EMBEDDINGS OF $\mathbb{C}^*$-SURFACES INTO WEIGHTED PROJECTIVE SPACES

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Abstract. Let $V$ be a normal affine surface which admits a $\mathbb{C}^*$- and a $\mathbb{C}^+$-action. Such surfaces were classified e.g., in [FlZa1, FlZa2], see also the references therein. In this note we show that in many cases $V$ can be embedded as a principal Zariski open subset into a hypersurface of a weighted projective space. In particular, we recover a result of D. Daigle and P. Russell, see Theorem A in [DR].

1. Introduction

If $V = \text{Spec} \, A$ is a normal affine surface equipped with an effective $\mathbb{C}^*$-action, then its coordinate ring $A$ carries a natural structure of a $\mathbb{Z}$-graded ring $A = \bigoplus_{i \in \mathbb{Z}} A_i$. As was shown in [FlZa1], such a $\mathbb{C}^*$-action on $V$ has a hyperbolic fixed point if and only if $C = \text{Spec} \, A_0$ is a smooth affine curve and $A_0 \neq 0$. In this case the structure of the graded ring $A$ can be elegantly described in terms of a pair $(D_+, D_-)$ of $\mathbb{Q}$-divisors on $C$ with $D_+ + D_- \leq 0$. More precisely, $A$ is the graded subring $A = A_0[D_+, D_-] \subseteq K_0[u, u^{-1}]$, $K_0 := \text{Frac} \, A_0$,

where for $i \geq 0$

$$A_i = \{f \in K_0 \mid \text{div} f + iD_+ \geq 0\} u^i \quad \text{and} \quad A_{-i} = \{f \in K_0 \mid \text{div} f + iD_- \geq 0\} u^{-i}.$$ 

This presentation of $A$ (or $V$) is called in [FlZa1] the DPD-presentation. Furthermore two pairs $(D_+, D_-)$ and $(D'_+, D'_-)$ define equivariantly isomorphic surfaces over $C$ if and only if they are equivalent that is,

$$D_+ = D'_+ + \text{div} f \quad \text{and} \quad D_- = D'_- - \text{div} f \quad \text{for some } f \in K^*_0.$$

In this note we show that if such a surface $V$ admits also a $\mathbb{C}^+$-action then it can be $\mathbb{C}^*$-equivariantly embedded (up to normalization) into a weighted projective space as a hypersurface minus a hyperplane; see Theorem 2.3 and Corollary 2.5 below. In particular we recover the following result of Daigle and Russell [DR].

**Theorem 1.1.** Let $V$ be a normal Gizatullin surface$^1$ with a finite divisor class group. Then $V$ can be embedded into a weighted projective plane $\mathbb{P}(a, b, c)$ minus a hypersurface. More precisely:

(a) If $V = V_{d,e}$ is toric$^2$ then $V$ is equivariantly isomorphic to the open part$^3$ $\mathbb{D}_+(z)$ of the weighted projective plane $\mathbb{P}(1, e, d)$ equipped with homogeneous coordinates $(x : y : z)$ and with the 2-torus action $(\lambda_1, \lambda_2)(x : y : z) = (\lambda_1 x : \lambda_2 y : z)$.

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$^1$That is, $V$ admits a completion by a linear chain of smooth rational curves; see Section 3 below.

$^2$See [FlZa1] below.

$^3$We use the standard notation $V_+(f) = \{f = 0\}$ and $\mathbb{D}_+(f) = \{f \neq 0\}$. 


(b) If \( V \) is non-toric then \( V \cong \mathbb{D}_+(xy - zm) \subseteq \mathbb{P}(a, b, c) \) for some positive integers \( a, b, c \) satisfying \( a + b = cm \) and \( \gcd(a, b) = 1 \).

2. Embeddings of \( \mathbb{C}^* \)-surfaces into weighted projective spaces

According to Proposition 4.8 in [FlZa1] every normal affine \( \mathbb{C}^* \)-surface \( V \) is equivariantly isomorphic to the normalization of a weighted homogeneous surface \( V' \) in \( \mathbb{A}^4 \). In some cases (described in loc.cit.) \( V' \) can be chosen to be a hypersurface in \( \mathbb{A}^3 \). Cf. also [Du] for affine embeddings of some other classes of surfaces.

In Theorem 2.3 below we show that any normal \( \mathbb{C}^* \)-surface \( V \) with a \( \mathbb{C}_+ \)-action is the normalization of a principal Zariski open subset of some weighted projective hypersurface.

In the proofs we use the following observation from [Fl].

Proposition 2.1. Let \( R = \bigoplus_{i \geq 0} R_i \) be a graded \( R_0 \)-algebra of finite type containing the field of rational numbers \( \mathbb{Q} \). If \( z \in R_d, d > 0 \), is an element of positive degree then the group of \( d \)th roots of unity \( E_d \) acts on \( R \) and then also on \( R/(z - 1) \) via
\[
\zeta \cdot a = \zeta^i \cdot a \quad \text{for} \quad a \in R_i, \; \zeta \in E_d,
\]
with ring of invariants \( (R/(z - 1))^{E_d} \cong (R[1/z])_0 \). Consequently
\[
(\text{Spec } R/(z - 1))/E_d \cong \mathbb{D}_+(z)
\]
is isomorphic to the complement of the hyperplane \( \{z = 0\} \) in \( \text{Proj}(R) \).

Let us fix the notations.

2.2. Let \( V = \text{Spec } A \) be a normal \( \mathbb{C}^* \)-surface with DPD-presentation
\[
A = \mathbb{C}[t][D_+, D_-] \subseteq \mathbb{C}(t)[u, u^{-1}].
\]
If \( V \) carries a \( \mathbb{C}_+ \)-action then according to [FlZa2], after interchanging \( (D_+, D_-) \) and passing to an equivalent pair, if necessary, we may assume that
\[
D_+ = -\frac{e_+}{d} [0] \quad \text{with} \quad 0 < e_+ \leq d,
D_- = -\frac{e_-}{d} [0] - \frac{1}{k} D_0
\]
with an integral divisor \( D_0 \), where \( D_0(0) = 0 \). We choose a polynomial \( Q \in \mathbb{C}[t] \) with \( D_0 = \text{div}(Q) \); so \( Q(0) \neq 0 \).

Theorem 2.3. Let \( F \) be the polynomial
\[
F = x^k y - s^{k(e_+ + e_-)} Q(s^d/z) z^d Q \in \mathbb{C}[x, y, z, s],
\]
which is weighted homogeneous of degree
\[
k(e_+ + e_-) + d \deg Q \quad \text{with respect to the weights}
\]
\[
\deg x = e_+, \quad \deg y = ke_+ + d \deg Q, \quad \deg z = d, \quad \deg s = 1.
\]
Then the surface \( V \) as in 2.2 above is equivariantly isomorphic to the normalization of the principal Zariski open subset \( \mathbb{D}_+(z) \) of the hypersurface \( \mathbb{V}_+(F) \) in the weighted projective 3-space
\[
\mathbb{P} = \mathbb{P}(e_+, ke_+ + d \deg Q, d, 1).
\]
\footnote{We note that \( e_+ + e_- = d(-D_+(0) - D_-(0)) \geq 0 \).}
Example 4.10 in [FlZa] and by \( \zeta \cdot s = \zeta \cdot s \) if \( \zeta \in E_d \). Let \( A' \) be the normalization of \( A \) in \( L \). According to Proposition 4.12 in [FlZa]

\[ A' = \mathbb{C}[s][D'_+, D'_-] \subseteq \mathbb{C}(s)[u, u^{-1}] \]

with \( D'_\pm = \pi_d(D_{\pm}) \), where \( \pi_d : \mathbb{A}^1 \rightarrow \mathbb{A}^1 \) is the covering \( s \mapsto s^d \). Thus

\[ (D'_+, D'_-) = \left(-e_+[0], -e_-[0] - \frac{1}{k}\pi_d(D_0)\right) = \left(-e_+[0], -e_-[0] - \frac{1}{k}\text{div}(Q(s^d))\right). \]

The element \( x = s^{e_+}u \in A'_1 \) is a generator of \( A'_1 \) as a \( \mathbb{C}[s] \)-module. According to Example 4.10 in [FlZa] the graded algebra \( A' \) is isomorphic to the normalization of

\[ B = \mathbb{C}[x, y, s]/(x^ky - s^{k(e_++e_-)}Q(s^d)). \]

The cyclic group \( E_d \) acts on \( A' \) via

\[ \zeta \cdot x = \zeta^{e_+}x, \quad \zeta \cdot y = \zeta^{ke_-}y, \quad \zeta \cdot s = \zeta s \]

with invariant ring \( A \). Clearly this action stabilizes the subring \( B \). Assigning to \( x, y, z, s \) the degrees as in (4), \( F \) as in (3) is indeed weighted homogeneous. Since \( F(x, y, 1, s) = x^ky - s^{k(e_++e_-)}Q(s^d) \), the graded algebra

\[ R = \mathbb{C}[x, y, z, s]/(F) \]

satisfies \( R/(z - 1) \cong B \). Applying Proposition 2.1, \( V = \text{Spec} \ A \) is isomorphic to the normalization of \( \mathbb{D}_+(z) \cap \mathbb{V}_+(F) \) in the weighted projective space \( \mathbb{P} \).

**Remark 2.4.** In general not all weights of the weighted projective space \( \mathbb{P} \) in (5) are positive. Indeed it can happen that \( ke_- + d\deg Q \leq 0 \). In this case we can choose \( \alpha \in \mathbb{N} \) with \( ke_- + d(\deg Q + \alpha) > 0 \) and consider instead of \( F \) the polynomial

\[ \tilde{F} = x^ky - s^{k(e_++e_-)}Q(s^d/z)z^{\deg Q+\alpha} \in \mathbb{C}[x, y, z, s], \]

which is now weighted homogeneous of degree \( k(e_++e_-) + d\deg Q + \alpha \) with respect to the positive weights

\[ \deg x = e_+, \quad \deg y = ke_- + d(\deg Q + \alpha), \quad \deg z = d, \quad \deg s = 1. \]

As before \( V = \text{Spec} \ A \) is isomorphic to the normalization of the principal open subset \( \mathbb{D}_+(z) \) of the hypersurface \( \mathbb{V}_+(F) \) in the weighted projective space

\[ \mathbb{P} = \mathbb{P}(e_+, ke_- + d(\deg Q + \alpha), d, 1). \]

In certain cases it is unnecessary in Theorem 2.3 to pass to normalization.

**Corollary 2.5.** Assume that in (2) one of the following conditions is satisfied.

(i) \( k = 1 \);

(ii) \( e_+ + e_- = 0 \), and \( D_0 \) is a reduced divisor.

Then \( V = \text{Spec} \ A \) is equivariantly isomorphic to the principal open subset \( \mathbb{D}_+(z) \) of the weighted projective hypersurface \( \mathbb{V}_+(F) \) as in (3) in the weighted projective space \( \mathbb{P} \) from (2).
Proof. In case (i) the hypersurface in $\mathbb{A}^3$ with equation
\[ F(x, y, 1, s) = xy - s^{e_+ + e_-}Q(s^d) = 0 \]
is normal. In other words, the quotient $R/(z - 1)$ of the graded ring $R = \mathbb{C}[x, y, z, s]/(F)$ is normal and so is its ring of invariants $(R/(z - 1))^{\mathbb{C}^*}$. Comparing with Theorem 2.3 the result follows.

Similarly, in case (ii)
\[ F(x, y, 1, s) = x^ky - Q(s^d). \]
Since the divisor $D_0$ is supposed to be reduced and $D_0(0) = 0$, the polynomials $Q(t)$ and then also $Q(s^d)$ both have simple roots. Hence the hypersurface $F(x, y, 1, s) = 0$ in $\mathbb{A}^3$ is again normal, and the result follows as before. \qed

Remark 2.6. The surface $V$ as in (2.2) is smooth if and only if the divisor $D_0$ is reduced and $-m_+m_-(D_+(0) + D_-(0)) = 1$, where $m_+ > 0$ is the denominator in the irreducible representation of $D_+(0)$, see Proposition 4.15 in [FKZ]. It can happen, however, that $V$ is smooth but the surface $V_+(F) \cap \mathbb{D}_+(z) \subseteq \mathbb{P}$ has non-isolated singularities. For instance, if in (2.2) $D_0 = 0$ (and so $Q = 1$), then $V$ is an affine toric surface. In fact, every affine toric surface different from $(\mathbb{A}^1)^2$ or $\mathbb{A}^1 \times \mathbb{A}^1$ appears in this way, see Lemma 4.2(b) in [FKZ].

In this case the integer $k > 0$ can be chosen arbitrarily. For any $k > 1$, the affine hypersurface $V_+(F) \cap \mathbb{D}_+(z) \subseteq \mathbb{P}$ with equation $x^ky - s^{k(e_+ + e_-)} = 0$ has non-isolated singularities and hence is non-normal. Its normalization $V = \text{Spec} A$ can be given as the Zariski open part $\mathbb{D}_+(z)$ of the hypersurface $V_+(xy - s^{e_+ + e_-})$ in $\mathbb{P}' = \mathbb{P}(e_+, e_-, d, 1)$ (which corresponds to the choice $k = 1$). Indeed, the element $y' = s^{e_+ + e_-}/x \in K$ with $y'^k = y$ is integral over $A$. However cf. Theorem 1.1(a).

Example 2.7. (Danilov-Gizatullin surfaces) We recall that a Danilov-Gizatullin surface $V(n)$ of index $n$ is the complement to a section $S$ in a Hirzebruch surface $\Sigma_d$, where $S^2 = n > d$. By a remarkable result of Danilov and Gizatullin up to an isomorphism such a surface only depends on $n$ and neither on $d$ nor on the choice of the section $S$, see e.g., [DaGi], [CNR], [GMMR] for a proof.

According to [FKZ] §5, up to conjugation $V(n)$ carries exactly $(n - 1)$ different $\mathbb{C}^*$-actions. They admit DPD-presentations
\[(D_+, D_-) = \left(-\frac{1}{d}[0], -\frac{1}{n-d}[1]\right), \quad \text{where} \quad d = 1, \ldots, n-1.\]

Applying Theorem 2.3 with $e_+ = 1$, $e_- = 0$, and $k = n - d$, the $\mathbb{C}^*$-surface $V(n)$ is the normalization of the principal open subset $\mathbb{D}_+(z)$ of the hypersurface $V_+(F_{n,d}) \subseteq \mathbb{P}(1, d, d, 1)$ of degree $n$, where
\[ F_{n,d}(x, y, z, s) = x^{n-d}y - s^{n-d}(s^d - z). \]

Taking here $d = 1$ it follows that $V(n)$ is isomorphic to the normalization of the hypersurface $x^{n-1}y - (s - 1)s^{n-1} = 0$ in $\mathbb{A}^3$.

As our next example, let us consider yet another remarkable class of surfaces. These were studied from different viewpoints in [MM, Theorem 1.1], [FKZ3, Theorem 1.1(iii)], [GMMR, 3.8-3.9], [KK, Theorem 1.1, and Example 1], [Za, Theorem 1(b) and Lemma

\[5\text{See 3.1(a) below.}\]
Theorem 2.8. For a smooth affine surface \( V \), the following conditions are equivalent.

(i) \( V \) is not Gizatullin and admits an effective \( \mathbb{C}^* \)-action and an \( \mathbb{A}^1 \)-fibration \( V \to \mathbb{A}^1 \) with exactly one degenerate fiber, which is irreducible.\(^6\)

(ii) \( V \) is \( \mathbb{Q} \)-acyclic, \( \bar{k}(V) = -\infty \) and \( V \) carries a curve \( \Gamma \cong \mathbb{A}^1 \) with \( \bar{k}(V \setminus \Gamma) \geq 0 \).

(iii) \( V \) is \( \mathbb{Q} \)-acyclic and admits an effective \( \mathbb{C}^* \) - and \( \mathbb{C}_+ \)-actions. Furthermore, the \( \mathbb{C}^* \)-action possesses an orbit closure \( \Gamma \cong \mathbb{A}^1 \) with \( \bar{k}(V \setminus \Gamma) \geq 0 \).

(iv) The universal cover \( \tilde{V} \to V \) is isomorphic to a surface \( x^ky - (s^d - 1) = 0 \) in \( \mathbb{A}^3 \), with the Galois group \( \pi_1(V) \cong E_d \) acting via \( \zeta \cdot (x, y, s) = (\zeta x, \zeta^{-k}y, \zeta^e s) \), where \( k > 1 \) and \( \gcd(e, d) = 1 \).

(v) \( V \) is isomorphic to the \( \mathbb{C}^* \)-surface with DPD presentation \( \text{Spec} \mathbb{C}[t][D_+, D_-] \), where

\[
(D_+, D_-) = \left( -\frac{e}{d}[0], \frac{e}{d}[0] - \frac{1}{k}[1] \right) \quad \text{with} \quad 0 < e \leq d \quad \text{and} \quad k > 1.
\]

(vi) \( V \) is isomorphic to the Zariski open subset

\[
\mathbb{D}_+(x^ky - s^d) \subseteq \mathbb{P}(e, d - ke, 1), \quad \text{where} \quad 0 < e \leq d \quad \text{and} \quad k > 1.
\]

Proof. In view of the references cited above it remains to show that the surfaces in (v) and (vi) are isomorphic. By Corollary 2.5(ii) with \( e_+ = -e_- = e \), the surface \( V \) as in (v) is isomorphic to the principal open subset \( \mathbb{D}_+(z) \) in the weighted projective hypersurface

\[
V_+(x^ky - (s^d - z)) \subseteq \mathbb{P}(e, d - ke, d, 1).
\]

Eliminating \( z \) from the equation \( x^ky - (s^d - z) = 0 \) yields (vi). \( \square \)

These surfaces admit as well a constructive description in terms of a blowup process starting from a Hirzebruch surface, see \([\text{GMMR} \text{ 3.8}] \) and \([\text{KK} \text{ Example 1}] \).

An affine line \( \Gamma \cong \mathbb{A}^1 \) on \( V \) as in (ii) is distinguished because it cannot be a fiber of any \( \mathbb{A}^1 \)-fibration of \( V \). In fact there exists a family of such affine lines on \( V \), see \([\text{Za}] \).

Some of the surfaces as in Theorem 2.8 can be properly embedded in \( \mathbb{A}^3 \) as Bertin surfaces \( x^e y - x - s^d = 0 \), see \([\text{FlZa} \text{ Example 5.5}] \) or \([\text{Za} \text{ Example 1}] \).

3. Gizatullin surfaces with a finite divisor class group

A Gizatullin surface is a normal affine surface completed by a zigzag i.e., a linear chain of smooth rational curves. By a theorem of Gizatullin \([\text{Gi}] \) such surfaces are characterized by the property that they admit two \( \mathbb{C}_+ \)-actions with different general orbits.

In this section we give an alternative proof of the Daigle-Russell Theorem \([\text{11}] \) cited in the Introduction. It will be deduced from the following result proven in \([\text{FKZ2} \text{ Corollary 5.16}] \).

Proposition 3.1. Every normal Gizatullin surface with a finite divisor class group is isomorphic to one of the following surfaces.

\(^6\) Since \( V \) is not Gizatullin there is actually a unique \( \mathbb{A}^1 \)-fibration \( V \to \mathbb{A}^1 \). A surface \( V \) as in (i) is necessarily a \( \mathbb{Q} \)-homology plane (or \( \mathbb{Q} \)-acyclic) that is, all higher Betti numbers of \( V \) vanish.

\(^7\) As usual, \( k \) stands for the logarithmic Kodaira dimension.
(a) The toric surfaces $V_{d,e} = \mathbb{A}^2/E_d$, where the group $E_d \cong \mathbb{Z}_d$ of $d$-th roots of unity acts on $\mathbb{A}^2$ via
$$\zeta.(x,y) = (\zeta x, \zeta^e y) .$$

(b) The non-toric $\mathbb{C}^*$-surfaces $V = \text{Spec} \mathbb{C}[t][D_+, D_-]$, where
$$(D_+, D_-) = \left( -\frac{e}{m}[p], \frac{e}{m}[p] - c[q] \right) \quad \text{with} \quad c \geq 1, \quad p,q \in \mathbb{A}^1, \quad p \neq q ,$$
and with coprime integers $e, m$ such that $1 \leq e < m$.

Conversely, any normal affine $\mathbb{C}^*$-surface $V$ as in (a) or (b) is a Gizatullin surface with a finite divisor class group.

Let us now deduce Theorem 1.1.

Proof of Theorem 1.1. To prove (a), we note that according to 2.1 the cyclic group $E_d$ acts on the ring $\mathbb{C}[x,y,z]/(z - 1) \cong \mathbb{C}[x,y]$ via $\zeta.x = \zeta x$, $\zeta.y = \zeta^e y$, and $\zeta.z = z$, where
$$\deg x = 1, \quad \deg y = e, \quad \text{and} \quad \deg z = d .$$
Hence $D_+(z) = \text{Spec} \mathbb{C}[x,y]^{E_d} = V_{d,e}$, as required in (a).

To show (b) we consider $V = \text{Spec} A$ as in 3.1(b), where
$$A = \mathbb{C}[t][D_+, D_-] \subseteq \mathbb{C}[t][u,u^{-1}] .$$
By definition (11) the homogeneous pieces $A_{\pm 1}$ of $A$ are generated as $\mathbb{C}[t]$-modules by the elements
$$u_+ = tu \quad \text{and} \quad u_- = (t - 1)^e u^{-1} ,$$
and similarly $A_{\pm m}$ by
$$v_+ = t^e u^m \quad \text{and} \quad v_- = t^{-e}(t - 1)^m u^{-m} .$$
Thus
$$u_+^m = t^{m-e} v_+, \quad u_-^m = t^e v_-, \quad \text{and} \quad u_+ u_- = t(t - 1)^c .$$
The algebra $A$ is the integral closure of the subalgebra generated by $u_\pm, v_\pm$ and $t$.

Consider now the normalization $A'$ of $A$ in the field $L = \text{Frac}(A)[u'_+],\text{ where}$
$$(10) \quad u'_+ = \sqrt{v_+} \quad \text{with} \quad d = cm .$$
Clearly the elements $\sqrt{v_+} = t^{\frac{e-m}{m}} u_+$ and then also $t^{\frac{e-m}{m}}$ both belong to $L$. Since $e$ and $m$ are coprime we can choose $\alpha, \beta \in \mathbb{Z}$ with $\alpha(e - m) + \beta m = 1$. It follows that the element $\tau := t^\frac{1}{m} = t^{\alpha(e - m)} t^{\beta m}$ is as well in $L$ whence being integral over $A$ we have $\tau \in A'$.

The element $u'_+$ as in (10) also belongs to $A'$ and as well $u'_- = \sqrt{v_-} \in A'$. Now $v_+ v_- = (t - 1)^m$, so taking $d$th roots we get for a suitable choice of the root $u'_-$,
$$(11) \quad u'_+ u'_- = \tau^m - 1 .$$
We note that $u_\pm, v_\pm$ and $t$ are contained in the subalgebra $B = \mathbb{C}[u'_+, u'_-, \tau] \subseteq A'$. The equation (11) defines a smooth surface in $\mathbb{A}^3$. Hence $B$ is normal and so
$$A' = B \cong \mathbb{C}[u'_+, u'_-, \tau]/(u'_+ u'_- - (\tau^m - 1)) .$$
By Lemma 3.2 below, for a suitable $\gamma \in \mathbb{Z}$ the integers $a = e - \gamma m$ and $d$ are coprime. We may assume as well that $1 \leq a < d$. We let $E_d$ act on $A'$ via $\zeta.u'_+ = \zeta^a u'_+$ and
Since \( \gcd(a,d) = 1 \), \( A \) is the invariant ring of this action. We claim that the action of \( E_d \) on \((u'_+, u'_-, \tau)\) is given by

\[
\zeta u'_+ = \zeta^a u'_+, \quad \zeta u'_- = \zeta^{-a} u'_- = \zeta^b u'_- \quad \text{and} \quad \zeta \tau = \zeta^c \tau,
\]

where \( b = d - a \). Indeed, the equality \( u'^c_+ = t^{\frac{c}{m}} u_+ = \tau^{e-m} u_+ \) implies that \( \zeta \tau^{e-m} = \zeta^ac^{-e-m} \). Since \( \tau = \tau^d(e-m) \tau^d \) the element \( \zeta \in E_d \) acts on \( \tau \) via \( \zeta \tau = \zeta^e \tau \). In view of the congruence \( \alpha a \equiv 1 \mod m \) the last expression equals \( \zeta^c \tau \). Now the last equality in (12) follows. In the equation \( u'_+ u'_- = \tau^m - 1 \) the term on the right is invariant under \( E_d \). Hence also the term on the left is. This provides the second equality in (12).

The algebra \( B = \mathbb{C}[u'_+, u'_-, \tau] \) is naturally graded via

\[ \deg u'_+ = a, \quad \deg u'_- = b, \quad \text{and} \quad \deg \tau = c. \]

According to Proposition 2.1 Spec \( A = \text{Spec} A'^{E_d} \) is the complement of the hypersurface \( \mathbb{V}_+(f) \) of degree \( d = a + b \) in the weighted projective plane

\[ \text{Proj}(B) = \mathbb{P}(a, b, c), \quad \text{where} \quad f = u'_+ u'_- - \tau^m, \]

proving (b). \( \square \)

To complete the proof we still have to show the following elementary lemma.

**Lemma 3.2.** Assume that \( c, m \in \mathbb{Z} \) are coprime. Then for every \( c \geq 2 \) there exists \( \gamma \in \mathbb{Z} \) such that \( \gamma m - e \) and \( c \) are coprime.

**Proof.** Write \( c = c' \gamma \) such that \( c' \) and \( m \) have no common factor and every prime factor of \( \gamma \) occurs in \( m \). Then for every \( \gamma \in \mathbb{Z} \) the integers \( \gamma m - e \) and \( \gamma \) have no common prime factor. Indeed, such a prime must divide \( m \) and then also \( e = \gamma m - (\gamma m - e) \). Hence it is enough to establish the existence of \( \gamma \in \mathbb{Z} \) such that \( \gamma m - e \) and \( c' \) are coprime. However, the latter is evident since the residue classes of \( \gamma m, \gamma \in \mathbb{Z} \), in \( \mathbb{Z}_{c'} \) cover this group. \( \square \)

**Remark 3.3.** 1. Two triples \((1, e, d)\) and \((1, e', d)\) as in Theorem 1.1(a) define the same affine toric surface if and only if \( ee' \equiv 1 \mod d \), see [FlZa, Remark 2.5].

2. As follows from Theorem 0.2 in [FKZ], the integers \( c, m \) in Theorem 1.1(b) are invariants of the isomorphism type of \( V \). Indeed, the fractional parts of both divisors \( D_+ \) as in (9) being nonzero and concentrated at the same point, there is a unique DPD presentation for \( V \) up to interchanging \( D_+ \) and \( D_- \), passing to an equivalent pair and applying an automorphism of the affine line \( A^1 = \text{Spec} \mathbb{C}[t] \).

Furthermore, from the proof of Theorem 1.1 one can easily derive that

\[ a \equiv e \mod m \quad \text{and} \quad b = mc - a \equiv -e \mod m. \]

Therefore also the pair \((a, b)\) is uniquely determined by the isomorphism type of \( V \) up to a transposition and up to replacing \((a, b)\) by \((a', b') = (a - sm, b + sm)\), while keeping \( \gcd(a', b') = 1 \).

**References**


