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FINITENESS OF GORENSTEIN INJECTIVE DIMENSION OF MODULES

LEILA KHATAMI, MASSOUD TOUSI, AND SIAMAK YASSEMI

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ABSTRACT. The Chouinard formula for the injective dimension of a module over a noetherian ring is extended to Gorenstein injective dimension. Specifically, if M is a module of finite positive Gorenstein injective dimension over a commutative noetherian ring R, then its Gorenstein injective dimension is the supremum of $\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through all prime ideals of R. It is also proved that if M is finitely generated and non-zero, then its Gorenstein injective dimension is equal to the depth of the base ring. This generalizes the classical Bass formula for injective dimension.

1. INTRODUCTION

Throughout this paper all rings are assumed to be unitary, commutative and noetherian. In 1976, Chouinard gave a general formula for the injective dimension of a module when it is finite (see [5]).

Chouinard's Formula. Let M be an R-module of finite injective dimension. Then

$$\operatorname{id}_R M = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

Recall that the width of a module M over a local ring R, $\operatorname{width}_R M$, is defined as $\inf\{\, i \mid \operatorname{Tor}_i^R(k, M) \neq 0\}$, where k is the residue field of R. This formula can be considered as a general version of the Bass formula for injective dimension. In his paper "On the ubiquity of Gorenstein rings" [3], Bass proves that a non-zero finitely generated module over a local ring has either infinite injective dimension or injective dimension equal to the depth of the base ring. The main results of this paper extend both Bass's and Chouinard's formulas to the *Gorenstein injective dimension* of modules.

The Gorenstein injective dimension is a refinement of the classical notion of the injective dimension of a module, in the sense that it is always less than or equal to the injective dimension and equality holds when the injective dimension is finite. It was introduced by Enochs and Jenda in [9] as the dual notion to the G-dimension defined by Auslander-Bridger [1], [2] some twenty years earlier. Auslander and

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Bridger's G-dimension extends the notion of projective dimension, and its finiteness characterizes Gorenstein local rings (see [2]).

Our Theorem 2.3 extends Chouinard's formula to the Gorenstein injective dimension of modules, while Theorem 2.5 is a generalized Bass formula for Gorenstein injective dimension. A generalized Chouinard's formula has been proved in [8] for modules of finite Gorenstein injective dimension over a quotient of a Gorenstein ring. There have also been several partial extensions of the Bass formula to Gorenstein injective dimension: over a Gorenstein local ring in [10], over a Cohen-Macaulay local quotient of a Gorenstein local ring in [6], over any local quotient of a Gorenstein local ring in [8] and over an almost Cohen-Macaulay local ring in [13]. These can all be concluded from our results, which give ultimate generalizations of the classical formulas in the case of positive Gorenstein injective dimension.

2. Main results

Definition 2.1. An R-module G is said to be Gorenstein injective if and only if there exists an exact complex of injective R-modules

$$I = \cdots \to I_2 \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow I_{-2} \longrightarrow \cdots$$

such that the complex $\operatorname{Hom}_R(J, I)$ is exact for every injective R-module J and G is the kernel in degree 0 of I. The Gorenstein injective dimension of an R-module M, $\operatorname{Gid}_R(M)$, is defined to be the infimum of integers n such that there exists an exact sequence

$$0 \to M \to G_0 \to G_{-1} \to \cdots \to G_{-n} \to 0$$

with all G_i's Gorenstein injective.

Lemma 2.2. *Let R be a commutative noetherian ring. The following inequality holds for any Gorenstein injective R-module M and any prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$:*

$$\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq 0.$$

Proof. Since M is Gorenstein injective, there exists an exact sequence

$$E_\bullet : \ \cdots \to E_1 \to E_0 \to M \to 0$$

of R-modules, with all E_i's injective. Set

$$K_1 = \ker(E_0 \to M)$$

and

$$K_i = \ker(E_{i-1} \to E_{i-2}), \text{ for } i \geq 2.$$

Now assume that $\mathfrak{p}$ is a prime ideal of R and T is an $R_{\mathfrak{p}}$-module. We have $\operatorname{Ext}^i_{R_{\mathfrak{p}}}(T, M_{\mathfrak{p}}) \cong \operatorname{Ext}^{i+t}_{R_{\mathfrak{p}}}(T, (K_t)_{\mathfrak{p}})$ for any two positive integers i and t. In particular, if T has finite projective dimension over $R_{\mathfrak{p}}$, we can conclude that $\operatorname{Ext}^i_{R_{\mathfrak{p}}}(T, M_{\mathfrak{p}})$ has to be zero for any positive integer i. We finish the proof by using [7, Proposition 5.3(c)] for the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ to get the second inequality below:

$$\begin{aligned} 0 \ &\geq \sup\{\, i \mid \operatorname{Ext}^i_{R_{\mathfrak{p}}}(T, M_{\mathfrak{p}}) \neq 0, \text{ for some } R_{\mathfrak{p}}\text{−module } T \text{ with } \operatorname{pd}_{R_{\mathfrak{p}}} T < \infty\} \\ &\geq \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}. \end{aligned}$$

□

The lemma shows that the inequality

$$0 \geq \sup\{\operatorname{depth} R_{\mathfrak{q}} - \operatorname{width}_{R_{\mathfrak{q}}} M_{\mathfrak{q}} \mid \mathfrak{q} \in \operatorname{Spec}(R)\}$$

holds for any Gorenstein injective module M. In particular, the supremum is equal to zero if there is a prime ideal $\mathfrak{q}$ with $\operatorname{depth} R_{\mathfrak{q}} - \operatorname{width}_{R_{\mathfrak{q}}} M_{\mathfrak{q}} \geq 0$. Any finitely generated module is an example of a module with this property; therefore the equality $0 = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec}(R)\}$ holds for any finitely generated Gorenstein injective R-module M (in fact, this is a special case of the main theorem of [13] by the first and third authors of this paper).

Theorem 2.3. *Let R be a commutative noetherian ring and M an R-module of finite positive Gorenstein injective dimension. Then*

$$\operatorname{Gid}_R(M) = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

Proof. By [8, Lemma 2.18], there exists a short exact sequence

$$0 \to K \to L \to M \to 0,$$

where K is a Gorenstein injective R-module and $\operatorname{id}_R L = \operatorname{Gid}_R M$.

For any $\mathfrak{p} \in \operatorname{Spec}(R)$, the exact sequence $0 \to K_{\mathfrak{p}} \to L_{\mathfrak{p}} \to M_{\mathfrak{p}} \to 0$ induces the long exact sequence

$$\cdots \to \operatorname{Tor}_i^{R_{\mathfrak{p}}}(k(\mathfrak{p}), K_{\mathfrak{p}}) \to \operatorname{Tor}_i^{R_{\mathfrak{p}}}(k(\mathfrak{p}), L_{\mathfrak{p}}) \\ \to \operatorname{Tor}_i^{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}}) \to \operatorname{Tor}_{i-1}^{R_{\mathfrak{p}}}(k(\mathfrak{p}), K_{\mathfrak{p}}) \to \cdots$$

where $k(\mathfrak{p})$ denotes the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Thus the following inequalities hold:

$$\operatorname{width}_{R_{\mathfrak{p}}} L_{\mathfrak{p}} \geq \min\{\operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}, \operatorname{width}_{R_{\mathfrak{p}}} K_{\mathfrak{p}}\}$$

and

$$\operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \geq \min\{\operatorname{width}_{R_{\mathfrak{p}}} L_{\mathfrak{p}}, \operatorname{width}_{R_{\mathfrak{p}}} K_{\mathfrak{p}} + 1\}.$$

Now suppose that $\mathfrak{p} \in \operatorname{Spec}(R)$ is such that $\operatorname{width}_{R_{\mathfrak{p}}} K_{\mathfrak{p}} \leq \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$. Using the inequalities above, we get $\operatorname{width}_{R_{\mathfrak{p}}} K_{\mathfrak{p}} \leq \operatorname{width}_{R_{\mathfrak{p}}} L_{\mathfrak{p}}$.

By Lemma 2.2, $\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} K_{\mathfrak{p}} \leq 0$; therefore for any prime ideal $\mathfrak{p}$ with $\operatorname{width}_{R_{\mathfrak{p}}} K_{\mathfrak{p}} \leq \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ we have

$$\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq 0$$

and

$$\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} L_{\mathfrak{p}} \leq 0.$$

By Chouinard's equality for modules of finite injective dimension [5, Corollary 3.1],

$$\operatorname{id}_R L = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} L_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

Using the fact that $\operatorname{id}_R L = \operatorname{Gid}_R M > 0$, we observe that

$$\begin{aligned} 0 < \operatorname{Gid}_R(M) &= \operatorname{id}_R(L) \\ &= \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} L_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec}(R)\} \\ &= \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} L_{\mathfrak{p}} \mid \mathfrak{p} \text{ with } \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \operatorname{width}_{R_{\mathfrak{p}}} K_{\mathfrak{p}}\}. \end{aligned}$$

We use the aforementioned width inequalities once again to see that for any prime ideal $\mathfrak{p}$ with $\operatorname{width}_{R_\mathfrak{p}} K_\mathfrak{p} > \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p}$, we must have $\operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} = \operatorname{width}_{R_\mathfrak{p}} L_\mathfrak{p}$. Thus

$$\begin{aligned}\operatorname{Gid}_R(M) &= \operatorname{id}_R(L)\\ &= \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} L_\mathfrak{p} \mid \mathfrak{p} \text{ with } \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} < \operatorname{width}_{R_\mathfrak{p}} K_\mathfrak{p}\}\\ &= \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \text{ with } \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} < \operatorname{width}_{R_\mathfrak{p}} K_\mathfrak{p}\}\\ &= \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec}(R)\}.\end{aligned}$$

□

The following consequence of Theorem 2.3 shows that Gorenstein injective dimension does not grow under localization.

Corollary 2.4. *Let M be an R-module and let $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals of R. Assume that $\operatorname{Gid}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$; then*

$$\operatorname{Gid}_{R_\mathfrak{p}} M_\mathfrak{p} \leq \operatorname{Gid}_{R_\mathfrak{q}} M_\mathfrak{q}.$$

Proof. The inequality is clear if $\operatorname{Gid}_{R_\mathfrak{q}} M_\mathfrak{q} = \infty$ or $\mathfrak{q} \notin \operatorname{Supp}(M)$. Now assume that $M_\mathfrak{q} \neq 0$ and $\operatorname{Gid}_{R_\mathfrak{q}} M_\mathfrak{q} < \infty$.

If $\operatorname{Gid}_{R_\mathfrak{q}} M_\mathfrak{q} = 0$, then Lemma 2.2 gives the last inequality below:

$$\begin{aligned}&\sup\{\operatorname{depth} R_Q - \operatorname{width}_{R_Q} M_Q \mid Q \in \operatorname{Spec}(R) \text{ and } Q \subseteq \mathfrak{p}\}\\ \leq &\sup\{\operatorname{depth} R_Q - \operatorname{width}_{R_Q} M_Q \mid Q \in \operatorname{Spec}(R) \text{ and } Q \subseteq \mathfrak{q}\} \quad \leq 0.\end{aligned}$$

Thus by Theorem 2.3, $\operatorname{Gid}_{R_\mathfrak{p}} M_\mathfrak{p}$ cannot be positive.

If $\operatorname{Gid}_{R_\mathfrak{q}} M_\mathfrak{q} > 0$, there is nothing to prove when $\operatorname{Gid}_{R_\mathfrak{p}} M_\mathfrak{p} = 0$, and the desired inequality is a direct consequence of Theorem 2.3 otherwise. □

We conclude this paper with an ultimate generalization of the classical Bass formula to Gorenstein injective dimension.

Theorem 2.5. *Let $(R, \mathfrak{m})$ be a local ring and let M be a non-zero finitely generated R-module with $\operatorname{Gid}_R(M) < \infty$. Then*

$$\operatorname{Gid}_R(M) = \operatorname{Gid}_{\widehat{R}}(M \otimes_R \widehat{R}) = \operatorname{depth} R.$$

Proof. First note that by Lemma 2.2 and Theorem 2.3, it is clear that $\operatorname{depth} R \leq \operatorname{Gid}_R M$.

On the other hand, by [11, Theorem 3.6], the finiteness of Gorenstein injective dimension of M over R guarantees the finiteness of Gorenstein injective dimension of $M \otimes_R \widehat{R}$ over $\widehat{R}$. Therefore, using the remark after Lemma 2.2 and Theorem 2.3 again, we get

$$\operatorname{Gid}_R M = \sup\{\operatorname{depth} R_\mathfrak{p} | \mathfrak{p} \in \operatorname{Supp}_R(M)\}$$

and

$$\operatorname{Gid}_{\widehat{R}}(M \otimes_R \widehat{R}) = \sup\{\operatorname{depth}(\widehat{R})_\mathfrak{q} | \mathfrak{q} \in \operatorname{Supp}_{\widehat{R}}(M \otimes_R \widehat{R})\}.$$

For any $\mathfrak{p} \in \operatorname{Supp}_R(M)$, let $\mathfrak{q}$ be a prime ideal of $\operatorname{Spec}(\widehat{R})$ minimally containing $\mathfrak{p}\widehat{R}$. We have $\mathfrak{q} \in \operatorname{Supp}_{\widehat{R}}(M \otimes_R \widehat{R})$ and $\operatorname{depth}(\widehat{R})_\mathfrak{q} = \operatorname{depth} R_\mathfrak{p}$. Therefore, the inequality $\operatorname{Gid}_R M \leq \operatorname{Gid}_{\widehat{R}}(M \otimes_R \widehat{R})$ holds.

Finally, since every complete local ring is a quotient of a regular local ring, we can use [8, Theorem 6.3] to get the equality $\operatorname{Gid}_{\widehat{R}} M \otimes_R \widehat{R} = \operatorname{depth} \widehat{R}$, which finishes the proof of the theorem. □

The i-th local cohomology module of M with respect to an ideal $\mathfrak{a}$ of R is denoted by $\mathrm{H}^i_{\mathfrak{a}}(M)$. These modules were introduced by Grothendieck and many people have worked on understanding their structure and finiteness properties. Grothendieck himself conjectured that the part of a local cohomology module $\mathrm{H}^i_{\mathfrak{a}}(M)$ that is killed by $\mathfrak{a}$ is a finitely generated module. In general, this is not true. The best-known and first counterexample was due to Hartshorne: set $R = \mathbb{Q}[[x,y,z,w]]$, $M = R$ and $\mathfrak{a} =< x, y >$. In this case, if $\mathfrak{m}$ denotes the maximal ideal of R, even $\mathrm{Hom}_R(R/\mathfrak{m}, \mathrm{H}^1_{\mathfrak{a}}(\mathrm{M}))$ is not finitely generated.

In view of his example, Hartshorne defined a (not necessarily finitely generated) module N to be $\mathfrak{a}$-cofinite if the support of N is contained in the variety of $\mathfrak{a}$ and in addition $\mathrm{Ext}^i_R(R/\mathfrak{a}, N)$ is a finitely generated R-module for all i.

For a non-zero finitely generated R-module M there is an inequality, $\dim_R M \leq \mathrm{id}\,_R M$; see e.g. [4, Theorem 3.1.17]. This result was extended to cofinite modules by Hellus [12, Theorem 2.3]. In another direction, [16, Proposition 1.2] provides a similar inequality for modules with finite Gorenstein injective dimension. All these statements can be concluded from our following theorem.

Theorem 2.6. *Let $(R, \mathfrak{m})$ be a local ring. If $\mathfrak{a}$ is an ideal of R and M is a non-zero $\mathfrak{a}$-cofinite R-module with finite Gorenstein injective dimension, then*

$$\dim_R M \leq \mathrm{Gid}_R M.$$

Proof. By [16, Lemma 1.1], the module $\mathrm{H}^i_{\mathfrak{m}}(M)$ is zero for all $i > \mathrm{Gid}_R M$. On the other hand, by [14, Theorem 2.9], the module $\mathrm{H}^{\mathrm{dimM}}_{\mathfrak{m}}(M)$ is non-zero. Now the assertion holds. □

Let $(R, \mathfrak{m})$ be a commutative noetherian local ring. In [16, Theorem 1.3], it is shown that if R admits a finite module of finite Gorenstein injective dimension and maximal Krull dimension, then R is Cohen-Macaulay. This generalizes a theorem of Bass and improves upon a result of Takahashi [15, Theorem 3.5(1)], where the ring is assumed to have a dualizing complex. Now we are ready to give an extension of this result for cofinite modules, as an application of our Theorem 2.3.

Theorem 2.7. *Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{a}$ an ideal of R. If R admits a non-zero $\mathfrak{a}$-cofinite R-module M with finite Gorenstein injective dimension and $\dim_R M = \dim R$, then R is Cohen-Macaulay.*

Proof. By Theorem 2.3, there exists $\mathfrak{q} \in \mathrm{Supp}\, M$ with $\mathrm{Gid}_R M \leq \mathrm{depth}\, R_{\mathfrak{q}}$. Then one has

$$\dim R = \dim_R M \leq \mathrm{Gid}_R M \leq \mathrm{depth}\, R_{\mathfrak{q}} \leq \dim R_{\mathfrak{q}} = \mathrm{ht}\, \mathfrak{q}.$$

Hence $\mathfrak{q}$ must be the maximal ideal $\mathfrak{m}$, and therefore

$$\dim R \leq \mathrm{depth}\, R_{\mathfrak{m}} = \mathrm{depth}\, R.$$

It follows that R is Cohen-Macaulay. □

Note that Theorem 2.7 is not valid in general. For example, let $(R, \mathfrak{m})$ be a local non-Cohen-Macaulay ring. Set $M = \mathrm{E}(R/\mathfrak{m})$, the injective envelope of $R/\mathfrak{m}$. Then M is Artinian and thus $\mathfrak{m}$-cofinite with finite Gorenstein injective dimension.

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Department of Mathematics, Northeastern University, 360 Huntington Avenue, Boston, Massachusetts 02115

E-mail address: l.khatami@neu.edu

Department of Mathematics, Shahid Beheshti University, Tehran, Iran — and — School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P. O. Box 19395-5746, Tehran, Iran

E-mail address: mtousi@ipm.ir

Department of Mathematics, University of Tehran, Tehran, Iran — and — School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P. O. Box 19395-5746, Tehran, Iran

E-mail address: yassemi@ipm.ir