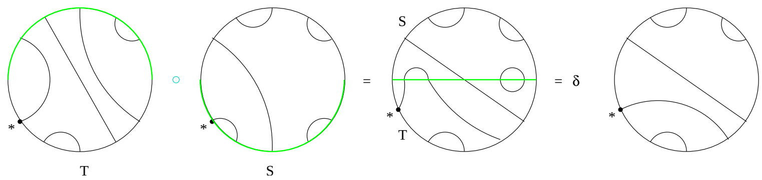

FIGURE 7. The product of tangles S and T

Temperley-Lieb algebras (modulo the trace). For instance, if we consider the case with $d = 4$, we obtain the tangles that generate TL_1, TL_2 and TL_3; see Figure 6 for the bases of TL_3.

The product of two such tangles is obtained by sewing the boundary segments between the kth and the $2k$th point of the first tangle and the 1st and the kth point of the second tangle. Figure 7 illustrates an example of a such a composition of tangles.

Remark 4.4. There is an equivalent description of Temperley-Lieb algebras by using the moduli space of M-maps of genus one. In this case, one needs to consider the tangles in a cylinder Σ^+, and the composition is given by sewing the boundary components. However, the numbers of critical points at each boundary of the cylinder need not be equal. Therefore, either one has to consider a more general algebra (which contains Temperley-Lieb algebra as a subalgebra) or one needs to restrict oneself to a subspace of $R_1(\mathbb{P}^1, d)$ that contains only those components C_T that correspond to the tangles of the Temperley-Lieb algebra.

5. ALGEBRAS OVER PLANAR OPERAD

In this section, we discuss two possible geometric constructions of algebras over the planar operad $\mathcal{P}$. The first one follows closely the approach of Cohen and Godin's in contrucing string topology operations [4]. The second one is in the spirit of Gromov-Witten theory [10], in the real algebraic geometry setting (see for instance [3]).

5.1. **Planar algebras in string topology.** As we have already observed in §4.2, the moduli space $R_g(\mathbb{P}^1, d)$ admits evaluation maps

$$ev_i : R_g(\mathbb{P}^1, d) \to L^{alg}(\mathbb{P}^1_R), \quad i = 1, \ldots, g+1.$$

By using these evaluations, we can pull back classes from $H^*(L^{alg}(\mathbb{P}^1_R))$. Note that, like the usual topological loop space LS^1, the algebraic loop space $L^{alg}(\mathbb{P}^1_R)$ has infinitely many connected components, each determined by the degree of the loops. However, in contrast to the topological case, the components of $L^{alg}(\mathbb{P}^1_R)$ are finite dimensional and are not contractible. Therefore they carry nontrivial cohomology classes.

Let $\gamma_1, \ldots, \gamma_{g+1} \in H^*_c(L^{alg}(\mathbb{P}^1_R))$. Then, we define correlators

$$\langle \gamma_1, \ldots, \gamma_{g+1} \rangle_T := ev^*(\gamma_1) \wedge \cdots \wedge ev^*(\gamma_{g+1}) \tag{5.1}$$

which take values in the cohomology with compact support $H^*_c(C_T)$ of the component $C_T \subset R_g(\mathbb{P}^1, d)$ corresponding to the tangle T.

Let $\{e_a\}$ be a basis for $H^*_c(L^{alg}(\mathbb{P}^1_R))$ and let g^{ab} be the intersection form. We use the above invariants of the loop space $L^{alg}(\mathbb{P}^1_R)$ to give a representation of the operad $\mathcal{P}$ of planar tangles in the following way. The image of T under the morphims $Z : \mathcal{P} \to \mathrm{Hom}$ is

$$T \mapsto Z(T) := \left(\gamma_1 \otimes \cdots \otimes \gamma_g \mapsto \sum_{a,b} \langle \gamma_1, \ldots, \gamma_g, e_a \rangle_T \; g^{ab} \; e_b \right). \tag{5.2}$$

The composition of these higher products are evident from the definitions of these products in (5.2) and from the compositions of tangles described in §4.2.

In this construction, the trace can be given by using the procedure described in §4.3. Let T be a planar tangle and let S be closed in T. In such a case, we think of T as a composition of two tangles T_1 and T_2 as in §4.3. Then, we obtain this composition as in equation (3.2). In particular, if the closed string bounds a disc (rather than a more complicated weighted pant), then it plays the role of trace, and it is given as in (3.4).

5.2. **Planar algebras via Gromov-Witten theory.** An alternative way to obtain representations of $\mathcal{P}$ is to use ideas from Gromov-Witten theory of $\mathbb{P}^1$. Let T be a tangle having at least one marked point at each boundary disc. In this case, the evalution map

$$ev : C_T \to Conf(\mathbb{P}^1_{\mathbb{R}}, g+1) \times \mathbb{R}Conf(\mathbb{P}^1, k), \quad \text{where} \quad k = n - g + 1,$$

maps (Σ^+, f^+) to its critical values in $\mathbb{P}^1$. The image of the evalution map is the space of $\mathbb{Z}/2\mathbb{Z}$-equivariant distinct (unordered) point configurations in $\mathbb{P}^1$ which we denote by $\mathbb{R}Conf(\mathbb{P}^1, k)$. Moreover, if we restrict ourselves to the specified critical points at each boundary component, the evalution map takes its values in the configuration space of ordered point configurations $Conf(\mathbb{P}^1_{\mathbb{R}}, g+1)$.

Let $\alpha_1, \dots, \alpha_k \in H^*_c(\mathbb{R}Conf(\mathbb{P}^1, k))$ and

$$\gamma_1, \dots, \gamma_{g+1} \in H^*(Conf(\mathbb{P}^1_{\mathbb{R}}, g+1)).$$

We then define correlators by setting

$$\langle\langle \alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_{g+1} \rangle\rangle_T := ev^*(\alpha_1 \wedge \cdots \wedge \alpha_k \wedge \gamma_1 \wedge \cdots \wedge \gamma_{g+1}).$$

These take values in the cohomology with compact support $H^*_c(C_T)$ of the component $C_T \subset R_g(\mathbb{P}^1, d)$ corresponding to the tangle T. Then, by using the same idea as in (5.2), we define the higher products corresponding to the tangles T, which depend on the classes α_i, by setting

$$T \mapsto Z(T) := \left(\gamma_1 \otimes \cdots \otimes \gamma_g \mapsto \sum_{a,b} \langle\langle \alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_g, e_a \rangle\rangle_T \ \eta^{ab} \ e_b \right).$$

Here η^{ab} is the intersection matrix of $H^*(Conf(\mathbb{P}^1_{\mathbb{R}}, g+1))$. The composition of these higher products and the trace operators are evident and are given as in the previous case described in §5.1

Remark 5.1. The requirement on the existence of marked points may seem artificial, but at present we do not have a convenient substitute for it. However, the above setting adapts well to the examples of Temperley–Lieb algebras and Fuss–Catalan algebras (see [2]).

Remark 5.2. Both situations discussed above, based on string topology or on Gromov-Witten theory, can be further enriched by using the tautological classes of the moduli space $R_g(\mathbb{P}^1, d)$. This might lead to additional interesting examples.

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