

The Nicolas and Robin inequalities with sums of two squares

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Abstract In 1984, G. Robin proved that the Riemann hypothesis is true if and only if the *Robin inequality* $\sigma(n) < e^\gamma n \log \log n$ holds for every integer $n > 5040$, where $\sigma(n)$ is the sum of divisors function, and γ is the Euler–Mascheroni constant. We exhibit a broad class of subsets \mathcal{S} of the natural numbers such that the Robin inequality holds for all but finitely many $n \in \mathcal{S}$. As a special case, we determine the finitely many numbers of the form $n = a^2 + b^2$ that do not satisfy the Robin inequality. In fact, we prove our assertions with the *Nicolas inequality* $n/\varphi(n) < e^\gamma \log \log n$; since $\sigma(n)/n < n/\varphi(n)$ for $n > 1$ our results for the Robin inequality follow at once.

Keywords Nicolas inequality · Robin inequality · Sums of two squares

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1 Introduction

Let $\varphi(n)$ denote the *Euler function*. In 1903 Landau (see [4, pp. 217–219]) showed that

$$\overline{\lim}_{n \rightarrow \infty} \frac{n}{\varphi(n) \log \log n} = e^\gamma, \quad (1)$$

where γ is the *Euler–Mascheroni constant*. Eighty years later, in a highly interesting work, Nicolas [5] proved that the inequality

$$\frac{n}{\varphi(n)} > e^\gamma \log \log n$$

holds for infinitely many natural numbers n . Moreover, if N_k denotes the product of the first k primes, he proved that

$$\frac{N_k}{\varphi(N_k)} > e^\gamma \log \log N_k$$

holds for every $k \geq 1$ on the *Riemann hypothesis* (RH). Assuming RH is false, he also showed there are both infinitely many k for which this inequality holds and infinitely many k for which it does not hold. To acknowledge the many contributions of Nicolas to this subject, we denote by \mathcal{N} the set of numbers $n \in \mathbb{N}$ that satisfy the *Nicolas inequality*:

$$\frac{n}{\varphi(n)} < e^\gamma \log \log n. \quad (2)$$

The principal aim of this paper is to exhibit a broad class of infinite subsets $\mathcal{S} \subset \mathbb{N}$ such that this inequality holds for all but finitely many $n \in \mathcal{S}$. This class includes a set that contains all natural numbers which can be expressed as a sum of two squares.

Let $\sigma(n)$ be the *sum of divisors function*. The analogue of (1) for this function was obtained by Gronwall [2], who proved that

$$\overline{\lim}_{n \rightarrow \infty} \frac{\sigma(n)}{n \log \log n} = e^\gamma.$$

Robin [7] showed that if RH is true, then the *Robin inequality*:

$$\frac{\sigma(n)}{n} < e^\gamma \log \log n \quad (3)$$

holds for every integer $n > 5040$, whereas if RH is false, then this inequality fails for infinitely many n . We denote by \mathcal{R} the set of numbers $n \in \mathbb{N}$ that satisfy (3). In view of the elementary inequality

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} \quad (n > 1),$$

it is clear that $\mathcal{N} \subset \mathcal{R}$. Thus, for the class of subsets $\mathcal{S} \subset \mathbb{N}$ considered in the present paper, the Robin inequality holds for all but finitely many $n \in \mathcal{S}$.

Our work was originally inspired by a recent paper of Choie et al. [1], which establishes the inclusion in \mathcal{R} of various infinite subsets of the natural numbers \mathbb{N} . In particular, in [1] it is shown that \mathcal{R} contains every square-free number $n > 30$, every odd integer $n > 9$, every powerful number $n > 36$, and every integer $n > 1$ not divisible by the fifth power of some prime. As a consequence it follows that the RH holds iff the Robin inequality holds for all natural numbers n divisible by the fifth power of some prime. Note that this criterion does not have the restriction $n \geq 5041$. Another “5041-free” criterion was given earlier by Lagarias [3], who showed that RH is true iff

$$\sigma(n) \leq H_n + e^{H_n} \log H_n,$$

where

$$H_n = \sum_{j \leq n} \frac{1}{j} \quad (n \geq 1).$$

To state our results more precisely, let \mathbb{P} denote the set of prime numbers, and for any subset $\mathcal{A} \subset \mathbb{P}$, put

$$\pi_{\mathcal{A}}(x) = \#\{p \leq x : p \in \mathcal{A}\}.$$

Let \mathcal{P} be an arbitrary (fixed) subset of \mathbb{P} such that

$$\bar{\delta} = \overline{\lim}_{x \rightarrow \infty} \frac{\pi_{\mathcal{P}}(x)}{\pi(x)} < 1 \quad \text{and} \quad \underline{\delta} = \underline{\lim}_{x \rightarrow \infty} \frac{\pi_{\mathcal{P}}(x)}{\pi(x)} > 0, \tag{4}$$

where $\pi(x) = \#\{p \leq x\}$ as usual. Let \mathcal{Q} denote the complementary set of primes (i.e., $\mathcal{Q} = \mathbb{P} \setminus \mathcal{P}$), and note that

$$\overline{\lim}_{x \rightarrow \infty} \frac{\pi_{\mathcal{Q}}(x)}{\pi(x)} = 1 - \underline{\delta} < 1 \quad \text{and} \quad \underline{\lim}_{x \rightarrow \infty} \frac{\pi_{\mathcal{Q}}(x)}{\pi(x)} = 1 - \bar{\delta} > 0. \tag{5}$$

In this paper, we work with the set $\mathcal{S} = \mathcal{S}(\mathcal{P})$ defined by

$$\mathcal{S} = \left\{ n \in \mathbb{N} : \text{if } p \in \mathcal{Q} \text{ and } p \mid n, \text{ then } p^2 \mid n \right\}. \tag{6}$$

Our main result is the following:

Theorem 1 *The set \mathcal{N} contains all but finitely many of the numbers in \mathcal{S} .*

Corollary 1 *Of the numbers n which do not satisfy the Nicolas inequality, all but finitely many are divisible by a prime $q \in \mathcal{Q}$ such that $q^2 \nmid n$.*

In particular, for any fixed $a, m \in \mathbb{N}$ with $\gcd(a, m) = 1$, one can put

$$\mathcal{P} = \{p \in \mathbb{P} : p \not\equiv a \pmod{m}\}$$

and apply Corollary 1 to deduce the following:

Corollary 2 *Of the numbers n which do not satisfy the Nicolas inequality, all but finitely many are divisible by a prime $q \equiv a \pmod{m}$ such that $q^2 \nmid n$.*

In Sect. 3 we examine more closely the special case

$$\mathcal{P} = \{p \in \mathbb{P} : p \equiv 1 \pmod{4}\} \cup \{2\}.$$

Note that the corresponding set \mathcal{S} contains all natural numbers of the form $n = a^2 + b^2$ (since, by a theorem of Fermat, every prime $q \equiv 3 \pmod{4}$ appears with even multiplicity in the prime factorization of n if and only if n can be written as a sum of two squares). Using effective bounds from [6] on the number of primes in arithmetic progressions modulo 4, we are able to determine the set $\mathcal{S} \setminus \mathcal{N}$ completely, leading to:

Theorem 2 *The set $\mathcal{S} \setminus \mathcal{N}$ contains precisely 347 natural numbers. In particular, there are precisely 246 numbers which can be expressed as a sum of two squares and such that the Nicolas inequality (2) does not hold, the largest of which is the number 52509581344222812810.*

As an application, we obtain the unconditional result that

$$\{1, 2, 4, 5, 8, 9, 10, 16, 18, 20, 36, 72, 180, 360, 720\}$$

is a complete list of those natural numbers which can be expressed as a sum of two squares and such that the Robin inequality (3) does not hold; this result is consistent with the truth of the Riemann Hypothesis.

Results like those of Theorem 2 can be established for certain quadratic forms other than $a^2 + b^2$. For example, using similar techniques one finds that there are precisely 261 numbers that can be expressed in the form $n = a^2 + 3b^2$ and for which the Nicolas inequality (2) does not hold, the largest of which is the number 397999936131188090700.

Throughout the paper, any implied constants in the symbols O , \ll , \gg and \asymp depend (at most) on the set \mathcal{P} and are absolute otherwise. We recall that for positive functions f, g the notations $f = O(g)$, $f \ll g$ and $g \gg f$ are all equivalent to the assertion that $f \leq cg$ for some constant $c > 0$, and the notation $f \asymp g$ means that $f \ll g$ and $g \ll f$.

2 Proof of Theorem 1

For every natural number n we put

$$F(n) = \frac{n}{\varphi(n)} = \prod_{p|n} \frac{p}{p-1}.$$

Note that

$$F(n) = F(\kappa(n)) \quad \text{and} \quad \omega(n) = \omega(\kappa(n)), \tag{7}$$

where $\omega(n)$ is the number of distinct prime divisors of n , and $\kappa(n)$ is the square-free kernel of n :

$$\kappa(n) = \prod_{p|n} p.$$

Let

$$\mathcal{N}^\circ = \mathbb{N} \setminus \mathcal{N} = \{n \in \mathbb{N} : F(n) \geq e^\gamma \log \log n\},$$

and for every integer $k \geq 0$, let

$$\mathcal{V}_k = \{n \in \mathbb{N} : \omega(n) \geq k\} \quad \text{and} \quad \mathcal{W}_k = \mathcal{S} \cap \mathcal{N}^\circ \cap \mathcal{V}_k.$$

Since $\mathcal{V}_0 = \mathbb{N}$, Theorem 1 is the assertion that $\mathcal{W}_0 = \mathcal{S} \cap \mathcal{N}^\circ$ is a finite set. In view of the next lemma, it suffices to show that $\mathcal{W}_k = \emptyset$ for some k .

Lemma 1 *For every $k \geq 0$, $\mathcal{W}_0 \setminus \mathcal{W}_k$ is a finite set.*

Since $\omega(n) < k$ and $F(n) \geq e^\gamma \log \log n$ for all $n \in \mathcal{W}_0 \setminus \mathcal{W}_k$, Lemma 1 is an immediate consequence of the following:

Lemma 2 *For every constant $K > 0$, there are at most finitely many natural numbers n such that $\omega(n) \leq K$ and $F(n) \geq e^\gamma \log \log n$.*

Proof If $\bar{p}_1, \bar{p}_2, \dots$ is the sequence of consecutive prime numbers, then for any such number n we have

$$\prod_{j \leq K} \frac{\bar{p}_j}{\bar{p}_j - 1} \geq \prod_{p|n} \frac{p}{p-1} = F(n) \geq e^\gamma \log \log n;$$

this shows that n is bounded by a constant which depends only on K . □

For every natural number n , let

$$s(n) = \left(\prod_{\substack{p|n \\ p \in \mathcal{P}}} p \right) \left(\prod_{\substack{q|n \\ q \in \mathcal{Q}}} q^2 \right),$$

and put

$$\mathcal{Y} = \{n \in \mathbb{N} : n = s(n)\}.$$

Note that $\mathcal{Y} \subset \mathcal{S}$. The following statements are elementary:

- (\mathcal{E}_1) if $n = pm$ with $p \in \mathcal{P}$ and $p \nmid m$, then $n \in \mathcal{Y}$ if and only if $m \in \mathcal{Y}$;
- (\mathcal{E}_2) if $n = q^2m$ with $q \in \mathcal{Q}$ and $q \nmid m$, then $n \in \mathcal{Y}$ if and only if $m \in \mathcal{Y}$;
- (\mathcal{E}_3) $s(n) \in \mathcal{S}$ for all n ;
- (\mathcal{E}_4) $\kappa(s(n)) = \kappa(n)$ for all n ;
- (\mathcal{E}_5) $s(n) | n$ for all $n \in \mathcal{S}$; in particular, $s(n) \leq n$.

Lemma 3 *If $\mathcal{W}_k \neq \emptyset$ and m_k is the least integer in \mathcal{W}_k , then $m_k \in \mathcal{Y}$.*

Proof Clearly, $s(m_k) \in \mathcal{S}$ by (\mathcal{E}_3). Combining (\mathcal{E}_4) with (7) one sees that

$$F(s(n)) = F(n) \quad \text{and} \quad \omega(s(n)) = \omega(n) \quad (n \in \mathbb{N}).$$

Then, using (\mathcal{E}_5) it follows that

$$F(s(m_k)) = F(m_k) \geq e^\gamma \log \log m_k \geq e^\gamma \log \log s(m_k),$$

which shows that $s(m_k) \in \mathcal{N}^\circ$. Finally, $s(m_k) \in \mathcal{V}_k$ since

$$\omega(s(m_k)) = \omega(m_k) \geq k.$$

Thus, we have shown that $s(m_k) \in \mathcal{S} \cap \mathcal{N}^\circ \cap \mathcal{V}_k = \mathcal{W}_k$. Since m_k is the *least* integer in \mathcal{W}_k , the equality $m_k = s(m_k)$ follows from (\mathcal{E}_5), hence $m_k \in \mathcal{Y}$. □

Next, for every integer $k \geq 0$ let

$$\mathcal{Z}_k = \{n \in \mathbb{N} : \Omega(n) = k\} \quad \text{and} \quad \mathcal{T}_k = \mathcal{N}^\circ \cap \mathcal{Y} \cap \mathcal{Z}_k.$$

Here, $\Omega(n)$ is the number of prime divisors of n , counted with multiplicity. Using Lemma 3 one sees that if $\mathcal{W}_\ell \neq \emptyset$ and m_ℓ is the least integer in \mathcal{W}_ℓ , then $m_\ell \in \mathcal{T}_k$ for some $k \geq \ell$; in particular,

$$\bigcup_{k \geq \ell} \mathcal{T}_k = \emptyset \implies \mathcal{W}_\ell = \emptyset.$$

As we mentioned earlier, in order to prove Theorem 1 it suffices to show that $\mathcal{W}_\ell = \emptyset$ for some ℓ , hence it is enough to show that $\mathcal{T}_k \neq \emptyset$ for at most finitely many integers $k \geq 0$.

When $\mathcal{T}_k \neq \emptyset$ we shall use the following notation. Let n_k denote the least integer in \mathcal{T}_k . Let \widehat{p}_k be the largest prime $p \in \mathcal{P}$ that divides n_k , and put $\widehat{p}_k = 1$ if no such prime exists. Similarly, let \widehat{q}_k be the largest prime $q \in \mathcal{Q}$ that divides n_k , and set $\widehat{q}_k = 1$ if no such prime exists. Finally, let

$$P_k^+ = \max\{\widehat{p}_k, \widehat{q}_k\} \quad \text{and} \quad P_k^- = \min\{\widehat{p}_k, \widehat{q}_k\}. \tag{8}$$

Note that P_k^+ is the largest prime factor of n_k .

Lemma 4 *Suppose $\mathcal{T}_k \neq \emptyset$:*

- (i) *if $p \in \mathcal{P}$ with $p < \widehat{p}_k$, then $p \mid n_k$;*
- (ii) *if $q \in \mathcal{Q}$ with $q < \widehat{q}_k$, then $q \mid n_k$.*

Proof Suppose on the contrary that $p \in \mathcal{P}$ with $p < \widehat{p}_k$ and $p \nmid n_k$. Since $n_k = s(n_k)$ we can write $n_k = \widehat{p}_k m$ with $\widehat{p}_k \nmid m$. Put $n^* = pm$. Since $n_k \in \mathcal{N}^\circ$, $F(p) > F(\widehat{p}_k)$, and $n^* < n_k$, it follows that

$$F(n^*) = F(p) F(m) > F(\widehat{p}_k) F(m) = F(n_k) \geq e^\gamma \log \log n_k > e^\gamma \log \log n^*,$$

where we have used the fact that F is multiplicative; this shows that $n^* \in \mathcal{N}^\circ$. As $n_k \in \mathcal{Y}$, (\mathcal{C}_1) implies that $n^* \in \mathcal{Y}$. Finally, since Ω is (completely) additive, we see that

$$\Omega(n^*) = \Omega(m) + 1 = \Omega(n_k) = k,$$

which shows that $n^* \in \mathcal{Z}_k$, and thus $n^* \in \mathcal{N}^\circ \cap \mathcal{Y} \cap \mathcal{Z}_k = \mathcal{T}_k$. But this is impossible since $n^* < n_k$ (the least number in \mathcal{T}_k), and this contradiction completes our proof of (i). Using (\mathcal{C}_2) , the proof of (ii) is similar; we omit the details. \square

Lemma 5 *Suppose that $\mathcal{T}_k \neq \emptyset$ and $\widehat{p}_k < \widehat{q}_k$. Then there is at most one prime $p \in \mathcal{P}$ such that $\widehat{p}_k < p < \widehat{q}_k$.*

Proof Suppose on the contrary that there are two primes $p_1, p_2 \in \mathcal{P}$ such that $\widehat{p}_k < p_1 < p_2 < \widehat{q}_k$. Since $n_k = s(n_k)$ we can write $n_k = \widehat{q}_k^2 m$, and it is clear that $\gcd(m, p_1 p_2 \widehat{q}_k) = 1$. Put $n^* = p_1 p_2 m$. Since $n_k \in \mathcal{N}^\circ$, $F(p_1 p_2) > F(\widehat{q}_k^2)$, and $n^* < n_k$, we have

$$F(n^*) = F(p_1 p_2) F(m) > F(\widehat{q}_k^2) F(m) = F(n_k) \geq e^\gamma \log \log n_k > e^\gamma \log \log n^*,$$

which shows that $n^* \in \mathcal{N}^\circ$. As $n_k \in \mathcal{Y}$, (\mathcal{C}_1) implies that $n^* \in \mathcal{Y}$. Finally, since

$$\Omega(n^*) = \Omega(m) + 2 = \Omega(n_k) = k,$$

we see that $n^* \in \mathcal{Z}_k$, and thus $n^* \in \mathcal{N}^\circ \cap \mathcal{Y} \cap \mathcal{Z}_k = \mathcal{T}_k$. But this is impossible since $n^* < n_k$, and this contradiction implies the result. \square

Lemma 6 *Suppose that $\mathcal{T}_k \neq \emptyset$ and $\widehat{p}_k > \widehat{q}_k$. Let p be the largest prime in \mathcal{P} that is less than \widehat{p}_k , and let q be the smallest prime in \mathcal{Q} that is greater than \widehat{q}_k . Then $q > p/2$.*

Proof Suppose on the contrary that $q \leq p/2$. Since $n_k = s(n_k)$ and $p \mid n_k$ (by Lemma 4) but $q \nmid n_k$ (since $q > \widehat{q}_k$), we can write $n_k = p\widehat{p}_k m$, where $\gcd(m, p\widehat{p}_k q) = 1$. Put $n^* = q^2 m$. As in the proofs of Lemmas 4 and 5, we see that $n^* \in \mathcal{Y} \cap \mathcal{Z}_k$. Since $p < \widehat{p}_k$ and $q \leq p/2$, we have

$$F(p\widehat{p}_k) = \frac{p\widehat{p}_k}{(p-1)(\widehat{p}_k-1)} < \frac{p^2}{(p-1)^2} < \frac{q}{q-1} = F(q^2);$$

therefore,

$$F(n^*) = F(q^2) F(m) > F(p\widehat{p}_k) F(m) = F(n_k) \geq e^\gamma \log \log n_k > e^\gamma \log \log n^*,$$

which shows that $n^* \in \mathcal{N}^\circ$. Thus, $n^* \in \mathcal{N}^\circ \cap \mathcal{Y} \cap \mathcal{Z}_k = \mathcal{T}_k$. But this is impossible since $n^* < n_k$, and this contradiction implies the result. \square

As mentioned above, in order to prove Theorem 1 it suffices to show that $\mathcal{T}_k \neq \emptyset$ for at most finitely many integers $k \geq 0$. Arguing by contradiction, we shall assume that the set

$$\mathcal{K} = \{k \geq 0 : \mathcal{T}_k \neq \emptyset\}$$

has infinitely many elements.

Since $\Omega(n_k) = k$, we see that $n_k \rightarrow \infty$ as $k \rightarrow \infty$ with $k \in \mathcal{K}$; using Lemma 2 it follows that $\omega(n_k) \rightarrow \infty$ as well, and therefore $P_k^+ \rightarrow \infty$.

We claim that

$$\widehat{p}_k \asymp \widehat{q}_k \quad (k \in \mathcal{K}), \tag{9}$$

which by (8) is equivalent to

$$P_k^+ \asymp P_k^- \quad (k \in \mathcal{K}). \tag{10}$$

To see this, we express \mathcal{K} as a disjoint union $\mathcal{A} \cup \mathcal{B}$, where \mathcal{A} [resp. \mathcal{B}] is the set of numbers $k \in \mathcal{K}$ for which $\widehat{p}_k < \widehat{q}_k$ [resp. $\widehat{p}_k > \widehat{q}_k$]. To prove (9) it suffices to show:

- (\mathcal{D}_1) $\widehat{p}_k \gg \widehat{q}_k$ for all $k \in \mathcal{A}$;
- (\mathcal{D}_2) $\widehat{p}_k \ll \widehat{q}_k$ for all $k \in \mathcal{B}$.

We use the following result, which is an easy consequence of the prime number theorem:

Lemma 7 *Let $c_{\mathcal{P}} = \overline{\delta} / \underline{\delta}$ and $c_{\mathcal{Q}} = (1 - \underline{\delta}) / (1 - \overline{\delta})$. For every $\varepsilon > 0$ there is a number $x_0(\varepsilon)$ such that for all $x > x_0(\varepsilon)$:*

- (i) if p is the smallest prime in \mathcal{P} greater than x , then $p \leq (c_{\mathcal{P}} + \varepsilon)x$;
- (ii) if q is the smallest prime in \mathcal{Q} greater than x , then $q \leq (c_{\mathcal{Q}} + \varepsilon)x$;
- (iii) if p is the largest prime in \mathcal{P} less than x , then $p \geq (c_{\mathcal{P}}^{-1} - \varepsilon)x$;
- (iv) if q is the largest prime in \mathcal{Q} less than x , then $q \geq (c_{\mathcal{Q}}^{-1} - \varepsilon)x$.

To prove (\mathcal{D}_1) we can assume that \mathcal{A} is an infinite set. Let $k \in \mathcal{A}$, so that $\widehat{p}_k < \widehat{q}_k$. Since $\widehat{q}_k = P_k^+ \rightarrow \infty$ as $k \rightarrow \infty$ with $k \in \mathcal{A}$, the assertion (\mathcal{D}_1) then follows from Lemmas 5 and 7.

To prove (\mathcal{D}_2) we can assume that \mathcal{B} is an infinite set. Let $k \in \mathcal{B}$, so that $\widehat{p}_k > \widehat{q}_k$. Let p, q be defined as in Lemma 6. Since $\widehat{p}_k = P_k^+ \rightarrow \infty$ as $k \rightarrow \infty$ with $k \in \mathcal{B}$, on combining Lemmas 6 and 7 it follows that

$$\widehat{p}_k \ll p \ll q \ll \widehat{q}_k,$$

which proves (\mathcal{D}_2) and completes our Proof of (9).

Next, for every $n \in \mathbb{N}$ let

$$\omega_{\mathcal{P}}(n) = \#\{p \in \mathcal{P} : p \mid n\} \quad \text{and} \quad \omega_{\mathcal{Q}}(n) = \#\{q \in \mathcal{Q} : q \mid n\}.$$

We claim that

$$\omega_{\mathcal{P}}(n_k) \asymp \omega_{\mathcal{Q}}(n_k) \quad (k \in \mathcal{K}). \tag{10}$$

Indeed, by Lemma 4 it follows that $\omega_{\mathcal{P}}(n_k) = \pi_{\mathcal{P}}(\widehat{p}_k)$ and $\omega_{\mathcal{Q}}(n_k) = \pi_{\mathcal{Q}}(\widehat{q}_k)$. Therefore, using the prime number theorem together with (4), (5) and (9) we have

$$\omega_{\mathcal{P}}(n_k) = \pi_{\mathcal{P}}(\widehat{p}_k) \asymp \frac{\widehat{p}_k}{\log \widehat{p}_k} \asymp \frac{\widehat{q}_k}{\log \widehat{q}_k} \asymp \pi_{\mathcal{Q}}(\widehat{q}_k) = \omega_{\mathcal{Q}}(n_k),$$

which proves (11).

Finally, we need the following relation:

$$\log \kappa(n_k) \asymp \omega(n_k) \log \omega(n_k) \quad (k \in \mathcal{K}). \tag{12}$$

To prove this, observe that Definition (8) and Lemma 4 together imply

$$\prod_{p \leq P_k^-} p \mid \kappa(n_k) \quad \text{and} \quad \kappa(n_k) \mid \prod_{p \leq P_k^+} p.$$

Consequently,

$$\sum_{p \leq P_k^-} \log p \leq \log \kappa(n_k) \leq \sum_{p \leq P_k^+} \log p,$$

and also

$$\pi(P_k^-) \leq \omega(n_k) \leq \pi(P_k^+).$$

By the prime number theorem, for either choice of the sign \pm we have

$$\sum_{p \leq P_k^\pm} \log p \sim P_k^\pm \quad \text{and} \quad \pi(P_k^\pm) \sim \frac{P_k^\pm}{\log P_k^\pm} \quad (k \rightarrow \infty, k \in \mathcal{K}),$$

therefore in view of (10) we see that

$$\log \kappa(n_k) \asymp P_k^+ \quad \text{and} \quad \omega(n_k) \asymp \frac{P_k^+}{\log P_k^+},$$

and (12) follows immediately.

Now we come to the heart of the argument. To complete the proof of Theorem 1, we seek a contradiction to our assumption that \mathcal{K} is an infinite set. For this, it is enough to prove both of the following statements with a suitably chosen real number $\varepsilon > 0$:

- (\mathcal{E}_1) the inequality $n_k \leq \kappa(n_k)^{1+\varepsilon}$ holds for at most finitely many $k \in \mathcal{K}$;
- (\mathcal{E}_2) the inequality $n_k > \kappa(n_k)^{1+\varepsilon}$ holds for at most finitely many $k \in \mathcal{K}$.

In view of (11) and (12), there is a constant $C > 1$ such that the inequalities

$$\omega_{\mathcal{P}}(n_k) \leq (C - 1) \omega_{\mathcal{Q}}(n_k) \tag{13}$$

and

$$\log \kappa(n_k) \leq C \omega(n_k) \log \omega(n_k) \tag{14}$$

both hold if k is sufficiently large. Let C be fixed, and put $\varepsilon = C^{-3}$.

To prove (\mathcal{E}_1), we suppose on the contrary that $n_k \leq \kappa(n_k)^{1+\varepsilon}$ holds for infinitely many $k \in \mathcal{K}$. Let k be large, and put

$$r = \omega_{\mathcal{P}}(n_k) = \pi_{\mathcal{P}}(\widehat{p}_k) \quad \text{and} \quad s = \omega_{\mathcal{Q}}(n_k) = \pi_{\mathcal{Q}}(\widehat{q}_k)$$

By what we have already seen it is clear that $\min\{r, s\} \rightarrow \infty$ as $k \rightarrow \infty$ with $k \in \mathcal{K}$, thus by (13) we have

$$r \leq (C - 1)s \tag{15}$$

if k is large enough. By Lemma 4 and the fact that $n_k \in \mathcal{Y}$, it follows that

$$n_k = \left(\prod_{\substack{p \leq \widehat{p}_k \\ p \in \mathcal{P}}} p \right) \left(\prod_{\substack{q \leq \widehat{q}_k \\ q \in \mathcal{Q}}} q^2 \right) \quad \text{and} \quad \kappa(n_k) = \left(\prod_{\substack{p \leq \widehat{p}_k \\ p \in \mathcal{P}}} p \right) \left(\prod_{\substack{q \leq \widehat{q}_k \\ q \in \mathcal{Q}}} q \right).$$

Hence, our assumption that $n_k \leq \kappa(n_k)^{1+\varepsilon}$ implies that

$$\kappa(n_k) \geq \left(\frac{n_k}{\kappa(n_k)}\right)^{1/\varepsilon} = \left(\prod_{\substack{q \leq \widehat{q}_k \\ q \in \mathcal{Q}}} q\right)^{1/\varepsilon}. \tag{16}$$

If $\overline{p}_1, \overline{p}_2, \dots$ is the sequence of consecutive prime numbers, then by the prime number theorem (and recalling our choice of ε) we derive that

$$\log \kappa(n_k) \geq C^3 \sum_{\substack{q \leq \widehat{q}_k \\ q \in \mathcal{Q}}} \log q \geq C^3 \sum_{p \leq \overline{p}_s} \log p \sim C^3 \overline{p}_s \sim C^3 s \log s$$

as $k \rightarrow \infty$ with $k \in \mathcal{K}$. On the other hand, using (14), (15) and the fact that $\omega(n_k) = r + s$, it follows that

$$\log \kappa(n_k) \leq C(r + s) \log(r + s) \leq C^2 s \log(Cs) \sim C^2 s \log s.$$

Since $C^3 > C^2$, these two inequalities for $\log \kappa(n_k)$ lead to a contradiction once k is sufficiently large, and this completes the proof of (\mathcal{E}_1) .

To prove (\mathcal{E}_2) we use some ideas from Choie et al. [1]. Suppose that $n_k > \kappa(n_k)^{1+\varepsilon}$, and put $t = \omega(n_k)$. We claim that either

$$\sum_{p \leq \overline{p}_t} \log p < (1 + \varepsilon)^{-1/2} \overline{p}_t, \tag{17}$$

or

$$\overline{p}_t \leq \exp(2/\log(1 + \varepsilon)). \tag{18}$$

Assuming the claim, it is easy to see that $\omega(n_k)$ is bounded above by a constant K that depends only on ε . By Lemma 2, n_k can take only finitely many distinct values, which implies (\mathcal{E}_2) .

To prove the claim, assume that (17) fails:

$$\log(\overline{p}_1 \cdots \overline{p}_t) = \sum_{p \leq \overline{p}_t} \log p \geq (1 + \varepsilon)^{-1/2} \overline{p}_t.$$

Thanks to Rosser and Schoenfeld [8] it is known that

$$\prod_{p \leq x} \frac{p}{p-1} \leq e^\gamma \left(\log x + \frac{1}{\log x}\right) \quad (x > 1).$$

Therefore, taking $x = \bar{p}_t$ and noting that $\kappa(n_k) \geq \bar{p}_1 \cdots \bar{p}_t$, we derive that

$$\begin{aligned} e^\gamma \left(\log \bar{p}_t + \frac{1}{\log \bar{p}_t} \right) &\geq \prod_{j=1}^t \frac{\bar{p}_j}{\bar{p}_j - 1} \geq \frac{n_k}{\varphi(n_k)} \geq e^\gamma \log \log n_k \\ &> e^\gamma \log((1 + \varepsilon) \log \kappa(n_k)) \\ &\geq e^\gamma \log((1 + \varepsilon) \log(\bar{p}_1 \cdots \bar{p}_t)) \\ &\geq e^\gamma \log\left((1 + \varepsilon)^{1/2} \bar{p}_t\right) = e^\gamma (\log \bar{p}_t + 0.5 \log(1 + \varepsilon)); \end{aligned}$$

that is,

$$\frac{1}{\log \bar{p}_t} \geq 0.5 \log(1 + \varepsilon),$$

which is equivalent to (18). This proves the claim and completes our proof of Theorem 1.

3 Proof of Theorem 2

We continue to use the notation of the previous section, but we focus on the special case that

$$\begin{aligned} \mathcal{P} &= \{p \in \mathbb{P} : p \equiv 1 \pmod{4}\} \cup \{2\}, \\ \mathcal{Q} &= \{q \in \mathbb{P} : q \equiv 3 \pmod{4}\}. \end{aligned}$$

Note that the corresponding set \mathcal{S} contains every natural number that can be expressed as a sum of two squares. As before, we write

$$\mathcal{T}_k = \{n \in \mathbb{N} : F(n) \geq e^\gamma \log \log n, n = s(n), \text{ and } \Omega(n) = k\}$$

and put

$$\mathcal{K} = \{k \geq 0 : \mathcal{T}_k \neq \emptyset\}.$$

Lemma 8 *If $k \in \mathcal{K}$, then $P_k^- < 50000$.*

Proof For every real number $x \geq 10$, let

- $g_{\mathcal{P}}(x)$ = the smallest prime in \mathcal{P} greater than x ;
- $g_{\mathcal{Q}}(x)$ = the smallest prime in \mathcal{Q} greater than x ;
- $\ell_{\mathcal{P}}(x)$ = the largest prime in \mathcal{P} less than x ;
- $\ell_{\mathcal{Q}}(x)$ = the largest prime in \mathcal{Q} less than x .

Also, put

$$\vartheta_{\mathcal{P}}(x) = \sum_{\substack{p \leq x \\ p \in \mathcal{P}}} \log p \quad \text{and} \quad \vartheta_{\mathcal{Q}}(x) = \sum_{\substack{q \leq x \\ q \in \mathcal{Q}}} \log q.$$

Using the explicit bounds of Theorems 1 and 2 of Ramaré and Rumely [6], we see that the inequalities

$$0.49x < \vartheta_{\mathcal{P}}(x) < 0.51x \quad \text{and} \quad 0.49x < \vartheta_{\mathcal{Q}}(x) < 0.51x \tag{19}$$

hold for all $x \geq 45000$ (note that $\vartheta_{\mathcal{P}}(x) = \log 2 + \theta(x; 4, 1)$ and $\vartheta_{\mathcal{Q}}(x) = \theta(x; 4, 3)$ in the notation of [6]). Consequently, for any $x \geq 50000$ we have

$$\frac{49}{51}x < \ell_{\mathcal{P}}(x) < x < g_{\mathcal{P}}(x) < \frac{51}{49}x$$

and

$$\frac{49}{51}x < \ell_{\mathcal{Q}}(x) < x < g_{\mathcal{Q}}(x) < \frac{51}{49}x.$$

Now suppose that $P_k^- \geq 50000$. Using Lemma 5 and the preceding bounds we have

$$\widehat{q}_k < g_{\mathcal{P}}(g_{\mathcal{P}}(\widehat{p}_k)) < \left(\frac{51}{49}\right)^2 \widehat{p}_k.$$

On the other hand, by Lemma 6 we have

$$\frac{51}{49} \widehat{q}_k > g_{\mathcal{Q}}(\widehat{q}_k) > \frac{1}{2} \ell_{\mathcal{P}}(\widehat{p}_k) > \frac{49}{102} \widehat{p}_k.$$

Hence, it follows that

$$0.92 \widehat{q}_k < \widehat{p}_k < 2.2 \widehat{q}_k. \tag{20}$$

By Lemma 4 it is clear that

$$\log \kappa(n_k) = \sum_{\substack{p \leq \widehat{p}_k \\ p \in \mathcal{P}}} \log p + \sum_{\substack{q \leq \widehat{q}_k \\ q \in \mathcal{Q}}} \log q = \vartheta_{\mathcal{P}}(\widehat{p}_k) + \vartheta_{\mathcal{Q}}(\widehat{q}_k).$$

On the other hand, arguing as in the proof of Theorem 1, it follows from (16) that

$$\log \kappa(n_k) \geq \varepsilon^{-1} \vartheta_{\mathcal{Q}}(\widehat{q}_k)$$

if $\varepsilon > 0$ is fixed and $n_k \leq \kappa(n_k)^{1+\varepsilon}$. Combining the two preceding results with (19), we see that

$$0.51(\widehat{p}_k + \widehat{q}_k) \geq \vartheta_{\mathcal{P}}(\widehat{p}_k) + \vartheta_{\mathcal{Q}}(\widehat{q}_k) \geq \varepsilon^{-1} \vartheta_{\mathcal{Q}}(\widehat{q}_k) \geq 0.49 \varepsilon^{-1} \widehat{q}_k$$

since $P_k^- \geq 50000$; taking into account (20), we further have

$$0.51 (1 + 2.2) \widehat{q}_k \geq 0.51 (\widehat{p}_k + \widehat{q}_k) \geq 0.49 \varepsilon^{-1} \widehat{q}_k,$$

which implies that $\varepsilon \geq 0.3002$. Thus, for the smaller value $\varepsilon = 0.3$, we see that the condition $n_k \leq \kappa(n_k)^{1.3}$ implies $P_k^- < 50000$.

On the other hand, if $n_k > \kappa(n_k)^{1.3}$, we put $t = \omega(n_k)$ as in the proof of Theorem 1. Since $\varepsilon = 0.3$, we derive from (17) and (18) that either

$$\vartheta(\overline{p}_t) = \sum_{p \leq \overline{p}_t} \log p < (1.3)^{-1/2} \overline{p}_t < 0.88 \overline{p}_t, \tag{21}$$

or

$$\overline{p}_t \leq \exp(2/\log 1.3) < 2045.$$

Using again Theorems 1 and 2 of Ramaré and Rumely [6] (see also [8]), it is easy to see that the inequality (21) implies $\overline{p}_t < 300$, hence the inequality $\overline{p}_t < 2045$ holds in both cases. It follows that $t < 310$, and therefore,

$$\min\{\pi_{\mathcal{P}}(\widehat{p}_k), \pi_{\mathcal{Q}}(\widehat{q}_k)\} = \min\{\omega_{\mathcal{P}}(n_k), \omega_{\mathcal{Q}}(n_k)\} \leq \omega(n_k) = t < 310,$$

which implies that $P_k^- < 5000$. This completes the proof. □

Corollary 3 *If $k \in \mathcal{K}$, then $k < 10000$.*

Proof For any $k \in \mathcal{K}$ we have

$$k = \Omega(n_k) = \omega_{\mathcal{P}}(n_k) + 2 \omega_{\mathcal{Q}}(n_k) = \pi_{\mathcal{P}}(\widehat{p}_k) + 2 \pi_{\mathcal{Q}}(\widehat{q}_k).$$

If $P_k^- = \widehat{p}_k$ (i.e., $\widehat{p}_k < \widehat{q}_k$), then by Lemmas 5 and 8 it follows that

$$\begin{aligned} k &\leq \max_{p < 50000} \{\pi_{\mathcal{P}}(p) + 2 \pi_{\mathcal{Q}}(g_{\mathcal{P}}(g_{\mathcal{P}}(p)))\} \\ &\leq \pi_{\mathcal{P}}(50000) + 2 \pi_{\mathcal{Q}}(g_{\mathcal{P}}(g_{\mathcal{P}}(50000))) = 7718. \end{aligned}$$

If $P_k^- = \widehat{q}_k$ (i.e., $\widehat{q}_k < \widehat{p}_k$), then by Lemmas 6 and 8 it follows that

$$\begin{aligned} k &\leq \max_{q < 50000} \max_{\substack{p \in \mathbb{P} \\ \ell_{\mathcal{P}}(p) < 2g_{\mathcal{Q}}(q)}} \{\pi_{\mathcal{P}}(p) + 2 \pi_{\mathcal{Q}}(q)\} \\ &= \max_{q < 50000} \max_{\substack{p \in \mathbb{P} \\ \ell_{\mathcal{P}}(p) < 2g_{\mathcal{Q}}(q)}} \{1 + \pi_{\mathcal{P}}(\ell_{\mathcal{P}}(p)) + 2 \pi_{\mathcal{Q}}(q)\} \\ &\leq \max_{q < 50000} \{1 + \pi_{\mathcal{P}}(2g_{\mathcal{Q}}(q)) + 2 \pi_{\mathcal{Q}}(q)\} \\ &\leq 1 + \pi_{\mathcal{P}}(2g_{\mathcal{Q}}(50000)) + 2 \pi_{\mathcal{Q}}(50000) = 9951. \end{aligned}$$

The result follows. □

Now let $\overline{p}_1, \overline{p}_2, \dots$ be the sequence of consecutive primes in \mathcal{P} , and let $\overline{q}_1, \overline{q}_2, \dots$ be the consecutive primes in \mathcal{Q} . For any integers $r, s \geq 0$, let

$$N_{r,s} = \left(\prod_{i=1}^r \overline{p}_i \right) \left(\prod_{j=1}^s \overline{q}_j^2 \right).$$

It is easy to see that $N_{r,s} \in \mathcal{Y}$ for all $r, s \geq 0$, and for every $k \in \mathcal{K}$ one has

$$n_k = N_{r,s}, \quad \widehat{p}_k = \overline{p}_r, \quad \widehat{q}_k = \overline{q}_s \quad \text{and} \quad k = r + 2s,$$

where $r = \omega_{\mathcal{P}}(n_k)$ and $s = \omega_{\mathcal{Q}}(n_k)$. By a straightforward computation, one verifies the following:

Lemma 9 *If $r, s \geq 0$, then $N_{r,s} \in \mathcal{N}^\circ$ if and only if the pair (r, s) lies in the set*

$$\begin{aligned} \mathcal{X} = \{ & (0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (2, 1), (1, 2), (3, 1), (2, 2), (4, 1), \\ & (3, 2), (2, 3), (4, 2), (3, 3), (5, 2), (4, 3), (3, 4), (5, 3), (4, 4), (6, 3), \\ & (5, 4), (4, 5), (7, 3), (6, 4), (5, 5), (7, 4), (6, 5), (7, 5), (8, 5) \}. \end{aligned}$$

We remark that, in view of Corollary 3, it suffices to check the condition $N_{r,s} \in \mathcal{N}^\circ$ only for those pairs (r, s) with $r + 2s < 10000$.

Corollary 4 *If $k \in \mathcal{K}$, then $k \leq 18$.*

Corollary 5 *If $n \in \mathcal{S} \cap \mathcal{N}^\circ$, $r = \omega_{\mathcal{P}}(n)$ and $s = \omega_{\mathcal{Q}}(n)$, then $(r, s) \in \mathcal{X}$. In particular, $\omega(n) \leq 13$.*

Proof Since

$$F(N_{r,s}) = \left(\prod_{i=1}^r \frac{\overline{p}_i}{\overline{p}_i - 1} \right) \left(\prod_{j=1}^s \frac{\overline{q}_j}{\overline{q}_j - 1} \right) \geq \left(\prod_{\substack{p|n \\ p \in \mathcal{P}}} \frac{p}{p-1} \right) \left(\prod_{\substack{q|n \\ q \in \mathcal{Q}}} \frac{q}{q-1} \right) = F(n)$$

and

$$n \geq s(n) = \left(\prod_{\substack{p|n \\ p \in \mathcal{P}}} p \right) \left(\prod_{\substack{q|n \\ q \in \mathcal{Q}}} q^2 \right) \geq \left(\prod_{i=1}^r \overline{p}_i \right) \left(\prod_{j=1}^s \overline{q}_j^2 \right) = N_{r,s},$$

we have

$$F(N_{r,s}) \geq F(n) \geq e^\gamma \log \log n \geq e^\gamma \log \log N_{r,s},$$

which shows that $N_{r,s} \in \mathcal{N}^\circ$. □

We now turn to a description of our method for generating the elements of $\mathcal{S} \setminus \mathcal{N} = \mathcal{S} \cap \mathcal{N}^\circ$. For any given $n \in \mathcal{S} \cap \mathcal{N}^\circ$ with $r = \omega_{\mathcal{P}}(n)$ and $s = \omega_{\mathcal{Q}}(n)$, we can write

$$s(n) = p_1 \cdots p_r q_1^2 \cdots q_s^2,$$

where $p_1 < \cdots < p_r$ are primes in \mathcal{P} and $q_1 < \cdots < q_s$ are primes in \mathcal{Q} . For fixed $i = 1, \dots, r$, let γ_i be the largest non-negative integer such that the number

$$\left(\prod_{\ell=1}^{i-1} \overline{p}_\ell\right) \left(\prod_{\ell=i}^r \overline{p}_{\ell+\gamma_i}\right) \left(\prod_{j=1}^s \overline{q}_j^2\right)$$

lies in \mathcal{N}° , which exist by Lemma 2. Using an argument similar to that in the proof of Lemma 4, one can deduce that

$$\overline{p}_i \leq p_i \leq \overline{p}_{i+\gamma_i} \quad (i = 1, \dots, r). \tag{22}$$

Similarly, for fixed $j = 1, \dots, s$, let δ_j be the largest non-negative integer such that the number

$$\left(\prod_{i=1}^r \overline{p}_i\right) \left(\prod_{\ell=1}^{j-1} \overline{q}_\ell^2\right) \left(\prod_{\ell=j}^s \overline{q}_{\ell+\delta_j}\right)$$

lies in \mathcal{N}° . Then,

$$\overline{q}_j \leq q_j \leq \overline{q}_{j+\gamma_j} \quad (j = 1, \dots, s). \tag{23}$$

Therefore, for fixed $(r, s) \in \mathcal{X}$, if $n \in \mathcal{S} \cap \mathcal{N}^\circ$ with $r = \omega_{\mathcal{P}}(n)$ and $s = \omega_{\mathcal{Q}}(n)$, then the number $s(n)$ must lie in the finite set $\mathcal{A}_{r,s}$ of integers of the form

$$m = p_1 \cdots p_r q_1^2 \cdots q_s^2, \tag{24}$$

where $p_1 < \cdots < p_r$ are primes in \mathcal{P} , $q_1 < \cdots < q_s$ are primes in \mathcal{Q} , the primes p_i and q_j satisfy the bounds (22) and (23), and $m \in \mathcal{N}^\circ$. The set $\mathcal{A}_{r,s}$ can be explicitly determined by a numerical computation, and we obtain a finite list of ‘‘admissible’’ values for the quantity $s(n)$.

To determine explicitly all of the numbers $n \in \mathcal{S} \cap \mathcal{N}^\circ$ with $r = \omega_{\mathcal{P}}(n)$ and $s = \omega_{\mathcal{Q}}(n)$, for every $m \in \mathcal{A}_{r,s}$ we need to find all such numbers for which $s(n) = m$. To do this, factor m as in (24). For fixed $i = 1, \dots, r$, let α_i be the largest integer such that the number $mp_i^{\alpha_i-1}$ lies in \mathcal{N}° . Similarly, for fixed $j = 1, \dots, s$, let β_j be the largest integer such that the number $mq_j^{\beta_j-1}$ lies in \mathcal{N}° . Put

$$M = m \cdot p_1^{\alpha_1-1} \cdots p_r^{\alpha_r-1} q_1^{\beta_1-1} \cdots q_s^{\beta_s-1}.$$

Then, it is easy to see that $m \mid n$ and $n \mid M$ for any $n \in \mathcal{S} \cap \mathcal{N}^\circ$ such that $s(n) = m$. Hence, n can take only finitely many values which can be determined explicitly for each $m \in \mathcal{A}_{r,s}$.

For example, taking $r = s = 2$ we find that

$$\{4410, 8820, 10890, 13230, 17640, 21780, 22050, 26460, 30870, 35280, 39690, 44100, 52920, 61740, 66150, 70560, 79380, 88200, 92610, 105840, 110250\}$$

is a complete list of the numbers $n \in \mathcal{S} \setminus \mathcal{N}$ with $\omega_{\mathcal{P}}(n) = \omega_{\mathcal{Q}}(n) = 2$. Examining the lists generated as (r, s) varies over the pairs in \mathcal{X} , we are lead to the statement of Theorem 2.

4 Evaluation of $\overline{\lim}_{n \in \mathcal{S}} \frac{n}{\varphi(n) \log \log n}$ and $\overline{\lim}_{n \in \mathcal{S}} \frac{\sigma(n)}{n \log \log n}$

We conclude the paper by giving two propositions and two corollaries that yield the analogue of the work of Landau [4] and Gronwall [2] for any set \mathcal{S} of the form (6) and for the set of natural numbers equal to a sum of two squares. In fact, Corollary 6 shows that Theorem 1 is nontrivial in the sense that $F(n)/\log \log n$ cannot be bounded away from e^γ by any positive constant for all large $n \in \mathcal{S}$. We will use the notation $f(n) = o(g(n))$ to mean that $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$.

Proposition 1 *Let $\{a_n\}$ be an infinite sequence of positive integers such that if we write $a_n = \prod_p p^{v(p,n)}$ we have:*

- (i) $\kappa(a_n) = \prod_{p \leq n} p$ (i.e., $v(p, n) = 0 \iff p > n$);
- (ii) $a_n = \exp(n^{1+o(1)})$;
- (iii) $\lim_{n \rightarrow \infty} v(p, n) = \infty$ for each p .

Then,

$$\lim_{n \rightarrow \infty} \frac{\sigma(a_n)}{a_n \log \log a_n} = e^\gamma.$$

Proof For all $n \geq 1$, let

$$b_n = \prod_{p \leq n} p \quad \text{and} \quad c_n = \frac{\sigma(a_n)}{a_n} \frac{\varphi(b_n)}{b_n},$$

and observe that (i) implies

$$c_n = \left(\prod_{p \leq n} \frac{p^{v(p,n)+1} - 1}{p^{v(p,n)}(p - 1)} \right) \left(\prod_{p \leq n} \frac{p - 1}{p} \right) = \prod_{p \leq n} \left(1 - \frac{1}{p^{v(p,n)+1}} \right).$$

Since $v(p, n) + 1 \geq 2$ for every prime $p \leq n$, we have for any $m \leq n$:

$$1 \geq c_n > \prod_{p \leq m} \left(1 - \frac{1}{p^{v(p,n)+1}}\right) \prod_{p > m} \left(1 - \frac{1}{p^2}\right).$$

Using (iii) we have for every fixed integer m :

$$1 \geq \overline{\lim}_{n \rightarrow \infty} c_n \geq \underline{\lim}_{n \rightarrow \infty} c_n \geq \prod_{p > m} \left(1 - \frac{1}{p^2}\right).$$

The product on the right tends to one as $m \rightarrow \infty$, hence $\lim_{n \rightarrow \infty} c_n = 1$; therefore,

$$\lim_{n \rightarrow \infty} \frac{\sigma(a_n)}{a_n \log n} = \lim_{n \rightarrow \infty} \frac{b_n}{\varphi(b_n) \log n}.$$

Our assumption (ii) implies that $\log \log a_n = (1 + o(1)) \log n$, and using Mertens' theorem (see, e.g., [8]) we have

$$\frac{\varphi(b_n)}{b_n} = \prod_{p \leq n} \left(1 - \frac{1}{p}\right) = (1 + o(1)) \frac{e^{-\gamma}}{\log n},$$

and the result follows. □

Using similar ideas (and an easier argument) one can obtain the following analogue of Proposition 1 for the Euler totient function:

Proposition 2 *Let $\{a_n\}$ be an infinite sequence of positive integers such that:*

- (i) $\kappa(a_n) = \prod_{p \leq n} p$;
- (ii) $a_n = \exp(n^{1+o(1)})$.

Then,

$$\lim_{n \rightarrow \infty} \frac{a_n}{\varphi(a_n) \log \log a_n} = e^\gamma.$$

Corollary 6 *For any set \mathcal{S} defined by (6), we have*

$$\overline{\lim}_{n \in \mathcal{S}} \frac{\sigma(n)}{n \log \log n} = \overline{\lim}_{n \in \mathcal{S}} \frac{n}{\varphi(n) \log \log n} = e^\gamma.$$

Proof Since

$$\overline{\lim}_{n \rightarrow \infty} \frac{\sigma(n)}{n \log \log n} = \overline{\lim}_{n \rightarrow \infty} \frac{n}{\varphi(n) \log \log n} = e^\gamma$$

by [2] and [4], respectively, it suffices to show that there is a sequence $\{a_n\}$ in \mathcal{S} such that

$$\lim_{n \rightarrow \infty} \frac{\sigma(a_n)}{a_n \log \log a_n} = \lim_{n \rightarrow \infty} \frac{a_n}{\varphi(a_n) \log \log a_n} = e^\gamma.$$

Let $a_1 = 1$, and for every integer $n \geq 2$, let

$$b_n = \prod_{p \leq n} p, \quad d_n = \left\lfloor n^{(\log n)^{-1/2}} \right\rfloor \quad \text{and} \quad a_n = b_n^{d_n}.$$

It is easy to see that $d_n \geq 2$ for $n \geq 2$, $d_n = n^{o(1)}$, and d_n tends to infinity with n . Clearly, $a_n \in \mathcal{S}$ for all $n \geq 1$, and by the Prime Number Theorem in the form $\sum_{p \leq x} \log p = x(1 + o(1))$ as $x \rightarrow \infty$ we see that

$$\log a_n = d_n \log b_n = n^{o(1)} \sum_{p \leq n} \log p = n^{1+o(1)} \quad (n \rightarrow \infty).$$

The sequence $\{a_n\}$ therefore satisfies the hypotheses of Propositions 1 and 2, and the result follows. □

Corollary 7 *We have*

$$\overline{\lim}_{n=a^2+b^2} \frac{\sigma(n)}{n \log \log n} = \overline{\lim}_{n=a^2+b^2} \frac{n}{\varphi(n) \log \log n} = e^\gamma.$$

Proof Defining a_n for all $n \geq 1$ as in the proof of Corollary 6, it is easy to see that the sequence $\{a_n^2\}$ satisfies the hypotheses of Propositions 1 and 2; it follows that

$$\overline{\lim}_{n=a^2} \frac{\sigma(n)}{n \log \log n} = \overline{\lim}_{n=a^2} \frac{n}{\varphi(n) \log \log n} = e^\gamma,$$

and this implies the stated result. □

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