LOCAL B-MODEL AND MIXED HODGE STRUCTURE

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ABSTRACT. We study the mixed Hodge theoretic aspects of the B-model side of local mirror symmetry. Our main objectives are to define an analogue of the Yukawa coupling in terms of the variations of the mixed Hodge structures and to study its properties. We also describe a local version of Bershadsky–Cecotti–Ooguri–Vafa's holomorphic anomaly equation.

1. INTRODUCTION

1.1. **Local mirror symmetry.** Mirror symmetry states a relationship between the genus zero Gromov–Witten theory ("A-model") of a Calabi–Yau threefold X and the Hodge theory ("B-model") of its mirror Calabi–Yau threefold $X^\vee$. After the first example of a quintic hypersurface in $\mathbb{P}^4$ and its mirror [18, 9], Batyrev [6] showed that a mirror pair of Calabi–Yau hypersurfaces in toric varieties can be constructed from a reflexive polyhedron[1]. Local mirror symmetry was derived from mirror symmetry for toric Calabi–Yau hypersurfaces by considering a certain limit in the Kähler and complex moduli spaces[2] [28] [11]. Chiang–Klemm–Yau–Zaslow [11] gave quite a thorough mathematical treatment to it. Their result can be summarized as follows.

Take a 2-dimensional reflexive polyhedron Δ (see Figure 1 for examples). On one side ("lcoal A-model" side), one considers the genus zero local Gromov–Witten (GW) invariants of a smooth weak Fano toric surface $\mathbb{P}_{\Sigma(\Delta^*)}$ which is determined by the 2-dimensional complete fan $\Sigma(\Delta^*)$ generated by integral points of Δ. On the other side ("local B-model" side), one considers a system of differential equations associated to Δ called the A-hypergeometric system with parameter zero due to Gel'fand–Kapranov–Zelevinsky [16, 17]. Then the statement of local mirror symmetry is that the genus zero local GW invariants can be obtained from solutions of the A-hypergeometric system.

Remark 1.1. The problem of computing the local GW invariants, not only at genus zero but also at all genera, is solved completely by the method of the topological vertex [1].

1.2. **Local B-model and the mixed Hodge structure.** When one compares local mirror symmetry with mirror symmetry, it is easy to see an analogy between the A-model (GW invariants) and the local A-model (local GW invariants). To compare the B-model and the

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[1] See §3.5 for the definition of reflexive polyhedra.

[2] This limit typically corresponds to a situation on the A-model side where one considers the effect of a *local* geometry of a weak Fano surface within a Calabi–Yau threefold. Hence the term "local mirror symmetry". See [28, §4].

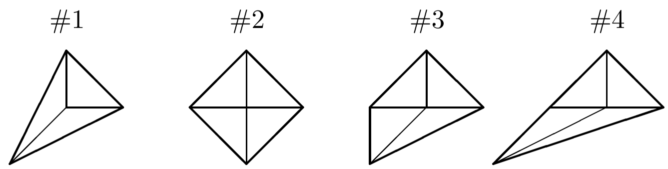

FIGURE 1. Examples of 2-dimensional reflexive polyhedra. ($\mathbb{P}^2$, $\mathbb{F}_0, \mathbb{F}_1, \mathbb{F}_2$ cases.)

local B-model, let us look into them in more detail. A natural framework for the B-model is the variation of polarized Hodge structures on $H^3(X^\vee)$ [3] (cf. [12, Ch.5], [36, Ch.1, Ch.3]). One considers

(i) the family $\pi : \mathcal{X} \to B$ of complex deformations of the Calabi–Yau threefold $X^\vee$,
(ii) a relative holomorphic three form $\Omega_{\mathcal{X}/B}$ which, together with the elements obtained by successive applications of the Gauss–Manin connection ∇, spans $H^3(X^\vee)$,
(iii) the Picard–Fuchs system for period integrals of $\Omega_{\mathcal{X}/B}$,
(iv) an $\mathcal{O}_B$-multilinear symmetric map from $TB \times TB \times TB$ to $\mathcal{O}_B$ called the Yukawa coupling:

$$Y_b(A_1, A_2, A_3) = \int_{X_b^\vee} \nabla_{A_1} \nabla_{A_2} \nabla_{A_3} \Omega_{\mathcal{X}/B} \wedge \Omega_{\mathcal{X}/B}\,, \qquad (b \in B)\ .$$

Let us turn to the local B-model. Our proposal in this paper is that a natural language for the local B-model is the mixed Hodge structures and their variations. The mixed Hodge structure (MHS) due to Deligne [13] is a generalization of the Hodge structure with the extra data $\mathcal{W}_\bullet$ called the weight filtration. See §2. Although the cohomology $H^*(V^\circ)$ of an open smooth variety V° does not have a Hodge structure in general, it does have a canonical MHS [13, 14]. There is also a canonical one on the relative cohomology $H^*(U^\circ, V^\circ)$.

Now, let us explain what are the counterparts of (i)–(iv) in the local B-model. Let Δ be a 2-dimensional reflexive polyhedron as above and F_a be a Δ-regular Laurent polynomial, i.e., a Laurent polynomial of the form

$$F_a(t_1, t_2) = \sum_{m \in \Delta \cap \mathbb{Z}^2} a_m t^m \quad \in \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$$

which satisfies a certain regularity condition (cf. Definition 3.1). In the literature, two closely related manifolds associated to F_a are considered: the one is the affine curve C_a° in the 2-dimensional algebraic torus $\mathbb{T}^2 = (\mathbb{C}^*)^2$ defined by $F_a(t_1, t_2) = 0$ [11, §6], and the other is the open threefold $Z_a^\circ \subset \mathbb{T}^2 \times \mathbb{C}^2$ defined by $F_a + xy = 0$ [23, §8]. As we shall see, they give the same result. By varying the parameter $a = (a_m)$, we have a family of affine curves $\mathcal{Z} \to \mathbb{L}_{\mathrm{reg}}(\Delta)$ and a family of open threefolds $\mathcal{Z}' \to \mathbb{L}_{\mathrm{reg}}(\Delta)$. By taking a quotient by the following action of $\mathbb{T}^3 = (\mathbb{C}^*)^3$,

$$F_a(t_1, t_2) \mapsto \lambda_0 F_a(\lambda_1 t_1, \lambda_2 t_2)\,, \quad (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{T}^3,$$

[3] Throughout the paper, the coefficient of the cohomology group is $\mathbb{C}$ unless otherwise specified.

we also have the quotient families $\mathcal{Z}/\mathbb{T}^3 \to \mathcal{M}(\Delta)$ and $\mathcal{Z}'/\mathbb{T}^3 \to \mathcal{M}(\Delta)$. These correspond to (i). As a counterpart of (ii), we consider, for the affine curve C_a°, the class

$$\omega_0 = \Big[\Big(\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2}, 0\Big)\Big] \in H^2(\mathbb{T}^2, C_a^\circ)\ ^4 ,$$

in the relative cohomology $H^2(\mathbb{T}^2, C_a^\circ)$, and for the open threefold Z_a°, the class of a holomorphic 3-form:

$$\omega_a = \Big[\mathrm{Res}\frac{1}{F_a + xy}\frac{dt_1}{t_1}\frac{dt_2}{t_2}dxdy\Big] \in H^3(Z_a^\circ) .$$

The counterpart of (iii) is the A-hypergeometric system as explained in [11]. Batyrev [5] and Stienstra [33] studied the variation of MHS (VMHS) on $H^2(\mathbb{T}^2, C_a^\circ)$ and showed the followings: $H^2(\mathbb{T}^2, C_a^\circ) \cong \mathbb{C}\omega_0 \oplus PH^1(C_a^\circ)$ is isomorphic to a certain vector space $\mathcal{R}_{F_a}$; ω_0 and elements obtained by successive applications of the Gauss–Manin connection span $H^2(\mathbb{T}^2, C_a^\circ)$; ω_0 satisfies the A-hypergeometric system considered in [11]. For the polyhedron #1 in Figure 1, Takahashi [34] independently showed that integrals

$$\int_\Gamma \omega_0 , \quad \Gamma \in H_2(\mathbb{T}^2, C_a^\circ, \mathbb{Z}) ,$$

over 2-chains Γ whose boundaries lie in C_a° satisfy the same differential equation. For the open threefold Z_a°, there is a result by Hosono [25] that integrals

$$\int_\gamma \omega_a , \quad \gamma \in H_3(Z_a^\circ, \mathbb{Z}) ,$$

satisfy exactly the same A-hypergeometric system. It has been known that $H^3(Z_a^\circ) \cong H^2(\mathbb{T}^2, C_a^\circ)$. Gross [20, §4] described the isomorphism and mentioned that the integration of ω_a over a 3-cycle reduces to that of ω_0 over a 2-chain under the isomorphism. In this paper, we shall study the (V)MHS of $H^3(Z_a^\circ)$ and show that it has the same description as $H^2(\mathbb{T}^2, C_a^\circ) \cong \mathcal{R}_{F_a}$ and that ω_a plays the same role as ω_0. This is one of the main results of this paper (cf. Theorem 5.1).

Remark 1.2. In [11], Chiang et al. considered the "1-form" $\mathrm{Res}_{F_a=0}(\log F_a)\omega_0$ on C_a° and argued that its period integrals satisfy the A-hypergeometric system. The result by Batyrev, Stienstra, and Takahashi implies that $\omega_0 \in H^2(\mathbb{T}^2, C_a^\circ)$ gives a rigorous definition of this "1-form". This point was mentioned in [20].

Remark 1.3. Calculation of the (V)MHS of $H^3(Z_a^\circ)$ in this paper closely follows the result by Batyrev on the MHS of affine hypersurfaces in algebraic tori [5].

1.3. **Weight filtration and the Yukawa coupling.** At this point, one may ask what is the role of the weight filtration. Our answer is that it is needed to define an analogue of the Yukawa coupling. It is the main motivation of the present work. In general, the lowest level subspace of the weight filtration in $H^*(V^\circ)$ is the image of the cohomology $H^*(V)$ of a smooth compactification V (see, e.g., [32, Proposition 6.30]). In our cases, it turns out that the lowest level subspace $\mathcal{W}_1 H^2(\mathbb{T}^2, C_a^\circ)$ (resp. $\mathcal{W}_3 H^3(Z_a^\circ)$) of the weight filtration on $H^2(\mathbb{T}^2, C_a^\circ)$ (resp. $H^3(Z^\circ)$) is isomorphic to $H^1(C_a)$ (resp. $H^3(Z_a)$), where C_a (resp. Z_a)

[4]Note that the class ω_0 depends on the parameter a, although it is not indicated in the notation.

is a smooth compactification of C_a° (resp. Z_a°). Thus we can use the intersection product on $H^1(C_a)$ or $H^3(Z_a)$ to define an analogue of the Yukawa coupling.

As a counterpart to (iv), we propose the following definition (Definition 6.2). Consider the family of affine curves $\mathcal{Z} \to \mathbb{L}_{\mathrm{reg}}(\Delta)$. Let $T^0\mathbb{L}_{\mathrm{reg}}(\Delta)$ be the subbundle of the holomorphic tangent bundle $T\mathbb{L}_{\mathrm{reg}}(\Delta)$ spanned by ∂_{a_0} [5]. Our Yukawa coupling is a multilinear map from $T\mathbb{L}_{\mathrm{reg}}(\Delta) \times T\mathbb{L}_{\mathrm{reg}}(\Delta) \times T^0\mathbb{L}_{\mathrm{reg}}(\Delta)$ to $\mathcal{O}_{\mathbb{L}_{\mathrm{reg}}(\Delta)}$. Take three vector fields $(A_1, A_2, A_3) \in T\mathbb{L}_{\mathrm{reg}}(\Delta) \times T\mathbb{L}_{\mathrm{reg}}(\Delta) \times T^0\mathbb{L}_{\mathrm{reg}}(\Delta)$. By the result on VMHS, we see that $\nabla_{A_3}\omega_0$ can be regarded as a $(1,0)$-form on C_a, and that although $\nabla_{A_1}\nabla_{A_2}\omega_0$ may not be in $\mathcal{W}_1$, we can associate a 1-form $(\nabla_{A_1}\nabla_{A_2}\omega_0)'$ on C_a (Lemma 6.1). We define

$$\mathrm{Yuk}(A_1, A_2; A_3) = \sqrt{-1}\int_{C_a} (\nabla_{A_1}\nabla_{A_2}\omega_0)' \wedge \nabla_{A_3}\omega_0 \ .$$

It is also possible to define the Yukawa coupling using the family of open threefolds. In fact they are the same up to multiplication by a nonzero constant. We also have a similar definition for the quotient family (§6.5).

In addition to the above geometric definition, we give an algebraic description of the Yukawa coupling via a certain pairing considered by Batyrev [5] (cf. §6.2, 6.3). We also derive the differential equations for them (Proposition 6.9, Lemma 6.12). These results enables us to compute the Yukawa couplings at least in the examples shown in Figure 1 (cf. Example 6.13, §8). They agree with the known results [29, 15, 2, 8, 22, 3]. We also see that they are mapped to the local A-model Yukawa couplings by the mirror maps (cf. Example 6.15, §8).

1.4. **Local B-model at higher genera.** If we are to pursue further the analogy between mirror symmetry and local mirror symmetry to higher genera, the first thing to do is to formulate an analogue of the so-called special Kähler geometry. It is a Kähler metric on the moduli B of complex deformations of a Calabi–Yau threefold $X^\vee$ whose curvature satisfies a certain equation called the special geometry relation. In the setting of the local B-model, we define a Hermitian metric on the rank one subbundle $T^0\mathcal{M}(\Delta)$ of $T\mathcal{M}(\Delta)$ and derive an equation similar to the special geometry relation (Lemma 7.1).

Next, we consider Bershadsky–Cecotti–Ooguri–Vafa's (BCOV's) holomorphic anomaly equation [7]. It is a partial differential equation for the B-model topological string amplitudes F_g's[6] which involves the Kähler metric, its Kähler potential and the Yukawa coupling. By making use of the result on the VMHS of $H^2(\mathbb{T}^2, C_a^\circ)$ (or $H^3(Z_a^\circ)$), we propose how to adapt the holomorphic anomaly equation to the setting of the local B-model (eqs. (7.4), (7.5)). We also explain it from Witten's geometric quantization approach [39].

In the examples shown in Figure 1, we checked that the solutions of this holomorphic anomaly equation with appropriate holomorphic ambiguities and with the holomorphic limit give the correct local GW invariants for $g = 1, 2$ at least for small degrees.

Remark 1.4. It is known that, in the local setting, the Kähler potential drops out from BCOV's holomorphic anomaly equation, and consequently the equation is solved by a certain

[5]Here a_0 is the parameter corresponds to the origin $(0,0) \in \Delta \cap \mathbb{Z}^2$.

[6] Its mathematical definition is yet unknown for $g \geq 2$.

Feynman rule with only one type of propagators $S^{i,j}$ with two indices [29, 24, 2, 22, 3]. Moreover, it is also known that essentially only one direction in $\mathcal{M}(\Delta)$ corresponding to the moduli of the elliptic curve C_a is relevant. Our description of BCOV's holomorphic anomaly equation is based on these results.

1.5. **Plan of the paper.** In §2, we recall the definition of the mixed Hodge structure. In §3, we define the vector space $\mathcal{R}_{F_a}$ following Batyrev [5] and recall Gel'fand–Kapranov–Zelevinsky's A-hypergeometric system [16, 17]. In §4, we give descriptions due to Batyrev [5] and Stienstra [33] of (V)MHS on the relative cohomology $H^n(\mathbb{T}^n, V_a^\circ)$ of the pair of the n-dimensional algebraic torus $\mathbb{T}^n$ and an affine hypersurface V_a° in terms of $\mathcal{R}_{F_a}$ (Theorem 4.2). In §5, we state the result on the MHS on the cohomology $H^3(Z_a^\circ)$ of the open threefold Z_a° and its relationship to that on $H^2(\mathbb{T}^2, C_a^\circ)$ (Theorem 5.1). The details for the calculation of $H^3(Z_a^\circ)$ are given in §A. In §6, we define the Yukawa coupling and study its properties. In §7 we propose a holomorphic anomaly equation for the local B-model.

We note that polyhedra dealt with in §3 and §4 are convex integral polyhedra, while in §5, §6 and §7, only 2-dimensional reflexive polyhedra are considered.

The examples treated in this article are listed in Figure 1. These will be sometimes called the cases of $\mathbb{P}^2$, $\mathbb{F}_0$, $\mathbb{F}_1$, $\mathbb{F}_2$ according to their local A-model toric surfaces. The $\mathbb{P}^2$ case appears in the course of the paper. The others are summarized in §8.

1.6. **Notations.** Throughout the paper, we use the following notations. $\mathbb{T}^n$ denotes the n-dimensional algebraic torus $(\mathbb{C}^*)^n$. For $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$, t^m stands for the Laurent monomial $t_1^{m_1} \cdots t_n^{m_n} \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. For a variable x, θ_x is the logarithmic derivative $x\partial_x$.

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2. Preliminaries on the mixed Hodge structures

In this section we recall the mixed Hodge structure of Deligne [13, 14]. See also [37, 38] [32].

A mixed Hodge structure (MHS) H is the triple $H = (H_{\mathbb{Z}}, \mathcal{W}_\bullet, \mathcal{F}^\bullet)$, where $H_{\mathbb{Z}}$ is a finitely generated abelian group, $\mathcal{W}_\bullet$ is an increasing filtration (called *the weight filtration*) on $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes \mathbb{Q}$, and $\mathcal{F}^\bullet$ is a decreasing filtration (called *the Hodge filtration*) on $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$ such that for each graded quotient

$$\mathrm{Gr}_n^{\mathcal{W}}(H_{\mathbb{Q}}) := \mathcal{W}_n / \mathcal{W}_{n-1} ,$$

with respect to the weight filtration $\mathcal{W}_\bullet$, the Hodge filtration $\mathcal{F}^\bullet$ induces a decomposition

$$\mathrm{Gr}_n^{\mathcal{W}}(H_{\mathbb{C}}) = \bigoplus_{p+q=n} H^{p,q}\,,$$

with $\overline{H^{p,q}} = H^{q,p}$, where $H^{p,q} := \mathrm{Gr}_{\mathcal{F}}^p \mathrm{Gr}_{p+q}^{\mathcal{W}}(H_{\mathbb{C}})$ and the bar denotes the complex conjugation. We say that the weight $m \in \mathbb{Z}$ occurs in H if $\mathrm{Gr}_m^{\mathcal{W}} \neq 0$, and that H is a pure Hodge structure of weight m if m is the only weight which occurs in H. By the classical Hodge theory, if X is a compact Kähler manifold, then $H^k(X)$ carries a pure Hodge structure of weight k.

Let V° be a smooth open algebraic variety of dimension n. By Deligne [13, 14], there is a canonical MHS on $H^k(V^\circ)$. The weights of $H^k(V^\circ)$ may occur in the range $[k, 2k]$ if $k \leq n$ and in $[k, 2n]$ if $k \geq n$. The construction goes as follows. Take a smooth compactification V of V° such that the divisor $D = V \setminus V^\circ$ is simple normal crossing, and consider meromorphic differential forms on V which may have logarithmic poles along the divisor D. Then the Hodge filtration is given by the degree of logarithmic forms while the weight filtration is given by the pole order. The constructed MHS does not depend on the chosen compactification V and is functorial, i.e., any morphism $f : V^\circ \to U^\circ$ of varieties induces a morphism $f^* : H^*(U^\circ) \to H^*(V^\circ)$ of MHS's.

Let $\iota : V^\circ \hookrightarrow U^\circ$ be an immersion between two smooth open algebraic varieties. By [14], there exists a canonical MHS on the relative cohomology $H^k(U^\circ, V^\circ)$. The construction is similar to the one above (cf. [32, Ch. 5], [38, Ch. 8]). The long exact sequence

$$\cdots \xrightarrow{\iota^*} H^{k-1}(V^\circ) \longrightarrow H^k(U^\circ, V^\circ) \longrightarrow H^k(U^\circ) \xrightarrow{\iota^*} H^k(V^\circ) \longrightarrow \cdots\,, \tag{2.1}$$

is an exact sequence of MHS's. The weights of $H^k(U^\circ, V^\circ)$ may occur in $[k-1, 2k]$.

The m-th Tate structure $T(m)$ is the pure Hodge structure of weight $-2m$ on the lattice $(2\pi\sqrt{-1})^m\mathbb{Z} \subset \mathbb{C}$ which is of type $(-m, -m)$, i.e., $T(m)_{\mathbb{C}} = T(m)^{-m,-m}$.

Example 2.1. The MHS on $H^m(\mathbb{T}^n)$ is $T(-m)^{\oplus\binom{n}{m}}$ for $0 \leq m \leq n$. See [5, Example 3.9].

Example 2.2. Let C be a smooth projective curve and $C^\circ = C \setminus D$ be an affine curve, where $D = \{p_1, \cdots, p_m\}$ is a set of distinct m-points on C. We describe the MHS on $H^1(C^\circ)$. The weight filtration $\mathcal{W}_\bullet$ and the Hodge filtration $\mathcal{F}^\bullet$ are:

$$0 \subset \mathcal{W}_1 = H^1(C) \subset \mathcal{W}_2 = H^1(C^\circ)\,, \quad 0 \subset \mathcal{F}^1 = H^0(\Omega_C^1(\log D)) \subset \mathcal{F}^0 = H^1(C^\circ)\,,$$

where $\Omega_C^1(\log D)$ is the sheaf of logarithmic 1-forms on (C, D). The "Hodge decomposition" of $\mathrm{Gr}_1^{\mathcal{W}}\left(H^1(C^\circ)\right) = H^{1,0} \oplus H^{0,1}$ is the same as that of $H^1(C)$:

$$H^{1,0} = H^{1,0}(C)\,, \quad H^{0,1} = H^{0,1}(C)\,.$$

That of $\mathrm{Gr}_2^{\mathcal{W}}\left(H^1(C^\circ)\right) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$ is,

$$H^{1,1} = H^0(\Omega_C^1(\log D))/H^0(\Omega_C^1)\,, \quad H^{2.0} = H^{0,2} = 0\,.$$

Hence the "Hodge numbers" of $H^1(C^\circ)$ are:

$$\begin{array}{c|ccc} & p=0 & 1 & 2 \\ \hline q=0 & & g & 0 \\ 1 & g & m-1 & \\ 2 & 0 & & \end{array} ,$$

where g is the genus of C. The MHS on $H^1(C^\circ)$ is an extension of $T(-1)^{\oplus(m-1)}$ by $H^1(C)$ in the sense of [10]:

$$0 \to H^1(C) \to H^1(C^\circ) \to T(-1)^{\oplus(m-1)} \to 0\,.$$

3. Polyhedron, Jacobian ring, $\mathcal{R}_F$ and A-hypergeometric system

A convex integral polyhedron $\Delta \subset \mathbb{R}^n$ is the convex hull of some finite set in $\mathbb{Z}^n$. The set of integral points in Δ is denoted by $A(\Delta)$ and its cardinality by $l(\Delta) := \#A(\Delta)$.

3.1. **Δ-regularity.** Let Δ be an n-dimensional integral convex polyhedron. Equip the ring $\mathbb{C}[t_0, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ with the grading given by $\det t_0^k t^m = k$. Define $\mathbf{S}_\Delta$ to be its subring:

$$\mathbf{S}_\Delta = \bigoplus_{k\geq 0} \mathbf{S}_\Delta^k\ , \qquad \mathbf{S}_\Delta^k = \bigoplus_{m\in\Delta(k)} \mathbb{C}\, t_0^k t^m\ ,$$

where

$$\Delta(k) := \left\{ m \in \mathbb{R}^n \mid \frac{m}{k} \in \Delta \right\}\ (k \geq 1)\ , \quad \Delta(0) := \{0\} \subset \mathbb{R}^n\ . \tag{3.1}$$

Recall that the Newton polyhedron of a Laurent polynomial

$$F = \sum_m a_m t^m \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

is the convex hull of $\{m \in \mathbb{Z}^n \mid a_m \neq 0\}$ in $\mathbb{R}^n$. Denote by $\mathbb{L}(\Delta)$ the space of Laurent polynomials whose Newton polyhedra are Δ.

Definition 3.1. A Laurent polynomial F is said to be Δ-regular if $F \in \mathbb{L}(\Delta)$ and, for every l-dimensional face $\Delta' \subset \Delta$ $(0 < l \leq n)$, the equations

$$F^{\Delta'} := \sum_{m\in\Delta'\cap\mathbb{Z}^n} a_m t^m\ = 0\ , \quad \frac{\partial F^{\Delta'}}{\partial t_1} = 0\ , \ldots , \frac{\partial F^{\Delta'}}{\partial t_n} = 0\ ,$$

have no common solutions in $\mathbb{T}^n$. Denote by $\mathbb{L}_{\rm reg}(\Delta)$ the space of Δ-regular Laurent polynomials.

Example 3.2. Let $\Delta \subset \mathbb{R}^2$ be the polyhedron #1 in Figure 1, which is the convex hull of $\{(1,0), (0,1), (-1,-1)\}$. Let $F \in \mathbb{L}(\Delta)$ which is of the form:

$$F = a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1 t_2} + a_0\ , \quad (a_0, a_1, a_2, a_3 \in \mathbb{C})\ . \tag{3.2}$$

We wrote $a_{(1,0)} = a_1$, $a_{(0,1)} = a_2$, $a_{(-1,-1)} = a_3$ for simplicity. Then we have

$$F \in \mathbb{L}_{\rm reg}(\Delta) \iff \frac{a_0^3}{a_1 a_2 a_3} + 27 \neq 0\ ,\ a_1 a_2 a_3 \neq 0\ .$$

3.2. **Jacobian ring, $\mathcal{R}_F$ and filtrations.** For $F \in \mathbb{L}(\Delta)$, let $\mathcal{D}_i$ $(i = 0, \dots, n)$ be the following differential operators acting on $\mathbf{S}_\Delta$:

$$\mathcal{D}_0 := \theta_{t_0} + t_0 F \ , \quad \mathcal{D}_i := \theta_{t_i} + t_0 \theta_{t_i} F \ (i = 1, \dots, n) \ . \tag{3.3}$$

Definition 3.3. Define $\mathbb{C}$-vector spaces $\mathcal{R}_F$ and $\mathcal{R}_F^+$ by

$$\mathcal{R}_F := \mathbf{S}_\Delta \Big/ \sum_{i=0}^{n} \mathcal{D}_i \mathbf{S}_\Delta \ , \quad \mathcal{R}_F^+ := \mathbf{S}_\Delta^+ \Big/ \sum_{i=0}^{n} \mathcal{D}_i \mathbf{S}_\Delta \ , \tag{3.4}$$

where $\mathbf{S}_\Delta^+ = \sum_{k \geq 1} \mathbf{S}_\Delta^k$.

Obviously, $\mathcal{R}_F = \mathcal{R}_F^+ \oplus \mathbf{S}_\Delta^0$.

We consider two filtrations on the vector spaces $\mathcal{R}_F$. The $\mathcal{E}$-filtration on $\mathbf{S}_\Delta$ is a decreasing filtration

$$\mathcal{E} : \cdots \supset \cdots \supset \mathcal{E}^{-k} \supset \cdots \mathcal{E}^{-1} \supset \mathcal{E}^0 \supset \cdots$$

where $\mathcal{E}^{-k}$ is the subspace spanned by all monomials of the degree $\leq k$. This induces filtrations on $\mathcal{R}_F$ and $\mathcal{R}_F^+$ which are denoted by $\mathcal{E}$ and $\mathcal{E}_+$ respectively. It holds that $\mathcal{E}^{-n} = \mathcal{R}_F$.

Definition 3.4. Let J_F be the ideal in $\mathbf{S}(\Delta)$ generated by $t_0 F, t_0 \theta_{t_1} F, \dots, t_0 \theta_{t_n} F$. The Jacobian ring R_F is defined as $\mathbf{S}_\Delta / J_F$. Denote by R_F^i the i-th homogeneous piece of R_F.

The graded quotient of $\mathcal{R}_F$ with respect to the $\mathcal{E}$-filtration is given by the Jacobian ring:

$$\mathrm{Gr}_{\mathcal{E}}^{-i} \mathcal{R}_F = R_F^i \, .$$

Denote by $I_\Delta^{(j)} (0 \leq j \leq n+1)$ the homogeneous ideals in $\mathbf{S}_\Delta$ generated as $\mathbb{C}$-subspaces by all monomials $t_0^k t^m$ where $k \geq 1$ and $m \in \Delta(k)$ which does not belong to any face of codimension j. We set $I_\Delta^{(n+2)} = \mathbf{S}_\Delta$. These form an increasing chain of ideals in $\mathbf{S}_\Delta$:

$$0 = I_\Delta^{(0)} \subset I_\Delta^{(1)} \subset I_\Delta^{(2)} \subset \cdots \subset I_\Delta^{(n+1)} = \mathbf{S}_\Delta^+ \subset I_\Delta^{(n+2)} = \mathbf{S}_\Delta \, . \tag{3.5}$$

Let $\mathcal{I}_j \subset \mathcal{R}_F$ be the image of $I_\Delta^{(j)}$. These subspaces define an increasing filtration $\mathcal{I}$ on $\mathcal{R}_F$:

$$0 = \mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots \subset \mathcal{I}_{n+1} = \mathcal{R}_F^+ \subset \mathcal{I}_{n+2} = \mathcal{R}_F \ .$$

Later we will see that $\mathcal{R}_F$ is isomorphic to the cohomology mentioned in §1.2. The $\mathcal{I}$-(resp. $\mathcal{E}$-)filtration describes the weight (resp. Hodge) filtration of the MHS on it.

Example 3.5. Let Δ be the polyhedron #1 in Figure 1. Assume that $F \in \mathbb{L}_{\mathrm{reg}}(\Delta)$. Then we have

$$\mathcal{R}_F \cong \mathbb{C}\, 1 \oplus \mathbb{C}\, t_0 \oplus \mathbb{C}\, t_0^2 \ .$$

The $\mathcal{I}$- and the $\mathcal{E}$-filtrations are

$$\mathcal{I}_3 = \mathcal{I}_2 = \mathcal{I}_1 \cong \mathbb{C}\, t_0 \oplus \mathbb{C}\, t_0^2 \ , \quad \mathcal{I}_4 = \mathcal{R}_F \ .$$

$$\mathcal{E}^0 = \mathbb{C}\, 1 \ , \quad \mathcal{E}^{-1} = \mathbb{C}\, 1 \oplus \mathbb{C}\, t_0 \ , \quad \mathcal{E}^{-2} = \mathcal{R}_F \ .$$

3.3. **Derivations with respect to parameters.** Let $a = (a_m)_{m\in A(\Delta)}$ be algebraically independent coefficients. Consider the $\mathbb{C}[a]$-module $\mathbf{S}_\Delta[a] := \mathbf{S}_\Delta \otimes \mathbb{C}[a]$. Let

$$\mathcal{R}_F[a] := \mathbf{S}_\Delta[a] \Big/ \Big(\sum_{i=0}^{n} \mathcal{D}_i \mathbf{S}_\Delta[a] \Big) \ . \tag{3.6}$$

Define the action of differential operators $\mathcal{D}_{a_m}$ ($m \in A(\Delta)$) on $\mathbf{S}_\Delta[a]$ by

$$\mathcal{D}_{a_m} = \frac{\partial}{\partial a_m} + t_0 t^m \ .$$

Since this action commutes with that of $\mathcal{D}_i$ ($i = 0, 1, \dots, n$), it induces an action of $\mathcal{D}_{a_m}$ on $\mathcal{R}_F[a]$.

We shall see that the operator $\mathcal{D}_{a_m}$ corresponds to the Gauss–Manin connection ∇_{a_m} on the cohomology of our interest (cf. §4.3, §5). Note that $\mathcal{D}_{a_m}$ preserves the $\mathcal{I}$-filtration: $\mathcal{D}_{a_m}\mathcal{I}^j \subset \mathcal{I}^j$. This corresponds to the fact that the weight filtration is preserved by the variation of MHS's. Note also that $\mathcal{D}_{a_m}$ decreases $\mathcal{E}$-filtration by one: $\mathcal{D}_{a_m}\mathcal{E}^{-k} \subset \mathcal{E}^{-k-1}$. This corresponds to the Griffiths transversality [35].

3.4. ***A*-hypergeometric system.** We briefly recall the A-hypergeometric system of Gel'fand–Kapranov–Zelevinsky [16] [17] in a form suitable to our situation. Let Δ be an n-dimensional integral convex polyhedron. For Δ, the lattice of relations is defined by

$$L(\Delta) := \Big\{ l = (l_m)_{m\in A(\Delta)} \in \mathbb{Z}^{l(\Delta)} \mid \sum_{m\in A(\Delta)} l_m m = 0 \ , \ \sum_{m\in A(\Delta)} l_m = 0 \Big\} \ . \tag{3.7}$$

The A-hypergeometric system associated to Δ (with parameters $(0, \dots, 0) \in \mathbb{C}^{n+1}$) is the following system of linear differential equations for $\Phi(a)$:

$$\mathcal{T}_i \Phi(a) = 0 \ (i = 0, 1, \dots, n) \ , \quad \Box_l \Phi(a) = 0 \ (l \in L(\Delta)) \ , \tag{3.8}$$

where

$$\mathcal{T}_0 = \sum_{m\in A(\Delta)} \theta_{a_m} \ , \quad \mathcal{T}_i = \sum_{m\in A(\Delta)} m_i \theta_{a_m} \quad (1 \le i \le n) \ ,$$
$$\Box_l = \prod_{\substack{m\in A(\Delta);\\ l_m>0}} \partial_{a_m}^{l_m} - \prod_{\substack{m\in A(\Delta);\\ l_m<0}} \partial_{a_m}^{-l_m} \ .$$

The number of independent solutions is equal to the volume[7] of the polyhedron Δ [16].

Example 3.6. In the case when Δ is the polyhedron #1 in Figure 1, the lattice of relations $L(\Delta)$ has rank one and generated by $(-3, 1, 1, 1)$. For simplicity we write $a_{(1,0)} = a_1$, $a_{(0,1)} = a_2$, $a_{(-1,-1)} = a_3$. The A-hypergeometric system is

$$(\theta_{a_1} - \theta_{a_3})\Phi(a) = 0 \ , \quad (\theta_{a_2} - \theta_{a_3})\Phi(a) = 0 \ ,$$
$$(\theta_{a_0} + \theta_{a_1} + \theta_{a_2} + \theta_{a_3})\Phi(a) = 0 \ ,$$
$$(\partial_{a_1}\partial_{a_2}\partial_{a_3} - \partial_{a_0}^3)\Phi(a) = 0 \ .$$

It is equivalent to $\Phi(a) = f(z)$, $z = \frac{a_1 a_2 a_3}{a_0^3}$ and

$$\big[\theta_z^3 + 3z\,\theta_z(3\theta_z + 1)(3\theta_z + 2)\big] f(z) = 0 \ .$$

[7] Here the volume is normalized so that the fundamental simplex in $\mathbb{R}^n$ has volume one.

Solutions about $z = 0$ are obtained in [11, eq.(6.22)]:

$$\begin{aligned} &\varpi(z;0) = 1 \ , \quad t := \partial_\rho \varpi(z;\rho)|_{\rho=0} = \log z + 3H(z) \ , \\ &\partial_S F = \partial_\rho^2 \varpi(z;\rho)|_{\rho=0} = (\log z)^2 + \cdots \end{aligned} \tag{3.9}$$

where

$$\varpi(z;\rho) = \sum_{n\geq 0} \frac{(3\rho)_{3n}}{(1+\rho)_n^3} (-1)^n z^{n+\rho} \ , \quad H(z) = \sum_{n\geq 1} \frac{(3n-1)!}{n!^3} (-z)^n \ .$$

Here $(\alpha)_n$ denotes the Pochhammer symbol: $(\alpha)_n = (\alpha)(\alpha+1)\cdots(\alpha+n-1)$ for $n \geq 1$, $(\alpha)_n = 1$ for $n = 0$.

Proposition 3.7. *1. For each $F \in \mathbb{L}_{\mathrm{reg}}(\Delta)$, $\mathcal{R}_F$ is spanned by $\mathcal{D}_{a_{m_1}} \cdots \mathcal{D}_{a_{m_k}} 1$ $(0 \leq k \leq n$, $m_1, \ldots, m_k \in A(\Delta))$.*
2. In $\mathcal{R}_F[a]$, the element 1 satisfies the A-hypergeometric system (3.8) with ∂_{a_m} $(m \in A(\Delta))$ replaced by $\mathcal{D}_{a_m}$.

Proof. 1. This follows because $\mathbf{S}_\Delta$ is spanned by monomials obtained by successive applications of $\mathcal{D}_{a_m}$ to 1 and because $\mathcal{R}_F = \mathcal{E}^{-n}$.
2. In the ring $\mathbf{S}_\Delta[a]$, it holds that

$$\begin{aligned} &\left(\mathcal{T}_i|_{\theta_{a_m} \to a_m \mathcal{D}_{a_m}}\right) 1 = \mathcal{D}_i \, 1 \quad (0 \leq i \leq n) \ , \\ &\left(\Box_l|_{\partial_{a_m} \to \mathcal{D}_{a_m}}\right) 1 = \prod_{m:l_m>0} (t^m)^{l_m} - \prod_{m:l_m<0} (t^m)^{-l_m} = 0 \ . \end{aligned} \tag{3.10}$$

□

3.5. **Reflexive polyhedra.** Recall from [6, §4] the following

Definition 3.8. An n-dimensional convex integral polyhedron $\Delta \subset \mathbb{R}^n$ is reflexive if $0 \in \Delta$ and the distance between 0 and the hypersurface generated by each codimension-one face Δ' is equal to 1, i.e., for each codimension-one face Δ' of Δ, there exists an integral primitive vector $v_{\Delta'} \in \mathbb{Z}^n$ such that $\Delta' = \{m \in \Delta \mid \langle v_{\Delta'}, m \rangle = 1\}$.

In the case when $n = 2$, it is known that there are exactly sixteen 2-dimensional reflexive polyhedra (see [11, Fig.1]). Let Δ be a 2-dimensional reflexive polyhedron and $F \in \mathbb{L}_{\mathrm{reg}}(\Delta)$. Then it is known that

$$\dim \mathrm{Gr}_{\mathcal{E}}^{-k} \mathcal{R}_F = \dim R_F^k = \begin{cases} 1 & (k = 0, 2) \, , \\ l(\Delta) - 3 & (k = 1) \, , \\ 0 & (k \geq 3). \end{cases}$$

See Theorem 4.8 in [5]. The $\mathcal{I}$- and the $\mathcal{E}$-filtrations on the vector space $\mathcal{R}_F$ are described as follows. Let $A'(\Delta) = A(\Delta) \setminus \{0, m^{(1)}, m^{(2)}, m^{(3)}\}$ where $m^{(1)}, m^{(2)}, m^{(3)}$ are any three vertices of Δ. Then we have

$$\begin{aligned} &\mathcal{R}_F = \mathcal{I}_4 = \mathcal{E}^{-2} \cong \mathbb{C}1 \oplus \bigoplus_{m \in A'(\Delta)} \mathbb{C} t_0 t^m \oplus \mathbb{C} t_0 \oplus \mathbb{C} t_0^2 \ , \\ &\mathcal{I}_1 \cong \mathbb{C} t_0 \oplus \mathbb{C} t_0^2 \ , \quad \mathcal{I}_3 \cong \mathcal{I}_1 \oplus \bigoplus_{m \in A'(\Delta)} \mathbb{C} t_0 t^m \ , \\ &\mathcal{E}^0 = \mathbb{C}1 \ , \quad \mathcal{E}^{-1} \cong \mathcal{E}^0 \oplus \bigoplus_{m \in A'(\Delta)} \mathbb{C} t_0 t^m \oplus \mathbb{C} t_0 \ . \end{aligned} \tag{3.11}$$

As to $\mathcal{I}_2$, it depends on the polyhedron Δ. For example, $\mathcal{I}_2 = \mathcal{I}_1$ for the polyhedra #2, #3 in Figure 1 while $\mathcal{I}_2 = \mathcal{I}_3$ for the polyhedra #4. See §8.

For a 2-dimensional reflexive polyhedron Δ, there are $l(\Delta) - 1$ independent solutions to the A-hypergeometric system associated to Δ. Explicit expressions for them can be found in [11, eq.(6.22)].

4. Mixed Hodge structures on $H^n(\mathbb{T}^n, V_a^\circ)$

Let $\Delta \subset \mathbb{R}^n$ be an n-dimensional convex integral polyhedron and $F_a = \sum a_m t^m \in \mathbb{L}_{\mathrm{reg}}(\Delta)$. We denote by V_a° the smooth affine hypersurface in $\mathbb{T}^n$ defined by F_a. We state the result on the (V)MHS on $H^n(\mathbb{T}^n, V_a^\circ)$ due to Batyrev [5] and Stienstra [33]. We remark that $H^k(\mathbb{T}^n, V_a^\circ) = 0$ if $k \neq n$ (cf. [5, Theorem 3.4]).

The cokernel of the pull-back $H^{n-1}(\mathbb{T}^n) \to H^{n-1}(V_a^\circ)$ is called the primitive part of the cohomology of V_a° and denoted by $PH^{n-1}(V_a^\circ)$. From the long exact sequence (2.1), we obtain the following short exact sequence of MHS's:

$$0 \longrightarrow PH^{n-1}(V_a^\circ) \longrightarrow H^n(\mathbb{T}^n, V_a^\circ) \longrightarrow H^n(\mathbb{T}^n) \longrightarrow 0\,. \tag{4.1}$$

Recall that the MHS on $H^n(\mathbb{T}^n)$ is the Tate structure $T(-n)$ (cf. Example 2.1). Therefore $H^n(\mathbb{T}^n, V_a^\circ)$ is an extension of $T(-n)$ by $PH^{n-1}(V_a^\circ)$.

4.1. **MHS on the primitive part $PH^{n-1}(V_a^\circ)$.** Let $R^+ : S_\Delta^+ \to \Gamma\Omega^{n-1}_{V_a^\circ}$ be the linear map given by

$$R^+(t_0^k t^m) = \mathrm{Res}_{V_a^\circ}\left(\frac{(-1)^k (k-1)! \cdot t^m}{F^k}\omega_0\right), \quad \omega_0 := \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}\,.$$

Theorem 4.1 (Batyrev)**.** *(i) R^+ induces an isomorphism*

$$\rho^+ : \mathcal{R}^+_{F_a} \xrightarrow{\cong} PH^{n-1}(V_a^\circ)\,.$$

(ii) Let $\mathcal{W}^+_\bullet$ be the weight filtration on $PH^{n-1}(V_a^\circ)$. Then, for $0 < i \le n-1$, we have

$$\rho^+(\mathcal{I}_i) = \mathcal{W}^+_{n-2+i}\,.$$

(iii) Let $\mathcal{F}^\bullet_+$ be the Hodge filtration on $PH^{n-1}(V_a^\circ)$. Then, for $0 \le i \le n-1$, we have

$$\rho^+(\mathcal{E}_+^{i-n}) = \mathcal{F}_+^i\,.$$

4.2. **MHS on the middle relative cohomology $H^n(\mathbb{T}^n, V_a^\circ)$.** Let $R^0 : S^0_\Delta \to \Gamma\Omega^n_{\mathbb{T}^n}$ be the linear map given by $R^0(1) = \omega_0$. Consider the map $R := R^+ \oplus R^0 : \mathbf{S}_\Delta \to \Gamma\Omega^n_{\mathbb{T}^n} \oplus \Gamma\Omega^{n-1}_{V_a^\circ}$. Then the following theorem follows from Theorem 4.1 (cf. [33, Theorem 7]).

Theorem 4.2 (Batyrev, Stienstra)**.** *(i) R induces an isomorphism*

$$\rho : \mathcal{R}_{F_a} \xrightarrow{\cong} H^n(\mathbb{T}^n, V_a^\circ)\,.$$

(ii) Let $\mathcal{W}_\bullet$ be the weight filtration on $H^n(\mathbb{T}^n, V_a^\circ)$. Then we have

$$\rho(\mathcal{I}_i) = \mathcal{W}_{n-2+i} \quad (0 < i \le n-1)\,, \quad \rho(\mathcal{I}_{n+1}) = \mathcal{W}_{2n-2} = \mathcal{W}_{2n-1}\,, \quad H^n(\mathbb{T}^n, V_a^\circ) = \mathcal{W}_{2n}\,.$$

(iii) Let $\mathcal{F}^\bullet$ be the Hodge filtration on $H^n(\mathbb{T}^n, V_a^\circ)$. Then, for $0 \le i \le n$, we have

$$\rho(\mathcal{E}^{i-n}) = \mathcal{F}^i\,.$$

4.3. **Gauss–Manin connection on $H^n(\mathbb{T}^n, V_a^\circ)$.** Consider the variation of MHS on $H^n(\mathbb{T}^n, V_a^\circ)$ over $\mathbb{L}_{\mathrm{reg}}(\Delta)$. It was studied by Stienstra [33, §6].

Lemma 4.3 (Stienstra)**.** *The Gauss–Manin connection $\nabla_{\frac{\partial}{\partial a_i}}$ on $H^n(\mathbb{T}^n, V_a^\circ)$ corresponds to the operator $\mathcal{D}_{a_i}$ on $\mathcal{R}_F[\mathbf{a}]$.*

Stienstra proved this by considering the de Rham complex $(\Omega^\bullet(\mathbb{T}^n), d + dF_a\wedge)$ [33, §6]. Here we give a different proof. This is a generalization of Takahashi's argument [34, Lemma 1.8].

Proof. Since F_a is Δ-regular, there exists a holomorphic $(n-1)$-form ψ_a in an open neighborhood of V_a° in $\mathbb{T}^n$ such that $\omega_0 = dF_a \wedge \psi_a$. The restriction of ψ_a to V_a° is equal to $\mathrm{Res}_{V_a^\circ} \frac{\omega_0}{F_a}$ and is denoted by $\frac{\omega_0}{dF_a}$. It is called the Gelfand–Leray form of ω_0 (cf. [4, Ch. 10]).

Let $\Gamma_a \in H_n(\mathbb{T}^n, V_a^\circ)$. Then one can show that

$$\frac{\partial}{\partial a_i} \int_{\Gamma_a} \omega_0 = - \int_{\partial \Gamma_a} \frac{\partial F_a}{\partial a_i} \frac{\omega_0}{dF_a} \, .$$

Namely, we have $\nabla_{\frac{\partial}{\partial a_i}} \rho(1) = \rho\left(\mathcal{D}_{a_i} 1\right)$. Since Batyrev [5] has shown that the Gauss–Manin connection $\nabla_{\frac{\partial}{\partial a_i}}$ on $PH^{n-1}(V_a^\circ)$ corresponds to $\mathcal{D}_{a_i}$ under ρ^+, the lemma follows. □

Lemma 4.3 and Proposition 3.7 imply the following

Corollary 4.4 (Stienstra)**.** *1. $H^n(\mathbb{T}^n, V_a^\circ)$ is spanned by $\nabla_{a_{m_1}} \cdots \nabla_{a_{m_k}} \omega_0$ $(0 \le k \le n$, $m_1, \ldots, m_k \in A(\Delta))$.*
2. ω_0 satisfies the A-hypergeometric system (3.8) *with ∂_{a_i} replaced by $\nabla_{\partial_{a_i}}$.*
3. The period integrals $\int_{\Gamma_a} \omega_0$ of the relative cohomology $H^n(\mathbb{T}^n, V_a^\circ)$ satisfies the $A(\Delta)$-hypergeometric system (3.8)*. Conversely, a solution of the A-hypergeometric system* (3.8) *is a period integral.*

5. Mixed Hodge structure on $H^3(Z_a^\circ)$

Throughout the section, Δ is a 2-dimensional reflexive polyhedron and $F_a \in \mathbb{L}_{\mathrm{reg}}(\Delta)$ is a Δ-regular Laurent polynomial.

5.1. **MHS on the cohomology of the threefold.** Consider the affine threefold Z_a° defined by

$$Z_a^\circ = \{(t, x, y) \in \mathbb{T}^2 \times \mathbb{C}^2 \mid F_a(t) + xy = 0\} \, . \tag{5.1}$$

We compute $H^3(Z_a^\circ)$ and its (V)MHS following Batyrev [5]. Let us briefly state the results. Details are relegated to §A.

The Poincaré residue map $\mathrm{Res} : H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z_a^\circ) \xrightarrow{\cong} H^3(Z_a^\circ)$ is an isomorphism (see eq. (A.2)). By Grothendieck [21], $H^\bullet(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z_a^\circ)$ is isomorphic to the cohomology of the global de Rham complex $(\Gamma\Omega^\bullet_{\mathbb{T}^2 \times \mathbb{C}^2}(*Z_a^\circ), d)$ of meromorphic differential forms on $\mathbb{T}^2 \times \mathbb{C}^2$ with poles of arbitrary order on Z_a°. We can show that the homomorphism:

$$R' : \mathbf{S}_\Delta \to \Gamma\Omega^4_{\mathbb{T}^2 \times \mathbb{C}^2}(*Z_a^\circ) \; ; \quad t_0^k t^m \mapsto \frac{(-1)^k k! \, t^m}{(F_a + xy)^{k+1}} \frac{dt_1}{t_1} \frac{dt_2}{t_2} dx dy \; ,$$

induces an isomorphism $\mathcal{R}_{F_a} \xrightarrow{\cong} H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z_a^\circ)$. Together with the residue map, we obtain an isomorphism $\rho' : \mathcal{R}_{F_a} \xrightarrow{\cong} H^3(Z_a^\circ)$. The Gauss–Manin connection $\nabla_{\partial_{a_m}}$ on $H^3(Z_a^\circ)$

corresponds to a differentiation by a_m on $\Gamma\Omega^4_{\mathbb{T}^2\times\mathbb{C}^2}(*Z^\circ_a)$, which in turn corresponds to the derivation $\mathcal{D}_{a_m}$ on $\mathcal{R}_{F_a}$ (§A.6).

To compute the weight and the Hodge filtrations, we compactify Z°_a as a smooth hypersurface in a toric variety (§A.2). Then we can work out calculation similar to [5, §6,§8]. (Since our Z°_a is a hypersurface in $\mathbb{T}^2\times\mathbb{C}^2$, not in $\mathbb{T}^4$, we need some modifications. Especially we need Mavlyutov's results on Hodge numbers of semiample hypersurfaces in a toric varieties [30].) It turns out that the weight and the Hodge filtrations are given by the $\mathcal{I}$ and the $\mathcal{E}$-filtrations on $\mathcal{R}_{F_a}$. The result on MHS of $H^3(Z^\circ_a)$ is summarized as follows (Theorems A.11, A.14).

$$\begin{aligned}&H^3(Z^\circ_a)=\mathcal{W}_6=\mathcal{F}^0=\mathcal{F}^1\cong\mathcal{R}_{F_a}\ ,\\ &\mathcal{W}_3\cong\mathcal{I}_1\ ,\quad \mathcal{W}_4=\mathcal{W}_5\cong\mathcal{I}_3\ ,\\ &\mathcal{F}^2\cong\mathcal{E}^{-1}\ ,\quad \mathcal{F}^3\cong\mathcal{E}^0\ .\end{aligned}\tag{5.2}$$

5.2. **Relationship to the relative cohomology.** Let C°_a be the affine curve in $\mathbb{T}^2$ defined by F_a. Since Δ is reflexive, it is an affine elliptic curve obtained by deleting $l(\Delta)-1$ points from an elliptic curve C_a. The MHS on the primitive part $PH^1(C^\circ_a)$ is an extension of $T(-1)^{\oplus(l(\Delta)-4)}$ by $H^1(C_a)$:

$$0\to H^1(C_a)\to PH^1(C^\circ_a)\to T(-1)^{\oplus(l(\Delta)-4)}\to 0\,.$$

This follows from the definition of primitive part and the description of $H^1(C^\circ_a)$ given in Example 2.2. The MHS on the relative cohomology $H^2(\mathbb{T}^2,C^\circ_a)$ is an extension (4.1) of $H^2(\mathbb{T}^2)=T(-2)$ by $PH^1(C_a)$ which turns out to be the trivial one (cf. Theorem 4.2).

Let $\rho:\mathcal{R}_{F_a}\overset{\cong}{\to}H^2(\mathbb{T}^2,C^\circ_a)$ be the isomorphism in Theorem 4.2 and $\omega_0=\rho(1)\in H^2(\mathbb{T}^2,C^\circ_a)$. Let

$$\omega_a=\rho'(1)=\left[\mathrm{Res}\frac{1}{F_a+xy}\frac{dt_1}{t_1}\frac{dt_2}{t_2}dxdy\right]\in H^3(Z^\circ_a)\,.$$

Theorem 5.1. *The composition of isomorphisms $H^3(Z^\circ_a)\overset{\rho'^{-1}}{\to}\mathcal{R}_{F_a}\overset{\rho}{\to}H^2(\mathbb{T}^2,C^\circ_a)$ gives an isomorphism*

$$\rho\circ\rho'^{-1}:H^3(Z^\circ_a)\overset{\cong}{\to}H^2(\mathbb{T}^2,C^\circ_a)$$

of $\mathbb{C}$-vector spaces which sends ω_a to ω_0. The filtrations correspond as follows:

$$\mathcal{F}^{i+1}H^3(Z^\circ_a)\overset{\cong}{\to}\mathcal{E}^{i-2}\overset{\cong}{\to}\mathcal{F}^iH^2(\mathbb{T}^2,C^\circ_a)\quad(i=0,1,2)\,,$$

$$\mathcal{W}_3H^3(Z^\circ_a)\overset{\cong}{\to}\mathcal{I}_1\overset{\cong}{\to}\mathcal{W}_1H^2(\mathbb{T}^2,C^\circ_a)\,,$$

$$\mathcal{W}_4H^3(Z^\circ_a)=\mathcal{W}_5H^3(Z^\circ_a)\overset{\cong}{\to}\mathcal{I}_3\overset{\cong}{\to}\mathcal{W}_2H^2(\mathbb{T}^2,C^\circ_a)=\mathcal{W}_3H^2(\mathbb{T}^2,C^\circ_a)\,.$$

Moreover, $\rho\circ\rho'^{-1}$ is compatible with the Gauss–Manin connections.

Note that $\mathcal{W}_1H^2(\mathbb{T}^2,C^\circ_a)=\mathcal{W}_1PH^1(C^\circ_a)\cong H^1(C_a)$. Therefore, it inherits a nondegenerate pairing. The same is true for $\mathcal{W}_3H^3(Z^\circ_a)$, since it is isomorphic to the cohomology $H^3(Z_a)$ of a certain smooth compactification Z_a of Z°_a (cf. §A.2)[8].

[8] The divisor $Z_a\setminus Z^\circ_a$ is not smooth but simple normal crossing. The pull-back $H^3(Z_a)\to\mathcal{W}_3H^3(Z^\circ_a)$, which is always surjective, turns out to be injective. This can be checked by comparing the dimension given in Lemma A.6 and that in Proposition A.12.

6. Analogue of Yukawa coupling

In this section, Δ is a 2-dimensional reflexive polyhedron unless otherwise specified.

6.1. **Definition of Yukawa coupling via affine curves or threefolds.** Let Δ be a 2-dimensional reflexive polyhedron. Let $T^0\mathbb{L}_{\mathrm{reg}}(\Delta)$ be the subbundle of the holomorphic tangent bundle $T\mathbb{L}_{\mathrm{reg}}(\Delta)$ of $\mathbb{L}_{\mathrm{reg}}(\Delta)$ generated by ∂_{a_0}. Consider the family of affine elliptic curves $p : \mathcal{Z} \to \mathbb{L}_{\mathrm{reg}}(\Delta)$:

$$\mathcal{Z} = \{(a,t) \in \mathbb{L}_{\mathrm{reg}}(\Delta) \times \mathbb{T}^2 \mid F_a(t) = 0\} \ .$$

Let C_a be the smooth compactification of the affine curve $C_a^\circ := p^{-1}(a)$. Note that we have $\mathrm{Gr}^0_{\mathcal{F}} H^2(\mathbb{T}^2, C_a^\circ) = \mathrm{Gr}^0_{\mathcal{F}} \mathcal{W}_1 H^2(\mathbb{T}^2, C_a^\circ)$.

Lemma 6.1. *For any $\alpha \in H^2(\mathbb{T}^2, C_a^\circ)$, there exists $\alpha' \in \mathcal{W}_1 H^2(\mathbb{T}^2, C_a^\circ)(\cong H^1(C_a))$ such that $[\alpha] = [\alpha']$ in $\mathrm{Gr}^0_{\mathcal{F}} H^2(\mathbb{T}^2, C_a^\circ) = \mathrm{Gr}^0_{\mathcal{F}} \mathcal{W}_1 H^2(\mathbb{T}^2, C_a^\circ)$.*

Proof. By (3.11), α is written as

$$\alpha = \alpha_{2,0}\rho(t_0^2) + \sum_{m \in A'(\Delta)} \alpha_{1,m}\rho(t_0 t^m) + \alpha_{1,0}\rho(t_0) + \alpha_{0,0}\omega_0 \ .$$

Take $\alpha' = \alpha_{2,0}\rho(t_0^2) + c\rho(t_0)$, where $c \in \mathbb{C}$ is arbitrary. □

The pairing

$$H^2(\mathbb{T}^2, C_a^\circ) \times \mathcal{F}^1\mathcal{W}_1 H^2(\mathbb{T}^2, C_a^\circ) \to \mathbb{C}\,; \quad (\alpha, \beta) \mapsto \int_{C_a} \alpha' \wedge \beta$$

is independent of the choice of α'. Recall that $\nabla_{a_0}\omega_0 \in \mathcal{F}^1\mathcal{W}_1 H^2(\mathbb{T}^2, C_a^\circ)$.

Definition 6.2. For $k \geq 1$, we define a map

$$\mathrm{Yuk}^{(k)} : \underbrace{T\mathbb{L}_{\mathrm{reg}}(\Delta) \times \cdots \times T\mathbb{L}_{\mathrm{reg}}(\Delta)}_{(k-1)\ \text{times}} \times T^0\mathbb{L}_{\mathrm{reg}}(\Delta) \to \mathcal{O}_{\mathbb{L}_{\mathrm{reg}}(\Delta)}$$

by

$$\mathrm{Yuk}^{(k)}(A_1, \ldots, A_{k-1}; A_k) = \int_{C_a} (\nabla_{A_1} \cdots \nabla_{A_{k-1}}\omega_0)' \wedge \nabla_{A_k}\omega_0 \ .$$

We call $\mathrm{Yuk}^{(3)}$ the Yukawa coupling and denote it by Yuk.

Remark 6.3. $\mathrm{Yuk}^{(1)} = \mathrm{Yuk}^{(2)} = 0$ by Griffiths' transversality. For $k \geq 4$, $\mathrm{Yuk}^{(k)}(A_1, \ldots, A_{k-1}; A_k)$ is $\mathcal{O}_{\mathbb{L}_{\mathrm{reg}}(\Delta)}$-linear in A_1, A_k and $\mathbb{C}$-linear in $A_2, \ldots, A_{k-1}$. For $k = 3$, $\mathrm{Yuk}^{(3)}$ is $\mathcal{O}_{\mathbb{L}_{\mathrm{reg}}(\Delta)}$-multilinear.

Remark 6.4. Instead of the relative cohomology $H^2(\mathbb{T}^2, C_a^\circ)$, we can use the cohomology $H^3(Z_a^\circ)$ of the open threefold Z_a° defined in (5.1), provided that the levels of Hodge and weight filtrations are shifted according to Theorem 5.1 and that the integration on C_a is replaced by that on the compact threefold Z_a defined in §A.2.

6.2. **Batyrev's paring.** We would like to give an algebraic description of the Yukawa coupling in terms of the Jacobian ring R_{F_a}. For that purpose, we recall Batyrev's pairing [5, §9]. Let Δ be an integral convex n-dimensional polyhedron and $F_a \in \mathbb{L}_{\mathrm{reg}}(\Delta)$ a Δ-regular Laurent polynomial. Denote by D_{F_a} the quotient

$$D_{F_a} := I_{\Delta}^{(1)} \Big/ (t_0 F_a, t_0\theta_{t_1} F_a, \ldots, t_0\theta_{t_n} F_a) \cdot I_{\Delta}^{(1)} \ .$$

It is a graded R_{F_a}-module consisting of the homogeneous pieces $D^i_{F_a}$ $(1 \leq i \leq n+1)$. We have $D^{n+1}_{F_a} \cong \mathbb{C}$. The multiplicative structure of R_{F_a}-module defines a nondegenerate pairing

$$\langle\ ,\ \rangle : R^i_{F_a} \times D^{n+1-i}_{F_a} \to D^{n+1}_{F_a} \cong \mathbb{C}\ .$$

Let H_{F_a} be the image of the homomorphism $D_{F_a} \to R_{F_a}$ induced by the inclusion $I_{\Delta}^{(1)} \hookrightarrow \mathbf{S}_{\Delta}$. Then the above pairing induces a nondegenerate pairing

$$\{\ ,\ \} : H^i_{F_a} \times H^{n+1-i}_{F_a} \to D^{n+1}_{F_a} \cong \mathbb{C};\ \{\alpha, \beta\} := \langle \alpha, \beta' \rangle\, ,$$

where $\beta' \in D^{n+1-i}_{F_a}$ is an element such that its image by the homomorphism $D^{n+1-i}_{F_a} \to H^{n+1-i}_{F_a}$ is β.

6.3. **Yukawa coupling in terms of Batyrev's pairing.** Now we come back to the case when Δ is a 2-dimensional reflexive polyhedron. In this case, we have $D_{F_a} \cong t_0 R_{F_a}$. We explain that the Yukawa coupling defined in Definition 6.2 is essentially Batyrev's pairing together with a choice (concerning the dependence on the parameter a) of the isomorphism

$$\xi_a : D^3_{F_a} \to \mathbb{C}\ .$$

First identify $\mathcal{I}_1$ with H_{F_a} so that it is compatible with the Hodge decomposition $H^1(C_a) = H^{1,0}(C_a) \oplus H^{0,1}(C_a)$ under the isomorphism $\rho : \mathcal{R}_{F_a} \to H^2(\mathbb{T}^2, C_a^\circ)$. Then Batyrev's pairing

$$\{\ ,\ \} :\ H^2_{F_a} \times H^1_{F_a} \to D^3_{F_a} \cong \mathbb{C}, \tag{6.1}$$

induces an antisymmetric pairing $\langle\ ,\ \rangle_{\mathcal{I}_1}$ on $\mathcal{I}_1$. Although we do not have an explicit description of such decomposition $\mathcal{I}_1 = H^1_{F_a} \oplus H^2_{F_a}$, the fact that $H^1_{F_a}$ and $H^2_{F_a}$ are one-dimensional makes it possible to find $\langle\ ,\ \rangle_{\mathcal{I}_1}$[9]. It is given by

$$\langle \alpha_{1,0} t_0 + \alpha_{2,0} t_0^2\, ,\, \beta_{1,0} t_0 + \beta_{2,0} t_0^2 \rangle_{\mathcal{I}_1} = (-\alpha_{1,0}\beta_{2,0} + \alpha_{2,0}\beta_{1,0})\, \xi_a(t_0^3)\ .$$

Our choice of ξ_a is as follows.

Proposition 6.5. *There exists a map $\xi_a : D^3_{F_a} \to \mathbb{C}$ which is holomorphic in $a \in \mathbb{L}_{\mathrm{reg}}(\Delta)$ and satisfies the following condition:*

$$\langle \mathcal{D}_{a_m} \alpha, \beta \rangle_{\mathcal{I}_1} + \langle \alpha, \mathcal{D}_{a_m} \beta \rangle_{\mathcal{I}_1} = \partial_{a_m} \langle \alpha, \beta \rangle_{\mathcal{I}_1}\ . \tag{6.2}$$

[9] An isomorphism $\mathcal{I}_1 \to H^1_{F_a} \oplus H^2_{F_a}$ compatible with the graded quotient is given by $\alpha_{1,0} t_0 + \alpha_{2,0} t_0^2 \mapsto (\alpha_{1,0} - u) t_0 \oplus \alpha_{2,0} t_0^2$ with some u. The induced antisymmetric pairing on $\mathcal{I}_1$ turns out to be independent of u.

Proof. Define α_m, $\beta_m \in \mathbb{C}(a)$ $(m \in A(\Delta))$ and γ, $\delta \in \mathbb{C}(a)$ by the following relations in $\mathcal{I}_1$:

$$t_0^2 t^m = \alpha_m t_0 + \beta_m t_0^2 \,, \quad t_0^3 = \gamma t_0 + \delta t_0^2 \,.$$

Then the condition (6.2) is equivalent to

$$\partial_{a_m} \xi_a(t_0^3) = -(2\alpha_m + \delta\beta_m + \partial_{a_0}\beta_m)\xi_a(t_0^3) \,. \tag{6.3}$$

The existence of a solution $\xi_a(t_0^3)$ to this equation is ensured by the equation

$$\partial_{a_n}(2\alpha_m + \delta\beta_m + \partial_{a_0}\beta_m) = \partial_{a_m}(2\alpha_n + \delta\beta_n + \partial_{a_0}\beta_n),$$

which follows from the relations in $\mathcal{I}_1$:

$$\mathcal{D}_{a_n} t_0^2 t^m - \mathcal{D}_{a_m} t_0^2 t^n = 0 \,, \quad \mathcal{D}_{a_n} t_0^3 t^m - \mathcal{D}_{a_m} t_0^3 t^n = 0 \,.$$

□

Remark 6.6. The condition (6.2) is equivalent to the following equation for the intersection product on $H^1(C_a)$ under the isomorphism $\rho : \mathcal{R}_{F_a} \to H^2(\mathbb{T}^2, C_a^\circ)$:

$$\int_{C_a} \nabla_{a_m} \alpha \wedge \beta + \int_{C_a} \alpha \wedge \nabla_{a_m} \beta = \partial_{a_m} \int_{C_a} \alpha \wedge \beta \,,$$

which is well-known in the context of variations of polarized Hodge structures.

Example 6.7. For the polyhedron #1 in Figure 1, solving (6.3), we obtain

$$\xi_a(t_0^3) = \frac{1}{27 a_1 a_2 a_3 + a_0^3} \times \text{a nonzero constant} \,.$$

Batyrev's pairing (6.1) together with the quotient map $\mathcal{R}_{F_a} \to H^2_{F_a} = R^2_{F_a} = \mathcal{E}^{-2}/\mathcal{E}^{-1}$ induces a pairing

$$(\ ,\ ) : \mathcal{R}_{F_a} \times H^1_{F_a} \to D^3_{F_a} \overset{\xi_a}{\cong} \mathbb{C} \,.$$

Then, by Remark 6.6, we have the equation

$$\mathrm{Yuk}^{(k)}(A_1, \ldots, A_{k-1}; A_k) = (\mathcal{D}_{A_1} \cdots \mathcal{D}_{A_{k-1}} 1, \mathcal{D}_{A_k} 1) \times \text{ a nonzero constant} \,. \tag{6.4}$$

Here $\mathcal{D}_A$ is the shorthand notation for

$$\mathcal{D}_A := \sum_{m \in A(\Delta)} A_m \mathcal{D}_{a_m}$$

where $A = \sum_{m \in A(\Delta)} A_m \partial_{a_m}$ is a vector field on $\mathbb{L}_{\mathrm{reg}}(\Delta)$.

Example 6.8. Let Δ be the polyhedron #1 in Figure 1. By (6.4), the Yukawa coupling $\mathrm{Yuk}(\partial_{a_0}, \partial_{a_0}; \partial_{a_0})$ is equal to $(\mathcal{D}_{a_0}, \mathcal{D}_{a_0} 1\,,\, \mathcal{D}_{a_0} 1) = \xi_a(t_0^3)$ up to non-zero multiplicative constant. Compare with Example 6.13 below.

6.4. **Yukawa coupling and the A-hypergeometric system.** Recall the A-hypergeometric system introduced in §3.4. The following proposition enables us to compute the Yukawa coupling by the A-hypergeometric system. (See also Lemma 6.12 in the next subsection.)

Proposition 6.9. *1. For $k \geq 3$ and $m_1, \dots, m_{k-1} \in A(\Delta)$,*

$$\mathcal{T}_i \mathrm{Yuk}^{(k)}(\theta_{a_{m_1}}, \dots, \theta_{a_{m_{k-1}}}; \theta_{a_0}) = 0 \quad (i = 0, 1, 2, 3). \tag{6.5}$$

2. For a vector $l = (l_m)_{m \in A(\Delta)} \in L(\Delta)$, let k be the order of the differential operator $\Box_l$. Let us write $\Box_l$ as

$$\Box_l = \partial_{a_{m_1}} \cdots \partial_{a_{m_k}} - \partial_{a_{n_1}} \cdots \partial_{a_{n_k}} \ .$$

Then we have

$$\mathrm{Yuk}^{(k+1)}(\partial_{a_{m_1}}, \dots, \partial_{a_{m_k}}; \partial_{a_0}) - \mathrm{Yuk}^{(k+1)}(\partial_{a_{n_1}}, \dots, \partial_{a_{n_k}}; \partial_{a_0}) = 0 \ .$$

Moreover, for $j_1, \dots, j_h \in A(\Delta)$, we have

$$\mathrm{Yuk}^{(k+h+1)}(\partial_{a_{j_1}}, \dots, \partial_{a_{j_h}}, \partial_{a_{m_1}}, \dots, \partial_{a_{m_k}}; \partial_{a_0}) - \mathrm{Yuk}^{(k+h+1)}(\partial_{a_{j_1}}, \dots, \partial_{a_{j_h}}, \partial_{a_{n_1}}, \dots, \partial_{a_{n_k}}; \partial_{a_0}) = 0 \ .$$

3. For $m, n \in A(\Delta)$,

$$\partial_{a_m} \mathrm{Yuk}^{(3)}(\partial_{a_0}, \partial_{a_n}; \partial_{a_0}) + \partial_{a_n} \mathrm{Yuk}^{(3)}(\partial_{a_0}, \partial_{a_m}; \partial_{a_0}) = 2\mathrm{Yuk}^{(4)}(\partial_{a_0}, \partial_{a_m}, \partial_{a_n}; \partial_{a_0})$$ [10].

Proof. Let $\Theta_{a_m} := a_m \mathcal{D}_{a_m}$.
1. Notice that for $\nabla_{\theta_{a_{m_1}}} \cdots \nabla_{\theta_{a_{m_{k-1}}}} \omega_0 = \rho(\Theta_{a_{m_1}} \cdots \Theta_{a_{m_{k-1}}} 1)$ is expressed in the form (cf. (3.11))

$$\alpha_{2,0}\rho(\Theta_{a_0}^2 1) + \sum_{m \in A'(\Delta)} \alpha_{1,m}\rho(\Theta_{a_0}\Theta_{a_m} 1) + \alpha_{1,0}\rho(\Theta_{a_0} 1) + \alpha_{0,0}\rho(1)$$

where the coefficients satisfy

$$\mathcal{T}_i \alpha_{2,0} = \mathcal{T}_i \alpha_{1,0} = \mathcal{T}_i \alpha_{1,m} = \mathcal{T}_i \alpha_{0,0} = 0 \quad (i = 0, 1, 2).$$

By Definition 6.2, we have

$$\mathrm{Yuk}^{(k)}(\theta_{a_{m_1}}, \dots, \theta_{a_{m_{k-1}}}; \theta_{a_0}) = \int_{C_a} \alpha_{2,0}\rho(\Theta_{a_0}^2 1) \wedge \rho(\Theta_{a_0} 1).$$

Then the statement follows from Proposition 3.7-2.
The statements 2 and 3 follow from Proposition 3.7-2 and Definition 6.2. □

6.5. **Yukawa coupling for Quotient Family.** Consider the action of $\mathbb{T}^3$ on $\mathbb{L}_{\mathrm{reg}}(\Delta)$:

$$\mathbb{T}^3 \times \mathbb{L}_{\mathrm{reg}}(\Delta) \to \mathbb{L}_{\mathrm{reg}}(\Delta) \ , \quad (\lambda_0, \lambda_1, \lambda_2) \cdot F_a(t_1, t_2) \mapsto \lambda_0 F_a(\lambda_1 t_1, \lambda_2 t_2) \ .$$

Let $\mathcal{M}(\Delta)$ be the geometric invariant theory quotient of $\mathbb{L}_{\mathrm{reg}}(\Delta)$ by this action[11]. Denote the quotient map by $q : \mathbb{L}_{\mathrm{reg}}(\Delta) \to \mathcal{M}(\Delta)$.

Since $\mathbb{T}^3$ acts as automorphisms on $\mathcal{Z}$, we also have a family of affine curves

$$\pi : \mathcal{Z}/\mathbb{T}^3 \to \mathcal{M}(\Delta) \ . \tag{6.6}$$

(Similarly we can construct the quotient family for the open threefold Z_a°.)

[10] This equation is analogous to the case of the compact Calabi–Yau threefold. See [26].

[11] Any $a \in \mathbb{L}_{\mathrm{reg}}(\Delta)$ is stable in the sense of the geometric invariant theory (cf. [5, Definition 10.5]).

The differential equation (6.5) implies that $\mathrm{Yuk}^{(k)}(\theta_{a_{m_1}},\ldots,\theta_{a_{m_{k-1}}};\theta_{a_0})$ depends on the parameter a only through $\mathbb{T}^3$-invariant combinations. Thus we can define the Yukawa coupling for the quotient family as follows. Let $T^0\mathcal{M}(\Delta)$ be the subbundle of the holomorphic tangent bundle $T\mathcal{M}(\Delta)$ generated by $q_*\theta_{a_0}$.

Definition 6.10. We define a map

$$\mathrm{Yuk}^{(k)}_{\mathcal{M}(\Delta)}:\underbrace{T\mathcal{M}(\Delta)\times\cdots\times T\mathcal{M}(\Delta)}_{(k-1)\text{ times}}\times T^0\mathcal{M}(\Delta)\to\mathcal{O}_{\mathcal{M}(\Delta)}$$

by

$$\mathrm{Yuk}^{(k)}_{\mathcal{M}(\Delta)}(A_1,\ldots,A_{k-1};A_k)=\mathrm{Yuk}^{(k)}(A'_1,\ldots,A'_{k-1};A'_k),$$

where A'_i are $\mathbb{T}^3$-invariant vector fields on $\mathbb{L}_{\mathrm{reg}}(\Delta)$ such that $q_*A'_i=A_i$. The case $k=3$ is called the Yukawa coupling and denoted by $\mathrm{Yuk}_{\mathcal{M}(\Delta)}$. (We may omit the subscript $\mathcal{M}(\Delta)$.)

In the rest of this subsection, we rewrite the differential equations for the Yukawa coupling (Proposition 6.9) obtained in the previous section to the setting of the quotient family. We fix a local coordinates of $\mathcal{M}(\Delta)$ of a particular class: take a basis $l^{(i)}$ $(1\le i\le l(\Delta)-3))$ of the lattice of relations $L(\Delta)$. Then

$$z_i=a^{l^{(i)}}\qquad(1\le i\le l(\Delta)-3)$$

form a local coordinate system on some open subset in $\mathcal{M}(\Delta)$. We use the shorthand notation

$$\theta_i:=\theta_{z_i}\ ,\quad \theta_0:=q_*\theta_{a_0}=\sum_{i=1}^{l(\Delta)-3}l_0^{(i)}\theta_i\ ,\quad \nabla_i:=\nabla_{\theta_{z_i}}\ ,\quad \nabla_0:=\nabla_{\theta_0}\ .$$

Let $\mathbf{D}$ be the set of differential operators on (some open set of) $\mathcal{M}(\Delta)$, consisting of

$$\theta_{i_1}\cdots\theta_{i_k}\mathcal{L}_l\ ,\quad (k\ge 0,\ 1\le i_1,\ldots,i_k\le l(\Delta)-3,\ l\in L(\Delta)).$$

Here $\mathcal{L}_l$ is defined by

$$\mathcal{L}_l=q_*\Big(\prod_{m;l_m>0}a_m^{l_m}\Big)\Box_l\ .$$

Example 6.11. In the case of polyhedron #1 (see Example 3.6), we have the coordinate $z=a^{(-3,1,1,1)}=\frac{a_1a_2a_3}{a_0^3}$ and $\theta_0:=q_*\theta_{a_0}=-3\theta_z$. Then

$$\mathcal{L}_{(-3,1,1,1)}=\theta_z^3+3z\theta_z(3\theta_z+1)(3\theta_z+2)\ ,\tag{6.7}$$

and $\mathbf{D}$ is generated by $\theta_z^k\mathcal{L}_{(-3,1,1,1)}$ $(k\ge 0)$.

For $0\le i_1,\ldots,i_k\le l(\Delta)-3$, we define

$$Y_{i_1\ldots i_k\,;\,0}:=\mathrm{Yuk}^{(k+1)}(\theta_{i_1},\ldots,\theta_{i_k};\theta_0)\ .\tag{6.8}$$

Proposition 6.9 implies the following

Lemma 6.12. *1. Let $\mathcal{L}\in\mathbf{D}$ and let $U_{i_1,\ldots,i_k}\in\mathbb{C}(z)$ be the coefficients of $\theta_{i_1}\ldots\theta_{i_k}$ in $\mathcal{L}$, i.e.*

$$\mathcal{L}=\sum_{k\ge1}\sum_{i_1,\ldots,i_k}U_{i_1,\ldots,i_k}\theta_{i_1}\cdots\theta_{i_k}\qquad(U_{i_1,\ldots,i_k}\in\mathbb{C}(z)).$$

Then the Yukawa coupling satisfies

$$\sum_{k\geq 2}\sum_{i_1,\dots,i_k} U_{i_1\dots i_k}\, Y_{i_1\dots i_k;0} = 0\ .$$

2. For $0 \leq i, j \leq l(\Delta) - 3$,

$$Y_{ij0;0} = \frac{1}{2}(\theta_i Y_{j0;0} + \theta_j Y_{i0;0})\ .$$

Example 6.13. Let Δ be the polyhedron #1 in Figure 1. Applying the above Lemma to the differential operator (6.7), we obtain the equation

$$(1 + 27z)\theta_z \mathrm{Yuk}(\theta_z, \theta_z; \theta_z) + 27z\mathrm{Yuk}(\theta_z, \theta_z; \theta_z) = 0\ , \tag{6.9}$$

which implies

$$\mathrm{Yuk}(\theta_z, \theta_z; \theta_z) = -\frac{c}{3(1 + 27z)}$$

where c is some nonzero constant. This result is the same as Example 6.8.

Remark 6.14. Let $t, \partial_S F$ be the solutions (3.9) of the A-hypergeometric system associated to the polyhedron #1 in Figure 1. Then we have

$$\mathrm{Yuk}(\partial_t, \partial_t; \partial_t) \propto \partial_t^2 \partial_S F\ . \tag{6.10}$$

This follows from the multilinearity of Yuk and the fact that

$$\mathrm{Wr}(t, \partial_S F) := \det\begin{pmatrix}\theta_z^2 t & \theta_z t\\ \theta_z^2\partial_S F & \theta_z\partial_S F\end{pmatrix} = -(\theta_z t)^3 \cdot \partial_t^2 \partial_S F$$

is proportional to $\mathrm{Yuk}(\theta_z, \theta_z; \theta_z)$ since it satisfies the same differential equation (6.9).

6.6. **Comments on Yukawa coupling in the local A-model and local mirror symmetry.** Let Δ be a 2-dimensional reflexive polyhedron. Consider the 2-dimensional nonsingular complete smooth fan $\Sigma(\Delta^*)$ whose generators of 1-cones are $A(\Delta) \setminus \{0\}$. Let $\mathbb{P}_{\Sigma(\Delta^*)}$ be the toric surface defined by $\Sigma(\Delta^*)$. For example, $\mathbb{P}_{\Sigma(\Delta^*)} = \mathbb{P}^2$ if Δ is the polyhedron #1 in Figure 1. Take a basis C_i $(1 \leq i \leq l(\Delta) - 3)$ of $H_2(\mathbb{P}_{\Sigma(\Delta^*)}, \mathbb{Z}) \cong L(\Delta)$ and let J_i $(1 \leq i \leq l(\Delta) - 3)$ be the dual basis. Denote by t_i $(1 \leq i \leq l(\Delta) - 3)$ the coordinates on $H^2(\mathbb{P}_{\Sigma(\Delta^*)})$ associated to this basis. Let c_i be the coefficients of J_i in $c_1(\mathbb{P}_{\Sigma(\Delta^*)}) = \sum_i c_i J_i$ and let $J_i \cdot J_j$ be the intersection numbers.

Let $N_{0,\beta}(\mathbb{P}_{\Sigma(\Delta^*)})$ be the genus zero local Gromov–Witten invariant of degree β, and define $F_{\mathrm{inst}}^{\mathbb{P}_{\Sigma(\Delta^*)}}(t)$ by

$$F_{\mathrm{inst}}^{\mathbb{P}_{\Sigma(\Delta^*)}}(t) = \sum_{\beta=\sum d_i C_i} N_\beta^{\mathbb{P}_{\Sigma(\Delta^*)}} e^{\sum d_i t_i}\ .$$

Note that $\dim H^2(\mathbb{P}_{\Sigma(\Delta^*)}) = \dim R^1_{F_a} = l(\Delta) - 3$. Let $t_i(z)$ be solutions of the A-hypergeometric system with a single logarithm, so-called the mirror maps, and let $\partial_S F$ be a solution with double logarithms. (See [11, eq.(6.22)] for definitions of $t_i, \partial_S F$. Π_i there is t_i here.) Local mirror symmetry [11] says that, under an appropriate identification between t_i's and $t_i(z)$'s, $\partial_S F$ is related to the local Gromov–Witten invariants by

$$\partial_S F = \sum_{i,j=1}^{l(\Delta)-3} \frac{J_i \cdot J_j}{2} t_i t_j - \sum_{i=1}^{l(\Delta)-3} c_i \partial_{t_i} F_{\mathrm{inst}}^{\mathbb{P}_{\Sigma(\Delta^*)}}(t)\ .$$

Let $T^0H^2(\mathbb{P}_{\Sigma(\Delta^*)})$ be the one-dimensional subspace of $TH^2(\mathbb{P}_{\Sigma(\Delta^*)})$ spanned by $\sum_i c_i\partial_{t_i}$. The local A-model Yukawa coupling Yuk_A may be defined as a multilinear map from $TH^2(\mathbb{P}_{\Sigma(\Delta^*)}) \times TH^2(\mathbb{P}_{\Sigma(\Delta^*)}) \times T^0H^2(\mathbb{P}_{\Sigma(\Delta^*)})$ to $\mathcal{O}_{H^2(\mathbb{P}_{\Sigma(\Delta^*)})}$ given by

$$\mathrm{Yuk}_A\Big(\partial_{t_i},\partial_{t_j};\sum_{l=1}^{l(\Delta)-3} c_l\partial_{t_l}\Big)=\partial_{t_i}\partial_{t_j}\partial_S F\ .$$

Example 6.15. Let Δ be the polyhedron #1 in Figure 1. As in Remark 6.14, the local A-model Yukawa coupling Yuk_A is proportional to the local B-model Yukawa coupling Yuk. To get the equality, we set $c = 1$ in Example 6.13.

We also see that for the other polyhedra in Figure 1, the Yukawa couplings coincide with the local A-model Yukawa couplings Yuk_A under the mirror maps t_1, t_2. See §8.

7. Holomorphic anomaly equation

7.1. **Analogue of Special Kähler Geometry.** We propose an analogue of the special geometry relation for $\mathcal{M}(\Delta)$. Consider the quotient family $\pi : \mathcal{Z}/\mathbb{T}^3 \to \mathcal{M}(\Delta)$. We use the same notations $z_i, \theta_0, \theta_i, \nabla_0, \nabla_i$ as in §6.5. Let

$$\phi := \nabla_0\omega_0 \in H^1(C_z)\ . \tag{7.1}$$

As in (6.8), we set

$$Y_{i\,0;0} = \sqrt{-1}\int_{C_z}\nabla_i\phi\wedge\phi\quad (0\le i\le l(\Delta)-3)\ .$$

We also set

$$G_{0\overline{0}} := -\sqrt{-1}\int_{C_z}\phi\wedge\overline{\phi}\ .$$

This defines a Hermitian metric on $T^0\mathcal{M}(\Delta)$ such that the norm of θ_0 is $G_{0\overline{0}}$.

By the definition of $G_{0\overline{0}}$, $Y_{00;0}$ and $Y_{i\,0;0}$, we have the following

Lemma 7.1.

$$(1)\quad \nabla_i\phi = \frac{\theta_i G_{0\overline{0}}}{G_{0\overline{0}}}\phi + \frac{Y_{i0;0}}{G_{0\overline{0}}}\overline{\phi}\ ,$$

$$(2)\quad \overline{\theta}_j\frac{\theta_i G_{0\overline{0}}}{G_{0\overline{0}}} = -\frac{Y_{i0;0}\overline{Y}_{j0;0}}{G_{0\overline{0}}^2}\ .$$

Let

$$\kappa := \theta_0\frac{\theta_0 G_{0\overline{0}}}{G_{0\overline{0}}} + \Big(\frac{\theta_0 G_{0\overline{0}}}{G_{0\overline{0}}}\Big)^2 - \frac{\theta_0 Y_{00;0}}{Y_{00;0}}\frac{\theta_0 G_{0\overline{0}}}{G_{0\overline{0}}}\ .$$

Then

$$(3)\quad \overline{\theta}_j\kappa = 0 \qquad (1\le j\le l(\Delta)-3)\ ,$$

$$(4)\quad \nabla_0^2\phi = \kappa\phi + \frac{\theta_0 Y_{00;0}}{Y_{00;0}}\nabla_0\phi\ .$$

The second equation is analogous to the special geometry equation [7]. The third equation is an analogue of [40, eq.(3.2)].

Example 7.2. Let Δ be the polyhedron #1 in Figure 1. By comparing the fourth equation of the above lemma and the differential operator (6.7), we have

$$Y_{00;0} = \frac{9}{1+27z} \ , \qquad \kappa = -\frac{54z}{1+27z} \ .$$

7.2. **Proposal of local holomorphic anomaly equation.** We propose how to adapt BCOV's holomorphic anomaly equation [7] to the local B-model. Let $\tilde{C}^g_n$ $(g, n \geq 0)$ be the n-point B-model topological string amplitude of genus g. For $2g - 2 + n \leq 0$, we set

$$\tilde{C}^0_0 = \tilde{C}^0_1 = \tilde{C}^0_2 = 0 \ , \quad \tilde{C}^1_0 = 0 \ . \tag{7.2}$$

For $2g - 2 + n \geq 1$, we put

$$\tilde{C}^g_{n+1} = \Big(\theta_0 - n\frac{\theta_0 G_{0\overline{0}}}{G_{0\overline{0}}}\Big)\tilde{C}^g_n \ .$$

For $(g, n) = (0, 3)$, let

$$\tilde{C}^0_3 = Y_{00;0} \ . \tag{7.3}$$

As a holomorphic anomaly equation for $(g, n) = (1, 1)$, we propose

$$\overline{\theta}_j \tilde{C}^1_1 = -\frac{1}{2}\overline{\theta}_j \frac{\theta_0 G_{0\overline{0}}}{G_{0\overline{0}}} \ , \quad \text{which implies that} \quad \tilde{C}^1_1 = -\frac{1}{2}\frac{\theta_0 G_{\overline{0}}}{G_{0\overline{0}}} + f^1_1(z) \ . \tag{7.4}$$

For $(g, n) = (g, 0)$ $(g \geq 2)$, we propose

$$\overline{\theta}_j \tilde{C}^g_0 = \frac{\overline{Y}_{j0;0}}{2G^2_{0\overline{0}}}\big(\tilde{C}^{g-1}_2 + \sum_{h_1+h_2=g} \tilde{C}^{h_1}_1 \tilde{C}^{h_2}_1\big) \ . \tag{7.5}$$

For $g \geq 2$, $\tilde{C}^g_0$ can be solved by the Feynman diagram method as in [7] or Yamaguchi–Yau's polynomial method as in [40].

7.3. **Solution by Feynman diagram** [7]**.** Define the propagator S^{00} by the differential equation $\overline{\theta}_j S^{00} = \frac{\overline{Y}_{j0;0}}{G^2_{0\overline{0}}}$. It is easily solved by Lemma 7.1-(2):

$$S^{00} = -\frac{1}{Y_{00;0}}\frac{\theta_0 G_{0\overline{0}}}{G_{0\overline{0}}} + f_s(z) \ ,$$

where $f_s(z)$ is a meromorphic function in z. Put $\Delta_{00} := -1/S^{00}$. Then assuming (7.3), (7.4) and (7.5), we can show that

$$\overline{\theta}_j \exp\Big[-\frac{1}{2\lambda^2}\Delta_{00}x^2 + \frac{1}{2}\log\frac{\Delta_{00}}{\lambda^2} + \sum_{n,g\geq 0}\frac{\lambda^{2g-2}}{n!}\tilde{C}^g_n x^n\Big] = 0 \ .$$

This implies that $\tilde{C}^g_0$ $(g \geq 2)$ can be computed as a sum over Feynman diagrams of genus g. The difference from the one given in [7] is that there is only one propagator, S^{00}.

7.4. **Solution by Yamaguchi–Yau's method** [40]**.** Let

$$A = \frac{\theta_0 G_{0\overline{0}}}{G_{0\overline{0}}} \ .$$

From the above Feynmann diagram method and the fact that $\theta_0 A \in \mathbb{C}(z)[A]$ (see Lemma 7.1-(3)), it follows that $\tilde{C}^g_n$ is a polynomial of degree $3g - 3 + n$ in $\mathbb{C}(z)[A]$. Moreover, it satisfies

$$\frac{\partial \tilde{C}^g_0}{\partial A} = -\frac{1}{2Y_{00;0}}\Big(\tilde{C}^{g-1}_2 + \sum_{h_1+h_2=g}\tilde{C}^{h_1}_1\tilde{C}^{h_2}_1\Big) \ . \tag{7.6}$$

Example 7.3. For $(g,n)=(1,1),(1,2)$ and $(2,0)$, we have

$$\begin{aligned}\tilde{C}^1_1&=-\frac{1}{2}A+f^1_1(z)\ ,\quad \tilde{C}^1_2=A^2+A\Big(-\frac{\theta_0 Y_{00;0}}{2Y_{00;0}}-f^1_1\Big)-\frac{\kappa}{2}+\theta_0 f^1_1\ ,\\ \tilde{C}^2_0&=-\frac{1}{2Y_{00;0}}\Big[\frac{5}{12}A^3-\Big(\frac{\theta_0 Y_{00;0}}{4Y_{00;0}}+f^1_1\Big)A^2+\Big(-\frac{\kappa}{2}+\theta_0 f^1_1+(f^1_1)^2\Big)A\Big]+f_2(z)\ .\end{aligned} \tag{7.7}$$

Example 7.4. Let Δ be the polyhedron #1 in Figure 1. We checked that $\tilde{C}^1_1,\tilde{C}^2_0$ give the correct local GW invariants of $\mathbb{P}^2$ at least in small degrees. The holomorphic ambiguities are

$$f^1_1(z)=\frac{1+54z}{4(1+27z)}\ ,\qquad f_2(z)=\frac{\frac{3}{40}z+\frac{783}{80}z^2+\frac{3645}{8}z^3}{(1+27z)^2}\ .$$

The holomorphic limit is

$$G_{0\overline{0}}\to\theta_z t\ .$$

7.5. **Witten's geometric quantization approach.** First recall Witten's geometric quantization and its implication for holomorphic anomaly equation [39]. Let $W=\mathbb{R}^{2N}$ be a vector space equipped with the standard symplectic form and let $L\to W$ be a complex line bundle whose connection 1-form is the canonical 1-form. Let $\mathcal{M}$ be the space of complex structures on W. To each complex structure $J\in\mathcal{M}$, associate the holomorphic polarization $\mathcal{H}_J$ which is a subspace of the space of square integrable sections $\Gamma(W,L)$ consisting of "holomorphic" ones. Then an infinite dimensional bundle $\mathcal{H}\to\mathcal{M}$ is obtained. Witten found a projectively flat connection on $\mathcal{H}$. His claim is that if this is applied to the case where $W=H^3(X^\vee,\mathbb{R})$ is the cohomology of a Calabi–Yau threefold $X^\vee$, then BCOV's holomorphic anomaly equation appears as the condition for the flatness of a section of $\mathcal{H}$.

We apply Witten's idea to the case when $W=\mathcal{W}_1H^1(C^\circ_z,\mathbb{R})=H^1(C_z,\mathbb{R})$ and $\mathcal{M}=\mathcal{M}(\Delta)$. (To be precise, $\mathcal{M}(\Delta)$ is not the space of complex structures of W but it is larger in general. However, this point does not matter in the following argument.) Take $\phi,\overline{\phi}$ defined in (7.1) as a basis of $W_{\mathbb{C}}=\mathcal{W}_1H^1(C^\circ_z)=H^1(C_z)$ and let $x,\overline{x}$ be the associated complex coordinates. W has a symplectic form $\sqrt{-1}G_{0\overline{0}}dx\wedge d\overline{x}$ given by the intersection product. Consider the trivial line bundle $L=\mathbb{C}\times W$ with the connection

$$\delta+\frac{1}{2}G_{0\overline{0}}(xd\overline{x}-\overline{x}dx)\ .$$

Here we use δ to denote the differential on W. Then the holomorphic polarization $\mathcal{H}_z$ ($z\in\mathcal{M}(\Delta)$) is as follows:

$$\begin{aligned}\mathcal{H}_z&=\Big\{\Phi\in\Gamma(W,L)\mid\Big(\overline{\delta}_{\overline{x}}+\frac{G_{0\overline{0}}}{2}x\Big)\Phi=0\Big\}\\ &=\big\{\Phi\in\Gamma(W,L)\mid\Phi=\varphi(x)e^{-\frac{G_{0\overline{0}}}{2}x\overline{x}}\ \big\}\ .\end{aligned}$$

Mimicking Witten's result, we can show that

$$\theta_j\mathcal{H}\subset\mathcal{H}\ ,\qquad\bigg(\overline{\theta}_j-\frac{\overline{Y}_{j0;0}}{2G^2_{0\overline{0}}}\Big(\delta_x-\frac{G_{0\overline{0}}}{2}\overline{x}\Big)^2\bigg)\mathcal{H}\subset\mathcal{H}\ .$$

Moreover these make a projectively flat connection on $\mathcal{H}$.

If we regard

$$\exp\Big[\sum_{n,g\ge 0}\frac{\lambda^{2g-2+n}}{n!}\tilde{C}^g_n x^n\Big]\times e^{-\frac{G_{0\overline{0}}}{2}x\overline{x}}$$

as a section of $\mathcal{H}$, then the condition that it is a flat section results in the following equation:

$$\overline{\theta}_j \tilde{C}^g_n = \frac{\overline{Y}_{j0;0}}{2G^2_{\overline{0}\overline{0}}} \left(\tilde{C}^{g-1}_{n+2} + \sum_{\substack{h_1+h_2=g,\\ 0\le m\le n}} \binom{n}{m} \tilde{C}^{h_1}_{m+1} \tilde{C}^{h_2}_{n-m+1} \right) .$$

8. Examples

In this section, we consider the polyhedra #2, 3, 4 in Figure 1.

8.1. $\mathbb{F}_0$ **case.** Let Δ be the polyhedron #2 in Figure 1:

$$\Delta = \text{ the convex hull of } \{(1,0),(0,1),(-1,0),(0,-1)\} .$$

Δ-regularity condition. The Δ-regularity condition for $F \in \mathbb{L}(\Delta)$ is as follows:

$$\begin{aligned} & F(t_1,t_2) = a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1} + \frac{a_4}{t_2} , \\ & a_1a_2a_3a_4 \neq 0 , \quad (a_0^2 - 4a_1a_3 - 4a_2a_4)^2 - 64 a_1a_2a_3a_4 \neq 0 . \end{aligned} \tag{8.1}$$

$\mathcal{R}_F$ and filtrations. We have

$$\mathcal{R}_F \cong \mathbb{C}\,1 \oplus \mathbb{C}\,t_0 \oplus \mathbb{C}\,t_0t_1 \oplus \mathbb{C}\,t_0^2 .$$

The $\mathcal{I}$-filtration and the $\mathcal{E}$-filtration are as follows.

$$\mathcal{I}_1 = \mathcal{I}_2 = \mathbb{C}\,t_0 \oplus \mathbb{C}\,t_0^2 , \quad \mathcal{I}_3 = \mathcal{I}_1 \oplus \mathbb{C}\,t_0t_1 , \qquad \mathcal{I}_4 = \mathcal{R}_F .$$

$$\mathcal{E}^0 = \mathbb{C}\,1 , \quad \mathcal{E}^{-1} = \mathcal{E}^0 \oplus \mathbb{C}\,t_0 \oplus \mathbb{C}\,t_0t_1 , \quad \mathcal{E}^{-2} = \mathcal{R}_F .$$

MHS. By Theorem 4.2 and (3.11),

$$H^2(\mathbb{T}^2, C_a^\circ) = \mathbb{C}\omega_0 \oplus PH^1(C_a^\circ) , \quad PH^1(C_a^\circ) = \mathbb{C}\,\rho(t_0) \oplus \mathbb{C}\rho(t_0t_1) \oplus \mathbb{C}\rho(t_0^2) .$$

$$\mathcal{W}_1 = \mathbb{C}\rho(t_0) \oplus \mathbb{C}\rho(t_0^2) , \quad \mathcal{W}_2 = \mathcal{W}_1 \oplus \mathbb{C}\rho(t_0t_1) , \qquad \mathcal{W}_4 = H^2(\mathbb{T}^2, \mathbb{C}) .$$

$$\mathcal{E}^0 = \mathbb{C}\omega_0 , \quad \mathcal{E}^{-1} = \mathcal{E}^0 \oplus \mathbb{C}\rho(t_0) \oplus \mathbb{C}\rho(t_0t_1) , \quad \mathcal{E}^{-2} = H^2(\mathbb{T}^2, \mathbb{C}) .$$

A-hypergeometric system. The lattice of relations $L(\Delta)$ (defined in (3.7)) is generated by two vectors

$$l^{(1)} = (-2,1,0,1,0) , \qquad l^{(2)} = (-2,0,1,0,1) .$$

The A-hypergeometric system is generated by the following differential operators:

$$\begin{aligned} & \theta_{a_1} - \theta_{a_3} , \quad \theta_{a_2} - \theta_{a_4} , \quad \theta_{a_1} + \theta_{a_2} + \theta_{a_3} + \theta_{a_4} + \theta_{a_0} , \\ & \partial_{a_1}\partial_{a_3} - \partial_{a_0}^2 , \quad \partial_{a_2}\partial_{a_4} - \partial_{a_0}^2 . \end{aligned}$$

Take

$$z_1 = a^{l^{(1)}} = \frac{a_1a_3}{a_0^2} , \qquad z_2 = a^{l^{(2)}} = \frac{a_2a_4}{a_0^2} .$$

These are coordinates of an open subset of $\mathcal{M}(\Delta)$. We have $\theta_0 := q_*\theta_{a_0} = -2\theta_{z_1} - 2\theta_{z_2}$. With these coordinates, the above A-hypergeometric system reduces to the following two differential operators of order 2:

$$\begin{aligned} \mathcal{L}_1 &= \theta_1^2 - z_1(-2\theta_1 - 2\theta_2)(-2\theta_1 - 2\theta_2 - 1) , \\ \mathcal{L}_2 &= \theta_2^2 - z_2(-2\theta_1 - 2\theta_2)(-2\theta_1 - 2\theta_2 - 1) . \end{aligned}$$

Solutions about $z_1 = 0, z_2 = 0$ are as follows.

$$\varpi(z;0) = 1 \ ,$$
$$t_1 := \partial_{\rho_1}\varpi(z;\rho)|_{\rho=0} = \log z_1 + 2H(z_1,z_2) \ ,$$
$$t_2 := \partial_{\rho_2}\varpi(z;\rho)|_{\rho=0} = \log z_2 + 2H(z_1,z_2) \ ,$$
$$\partial_S F := \partial_{\rho_1}\partial_{\rho_2}\varpi(z;\rho) = \log z_1 \log z_2 + \cdots \ ,$$

where

$$\varpi(z;\rho) = \sum_{n_1,n_2\geq 0} \frac{(2\rho_1+2\rho_2)_{2n_1+2n_2}}{(\rho_1+1)^2_{n_1}(\rho_2+1)^2_{n_2}} z_1^{n_1+\rho_1} z_2^{n_2+\rho_2} \ ,$$
$$H(z_1,z_2) = \sum_{\substack{n_1,n_2\geq 0\\(n_1,n_2)\neq(0,0)}} \frac{(2n_1+2n_2-1)!}{n_1!^2 n_2!^2} z_1^{n_1} z_2^{n_2} \ .$$

Yukawa coupling. In this case, $\mathbf{D}$ is generated by $\mathcal{L}_1, \mathcal{L}_2$. Applying Lemma 6.12 to $\mathcal{L}_1$, $\mathcal{L}_2$, $\theta_0\mathcal{L}_1$, $\theta_0\mathcal{L}_2$, we obtain first order partial differential equations for $Y_{i,j;0}$. Solving these equations, we obtain:

$$Y_{0,0;0} = \frac{8c}{d(z_1,z_2)} \ ,$$
$$Y_{1,1;0} = \frac{8cz_1}{d(z_1,z_2)} \ , \quad Y_{1,2;0} = \frac{c(1-4z_1-4z_2)}{d(z_1,z_2)} \ , \quad Y_{2,2;0} = \frac{8cz_2}{d(z_1,z_2)} \ ,$$

where $d(z_1,z_2) = (1-4z_1-4z_2)^2 - 64z_1z_2$ and $c \in \mathbb{C}$ is a nonzero constant.

Comparison with the local A-model Yukawa coupling. We show that the Yukawa coupling and the local A-model Yukawa coupling coincide under the mirror map:

$$\mathrm{Yuk}(\partial_{t_\alpha}, \partial_{t_\beta}; -2\partial_{t_1} - 2\partial_{t_2}) \propto \partial_{t_\alpha}\partial_{t_\beta}\partial_S F \ . \tag{8.2}$$

For this purpose, let us define the "Wronskian" of $t_1, t_2, \partial_S F$ by

$$\begin{aligned}\mathrm{Wr}_{i_1\dots i_k}(t_1,t_2,\partial_S F) &:= \det\begin{pmatrix} \theta_{i_1}\cdots\theta_{i_k}t_1 & \theta_1 t_1 & \theta_2 t_1 \\ \theta_{i_1}\cdots\theta_{i_k}t_2 & \theta_1 t_2 & \theta_2 t_2 \\ \theta_{i_1}\cdots\theta_{i_k}\partial_S F & \theta_1\partial_S F & \theta_2\partial_S F \end{pmatrix} \\ &= \det\begin{pmatrix} \theta_1 t_1 & \theta_2 t_1 \\ \theta_1 t_2 & \theta_2 t_2 \end{pmatrix} \cdot \sum_{\alpha,\beta=1}^{2} \partial_1 t_\alpha \cdot \partial_2 t_\beta \cdot \partial_{t_\alpha}\partial_{t_\beta}\partial_S F \ .\end{aligned} \tag{8.3}$$

We can show that Lemma 6.12 holds if we replace $\mathrm{Yuk}_{i_1,\dots,i_k;0}$ with $\mathrm{Wr}_{i_1\dots i_k}(t_1,t_2,\partial_S F)$[12]. Therefore $\mathrm{Wr}_{ij}(t_1,t_2,\partial_S F)$ must be proportional to $Y_{ij;0}$. Then (8.2) follows from the multilinearity of Yuk.

[12] The first statement follows from the cofactor expansion of the determinant and the fact that $t_1, t_2, \partial_S F$ are solutions of $\mathcal{L} = 0$ for $\mathcal{L} \in \mathbf{D}$:

$$\sum_{i_1,\dots,i_k} U_{i_1\dots i_k}\mathrm{Wr}_{i_1\dots i_k}(t_1,t_2,\partial_S F) = \det\begin{pmatrix} \theta_1 t_2 & \theta_2 t_2 \\ \theta_1\partial_S F & \theta_2\partial_S F \end{pmatrix}\mathcal{L}t_1 - \det\begin{pmatrix} \theta_1 t_1 & \theta_2 t_1 \\ \theta_1\partial_S F & \theta_2\partial_S F \end{pmatrix}\mathcal{L}t_2 + \det\begin{pmatrix} \theta_1 t_1 & \theta_2 t_1 \\ \theta_1 t_2 & \theta_2 t_2 \end{pmatrix}\mathcal{L}\partial_S F = 0 \ .$$

To prove the second statement, we first solve $\mathcal{L}_1 * = \mathcal{L}_2 * = 0$ and express $\theta_1^2 *, \theta_2^2 *$ in terms of $\theta_1\theta_2 *, \theta_1 *, \theta_2 *$ $(* = t_1, t_2, \partial_S F)$. Then if we substitute these into $\theta_1\mathrm{Wr}_{11}(t_1,t_2,\partial_S F) - \mathrm{Wr}_{110}(t_1,t_2,\partial_S F)$, terms cancell each other and we obtain zero. We can prove the other equations similarly.

Holomorphic ambiguities. The multiplication constant of $Y_{00;0}$ is $c=1$. From $\mathcal{L}_1,\mathcal{L}_2$, we obtain

$$\kappa=\frac{8(z_1+z_2-6(z_1^2+z_2^2)+12z_1z_2)}{d(z_1,z_2)} \ .$$

We checked that $\tilde{C}_1^1,\tilde{C}_0^2$ give the correct local GW invariants of $\mathbb{F}_0$ for small degrees. The holomorphic ambiguities are

$$f_1^1(z)=-\frac{1}{12}\frac{\theta_0 d(z_1,z_2)}{d(z_1,z_2)}+\frac{1}{6}\ ,\quad f_2(z)=\frac{1}{d(z_1,z_2)^2}\Big(\sum_{i,j=0}^{5} b_{ij}z_1^i z_2^j\Big)\ .$$

(The numerator of $f_2(z)$ is omitted because it is long.) As the holomorphic limit, we take

$$G_{0\bar{0}}\to 1-\theta_0 H(z_1,z_2).$$

8.2. $\mathbb{F}_1$ **case.** Let Δ be the polyhedron #3 in Figure 1:

$$\Delta=\text{ the convex hull of }\{(1,0),(0,1),(-1,0),(-1,-1)\}\ .$$

Δ-regularity. The Δ-regularity condition for $F\in\mathbb{L}(\Delta)$ is as follows:

$$F(t_1,t_2)=a_0+a_1t_1+a_2t_2+\frac{a_3}{t_1}+\frac{a_4}{t_1t_2}\ ,$$
$$a_1a_2a_3a_4\neq 0\ ,\quad a_3(a_0^2-4a_1a_3)^2-a_2a_4(a_0^3-36a_0a_1a_3+27a_1a_2a_4)\neq 0\ .$$

$\mathcal{R}_F$, $\mathcal{I}$-filtration, $\mathcal{E}$-filtration and MHS. These are the same as the $\mathbb{F}_0$-case.

A-hypergeometric system. The lattice of relations $L(\Delta)$ is generated by two vectors

$$l^{(1)}=(-2,1,0,1,0)\ ,\qquad l^{(2)}=(-1,0,1,-1,1)\ .$$

The A-hypergeometric system is generated by the following differential operators:

$$\theta_{a_1}-\theta_{a_3}-\theta_{a_4}\ ,\quad \theta_{a_2}-\theta_{a_4}\ ,\quad \theta_{a_1}+\theta_{a_2}+\theta_{a_3}+\theta_{a_4}+\theta_{a_0}\ ,$$
$$\partial_{a_1}\partial_{a_3}-\partial_{a_0}^2\ ,\quad \partial_{a_2}\partial_{a_4}-\partial_{a_0}\partial_{a_3}\ .$$

Take

$$z_1=a^{l^{(1)}}=\frac{a_1a_3}{a_0^2}\ ,\qquad z_2=a^{l^{(2)}}=\frac{a_2a_4}{a_0a_3}\ .$$

These are coordinates of an open subset of $\mathcal{M}(\Delta)$. We have $\theta_0:=q_*\theta_{a_0}=-2\theta_{z_1}-\theta_{z_2}$.

With these coordinates, the A-hypergeometric system reduces to the following two differential operators of order 2:

$$\mathcal{L}_1=\theta_1(\theta_1-\theta_2)-z_1(-2\theta_1-\theta_2)(-2\theta_1-\theta_2-1)\ ,$$
$$\mathcal{L}_2=\theta_2^2-z_2(-2\theta_1-\theta_2)(\theta_1-\theta_2)\ .$$

Solutions about $z_1=0,z_2=0$ are as follows.

$$\varpi(z;0)=1\ ,$$
$$t_1:=\partial_{\rho_1}\varpi(z;\rho)|_{\rho=0}=\log z_1+2H(z_1,z_2)\ ,$$
$$t_2:=\partial_{\rho_2}\varpi(z;\rho)|_{\rho=0}=\log z_2+H(z_1,z_2)\ ,$$
$$\partial_S F:=\Big(\frac{1}{2}\partial_{\rho_1}^2+\partial_{\rho_1}\partial_{\rho_2}\Big)\varpi(z;\rho)\ ,$$

where

$$\varpi(z;\rho)=\sum_{n_1,n_2\geq 0}\frac{(2\rho_1+\rho_2)_{2n_1+n_2}}{(\rho_1+1)_{n_1}(\rho_2+1)^2_{n_2}}\frac{\Gamma(1+\rho_1-\rho_2)}{\Gamma(1+\rho_1-\rho_2+n_1-n_2)}z_1^{n_1+\rho_1}z_2^{n_2+\rho_2}\ ,$$

$$H(z_1,z_2)=\sum_{\substack{n_1,n_2\geq 0\\ n_1\geq n_2}}\frac{(2n_1+n_2-1)!}{n_1!(n_1-n_2)!n_2!^2}(-1)^{n_2}z_1^{n_1}z_2^{n_2}\ .$$

Here $\Gamma(x)$ denotes the Gamma function.

Yukawa coupling.

$$\begin{aligned}Y_{0,0;0}&=\frac{c(8-9z_2)}{d(z_1,z_2)}\ ,\quad Y_{1,1;0}=\frac{c(1+4z_1-z_2-3z_1z_2)}{d(z_1,z_2)}\ ,\\ Y_{1,2;0}&=\frac{c(1-4z_1-z_2+6z_1z_2)}{d(z_1,z_2)}\ ,\quad Y_{2,2;0}=-\frac{c(z_2(1+12z_1))}{d(z_1,z_2)}\ ,\end{aligned}\tag{8.4}$$

where $d(z_1,z_2)=(1-4z_1)^2-z_2(1-36z_1+27z_1z_2)$ and $c\in\mathbb{C}$ is a nonzero constant.

Comparison with local A-model Yukawa coupling. As in the $\mathbb{F}_0$-case, we can show that

$$\mathrm{Yuk}(\partial_{t_\alpha},\partial_{t_\beta};-2\partial_{t_1}-\partial_{t_2})\propto\partial_{t_\alpha}\partial_{t_\beta}\partial_S F\qquad(1\leq\alpha,\beta\leq 2)\ .$$

Holomorphic ambiguities. The multiplication constant of $Y_{00;0}$ is $c=1$ and

$$\kappa=\frac{2z_1(-32+192z_1+282z_2-144z_1z_2-486z_2^2+243z_2^3)}{d(z_1,z_2)}\ .$$

We checked that $\tilde{C}_1^1,\tilde{C}_0^2$ give the correct local GW invariants of $\mathbb{F}_1$ for small degrees. The holomorphic ambiguities are

$$f_1^1(z)=-\frac{1}{12}\frac{\theta_0 d(z_1,z_2)}{d(z_1,z_2)}+\frac{1}{6}\ ,\quad f_2(z)=\frac{1}{d(z_1,z_2)^2}\Big(\sum_{i,j=0}^{7}b_{ij}z_1^iz_2^j\Big)\ .$$

(The numerator of $f_2(z)$ is omitted.) As the holomorphic limit, we take

$$G_{0\overline{0}}\to 1-\theta_0 H(z_1,z_2).$$

8.3. $\mathbb{F}_2$**-case.** Let Δ be the polyhedron #4 in Figure 1:

$$\Delta=\ \text{the convex hull of }\{(1,0),(0,1),(-1,0),(-2,-1)\}\ .$$

This Δ is different from previous examples in that there are one integral point lying on the middle of an edge. This case has several features different from the previous cases.

Δ-regularity. The Δ-regularity condition for $F\in\mathbb{L}(\Delta)$ is as follows[13]:

$$\begin{aligned}&F(t_1,t_2)=a_0+a_1t_1+a_2t_2+\frac{a_3}{t_1}+\frac{a_4}{t_1^2t_2}\ ,\\ &a_1a_2a_3a_4\neq 0\ ,\quad (a_3^2-4a_2a_4)\big((a_0^2-4a_1a_3)^2-64a_1^2a_2a_4\big)\neq 0\ .\end{aligned}$$

[13] In the last equation, the first factor comes from a 1-dimensional face and the second factor comes from the 2-dimensional face.

$\mathcal{R}_F$ and filtrations.

$$\mathcal{R}_F \cong \mathbb{C}\,1 \oplus \mathbb{C}\,t_0 \oplus \mathbb{C}\,\frac{t_0}{t_1} \oplus \mathbb{C}\,t_0^2\ .$$

The $\mathcal{I}$-filtration is

$$\mathcal{I}_1 \mathbb{C}\,t_0 \oplus \mathbb{C}\,t_0^2\ , \quad \mathcal{I}_2 = \mathcal{I}_3 = \mathcal{I}_1 \oplus \mathbb{C}\,\frac{t_0}{t_1}\ , \qquad \mathcal{I}_4 = \mathcal{R}_F\ .$$

The $\mathcal{E}$-filtration is

$$\mathcal{E}^0 = \mathbb{C}\,1\ , \quad \mathcal{E}^{-1} = \mathcal{E}^0 \oplus \mathbb{C}\,t_0 \oplus \mathbb{C}\,\frac{t_0}{t_1}\ , \quad \mathcal{E}^{-2} = \mathcal{R}_F\ .$$

MHS.

$$H^2(\mathbb{T}^2, C_a^\circ) = \mathbb{C}\omega_0 \oplus PH^1(C_a^\circ)\ , \quad PH^1(C_a^\circ) = \mathbb{C}\,\rho(t_0) \oplus \mathbb{C}\rho(t_0/t_1) \oplus \mathbb{C}\rho(t_0^2)\ .$$

$$\mathcal{W}_1 = \mathbb{C}\rho(t_0) \oplus \mathbb{C}\rho(t_0^2)\ , \quad \mathcal{W}_2 = \mathcal{W}_1 \oplus \mathbb{C}\rho(t_0/t_1)\ , \qquad \mathcal{W}_4 = H^2(\mathbb{T}^2, \mathbb{C})\ .$$

$$\mathcal{E}^0 = \mathbb{C}\omega_0\ , \quad \mathcal{E}^{-1} = \mathcal{E}^0 \oplus \mathbb{C}\rho(t_0) \oplus \mathbb{C}\rho(t_0/t_1)\ , \quad \mathcal{E}^{-2} = H^2(\mathbb{T}^2, \mathbb{C})\ .$$

A-hypergeometric system. The lattice of relations $L(\Delta)$ is generated by two vectors

$$l^{(1)} = (-2, 1, 0, 1, 0)\ , \qquad l^{(2)} = (0, 0, 1, -2, 1)\ ,$$

and the A-hypergeometric system is generated by the following differential operators:

$$\theta_{a_1} - \theta_{a_3} - 2\theta_{a_4}\ , \quad \theta_{a_2} - \theta_{a_4}\ , \quad \theta_{a_1} + \theta_{a_2} + \theta_{a_3} + \theta_{a_4} + \theta_{a_0}\ ,$$
$$\partial_{a_1}\partial_{a_3} - \partial_{a_0}^2\ , \quad \partial_{a_2}\partial_{a_4} - \partial_{a_3}^2\ .$$

Take the following local coordinates of $\mathcal{M}(\Delta)$:

$$z_1 = a^{l^{(1)}} = \frac{a_1 a_3}{a_0^2}\ , \qquad z_2 = a^{l^{(2)}} = \frac{a_2 a_4}{a_3^2}\ .$$

Then we have $\theta_0 := q_* \theta_{a_0} = -2\theta_{z_1}$.

With these coordinates, the A-hypergeometric system reduces to the following two differential operators of order 2:

$$\mathcal{L}_1 = \theta_1(\theta_1 - 2\theta_2) - z_1(-2\theta_1)(-2\theta_1 - 1)\ ,$$
$$\mathcal{L}_2 = \theta_2^2 - z_2(\theta_1 - 2\theta_2)(\theta_1 - 2\theta_2 - 1)\ .$$

Solutions about $z_1 = 0, z_2 = 0$ are as follows.

$$\varpi(z; 0) = 1\ ,$$
$$t_1 := \partial_{\rho_1}\varpi(z;\rho)|_{\rho=0} = \log z_1 + H(z_1, z_2) - G(z_2)\ ,$$
$$t_2 := \partial_{\rho_2}\varpi(z;\rho)|_{\rho=0} = \log z_2 + 2G(z_2)\ ,$$
$$\partial_S F := (\partial_{\rho_1}^2 + \partial_{\rho_1}\partial_{\rho_2})\varpi(z;\rho)\ ,$$

where

$$\varpi(z;\rho) = \sum_{n_1, n_2 \geq 0} \frac{(2\rho_1)_{2n_1}}{(\rho_1+1)_{n_1}(\rho_2+1)_{n_2}^2} \frac{\Gamma(1+\rho_1-2\rho_2)}{\Gamma(1+\rho_1-2\rho_2+n_1-2n_2)} z_1^{n_1+\rho_1} z_2^{n_2+\rho_2}\ ,$$

$$H(z_1, z_2) = 2\sum_{\substack{n_1, n_2 \geq 0 \\ n_1 \geq 2n_2}} \frac{(2n_1-1)!}{n_1!(n_1-2n_2)!n_2!^2} z_1^{n_1} z_2^{n_2}\ ,$$

$$G(z_2) = \sum_{n_2 \geq 1} \frac{(2n_2-1)!}{n_2!^2} z_2^{n_2}\ .$$

Yukawa coupling.

$$Y_{1,1;0}=\frac{2c}{d(z_1,z_2)}\ ,\quad Y_{1,2;0}=\frac{c(1-4z_1)}{d(z_1,z_2)}\ ,\quad Y_{2,2;0}=-\frac{2cz_2(1-8z_1)}{(1-4z_2)d(z_1,z_2)}\ , \tag{8.5}$$

and $Y_{00;0}=4Y_{11;0}$ where $d(z_1,z_2)=(1-4z_1)^2-64z_1^2z_2$ and $c\in\mathbb{C}$ is a nonzero constant.

Comparison with local A-model Yukawa coupling. We show that

$$\mathrm{Yuk}(\partial_{t_\alpha},\partial_{t_\beta};-2\partial_{t_1})\propto\partial_{t_\alpha}\partial_{t_\beta}\partial_S F\quad(1\le\alpha,\beta\le 2)\ . \tag{8.6}$$

Note that the Wronskian $\mathrm{Wr}_{i_1\dots i_k}(t_1,t_2,\partial_S F)$ defined as in (8.3) is divisible by $\theta_2 t_2$ due to the fact that t_2 does not depend on z_1. We define the modified Wronskian[14] by

$$\begin{aligned}\mathrm{Wr}'_{i_1\dots i_k}(t_1,t_2,\partial_S F)&:=\mathrm{Wr}_{i_1\dots i_k}(t_1,t_2,\partial_S F)/\theta_2 t_2\\ &=\theta_1 t_1\cdot\sum_{\alpha,\beta=1}^{2}\partial_1 t_\alpha\cdot\partial_2 t_\beta\cdot\partial_{t_\alpha}\partial_{t_\beta}\partial_S F\ .\end{aligned}$$

As in the $\mathbb{F}_0$-case, Lemma 6.12 holds if we replace $Y_{i_1\dots i_k;0}$ by $\mathrm{Wr}'_{i_1\dots i_k}(t_1,t_2,\partial_S F)$. Therefore $\mathrm{Wr}_{ij}(t_1,t_2,\partial_S F)$ is proportional to the Yukawa coupling $Y_{ij;0}$. Then (8.6) follows from the multi-linearity of Yuk.

Holomorphic ambiguities. The multiplication constant of $Y_{00;0}$ is $c=1$ and

$$\kappa=\frac{8z_1(1-6z_1+24z_1z_2)}{d(z_1,z_2)}\ .$$

We checked that $\tilde{C}_1^1,\tilde{C}_0^2$ give the correct local GW invariants of $\mathbb{F}_2$ for small degrees. The holomorphic ambiguities are

$$f_1^1(z)=-\frac{1}{12}\frac{\theta_0 d(z_1,z_2)}{d(z_1,z_2)}+\frac{1}{6}\ ,\quad f_2(z)=\frac{1}{d(z_1,z_2)^2}\Big(\sum_{i,j=0}^{7}b_{ij}z_1^i z_2^j\Big)\ .$$

(The numerator of $f_2(z)$ is omitted.) As the holomorphic limit, we take

$$G_{0\overline{0}}\to 1-\theta_0 H(z_1,z_2).$$

Appendix A. Mixed Hodge structure of an Open threefold

In this section, Δ is a 2-dimensional reflexive polyhedron. Let $F_a\in\mathbb{L}_{\mathrm{reg}}(\Delta)$ be a Δ-regular Laurent polynomial. Define $P_a\in\mathbb{C}[t_1^{\pm},t_2^{\pm},x,y]$ by

$$P_a(t_1,t_2,x,y)=F_a(t_1,t_2)+xy\ .$$

Let Z_a° be the affine hypersurface in $\mathbb{T}^2\times\mathbb{C}^2$ defined by P_a:

$$Z_a^\circ:=\{(t_1,t_2,x,y)\in\mathbb{T}^2\times\mathbb{C}^2\mid F_a(t_1,t_2)+xy=0\}\ . \tag{A.1}$$

It is easy to see that the Δ-regularity of F_a implies the smoothness of Z_a°.

The goal of the appendix is to give an explicit description of the MHS on $H^3(Z_a^\circ)$. First we show that $H^3(Z_a^\circ)\cong\mathcal{R}_{F_a}$. Next we compactify Z_a° as a hypersurface in a smooth toric variety. Then using this compactification, we compute the Hodge and weight filtrations on

[14] A reason to consider the modified Wronskian in the $\mathbb{F}_2$-case is that the Wronskians do not satisfy the statement corresponding to the second one in Lemma 6.12.

$H^3(Z_a^\circ)$. We use Batyrev's method for affine hypersurfaces in algebraic tori [5, §6–8] with some modifications.

A.1. **Middle cohomology $H^3(Z_a^\circ)$.** We have a long exact sequence

$$\cdots \to H^4(\mathbb{T}^2\times\mathbb{C}^2) \to H^4(\mathbb{T}^2\times\mathbb{C}^2\setminus Z_a^\circ) \xrightarrow{\mathrm{Res}} H^3(Z_a^\circ) \to H^5(\mathbb{T}^2\times\mathbb{C}^2) \to \cdots . \tag{A.2}$$

Since $H^4(\mathbb{T}^2\times\mathbb{C}^2)=H^5(\mathbb{T}^2\times\mathbb{C}^2)=0$, the *Poincaré residue map* $\mathrm{Res}: H^4(\mathbb{T}^2\times\mathbb{C}^2\setminus Z_a^\circ)\to H^3(Z_a^\circ)$ is an isomorphism.

In the rest of this subsection, t^m stands for $t_1^{m_1}t_2^{m_2}$. By Grothendieck [21], $H^\bullet(\mathbb{T}^2\times\mathbb{C}^2\setminus Z_a^\circ)$ is isomorphic to the cohomology of the global de Rham complex $(\Gamma\Omega^\bullet_{\mathbb{T}^2\times\mathbb{C}^2}(*Z_a^\circ),d)$ of meromorphic differential forms on $\mathbb{T}^2\times\mathbb{C}^2$ with poles of arbitrary order on Z_a°. Let R' be the homomorphism:

$$R':\mathbf{S}_\Delta\to\Gamma\Omega^4_{\mathbb{T}^2\times\mathbb{C}^2}(*Z_a^\circ)\ ,\quad t_0^kt^m\mapsto\frac{(-1)^kk!\,t^m}{{P_a}^{k+1}}\frac{dt_1}{t_1}\frac{dt_2}{t_2}dxdy\ .$$

Proposition A.1. *The map R' induces an isomorphism*

$$R':\mathcal{R}_{F_a}\xrightarrow{\cong}H^4(\mathbb{T}^2\times\mathbb{C}^2\setminus Z_a^\circ)\ .$$

Corollary A.2. *The map R' and the Poincaré residue map give an isomorphism*

$$\rho':\mathcal{R}_{F_a}\xrightarrow{\cong}H^3(Z_a^\circ)\ .$$

Remark A.3. For $i=0,1,2$, $\iota^*:H^i(\mathbb{T}^2\times\mathbb{C}^2)\to H^i(Z_a^\circ)$ is an isomorphism where $\iota:Z_a^\circ\to\mathbb{T}^2\times\mathbb{C}^2$ is the inclusion. For $i\geq4$, $H^i(Z_a^\circ)=0$ since Z_a° is affine.

Proof. (of Proposition A.1.) We would like to compute

$$\frac{\Gamma\Omega^4_{\mathbb{T}^2\times\mathbb{C}^2}(*Z_a^\circ)}{d\Gamma\Omega^3_{\mathbb{T}^2\times\mathbb{C}^2}(*Z_a^\circ)}\cong H^4(\mathbb{T}^2\times\mathbb{C}^2\setminus Z_a^\circ)\ . \tag{A.3}$$

Let

$$\mathbf{M}_0=\mathbb{C}[t_0,t_1^\pm,t_2^\pm,x,y]\ ,\quad \mathbf{M}=\mathbf{M}_0/\mathcal{D}_0'\mathbf{M}_0\ ,\quad \mathbf{L}=\mathbb{C}[t_1^\pm,t_2^\pm]\ ,$$

where $\mathcal{D}_0':\mathbf{M}_0\to\mathbf{M}_0$ is defined by

$$\mathcal{D}_0'(t_0^kt^mx^{m_3}y^{m_4}):=\begin{cases}(k+t_0P_a)t_0^kt^mx^{m_3}y^{m_4} & (k>0)\\ (1+t_0P_a)t_0^kt^mx^{m_3}y^{m_4} & (k=0)\end{cases}\ .$$

We first rewrite the left-hand-side of (A.3) using $\mathbf{M}$. Consider the homomorphism $\Psi_0:\mathbf{M}_0\to\Omega^0_{\mathbb{T}^2\times\mathbb{C}^2}(*Z_a^\circ)$ given by

$$\Psi_0(t_0^kt^mx^{m_3}y^{m_4})=\begin{cases}\frac{(-1)^{k-1}(k-1)!t^mx^{m_3}y^{m_4}}{P_a^k} & (k\geq1)\\ -t^mx^{m_3}y^{m_4} & (k=0)\end{cases}\ .$$

Then the kernel of Ψ_0 is $\mathcal{D}_0'\mathbf{M}_0$. Therefore Ψ_0 induces an isomorphism $\Psi:\mathbf{M}\xrightarrow{\cong}\Omega^0_{\mathbb{T}^2\times\mathbb{C}^2}(*Z_a^\circ)$.

Define the operators $\mathcal{D}'_i$ $(1 \le i \le 4)$ acting on $\mathbf{M}$ by

$$\mathcal{D}'_i(t_0^k t^m x^{m_3} y^{m_4}) := \begin{cases} (m_i + t_0\theta_{t_i} P_a) t_0^k t^m x^{m_3} y^{m_4} & (i = 1, 2,\ k > 0) \\ m_i t_0^k t^m x^{m_3} y^{m_4} & (i = 1, 2\ , k = 0) \end{cases}$$

$$\mathcal{D}'_3(t_0^k t^m x^{m_3} y^{m_4}) := \begin{cases} (m_3 + t_0\theta_x P_a) t_0^k t^m x^{m_3 - 1} y^{m_4} & (k > 0) \\ m_3 t_0^k t^m x^{m_3 - 1} y^{m_4} & (k = 0) \end{cases}$$

$$\mathcal{D}'_4(t_0^k t^m x^{m_3} y^{m_4}) := \begin{cases} (m_4 + t_0\theta_y P_a) t_0^k t^m x^{m_3} y^{m_4 - 1} & (k > 0) \\ m_4 t_0^k t^m x^{m_3} y^{m_4 - 1} & (k = 0) \end{cases} .$$

Let $e_1, \ldots, e_4$ be the standard basis on $\mathbb{C}^4$. For $I = \{i_1, \ldots, i_p\} \subset \{1, 2, 3, 4\}$, let $e_I := e_{i_1} \wedge \cdots \wedge e_{i_p}$. Then we have an isomorphism

$$\Psi_p : \mathbf{M} \otimes \wedge^p \mathbb{C}^4 \xrightarrow{\sim} \Omega^p_{\mathbb{T}^2 \times \mathbb{C}^2}(*Z_a^\circ)\ ; \quad \sum_{i=1}^4 f_i \otimes e_I \mapsto \sum_{i=1} \Psi(f_i)\gamma(e_I)\ ,$$

where γ is defined by

$$\gamma(e_i) = \begin{cases} \frac{dt_i}{t_i} & (i = 1, 2) \\ dx & (i = 3) \\ dy & (i = 4) \end{cases} , \quad \gamma(e_I) = \gamma(e_{i_1}) \wedge \cdots \wedge \gamma(e_{i_p})\ .$$

If we define $\mathcal{D}' : \mathbf{M} \otimes \wedge^3 \mathbb{C}^4 \to \mathbf{M} \otimes \wedge^4 \mathbb{C}^4$ by

$$\mathcal{D}'(f_I \otimes e_I) := \sum_{i=1}^4 \mathcal{D}'_i(f_I)\gamma(e_i \wedge e_I)\ ,$$

we have a commutative diagram

$$\begin{array}{ccc} \mathbf{M} \otimes \wedge^3 \mathbb{C}^4 & \xrightarrow{\mathcal{D}'} & \mathbf{M} \otimes \wedge^4 \mathbb{C}^4 \\ \Psi_3 \downarrow & & \downarrow \Psi_4 \\ \Omega^3_{\mathbb{T}^2 \times \mathbb{C}^2}(*Z_a^\circ) & \xrightarrow{d} & \Omega^4_{\mathbb{T}^2 \times \mathbb{C}^2}(*Z_a^\circ) \end{array} .$$

Thus we have

$$\frac{\Gamma\Omega^4_{\mathbb{T}^2 \times \mathbb{C}^2}(*Z_a^\circ)}{d\Gamma\Omega^3_{\mathbb{T}^2 \times \mathbb{C}^2}(*Z_a^\circ)} \cong \frac{\mathbf{M} \otimes \wedge^4 \mathbb{C}^4}{\mathcal{D}'\mathbf{M} \otimes \wedge^3 \mathbb{C}^4}\ .$$

Then the proposition follows from the next lemma.

Lemma A.4. *1. The homomorphism* $\mathbf{L}[t_0] \to \mathbf{M} \otimes \wedge^4 \mathbb{C}^4$ *given by* $t_0^k t^m \mapsto t_0^{k+1} t^m \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4$ *induces an isomorphism*

$$\mathbf{L}[t_0] / \sum_{i=0}^2 \mathcal{D}_i \mathbf{L}[t_0] \xrightarrow{\sim} \frac{\mathbf{M} \otimes \wedge^4 \mathbb{C}^4}{\mathcal{D}'(\mathbf{M} \otimes \wedge^3 \mathbb{C}^4)}\ .$$

Here $\mathcal{D}_i$ *are the same as those defined in* (3.3).
2. The inclusion $\mathbf{S}_\Delta \to \mathbf{L}[t_0]$ *induces an isomorphism*

$$\mathbf{S}_\Delta / \sum_{i=0}^2 \mathcal{D}_i \mathbf{S}_\Delta \xrightarrow{\sim} \mathbf{L}[t_0] / \sum_{i=0}^2 \mathcal{D}_i \mathbf{L}[t_0]\ .$$

Proof of the lemma is by brute force calculation. □

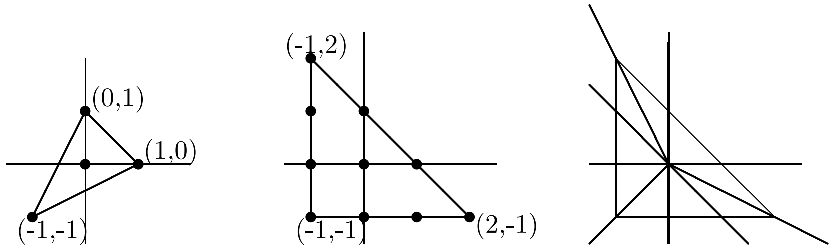

FIGURE 2. An example of a reflexive polyhedron Δ (left), its dual polytope Δ^* (middle) and the fan $\Sigma(\Delta)$ (right).

A.2. **Compactification Z_a of Z_a°.** In §A.2–§A.5, we omit the subscript a from F_a, P_a, Z_a, Z_a°, C_a° and C_a for simplicity. In §A.2 and §A.3, t^m stands for the Laurent monomial $t_1^{m_1}t_2^{m_2}t_3^{m_3}t_4^{m_4}$.

We construct a compactification of Z° as a semiample smooth hypersurface Z in a 4-dimensional toric variety $\mathbf{V}$ such that the divisor $D = Z \setminus Z^\circ$ is a simple normal crossing divisor:

$$\begin{array}{ccc} Z^\circ & \subset & Z \\ \cap & & \cap \\ \mathbb{T}^2 \times \mathbb{C}^2 & \subset & \mathbf{V} \end{array} .$$

The basic idea is to consider the following slightly modified expression for $P = F + xy$:

$$\tilde{P} := \frac{F(t_1,t_2)}{t_3t_4} + \frac{b_1}{t_4} + b_0 \ , \quad (b_1, b_0 \neq 0, F \in \mathbb{L}_{\mathrm{reg}}(\Delta)) \ . \tag{A.4}$$

The Newton polyhedron $\tilde{\Delta}$ of $\tilde{P}$ is given by

$$\begin{aligned} \tilde{\Delta} := \big\{ &(m_1, m_2, m_3, m_4) \in \mathbb{R}^4 \mid \\ &m_3 \leq 0,\ m_4 \geq -1,\ m_3 - m_4 \geq 0, (m_1, m_2) \in \Delta(-m_3) \big\} \ . \end{aligned} \tag{A.5}$$

Then by the general theory of the toric variety, we obtain a singular projective toric variety $\mathbf{V}' = \mathrm{Proj}\, \mathbf{S}_{\tilde{\Delta}}$ such that $H^0(\mathbf{V}', \mathcal{O}(1)) \cong \oplus_{m \in A(\tilde{\Delta})} \mathbb{C}t^m$. We blow up $\mathbf{V}'$ to obtain a smooth toric variety $\mathbf{V}$. A compactification of Z° can be obtained as a hypersurface defined by a generic section of the pull-back of $\mathcal{O}(1)$.

Such a $\mathbf{V}$ can be explicitly given as follows. First, let $v_i \in \mathbb{Z}^2$ $(1 \leq i \leq r,\ r := l(\Delta) - 1)$ be the primitive vectors lying on faces of the dual polyhedron Δ^* of Δ. Let $\Sigma(\Delta)$ be the 2-dimensional complete fan spanned by $v_1, \ldots, v_r$ (see Figure 2) and let $\mathbb{P}_{\Sigma(\Delta)}$ be the corresponding smooth toric surface. Then $\mathbb{P}_{\Sigma(\Delta)}$ is a resolution of the singular toric surface $\mathrm{Proj}\, \mathbf{S}_\Delta$, and C° can be compactified smoothly to C in $\mathbb{P}_{\Sigma(\Delta)}$. Next we set

$$\tilde{v}_i = \begin{pmatrix} v_i \\ -1 \\ 0 \end{pmatrix} \ (1 \leq i \leq r), \quad u_1 = \begin{pmatrix} \vec{0} \\ 1 \\ 0 \end{pmatrix} , \quad u_2 = \begin{pmatrix} \vec{0} \\ 0 \\ 1 \end{pmatrix} , \quad u_3 = \begin{pmatrix} \vec{0} \\ -1 \\ 0 \end{pmatrix} , \quad u_4 = \begin{pmatrix} \vec{0} \\ 1 \\ -1 \end{pmatrix} .$$

Then the 1-cones of the fan $\Sigma_{\mathbf{V}}$ are given by

$$\nu_i = \mathbb{R}_{\geq 0}\tilde{v}_i \ (1 \leq i \leq r) \ , \quad \mu_i = \mathbb{R}_{\geq 0}u_j \ (1 \leq j \leq 4) \ ,$$

and the 4-cones of $\Sigma_{\mathbf{V}}$ are given by

$$[i, i+1; j, j+1] := \mathbb{R}_{\geq 0}\tilde{v}_i + \mathbb{R}_{\geq 0}\tilde{v}_{i+1} + \mathbb{R}_{\geq 0}u_j + \mathbb{R}_{\geq 0}u_{j+1} \quad (1 \leq i \leq r,\ 1 \leq j \leq 4).$$

(For the sake of convenience, we set $\tilde{v}_{r+1} := \tilde{v}_1, u_5 := u_1$ and $\nu_{r+1} := \nu_1, \mu_5 := \mu_1$.) The 3-cones and the 2-cones are faces of the above 4-cones. The toric variety $\mathbf{V}$ associated to the fan $\Sigma_{\mathbf{V}}$ is a bundle over the toric surface $\mathbb{P}_{\Sigma(\Delta)}$ whose fiber is the Hirzebruch surface $\mathbb{F}_1$.

Let $\mathbb{D}_i$ $(1 \le i \le r)$ and $\mathbb{E}_j$ $(1 \le j \le 4)$ be the toric divisors of $\mathbf{V}$ corresponding to the 1-cones ν_i and μ_j respectively. By a standard computation in the theory of toric varieties (see e.g. [31]), we have

Lemma A.5.

$$H^0(\mathbf{V}, \mathcal{O}(\mathbb{E}_1 + \mathbb{E}_2)) = \bigoplus_{m \in A(\tilde{\Delta})} \mathbb{C}\, t^m \ .$$

Therefore $\tilde{P}$ in eq.(A.4) is a generic section of the line bundle corresponding to the divisor $\mathbb{E}_1 + \mathbb{E}_2$. We define Z to be the hypersurface in $\mathbf{V}$ defined by $\tilde{P}$. We show that if we assume F is Δ-regular and $b_0 b_1 \neq 0$, then (1) $Z^\circ \subset Z$; (2) Z is smooth; (3) $D = Z/Z^\circ$ is a (simple) normal crossing divisor. (1) can be shown as follows. Let $U_{\mu_1,\mu_2} = \mathrm{Spec}\,[t_1^{\pm 1}, t_2^{\pm 2}, t_3, t_4] \subset \mathbf{V}$ be the open set corresponding to the 2-cone spanned by μ_1, μ_2. It is isomorphic to $\mathbb{T}^2 \times \mathbb{C}^2$. The defining equation of Z on U_{μ_1,μ_2} is $F(t_1,t_2) + b_1 t_3 + b_0 t_3 t_4$ and this is equal to P if we identify $x = t_3, y = b_1 + b_0 t_4$. We can prove (2) and (3) by looking at the defining equation P_σ of Z on the open subset $U_\sigma \subset \mathbf{V}$ corresponding to each 4-cone σ.

We end this subsection by listing the Hodge numbers of Z.

Lemma A.6. *The Hodge numbers $h^{p,q}(Z) = \dim H^{p,q}(Z)$ are*

	$p=0$	1	2	3
$q=0$	1	0	0	0
1	0	$l(\Delta)-1$	1	0
2	0	1	$l(\Delta)-1$	0
3	0	0	0	1

Proof. By the formula on cohomology of semiample divisors on a toric variety due to Mavlyutov [30, Cor.2.7][15], we can explicitly compute the dimensions of $H^q(\mathbf{V}, \Omega^p(\mathbb{E}_1 + \mathbb{E}_2))$ and $H^q(\mathbf{V}, \Omega^p(2\mathbb{E}_1 + 2\mathbb{E}_2))$. Then we obtain $\dim H^q(Z, \Omega^p)$ by exact sequences (as in the proof of the Lefshetz hyperplane theorem [19, p.156]). □

A.3. **The Hodge filtration.** Let $\mathbb{D} = \sum_{i=1}^r \mathbb{D}_i + \mathbb{E}_3 + \mathbb{E}_4$. Note that $\mathbb{D} = \mathbf{V} \setminus \mathbb{T}^2 \times \mathbb{C}^2$ and $D = Z/Z^\circ = Z \cap \mathbb{D}$.

[15] For a semiample toric divisor X in a d-dimensional complete simplicial toric variety $\mathbb{P}_\Sigma$, Mavlyutov's formula is:

$$\dim H^k(\mathbb{P}_\Sigma, \Omega^l_{\mathbb{P}_\Sigma}(X)) = \sum_\delta l^*(\delta) \binom{\dim \delta}{l-k} \cdot \sum_{j=0}^{k} \binom{d - \dim\delta - j}{k-j} (-1)^{k-j} \#\Sigma_{\sigma_\delta}(j) \ .$$

The sum is over all faces δ of the polytope Δ_X associated to X, $l^*(\delta)$ is the number of interior integral points in the face δ, $\sigma_\delta \in \Sigma_X$ is a cone corresponding to the face δ in the fan Σ_X (which is the fan such that there is a morphism $s : \Sigma \to \Sigma_X$; X is a pull-back of an ample divisor by the induced morphism of toric varieties $\mathbb{P}_\Sigma \to \mathbb{P}_{\Sigma_X}$), and $\#\Sigma_{\sigma_\delta}(j)$ is the number of j-cones in $\Sigma_{\sigma_\delta} = \{s(\tau) \in \Sigma : \tau \in \sigma_\delta\}$. In the case of the threefold $Z \subset \mathbf{V}$, Σ_Z is generated by u_2, u_4 and 1-cones $\tilde{v}_i$ such that v_i are vertices of Δ^*.

Proposition A.7. *For $p = 1, 2, 3, 4$, the residue mapping*

$$H^{4-p}\big(\mathbf{V}, \Omega^p_{\mathbf{V}}(\log(Z + \mathbb{D}))\big) \overset{\mathrm{Res}_Z}{\to} H^{4-p}\big(Z, \Omega^{p-1}_Z(\log D)\big)$$

is an isomorphism.

Proof. Consider the exact sequence

$$0 \to \Omega^p_{\mathbf{V}}(\log \mathbb{D}) \to \Omega^p_{\mathbf{V}}(\log(Z + \mathbb{D})) \overset{\mathrm{Res}_Z}{\to} \Omega^{p-1}_Z(\log D) \to 0 \ ,$$

and take the cohomology. The vanishings $H^{4-p}(\mathbf{V}, \Omega^p_{\mathbf{V}}(\log \mathbb{D})) = H^{5-p}(\mathbf{V}, \Omega^p_{\mathbf{V}}(\log \mathbb{D})) = 0$ imply the proposition. □

We use the notation $\Omega^p_{\mathbf{V},\mathbb{D}}(k) := \Omega^p_{\mathbf{V}}(\log \mathbb{D}) \otimes \mathcal{O}(kZ)$ for integers $k, p \geq 0$.

Proposition A.8.

$$H^{4-p}\big(\mathbf{V}, \Omega^p(\log(Z + \mathbb{D}))\big) \cong \frac{H^0(\mathbf{V}, \Omega^4_{\mathbf{V},\mathbb{D}}(5-p))}{H^0(\mathbf{V}, \Omega^4_{\mathbf{V},\mathbb{D}}(4-p)) + dH^0(\mathbf{V}, \Omega^3_{\mathbf{V},\mathbb{D}}(4-p))} \qquad (p = 1, 2, 3),$$

$$H^0\big(\mathbf{V}, \Omega^4(\log(Z + \mathbb{D}))\big) \cong H^0(\mathbf{V}, \Omega^4_{\mathbf{V},\mathbb{D}}(1)) \ .$$

Proof. The proposition follows from the exact sequence

$$0 \to \Omega^p_{\mathbf{V}}(\log(Z + \mathbb{D})) \hookrightarrow \Omega^p_{\mathbf{V},\mathbb{D}}(1) \overset{d}{\to} \frac{\Omega^{p+1}_{\mathbf{V},\mathbb{D}}(2)}{\Omega^{p+1}_{\mathbf{V},\mathbb{D}}(1)} \overset{d}{\to} \cdots \overset{d}{\to} \frac{\Omega^4_{\mathbf{V},\mathbb{D}}(5-p)}{\Omega^4_{\mathbf{V},\mathbb{D}}(4-p)} \to 0 \ ,$$

and Lemma A.9 below. □

Lemma A.9. *Let k be a nonnegative integer, $p = 1, 2, 3, 4$.*

1. $H^q(\mathbf{V}, \Omega^p(\log(\mathbb{D} + \mathbb{E}_1 + \mathbb{E}_2)) \otimes \mathcal{O}(kZ)) = 0 \quad (q > 0)$,
2. $H^q(\mathbf{V}, \Omega^p(\log(\mathbb{D} + \mathbb{E}_i)) \otimes \mathcal{O}(kZ)) = 0 \quad (q > 0, i = 1, 2)$,
3. $H^q(\mathbf{V}, \Omega^p_{\mathbf{V},\mathbb{D}}(k)) = 0 \quad (q > 0)$.

.

Proof. 1. Let $\mathbb{D}_{\mathbb{T}} := \mathbb{D} + \mathbb{E}_1 + \mathbb{E}_2$. Note that this is the sum of all toric divisors in $\mathbf{V}$. It is well known that $\Omega^p_{\mathbf{V}}(\log \mathbb{D}_{\mathbb{T}}) \cong \mathcal{O}_{\mathbf{V}} \otimes \wedge^p M$ where M is the dual lattice of $N \cong \mathbb{Z}^4$ (cf. [31]). On the other hand, since $\mathbb{E}_1 + \mathbb{E}_2$ is semiample, we have $H^q(\mathbf{V}, \mathcal{O}_{\mathbf{V}}(kZ)) = 0$ for $q > 0$ [30]. Therefore

$$H^q(\mathbf{V}, \Omega^p_{\mathbf{V}}(\log \mathbb{D}_{\mathbb{T}}) \otimes \mathcal{O}_{\mathbf{V}}(kZ)) \cong H^q(\mathbf{V}, \mathcal{O}_{\mathbf{V}}(kZ)) \otimes \wedge^p M \ = 0 \quad (q > 0) \ .$$

2. As above, the following vanishing holds:

$$H^q(\mathbb{E}_2, \Omega^{p-1}_{\mathbb{E}_2}\big(\log(\mathbb{D} + \mathbb{E}_1)\big) \otimes \mathcal{O}_{\mathbb{E}_2}(kZ)) \cong H^q(\mathbb{E}_2, \mathcal{O}_{\mathbb{E}_2}(kZ)) \otimes \wedge^{p-1}\mathbb{Z}^3 = 0 \quad (q > 0).$$

Moreover, the map

$$H^0(\mathbf{V}, \Omega^p_{\mathbf{V}}(\log \mathbb{D}_{\mathbb{T}}) \otimes \mathcal{O}_{\mathbf{V}}(kZ)) \overset{\mathrm{Res}_{\mathbb{E}_2}}{\to} H^0(\mathbb{E}_2, \Omega^{p-1}_{\mathbb{E}_2}\big(\log(\mathbb{D} + \mathbb{E}_1)\big) \otimes \mathcal{O}_{\mathbb{E}_2}(kZ))$$

is surjective. Taking the exact sequence of cohomology of the exact sequence:

$$0 \to \Omega^p_{\mathbf{V}}\big(\log(\mathbb{D} + \mathbb{E}_1)\big) \otimes \mathcal{O}_{\mathbf{V}}(kZ) \to \Omega^p_{\mathbf{V}}(\log \mathbb{D}_{\mathbb{T}}) \otimes \mathcal{O}_{\mathbf{V}}(kZ)$$
$$\overset{\mathrm{Res}_{\mathbb{E}_2}}{\to} \Omega^{p-1}_{\mathbb{E}_2}\big(\log(\mathbb{D} + \mathbb{E}_1)\big) \otimes \mathcal{O}_{\mathbb{E}_2}(kZ) \to 0 \ ,$$

we obtain

$$H^q(\mathbf{V}, \Omega^p(\log(\mathbb{D}+\mathbb{E}_1))\otimes\mathcal{O}(kZ)) = 0 \quad (q>0)\ .$$

The proof for $\mathbb{E}_2$ is similar.

3. As above, we can show the vanishing

$$H^q(\mathbb{E}_1\cap\mathbb{E}_2, \Omega^{p-2}_{\mathbb{E}_1\cap\mathbb{E}_2}(\log\mathbb{D})\otimes\mathcal{O}(kZ)) \cong H^q(\mathbb{E}_1\cap\mathbb{E}_2, \mathcal{O}_{\mathbb{E}_1\cap\mathbb{E}_2}(kZ))\otimes\wedge^{p-2}\mathbb{Z}^2 = 0 \quad (q>0),$$

and the surjectivity of the map

$$\bigoplus_{i=1,2} H^0(\mathbf{V}, \Omega^p_{\mathbf{V}}\big(\log(\mathbb{D}+\mathbb{E}_i)\big)\otimes\mathcal{O}(kZ)) \overset{\mathrm{Res}_{\mathbb{E}_1,\mathbb{E}_2}}{\to} H^0(\mathbb{E}_1\cap\mathbb{E}_2, \Omega^{p-2}_{\mathbb{E}_1\cap\mathbb{E}_2}(\log\mathbb{D})\otimes\mathcal{O}(kZ)).$$

Consider the exact sequence

$$0\to\Omega^p_{\mathbf{V}}(\log\mathbb{D})\otimes\mathcal{O}(kZ)\to\bigoplus_{i=1,2}\Omega^p_{\mathbf{V}}\big(\log(\mathbb{D}+\mathbb{E}_i)\big)\otimes\mathcal{O}(kZ)$$
$$\to\Omega^p_{\mathbf{V}}(\log\mathbb{D}_{\mathbb{T}})\otimes\mathcal{O}(kZ)\overset{\mathrm{Res}_{\mathbb{E}_1,\mathbb{E}_2}}{\longrightarrow}\Omega^{p-2}_{\mathbb{E}_1\cap\mathbb{E}_2}(\log\mathbb{D})\otimes\mathcal{O}(kZ)\to 0\ .$$

Taking the cohomology, we obtain the statement 3. □

Let $\tilde{\Delta}(k)$ be the polyhedron defined by applying (3.1) to $\tilde{\Delta}$ defined in (A.5) and let $\tilde{\Delta}[k]$ be the following 4-dimensional polyhedron:

$$\tilde{\Delta}[k] := \big\{(m_1,m_2,m_3,m_4)\in\mathbb{R}^4 \mid$$
$$(m_1,m_2)\in\Delta(-m_3-1),\ -k+1\le m_3\le 0,\ m_4\ge -k,\ m_3-m_4\ge 0\ \big\}\ .$$

Proposition A.10. *Let k be a positive integer $k\ge 1$.*

1. $$H^0(\mathbf{V}, \Omega^4_{\mathbf{V},\mathbb{D}}(k)) = \bigoplus_{m\in\tilde{\Delta}(k-1)\cap\mathbb{Z}^4}\mathbb{C}\,\frac{t^m}{\tilde{P}^k}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}\frac{dt_4}{t_4}\ .$$
2. $$H^0(\mathbf{V}, \Omega^3_{\mathbf{V},\mathbb{D}}(k)) = \bigoplus_{m\in\tilde{\Delta}[k]\cap\mathbb{Z}^4}\mathbb{C}\,\frac{t^m}{\tilde{P}^k}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}$$
$$\oplus\bigoplus_{m\in\tilde{\Delta}(k-1)\cap\mathbb{Z}^4}\Big[\mathbb{C}\,\frac{t^m}{\tilde{P}^k}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_4}{t_4}\oplus\mathbb{C}\,\frac{t^m}{\tilde{P}^k}\frac{dt_1}{t_1}\frac{dt_3}{t_3}\frac{dt_4}{t_4}\oplus\mathbb{C}\,\frac{t^m}{\tilde{P}^k}\frac{dt_2}{t_2}\frac{dt_3}{t_3}\frac{dt_4}{t_4}\Big]\ .$$
3. $$\frac{H^0(\mathbf{V}, \Omega^4_{\mathbf{V},\mathbb{D}}(k+1))}{H^0(\mathbf{V}, \Omega^n_{\mathbf{V},\mathbb{D}}(k)) + dH^0(\mathbf{V}, \Omega^3_{\mathbf{V},\mathbb{D}}(k))}\cong R^k_F\ ,$$

where the isomorphism is induced from the map

$$\mathbf{S}^k_{\Delta}\to H^0(\mathbf{V}, \Omega^4_{\mathbf{V},\mathbb{D}}(k+1))\ ,\quad t_0^k t_1^{m_1}t_2^{m_2}\mapsto\frac{(-1)^k k!\,t_1^{m_1}t_2^{m_2}}{\tilde{P}^{k+1}(t_3t_4)^k}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}\frac{dt_4}{t_4}\ .$$

Proof. The statements 1 and 2 can be shown by calculation of the Cech cohomology associated to the open cover given by the toric fan. The third statement follows from the first and the second. □

Theorem A.11. *The Hodge filtration on $H^3(Z^\circ)$ satisfies*

$$\mathrm{Gr}^p_{\mathcal{F}}H^3(Z^\circ)\cong R^{3-p}_F\ ,\quad (0\le p\le 3)\ .$$

Proof. As is well known, there are canonical isomorphisms $\mathrm{Gr}^p_{\mathcal{F}}H^3(Z^\circ)\cong H^{3-p}(Z,\Omega^p_Z(\log D))$. The theorem follows from Propositions A.7, A.8 and A.10. □

A.4. **The weight filtrations.**

Proposition A.12.

$$\dim \mathrm{Gr}_6^{\mathcal{W}} = 1 \ , \quad \dim \mathrm{Gr}_5^{\mathcal{W}} = 0 \ ,$$
$$\dim \mathrm{Gr}_4^{\mathcal{W}} = l(\Delta) - 4 \ , \quad \dim \mathrm{Gr}_3^{\mathcal{W}} = 2 \ .$$

Proof. The divisor $D = Z \setminus Z^\circ$ consists of $r+2$ components. Define $D^{(k)}$ to be the disjoint union of intersections of k components for $k = 1,2,3$ and $D^{(0)} := Z$. Consider the spectral sequence ${}_{\mathcal{W}}E$ of the hypercohomology $\mathbb{H}^k(Z, \Omega_Z^\bullet(\log D))$ associated to the decreasing weight filtration $\mathcal{W}^{-l} := \mathcal{W}_l$. This spectral sequence degenerates at ${}_{\mathcal{W}}E_2$. We have ${}_{\mathcal{W}}E_1^{p,q} \cong H^{2p+q}(D^{(-p)}, \mathbb{C})$ and the differential $d_1 : H^{2p+q}(D^{(-p)}, \mathbb{C}) \to H^{2p+q+2}(D^{(-p-1)}, \mathbb{C})$ is given by the Gysin morphism (see e.g. [37, Corollary 8.33, Proposition 8.34]). Computing the cohomology of d_1, we obtain the following result.

$\dim {}_{\mathcal{W}}E_2^{p,q} =$

	$p=0$	-1	-2	-3
$q=0$	1	0	0	0
1	0	0	0	0
2	0	2	0	0
3	2	0	0	0
4	0	$l(\Delta)-4$	1	0
5	0	0	0	0
6	0	0	0	1

.

The dimensions of the graded quotients $\mathrm{Gr}_{-p+3}^{\mathcal{W}} H^3(Z^\circ) \cong {}_{\mathcal{W}}E_2^{p,3-p}$ can be read from this table. □

Proposition A.13. *The weight filtration on* $H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ)$ *is*

$$\mathcal{W}_8 H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ) = \mathcal{I}_4 = \mathcal{R}_F \ ,$$
$$\mathcal{W}_7 H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ) = \mathcal{W}_6 H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ) = \mathcal{I}_3 \ ,$$
$$\mathcal{W}_5 H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ) = \mathcal{I}_1 \ .$$

Proof. We consider three filtrations $\mathcal{V}, \mathcal{V}', \mathcal{V}''$ on $H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ)$ and compare them. First we define

$$\mathcal{V}_k \Omega_{\mathbf{V}}^i(\log(Z + \mathbb{D})) := \Omega_{\mathbf{V}}^{i-k} \wedge \Omega_{\mathbf{V}}^k(\log(Z + \mathbb{D})) \quad (0 \le k \le 4).$$

This induces the weight filtration $\mathcal{V}_k H^i(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ) = \mathcal{W}_{k+4} H^i(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ)$. We have already computed the dimension of graded quotients in Proposition A.12.

Second, let $U := \mathbf{V} \setminus (Z \cup \mathbb{E}_3 \cup \mathbb{E}_4)$. We define

$$\mathcal{V}'_k \Omega_U^i(\log(U \cap \mathbb{D})) := \Omega_U^{i-k} \wedge \Omega_U^k(\log(U \cap \mathbb{D})) \quad (k = 0,1,2).$$

This induces another filtration $\mathcal{V}'$ on $H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ)$. As in [5, §8], this is given by the $\mathcal{I}$-filtration on $\mathcal{R}_F$:

$$\mathcal{V}'_0 H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ) \cong \mathcal{I}_1 \ , \quad \mathcal{V}'_1 H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ) \cong \mathcal{I}_3 \ , \quad \mathcal{V}'_2 H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ) \cong \mathcal{I}_4 \ .$$

Third, let $j : U \to \mathbf{V}$ be the inclusion and define

$$\mathcal{V}''_k \Omega_{\mathbf{V}}^i(\log(Z + \mathbb{D})) := \Omega_{\mathbf{V}}^{i-k} \wedge \Omega_{\mathbf{V},\mathbb{D}}^k + (\mathcal{V}'_{k-1} j_* \Omega_U^i(\log(U \cap \mathbb{D}))) \cap \Omega_{\mathbf{V}}^i(\log(Z + \mathbb{D})) \ .$$

Since $H^4(\mathbb{T}^2 \times \mathbb{C}^2) = 0$, the first term does not contribute to $H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ)$. So the induced filtration is related to $\mathcal{V}'$ by

$$\mathcal{V}''_k H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ) = \mathcal{V}'_{k-1} H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ) \quad (k = 1, 2, 3) .$$

Moreover it holds that

$$\mathcal{V}_k \Omega^i_{\mathbf{V}}(\log(Z + \mathbb{D})) \subset \mathcal{V}''_k \Omega^i_{\mathbf{V}}(\log(Z + \mathbb{D})) \quad (k = 1, 2) ,$$
$$\mathcal{V}_4 \Omega^i_{\mathbf{V}}(\log(Z + \mathbb{D})) \subset \mathcal{V}''_3 \Omega^i_{\mathbf{V}}(\log(Z + \mathbb{D})) .$$

Therefore we have

$$\mathcal{V}_k H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ) \subset \mathcal{V}'_{k-1} H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ)\ (k = 1, 2) , \quad \mathcal{V}_4 H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ) \subset \mathcal{V}'_2 H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ) .$$

By the dimension consideration, we see that the proposition holds. □

Since taking the residue map $H^4(\mathbb{T}^2 \times \mathbb{C}^2 \setminus Z^\circ) \to H^3(Z^\circ)$ decreases the weight by 2, we obtain

Theorem A.14. *The weight filtration on $H^3(Z^\circ)$ is as follows.*

$$\mathcal{W}_6 H^3(Z^\circ) \cong \mathcal{R}_F ,$$
$$\mathcal{W}_5 H^3(Z^\circ) = \mathcal{W}_4 H^3(Z^\circ) \cong \mathcal{I}_3 ,$$
$$\mathcal{W}_3 H^3(Z^\circ) \cong \mathcal{I}_1 .$$

A.5. **Deformation and Obstruction.** By Kawamata's result [27], $H^1\big(Z, T_Z(-\log D))\big)$ and $H^2\big(Z, T_Z(-\log D))\big)$ are the set of infinitesimal logarithmic deformations and the set of obstructions respectively.

Let ω be the following global section of $K_Z(D) = \Omega^3_Z(\log D)$:

$$\omega = \mathrm{Res}_Z \frac{1}{P} \frac{dt_1}{t_1} \frac{dt_2}{t_2} dx dy .$$

Proposition A.15.

$$H^1(Z, T_Z(-\log D)) \xrightarrow{\sim} H^1(Z, \Omega^2_Z(\log D)) \cong R^1_F ,$$
$$H^2(Z, T_Z(-\log D)) \xrightarrow{\sim} H^2(Z, \Omega^2_Z(\log D)) = 0 ,$$

where the isomorphisms are given by the contraction with the three form ω.

Proof. The contraction with ω is an isomorphism since $K_Z(D) \cong \mathcal{O}_Z$ and $T_Z(-\log D)$ and $\Omega^2_Z(\log D)$ are locally free. For the rest, see Theorem A.11 and Remark A.3. □

A.6. **Variation of Mixed Hodge Structures.** Varying the parameter $a \in \mathbb{L}_{\mathrm{reg}}(\Delta)$, we obtain a family of threefolds Z°_a:

$$p' : \mathcal{Z}' \to \mathbb{L}_{\mathrm{reg}}(\Delta) .$$

We have

$$R^3 p'_* \mathbb{Z} \otimes \mathcal{O}_{\mathbb{L}_{\mathrm{reg}}(\Delta)} \cong \mathcal{R}_F[a] \otimes \mathcal{O}_{\mathbb{L}_{\mathrm{reg}}(\Delta)} .$$

The Gauss–Manin connection on ∇_{a_m} on $R^3 p'_* \mathbb{Z} \otimes \mathcal{O}_{\mathbb{L}_{\mathrm{reg}}(\Delta)}$ corresponds to the derivation $\mathcal{D}_{a_m}$ since it corresponds to the differentiation by a_m on $\Gamma \Omega^4_{\mathbb{T}^2 \times \mathbb{C}^2}(* Z^\circ_a)$.

Let ω_a be the relative holomorphic three form on $\mathcal{Z}'$ such that

$$\omega_a = \mathrm{Res}\frac{1}{F_a + xy}\frac{dt_1}{t_1}\frac{dt_2}{t_2}dxdy \ .$$

By Proposition 3.7, we obtain

Corollary A.16. *1.* $H^3(Z_a^\circ)$ *is spanned by* ω_a, $\nabla_{\partial_{a_m}}\omega_a$ *and* $\nabla_{\partial_{a_m}}\nabla_{\partial_{a_n}}\omega_a$ $(m, n \in A(\Delta))$.
2. ω_a *satisfies the A-hypergeometric system* (3.8) *with* ∂_{a_i} *replaced by* $\nabla_{\partial_{a_i}}$.
3. Period integrals of ω_a *satisfies the A-hypergeometric system* (3.8). *Conversely, a solution of the A-hypergeometric system* (3.8) *is a period integral.*

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