G-BILIAISON OF LADDER PFAFFIAN VARIETIES

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Abstract: The ideals generated by pfaffians of mixed size contained in a subladder of a skew-symmetric matrix of indeterminates define arithmetically Cohen-Macaulay, projectively normal, reduced and irreducible projective varieties. We show that these varieties belong to the G-biliaison class of a complete intersection. In particular, they are glicci.

Introduction

Pfaffian ideals and the varieties that they define have been studied both from the algebraic and from the geometric point of view. In [1] Avramov showed that the ideals generated by pfaffians of fixed size define reduced and irreducible, projectively normal schemes. In this article, we study the ideals generated by pfaffians of mixed size contained in a subladder of a skew-symmetric matrix of indeterminates. Ideals generated by pfaffians of the same size contained in a subladder of a skew-symmetric matrix of indeterminates were already studied by the first author. In [7] it is shown that they define irreducible projective varieties, which are arithmetically Cohen-Macaulay and projectively normal. A necessary and sufficient condition for these schemes to be arithmetically Gorenstein is given in terms of the vertices of the defining ladder. In [6], one-cogenerated ideals of pfaffians are studied. The deformation properties of schemes defined by pfaffians of fixed size of a skew-symmetric matrix are studied in [13] and [14].

In this paper, we study ladder ideals of pfaffians of mixed size from the point of view of liaison theory (see [16] for an introduction to the subject, definitions and main results). A central open question in liaison theory asks whether every arithmetically Cohen-Macaulay projective scheme is glicci (i.e. whether it belongs to the G-liaison class of a complete intersection of the same codimension). Migliore and Nagel have shown that the question has an affirmative answer up to deformation (see [17]). The main result of this paper is that ladder pfaffian varieties belong to the G-biliaison class of a linear variety. In particular they are glicci. The result is a natural extension to ideals of pfaffians of the results established by the second author in [9], [10], and [11] for ideals of minors.

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In the first section we study the ideals generated by pfaffians of mixed size contained in a subladder of a skew-symmetric matrix of indeterminates. We prove that they define reduced and irreducible projective schemes (see Proposition 1.9), that we call ladder determinantal varieties. These varieties are shown to be arithmetically Cohen-Macaulay and projectively normal in Proposition 1.9. A localization argument is crucial for extending these properties from the case of fixed size pfaffians of ladders to the case when the pfaffians have mixed size (see Proposition 1.8). In Proposition 1.10 we compute the codimension of ladder pfaffian varieties.

Section 2 contains the liaison results. In Theorem 2.3 we prove that ladder pfaffian varieties belong to the G-biliaison class of a linear variety. Using standard liaison results, we conclude in Corollary 2.4 that they are (evenly) G-linked to a complete intersection.

1. Pfaffian ideals of ladders and ladder pfaffian varieties

Let V be a variety in $\mathbb{P}^r = \mathbb{P}^r_K$, where K is an algebraically closed field of arbitrary characteristic. Let I_V be the saturated homogeneous ideal associated to V in the coordinate ring of $\mathbb{P}^r$. Let $\mathcal{I}_V \subset \mathcal{O}_{\mathbb{P}^r}$ be the ideal sheaf of V. Let W be a scheme that contains V. We denote by $\mathcal{I}_{V|W}$ the ideal sheaf of V restricted to W, i.e. the quotient sheaf $\mathcal{I}_V/\mathcal{I}_W$.

Let $X = (x_{ij})$ be an $n \times n$ skew-symmetric matrix of indeterminates. In other words, the entries x_{ij} with $i < j$ are indeterminates, $x_{ij} = -x_{ji}$ for $i > j$, and $x_{ii} = 0$ for all $i = 1, ..., n$. Let $K[X] = K[x_{ij} \mid 1 \leq i < j \leq n]$ be the polynomial ring associated to X. Given a nonempty subset $\mathcal{U} = \{u_1, ..., u_{2p}\}$ of $\{1, ..., n\}$ we denote by $[u_1, \cdots, u_{2p}]$ the *pfaffian* of the matrix $(x_{ij})_{i\in\mathcal{U}, j\in\mathcal{U}}$.

Definition 1.1. A *ladder* $\mathcal{Y}$ of X is a subset of the set $\{(i,j) \in \mathbb{N}^2 \mid 1 \leq i,j \leq n\}$ with the following properties :

(1) if $(i,j) \in \mathcal{Y}$ then $(j,i) \in \mathcal{Y}$,
(2) if $i < h, j > k$ and $(i,j),(h,k)$ belong to $\mathcal{Y}$, then also $(i,k),(i,h),(h,j),(j,k)$ belong to $\mathcal{Y}$.

We do not assume that a ladder $\mathcal{Y}$ is connected, nor that X is the smallest skew-symmetric matrix having $\mathcal{Y}$ as ladder.

It is easy to see that any ladder can be decomposed as a union of square subladders

$$\mathcal{Y} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_s \tag{1}$$

where

$$\mathcal{X}_k = \{(i,j) \mid a_k \leq i,j \leq b_k\},$$

for some integers $1 \leq a_1 \leq \ldots \leq a_s \leq n$ and $1 \leq b_1 \leq \ldots \leq b_s \leq n$ such that $a_k < b_k$ for all k. We say that $\mathcal{Y}$ is the ladder with *upper corners* $(a_1,b_1),\ldots,(a_s,b_s)$, and that $\mathcal{X}_k$ is the square subladder of $\mathcal{Y}$ with upper outside corner (a_k,b_k). We allow two upper corners to have the same first or second coordinate, however we assume that no two upper corners coincide. Notice that with this convention a ladder does not have a unique decomposition of the form (1). In other words, a ladder does not correspond uniquely to a set of upper

$\mathcal{Y} =$

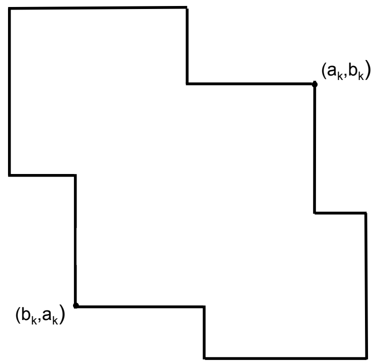

FIGURE 1. A Ladder

corners $(a_1, b_1), \ldots, (a_s, b_s)$. However, a ladder is determined by its upper corners as in (1). Moreover, the upper corners of a ladder $\mathcal{Y}$ determine both the subladders $\mathcal{X}_k$ and the smallest skew-symmetric submatrix of X that has $\mathcal{Y}$ as ladder.

Given a ladder $\mathcal{Y}$ we set $Y = \{x_{ij} \in X \mid (i,j) \in \mathcal{Y},\ i < j\}$, and denote by $K[Y]$ the polynomial ring $K[x_{ij} \mid x_{ij} \in Y]$. If p is a positive integer, we let $I_{2p}(Y)$ denote the ideal generated by the set of the $2p$-pfaffians of X which involve only indeterminates of Y. In particular $I_{2p}(X)$ is the ideal of $K[X]$ generated by the $2p$-pfaffians of X.

Whenever we consider a ladder $\mathcal{Y}$, we assume that it comes with its set of upper corners and the corresponding decomposition as a union of square subladders as in (1).

Notice that the set of upper corners as given in our definition contains all the usual upper outside corners, and may contain some of the usual upper inside corners, as well as other elements of the ladder which are not corners of the ladder in the usual sense.

Example 1.2. Consider the set of upper corners $\{(1,2),(1,4),(3,4),(3,6),(4,7)\}$. Then $k = 5$ and

$$\mathcal{X}_1 = \begin{matrix} (1,1) & (1,2) \\ (2,1) & (2,2) \end{matrix} \qquad \mathcal{X}_2 = \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,3) & (3,4) \\ (4,1) & (4,2) & (4,3) & (4,4) \end{matrix} \qquad \mathcal{X}_3 = \begin{matrix} (3,3) & (3,4) \\ (4,3) & (4,4) \end{matrix}$$

$$\mathcal{X}_4 = \begin{matrix} (3,3) & (3,4) & (3,5) & (3,6) \\ (4,3) & (4,4) & (4,5) & (4,6) \\ (5,3) & (5,4) & (5,5) & (5,6) \\ (6,3) & (6,4) & (6,5) & (6,6) \end{matrix} \qquad \mathcal{X}_5 = \begin{matrix} (4,4) & (4,5) & (4,6) & (4,7) \\ (5,4) & (5,5) & (5,6) & (5,7) \\ (6,4) & (6,5) & (6,6) & (6,7) \\ (7,4) & (7,5) & (7,6) & (7,7) \end{matrix}$$

The ladder determined by this choice of upper corners is

$$\mathcal{Y}=\mathcal{X}_1\cup\mathcal{X}_2\cup\mathcal{X}_3\cup\mathcal{X}_4\cup\mathcal{X}_5=\begin{array}{ccccccc}(1,1)&(1,2)&(1,3)&(1,4)&&&\\(2,1)&(2,2)&(2,3)&(2,4)&&&\\(3,1)&(3,2)&(3,3)&(3,4)&(3,5)&(3,6)&\\(4,1)&(4,2)&(4,3)&(4,4)&(4,5)&(4,6)&(4,7)\\&&(5,3)&(5,4)&(5,5)&(5,6)&(5,7)\\&&(6,3)&(6,4)&(6,5)&(6,6)&(6,7)\\&&&(7,4)&(7,5)&(7,6)&(7,7)\end{array}$$

$(1,2)$ is the upper outside corner of $\mathcal{X}_1$, $(1,4)$ is the upper outside corner of $\mathcal{X}_2$, $(3,4)$ is the upper outside corner of $\mathcal{X}_3$, $(3,6)$ is the upper outside corner of $\mathcal{X}_4$, and $(4,7)$ is the upper outside corner of $\mathcal{X}_5$.

Notice that our set of upper corners contains $(3,4)$, which in the usual terminology is referred to as an upper inside corner. However it does not contain the usual upper inside corner $(4,6)$. Moreover, our set of upper corners contains $(1,2)$ which is not a corner in the usual terminology. It contains also all the usual upper outside corners, namely $(1,4),(3,6)$, and $(4,7)$. Let

$$X=\begin{pmatrix}0&x_{1,2}&x_{1,3}&x_{1,4}&x_{1,5}&x_{1,6}&x_{1,7}\\-x_{1,2}&0&x_{2,3}&x_{2,4}&x_{2,5}&x_{2,6}&x_{2,7}\\-x_{1,3}&-x_{2,3}&0&x_{3,4}&x_{3,5}&x_{3,6}&x_{3,7}\\-x_{1,4}&-x_{2,4}&-x_{3,4}&0&x_{4,5}&x_{4,6}&x_{4,7}\\-x_{1,5}&-x_{2,5}&-x_{3,5}&-x_{4,5}&0&x_{5,6}&x_{5,7}\\-x_{1,6}&-x_{2,6}&-x_{3,6}&-x_{4,6}&-x_{5,6}&0&x_{6,7}\\-x_{1,7}&-x_{2,7}&-x_{3,7}&-x_{4,7}&-x_{5,7}&-x_{6,7}&0\end{pmatrix}$$

be the smallest skew-symmetric matrix having $\mathcal{Y}$ as ladder. The set of indeterminates corresponding to $\mathcal{Y}$ is

$$Y=\begin{array}{cccccc}x_{1,2}&x_{1,3}&x_{1,4}&&&\\&x_{2,3}&x_{2,4}&&&\\&&x_{3,4}&x_{3,5}&x_{3,6}&\\&&&x_{4,5}&x_{4,6}&x_{4,7}\\&&&&x_{5,6}&x_{5,7}\\&&&&&x_{6,7}\end{array}$$

Definition 1.3. Let $\mathcal{Y}=\mathcal{X}_1\cup\ldots\cup\mathcal{X}_s$ be a ladder as in Definition 1.1.
Let $X_k=\{x_{i,j}\mid(i,j)\in\mathcal{X}_k,\ i<j\}$ for $k=1,\ldots,s$. Fix a vector $t=(t_1,\ldots,t_s)$, $t\in\{1,\ldots,\lfloor\frac{n}{2}\rfloor\}^s$. The *pfaffian ideal* $I_{2t}(Y)$ is by definition the sum of pfaffian ideals $I_{2t_1}(X_1)+\ldots+I_{2t_s}(X_s)\subseteq K[Y]$. Sometimes we refer to these ideals as *pfaffian ideals of ladders.*

Example 1.4. Let $\mathcal{Y}$ be the ladder of Example 1.2, together with the same choice of upper corners. Let $t=(1,2,1,2,2)$, then the pfaffian ideal is

$$\begin{aligned}I_{2t}(Y)&=(x_{1,2},-x_{1,3}x_{2,4}+x_{1,4}x_{2,3},x_{3,4},-x_{3,5}x_{4,6}+x_{3,6}x_{4,5},x_{4,5}x_{6,7}-x_{4,6}x_{5,7}+x_{4,7}x_{5,6})\\&\subseteq K[x_{1,2},x_{1,3},x_{1,4},x_{2,3},x_{2,4},x_{3,4},x_{3,5},x_{3,6},x_{4,5},x_{4,6},x_{4,7},x_{5,6},x_{5,7},x_{6,7}].\end{aligned}$$

$I_{2t}(Y)$ is the saturated ideal of a variety of codimension 5 in $\mathbb{P}^{13}$.

Remarks 1.5. (1) Let $\mathcal{Z} \supseteq \mathcal{Y}$ be two ladders of X, and let Z, Y be the corresponding sets of indeterminates. We have an isomorphism of graded K-algebras

$$K[Y]/I_{2t}(Y) \cong K[Z]/I_{2t}(Y) + (x_{ij} \mid x_{ij} \in Z \setminus Y) \cong K[X]/I_{2t}(Y) + (x_{ij} \mid x_{ij} \in X \setminus Y).$$

Here $I_{2t}(Y)$ is an ideal in $K[Y], K[X]$, and $K[Z]$ respectively. Then the height of the ideal $I_{2t}(Y)$ does not depend of whether we regard it as an ideal of $K[Y], K[X]$, or $K[Z]$.

(2) We can assume without loss of generality that

$$2t_k \leq b_k - a_k + 1.$$

In fact, if $2t_k > b_k - a_k + 1$ then $I_{2t_k}(X_k) = 0$.

(3) Moreover, we can assume that

$$a_k - a_{k-1} > t_{k-1} - t_k \quad \text{and} \quad b_k - b_{k-1} > t_k - t_{k-1}, \qquad \text{for } k = 2, \ldots, s.$$

If $a_k - a_{k-1} \leq t_{k-1} - t_k$, by successively developing each $2t_{k-1}$-pfaffian of X_{k-1} with respect to the first $2(a_k - a_{k-1})$ rows and columns, we obtain an expression of the pfaffian as a combination of pfaffians of size $2t_{k-1} - 2(a_k - a_{k-1}) \geq 2t_k$ that involve only rows and columns from X_k. Therefore $I_{2t_k}(X_k) \supseteq I_{2t_{k-1}}(X_{k-1})$. Similarly, if $b_k - b_{k-1} \leq t_k - t_{k-1}$, by developing each $2t_k$-pfaffian of X_k with respect to the last $2(b_k - b_{k-1})$ rows and columns, we obtain an expression of the pfaffian as a combination of pfaffians of size $2t_k - 2(b_k - b_{k-1}) \geq 2t_{k-1}$ that involve only rows and columns from X_{k-1}. Therefore $I_{2t_k}(X_k) \subseteq I_{2t_{k-1}}(X_{k-1})$. In either case, we can remove a part of the ladder and reduce to the study of a proper subladder that corresponds to the same pfaffian ideal. For example, if $b_k - b_{k-1} \leq t_k - t_{k-1}$ we can consider the ladder

$$\tilde{\mathcal{Y}} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_{k-1} \cup \mathcal{X}_{k+1} \cup \ldots \cup \mathcal{X}_s$$

and let

$$t' = (t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_s).$$

Since $I_{2t_k}(X_k) \subseteq I_{2t_{k-1}}(X_{k-1})$, we have $I_{2t}(Y) = I_{2t'}(\tilde{Y})$, where $\tilde{Y}$ is the set of indeterminates corresponding to $\tilde{\mathcal{Y}}$.

The class of pfaffian ideals that we consider is very large. We now give examples of interesting families of ideals generated by pfaffians, which belong to the class of pfaffian ideals that we study.

Examples 1.6. (1) If $t = (t, \ldots, t) \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}^s$ then $I_{2t}(Y)$ is the ideal generated by the pfaffians of size $2t$ of X that involve only indeterminates from Y. In [7] it is proven that in this case $K[Y]/I_{2t}(Y)$ is a Cohen-Macaulay normal domain.

(2) If we choose all the upper corners on the same row, we obtain ideals of pfaffians of matrices which are contained one in the other. In this case $1 = a_1 = a_2 = \ldots = a_s$, hence $1 < b_1 < b_2 < \ldots < b_s = n$. By Remark 1.5 (3), this forces $1 \leq t_1 < t_2 < \ldots < t_s$. The ideal $I_{2t}(X)$ is generated by the $2t_i$-pfaffians of the submatrix of the first b_i rows and columns, $i = 1, ..., s$.

Similarly, if we choose all the upper corners on the same column, that is, $1 = a_1 < a_2 < \ldots < a_s$ and $b_1 = b_2 = \ldots = b_s = n$, we obtain the ideal generated by the $2t_i$-pfaffians of the last $n - a_i + 1$ rows and columns, $i = 1, ..., s$, with

$t_1 > t_2 > \ldots > t_s \geq 1$. Notice that these two choices of upper corners produce the same family of ideals.

(3) Consider the ladder with two upper corners $(1, b), (1, n)$, $b < n$, and the vector $(t_1, t_2) = (t, t + 1)$. Then the ideal $I_{2t}(Y)$ is generated by all the $2t + 2$-pfaffians and by the $2t$-pfaffians of the first b rows and columns, of an $n \times n$ skew-symmetric matrix of indeterminates. This ideals belong to a well-known class of ideals generated by pfaffians in a matrix, the cogenerated ideals, which have been studied in [6]. In fact $I_{2t}(Y)$ is the ideal cogenerated by the pfaffian $[1, 2, \ldots, 2t-1, b+1]$. Notice however that not every one-cogenerated pfaffian ideal is a pfaffian ideal of ladders.

We will show that for every vector t the ideals $I_{2t}(Y)$ are prime (see Proposition 1.9). Therefore they define reduced and irreducible projective varieties.

Definition 1.7. Let $V \subseteq \mathbb{P}^r$. V is a *pfaffian variety* if $I_V = I_{2t}(X)$, where X is a skew symmetric matrix of indeterminates of size $n \times n$. V is a *ladder pfaffian variety* if $I_V = I_{2t}(Y) = I_{2t_1}(X_1) + \ldots + I_{2t_s}(X_s)$ for some ladder $\mathcal{Y} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_s$ and some vector $t = (t_1, \ldots, t_s) \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}^s$.

Notice that every pfaffian variety is a ladder pfaffian variety. Therefore, from now on we will only consider ladder pfaffian varieties. Moreover, in view of Remark 1.5 (1) we will not distinguish between ladder pfaffian varieties and cones over them.

In this section we study ladder pfaffian varieties. We prove that their saturated ideals are generated by pfaffians of mixed size contained in a subladder of a skew-symmetric matrix of indeterminates, or in other words that the ideals in question are prime. We prove that the ladder pfaffian varieties are arithmetically Cohen-Macaulay and projectively normal, and we compute their codimension. We choose to follow a classical commutative algebra localization argument to approach the problem. Some of our results could be obtained also by using a Schubert calculus approach, at least for the case of ideals of pfaffians of fixed size in a matrix, which define Schubert varieties in orthogonal Grassmannians.

In order to establish the properties that we just mentioned, we will make use of a localization argument (analogous to that of Lemma 7.3.3 in [2]). The following proposition will be crucial in the sequel. We use the notation of Definition 1.1 and Definition 1.3, and refer to Fig. 2. Notice that if $t_l \geq 2$ for some l, then it is always possible to choose a k such $t_k \geq 2$, $a_{k+1} - 1 \geq a_k$, and $b_k \geq b_{k-1} + 1$. In fact, it suffices to choose k such that $t_k \geq t_l$ for all l, and the inequalities follow from Remark 1.5 (3). Notice moreover that for a classical ladder (i.e. a ladder for which no two vertices belong to the same row or column) these conditions are automatically satisfied.

Proposition 1.8. *Let $\mathcal{Y} = \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_s$ be a ladder of a skew-symmetric matrix X of indeterminates. Let $t = (t_1, \ldots, t_s) \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}^s$, and let $I_{2t}(Y)$ be the corresponding pfaffian ideal. Fix $k \in \{1, \ldots, s\}$ such that $t_k \geq 2$, $a_{k+1} - 1 \geq a_k$, and $b_k \geq b_{k-1} + 1$. Let $t' = (t_1, \ldots, t_{k-1}, t_k - 1, t_{k+1}, \ldots, t_s)$. Let $\mathcal{Y}'$ be the subladder of $\mathcal{Y}$ with outside corners*

$$(a_1, b_1), \ldots, (a_{k-1}, b_{k-1}), (a_k + 1, b_k - 1), (a_{k+1}, b_{k+1}), \ldots, (a_s, b_s).$$

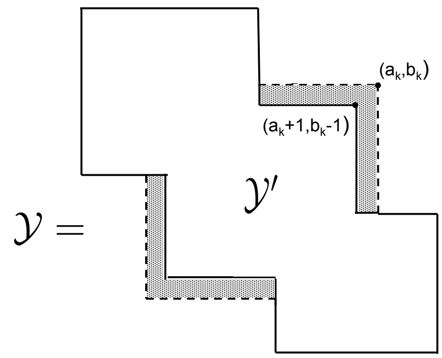

FIGURE 2.

Then there is an isomorphism

$$K[Y]/I_{2t}(Y)[x_{a_k,b_k}^{-1}] \cong K[Y']/I_{2t'}(Y')[x_{a_k,b_{k-1}+1},\dots,x_{a_k,b_k},x_{a_k+1,b_k},\dots,x_{a_{k+1}-1,b_k}][x_{a_k,b_k}^{-1}].$$

Proof. We prove the proposition in the case $s=k=2$. In the general case the proof works exactly the same way. Let $\mathcal{Y}=\mathcal{X}_1\cup\mathcal{X}_2$ be the ladder with upper corners $(1,c)$ and (a,b), $t=(t_1,t_2)$ with $t_2\geq 2$. Let $X_1=(x_{i,j})_{1\leq i,j\leq c}$ and $X_2=(x_{i,j})_{a\leq i,j\leq b}$ be two submatrices of X. Let $\tilde{X}_1=(\tilde{x}_{u,v})$ be a skew-symmetric matrix of indeterminates of size $c\times c$, whose entries have indexes $1\leq u,v\leq c$, where $\tilde{x}_{u,v}=x_{u,v}$ if $u<a$ or $v<a$, and $\tilde{x}_{u,v}$ are new indeterminates whenever $a\leq u<v\leq c$ (that is $(u,v)\in\mathcal{X}_2$).

Let $\tilde{Y}$ be the set of all the indeterminates in the matrices $\tilde{X}_1$ and X_2, and denote by L the ideal $(\{x_{u,v}-\tilde{x}_{u,v}\}_{a\leq u<v\leq c})$. Then

$$K[Y]/I_{2t_1}(X_1)+I_{2t_2}(X_2)\cong K[\tilde{Y}]/I_{2t_1}(\tilde{X}_1)+I_{2t_2}(X_2)+L.$$

Therefore

$$K[Y]/(I_{2t_1}(X_1)+I_{2t_2}(X_2))[x_{a,b}^{-1}]\cong K[\tilde{Y}]/(I_{2t_1}(\tilde{X}_1)+I_{2t_2}(X_2)+L)[x_{a,b}^{-1}]\cong$$
$$K[\tilde{Y}][x_{a,b}^{-1}]/I_{2t_1}(\tilde{X}_1)^e+I_{2t_2}(X_2)^e+L^e$$

where for any ideal I, I^e denotes the extension $IK[\tilde{Y}][x_{a,b}^{-1}]$. By a well-know localization argument (see for example [7, Proposition 3.2]) we have that

$$K[X_2][x_{a,b}^{-1}]/I_{2t_2}(X_2)^e\cong K[X_2][x_{a,b}^{-1}]/I_{2t_2-2}(X_2')^e,$$

where X_2' is the submatrix of X_2 obtained by removing the a-th and the b-th row and column. Therefore

$$K[\tilde{Y}][x_{a,b}^{-1}]/I_{2t_1}(\tilde{X}_1)^e+I_{2t_2}(X_2)^e+L^e\cong K[\tilde{Y}][x_{a,b}^{-1}]/I_{2t_1}(\tilde{X}_1)^e+I_{2t_2-2}(X_2')^e+L^e\cong$$
$$K[Y]/I_{2t'}(Y')[x_{a,b}^{-1}]\cong K[Y']/I_{2t'}(Y')[x_{a,c+1},\dots,x_{a,b},x_{a+1,b}\dots,x_{b-1,b}][x_{a,b}^{-1}].$$

□

Using Proposition 1.8 we can establish some properties of ladder pfaffian varieties.

Proposition 1.9. *Pfaffian ideals of ladders define reduced and irreducible, arithmetically Cohen-Macaulay projectively normal varieties.*

Proof. Let $I_{2t}(Y)$ be a pfaffian ideal. Let $t_{\max}$ be the maximum of $\{t_1, ..., t_s\}$. If $t_{\max} = 1$ then $I_{2t}(X)$ is generated by indeterminates, und we are done. Assume that $t_{\max} \geq 2$. Let $\widehat{\mathcal{Y}}$ be the ladder obtained by enlarging $\mathcal{Y}$ along its borders by the region which increases the size of every $\mathcal{X}_k$ by $t_{\max} - t_k$. Thus $\widehat{\mathcal{Y}}$ is the ladder with upper corners $(a_k - t_{\max} + t_k, b_k + t_{\max} - t_k)$, with $k = 1, \dots, s$. Let $\widehat{Y} = \{x_{ij} \in X \mid (i, j) \in \widehat{\mathcal{Y}},\ i < j\}$ and let $\Psi = \widehat{Y} \setminus Y$. By Proposition 1.8, we can repeatedly localize $K[\widehat{Y}]/I_{2t_{\max}}(\widehat{Y})$ at appropriate upper outside corners and obtain the original ladder $\mathcal{Y}$ and the pfaffians of size $t_1, \dots, t_s$. It follows that there exists a subset $\{z_1, ..., z_p\}$ of Ψ such that

$$K[\widehat{Y}]/I_{2t_{\max}}(\widehat{Y})[z_1^{-1}, \dots, z_p^{-1}] \cong K[Y]/I_{2t}(Y)[\Psi][z_1^{-1}, \dots, z_p^{-1}].$$

By [7, 1.2,2.1,3.5] one has that $K[\widehat{Y}]/I_{2t_{\max}}(\widehat{Y})$ is a Cohen-Macaulay normal domain, thus $K[Y]/I_{2t}(Y)[\Psi][u_1^{-1}, \dots, u_q^{-1}]$ is a Cohen-Macaulay normal domain. Since Ψ is a set of indeterminates over $K[Y]/I_{2t}(Y)$ and $\{u_1, ..., u_q\} \subset \Psi$, then also $K[Y]/I_{2t}(Y)[\Psi]$ is a Cohen-Macaulay normal domain. Hence $I_{2t}(Y)$ defines a reduced and irreducible, arithmetically Cohen-Macaulay normal projective variety. □

A standard argument allows us to compute the codimension of ladder pfaffian varieties. The notation is the same as in Proposition 1.8.

Proposition 1.10. *Let $\mathcal{Y}$ be a ladder with upper corners $(a_1, b_1), \dots, (a_s, b_s)$. Let*

$$\mathcal{L} = \{(i, j) \mid a_k + t_k - 1 \leq i, j \leq b_k - t_k + 1, \text{ for some } 1 \leq k \leq s\}$$

be a subset of $\mathcal{Y}$. Then $\mathcal{L}$ is a ladder and the height of $I_{2t}(Y)$ is equal to the cardinality of $\{(i, j) \in \mathcal{L} \mid i < j\}$.

Proof. Observe that

$$a_k + t_k - 1 > a_{k-1} + t_{k-1} - 1, \quad \text{and} \quad b_k - t_k + 1 > b_{k-1} - t_{k-1} + 1$$

by Remark 1.5 (3). Moreover by Remark 1.5 (2) we have $b_k - a_k > 2t_k - 2$. Then

$$b_k - t_k + 1 > a_k + t_k - 1$$

for all k. Therefore $\mathcal{L}$ is a ladder with upper corners $\{(a_k+t_k-1, b_k-t_k+1) \mid k = 1, \dots, s\}$. Notice that $\mathcal{L}$ has no two corners on the same row or column. Let $L = \{x_{i,j} \mid (i, j) \in \mathcal{L},\ i < j\}$.

We argue by induction on $\tau = t_1 + \ldots + t_s \geq s$. If $\tau = s$, then $t_1 = \ldots = t_s = 1$, and $\mathcal{L} = \mathcal{Y}$. Moreover,

$$I_2(Y) = (x_{ij} \mid x_{ij} \in Y,\ i < j) = (x_{ij} \mid x_{ij} \in L,\ i < j),$$

thus the thesis holds true.

Assume then that the thesis is true for $\tau - 1 \geq s$ and prove it for τ. Since $\tau > s$, then $t_k \geq 2$ for some k. By Proposition 1.8 we have an isomorphism

$$K[Y]/I_{2t}(Y)[x_{a_k,b_k}^{-1}] \cong K[Y']/I_{2t'}(Y')[x_{a_k,b_{k-1}+1},\ldots,x_{a_k,b_k},x_{a_k,b_k+1},\ldots,x_{a_k,b_{k+1}-1}][x_{a_k,b_k}^{-1}].$$

Since x_{a_k,b_k} does not divide zero modulo $I_{2t'}(Y')$ and $I_{2t}(Y)$, we have

$$\mathrm{ht}\ I_{2t}(Y) = \mathrm{ht}\ I_{2t'}(Y').$$

Notice that the same ladder $\mathcal{L}$ computes the height of both $I_{2t'}(Y')$ and $I_{2t}(Y)$, thus the thesis follows by the induction hypothesis. □

2. Linkage of ladder pfaffian varieties

In this section we prove that ladder pfaffian varieties belong to the G-biliaison class of a complete intersection. The biliaisons are performed on ladder pfaffian varieties, which are reduced and irreducible (hence generically Gorenstein), and arithmetically Cohen-Macaulay. Therefore we can conclude that ladder pfaffian varieties are glicci. Notice the analogy with determinantal varieties, symmetric determinantal varieties and mixed ladder determinantal varieties, that were treated by the second author with analogous techniques in [9], [10], and [11].

The following lemma due to De Concini and Procesi [5, 6.1] will be needed in the sequel.

Lemma 2.1. *Let A be a skew symmetric $n \times n$ matrix, $p, m \leq n$ even integers and $c_1, ..., c_p, d_1, ..., d_m$ elements of the set $\{1, ..., n\}$. Then*

$$[c_1,...,c_p][d_1,...,d_m] - \sum_{h=1}^{p}[c_1,...,c_{h-1},d_1,c_{h+1},...,c_p][c_h,d_2,...,d_m] =$$

$$\sum_{k=2}^{m}(-1)^{k-1}[d_k,d_1,c_1,...,c_p][d_2,...,d_{k-1},d_{k+1},...,d_m]$$

where [...] denotes a pfaffian of A.

The following result will also be needed in the proof. We will use it to construct the ladder pfaffian varieties on which we perform the G-biliaisons. We follow the notation established in Definitions 1.1 and 1.3.

Lemma 2.2. *Let $V \subseteq \mathbb{P}^r$ be a ladder pfaffian variety of codimension c. Let $\mathcal{Y}$ be the ladder corresponding to V, and assume that $t_k = \max\{t_1,\ldots,t_s\} \geq 2$. Let $\mathcal{Z}$ be the subladder of $\mathcal{Y}$ with upper corners*

$$(a_1,b_1),\ldots,(a_{k-1},b_{k-1}),(a_k,b_k-1),(a_k+1,b_k),(a_{k+1},b_{k+1}),\ldots,(a_s,b_s)$$

and let $u = (t_1,\ldots,t_{k-1},t_k,t_k,t_{k+1},\ldots,t_s)$. Then the ladder pfaffian variety $W \subseteq \mathbb{P}^r$ with $I_W = I_{2u}(\mathcal{Z})$ has codimension $c-1$.

Proof. We decompose the ladder $\mathcal{Z}$ as

$$\mathcal{Z} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_{k-1} \cup \mathcal{X}_k^{(1)} \cup \mathcal{X}_k^{(2)} \cup \mathcal{X}_{k+1} \cup \ldots \cup \mathcal{X}_s$$

where $\mathcal{X}_k^{(1)}$, $\mathcal{X}_k^{(2)}$ are the square subladders with upper outside corner $(a_k, b_k - 1)$ and $(a_k + 1, b_k)$, respectively. Let $u = (t_1, \ldots, t_{k-1}, t_k, t_k, t_{k+1}, \ldots, t_s)$, $\quad u \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}^{s+1}$.

If the ladder $\mathcal{Z}$ satisfies the inequalities of Remark 1.5 (2) and (3), then the codimension count follows from Proposition 1.10. In fact, the codimension c of V equals the cardinality of the subset $\{(i,j) \in \mathcal{L} \mid i < j\}$ where $\mathcal{L}$ is the ladder with upper corners $(a_1 + t_1 - 1, b_1 - t_1 + 1), \ldots, (a_s + t_s - 1, b_s - t_s + 1)$. The codimension of W equals cardinality of $\{(i,j) \in \mathcal{L}' \mid i < j\}$, where $\mathcal{L}'$ is the ladder obtained from $\mathcal{L}$ by removing $(a_k + t_k - 1, b_k - t_k + 1)$ and $(b_k - t_k + 1, a_k + t_k - 1)$. So we conclude that W has codimension $c - 1$.

Notice however that the ladder $\mathcal{Z}$ may not satisfy the inequalities of Remark 1.5 (2),(3) even under the assumption that the ladder $\mathcal{Y}$ does. In particular, the following three situations may occur:

(1) $2t_k = b_k - a_k + 1 > b_k - a_k = (b_k - 1) - a_k + 1 = b_k - (a_k + 1) + 1$,
(2) $a_{k+1} - (a_k + 1) = t_k - t_{k+1}$,
(3) $(b_k - 1) - b_{k-1} = t_k - t_{k-1}$.

In case (1) we delete the subladders $\mathcal{X}_k^{(1)}$ and $\mathcal{X}_k^{(2)}$, and let

$$\mathcal{Z} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_{k-1} \cup \mathcal{X}_{k+1} \cup \ldots \cup \mathcal{X}_s$$

and $u = (t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_s)$.

In case (2) we delete the subladder $\mathcal{X}_k^{(2)}$, and let

$$\mathcal{Z} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_{k-1} \cup \mathcal{X}_k^{(1)} \cup \mathcal{X}_{k+1} \cup \ldots \cup \mathcal{X}_s$$

and $u = (t_1, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_s)$.

In case (3) we delete the subladder $\mathcal{X}_k^{(1)}$, and let

$$\mathcal{Z} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_{k-1} \cup \mathcal{X}_k^{(2)} \cup \mathcal{X}_{k+1} \cup \ldots \cup \mathcal{X}_s$$

and $u = (t_1, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_s)$.

Notice that it may happen that more than one of the cases (1), (2) and (3) is verified for the ladder $\mathcal{Z}$. In this case, we behave as if we were in the situation (1). As we already observed, none of the operations above affects the ideal I_W.

If we are in situation (1), then $2t_k = b_k - a_k + 1$. Applying Proposition 1.10 to the ladder $\mathcal{Z} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_{k-1} \cup \mathcal{X}_{k+1} \cup \ldots \cup \mathcal{X}_s$ we obtain that the codimension of W equals the cardinality of $\{(i,j) \in \mathcal{L}' \mid i < j\}$, where $\mathcal{L}'$ is the ladder with upper corners

$$(a_1 + t_1 - 1, b_1 - t_1 + 1), \ldots, (a_{k-1} + t_{k-1} - 1, b_{k-1} - t_{k-1} + 1),$$

$$(a_{k+1} + t_{k+1} - 1, b_{k+1} - t_{k+1} + 1), \ldots, (a_s + t_s - 1, b_s - t_s + 1).$$

Since $a_k+t_k-1=b_k-t_k$ and $a_k+t_k=b_k-t_k+1$, the cardinality of $\{(i,j)\in\mathcal{L}'\mid i<j\}$ coincides with the cardinality of $\{(i,j)\in\mathcal{L}''\mid i<j\}$, where $\mathcal{L}''$ has upper corners

$$(a_1+t_1-1,b_1-t_1+1),\ldots,(a_{k-1}+t_{k-1}-1,b_{k-1}-t_{k-1}+1),(a_k+t_k-1,b_k-t_k),$$

$$(a_k+t_k,b_k-t_k+1),(a_{k+1}+t_{k+1}-1,b_{k+1}-t_{k+1}+1),\ldots,(a_s+t_s-1,b_s-t_s+1).$$

Equivalently, $\mathcal{L}''$ is obtained from $\mathcal{L}$ by removing (a_k+t_k-1,b_k-t_k+1) and its symmetric point (b_k-t_k+1,a_k+t_k-1). Hence the codimension of W is $c-1$.

Similarly, if we are in the situation that both (2) and (3) are verified, we apply Proposition 1.10 to the ladder $\mathcal{Z}$ of case (1). The codimension of W equals the cardinality $\{(i,j)\in\mathcal{L}'\mid i<j\}$, where $\mathcal{L}'$ is the same as in case (1). Since $a_k+t_k=a_{k+1}+t_{k+1}-1$ and $b_k-t_k=b_{k-1}-t_{k-1}+1$, the cardinality of $\{(i,j)\in\mathcal{L}'\mid i<j\}$ coincides with the cardinality of $\{(i,j)\in\mathcal{L}''\mid i<j\}$, where $\mathcal{L}''$ is the same as in case (1). In fact, the angles (a_k+t_k,b_k-t_k+1) and $(a_{k+1}+t_{k+1}-1,b_{k+1}-t_{k+1}+1)$ are on the same row. Moreover the angles $(a_{k-1}+t_{k-1}-1,b_{k-1}-t_{k-1}+1)$ and (a_k+t_k-1,b_k-t_k) are on the same column. We conclude that the codimension of W is $c-1$.

If we are in situation (2), then $a_{k+1}-(a_k+1)=t_k-t_{k+1}$. Assume that $(b_k-1)-b_{k-1}>t_k-t_{k-1}$. Apply Proposition 1.10 to the ladder $\mathcal{Z}=\mathcal{X}_1\cup\ldots\cup\mathcal{X}_{k-1}\cup\mathcal{X}_k^{(1)}\cup\mathcal{X}_{k+1}\cup\ldots\cup\mathcal{X}_s$. The codimension of W equals the cardinality of $\{(i,j)\in\mathcal{L}'\mid i<j\}$, where $\mathcal{L}'$ is the ladder with upper corners

$$(a_1+t_1-1,b_1-t_1+1),\ldots,(a_{k-1}+t_{k-1}-1,b_{k-1}-t_{k-1}+1),(a_k+t_k-1,b_k-t_k)$$

$$(a_{k+1}+t_{k+1}-1,b_{k+1}-t_{k+1}+1),\ldots,(a_s+t_s-1,b_s-t_s+1).$$

Since $a_k+t_k=a_{k+1}+t_{k+1}-1$, the cardinality of $\{(i,j)\in\mathcal{L}'\mid i<j\}$ coincides with the cardinality of $\{(i,j)\in\mathcal{L}''\mid i<j\}$, where $\mathcal{L}''$ is the same as in case (1). In fact, the angles (a_k+t_k,b_k-t_k+1) and $(a_{k+1}+t_{k+1}-1,b_{k+1}-t_{k+1}+1)$ are on the same row. We conclude that the codimension of W is $c-1$.

An analogous argument applies to situation (3), where we consider $\mathcal{Z}=\mathcal{X}_1\cup\ldots\cup\mathcal{X}_{k-1}\cup\mathcal{X}_k^{(2)}\cup\mathcal{X}_{k+1}\cup\ldots\cup\mathcal{X}_s$ and observe that $\mathcal{X}_k^{(1)}$ can be disregarded in the codimension count, as the angles $(a_{k-1}+t_{k-1}-1,b_{k-1}-t_{k-1}+1)$ and (a_k+t_k-1,b_k-t_k) are on the same column. □

The next theorem is the main result of this paper. The idea of the proof is as follows: starting from a ladder pfaffian variety V, we construct two ladder pfaffian varieties V' and W such that V and V' are generalized divisors on W. Then we show how V' can be obtained from V by an elementary G-biliaison on W.

Theorem 2.3. *Any ladder pfaffian variety can be obtained from a linear variety by a finite sequence of ascending elementary G-biliaisons.*

Proof. Let V be a ladder pfaffian variety. Let $\mathcal{Y}$ be the ladder corresponding to V,

$$I_V=I_{2t}(Y)=I_{2t_1}(X_1)+\cdots+I_{2t_s}(X_s)\subseteq K[Y].$$

We perform all the linkage steps in $\mathbb{P}^r=\mathrm{Proj}\,(K[Y])$. If $t_1=\ldots=t_s=1$ then V is a linear variety. Therefore we consider the case when $t_k=\max\{t_1,\ldots,t_s\}\geq 2$. It follows

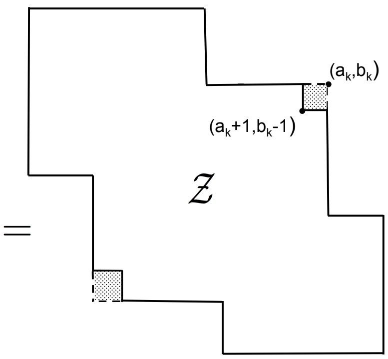

FIGURE 3.

that $a_{k+1}-a_k > t_k-t_{k+1} \geq 0$ and $b_k-b_{k-1} > t_k-t_{k-1} \geq 0$, therefore $a_{k-1} < a_k+1 \leq a_{k+1}$ and $b_{k-1} \leq b_k - 1 < b_{k+1}$. Let $\mathcal{Y}'$ be the ladder with upper corners

$$(a_1, b_1), \ldots, (a_{k-1}, b_{k-1}), (a_k + 1, b_k - 1), (a_{k+1}, b_{k+1}), \ldots, (a_s, b_s)$$

and let $t' = (t_1, \ldots, t_{k-1}, t_k - 1, t_{k+1}, \ldots, t_s)$. Let

$$\mathcal{Y}' = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_{k-1} \cup \mathcal{X}'_k \cup \mathcal{X}_{k+1} \cup \ldots \cup \mathcal{X}_s,$$

where $\mathcal{X}'_k$ is the square subladder with upper outside corner $(a_k + 1, b_k - 1)$ (see Figure 2). Notice that all the inequalities of Remark 1.5 (2),(3) are satisfied by $\mathcal{Y}', t'$. Let V' be the ladder pfaffian variety with saturated ideal $I_{V'} = I_{2t'}(Y')$. By Proposition 1.10 and Remark 1.5 (1), V and V' have the same codimension c in $\mathbb{P}^r = \mathrm{Proj}\,(K[Y])$. In fact, in both cases c equals the cardinality of the subset $\{(i,j) \in \mathcal{L} \mid i < j\}$ where $\mathcal{L}$ is the ladder with upper corners $(a_1 + t_1 - 1, b_1 - t_1 + 1), \ldots, (a_s + t_s - 1, b_s - t_s + 1)$.

Let $\mathcal{Z}$ be the ladder obtained from $\mathcal{Y}$ by removing (a_k, b_k) and (b_k, a_k) (see Figure 3), and let $u = (t_1, \ldots, t_{k-1}, t_k, t_k, t_{k+1}, \ldots, t_s)$.

Let W be the ladder pfaffian variety with $I_W = I_{2u}(Z)$. W has codimension $c - 1$ in $\mathbb{P}^r = \mathrm{Proj}\,(K[Y])$, by Lemma 2.2. Clearly $I_V \supseteq I_W$, therefore $V \subseteq W$. We claim that also $V' \subseteq W$. In order to show that $I_{2t'}(Y') \supseteq I_{2u}(Z)$, it suffices to show that $I_{2t_k}(X_k^{(1)}) + I_{2t_k}(X_k^{(2)}) \subseteq I_{2(t_k-1)}(X'_k)$. Let $f = [u_1, \ldots, u_{2t_k}]$ be a $2t_k$-pfaffian in $I_{2t_k}(X_k^{(1)}) + I_{2t_k}(X_k^{(2)})$, with $a_k \leq u_1 < u_2 < \cdots \leq u_{2t_k} \leq b_k$. If $a_k, b_k \notin \{u_1, \ldots, u_{2t_k}\}$, then $f \in I_{2t_k}(X'_k) \subseteq I_{2(t_k-1)}(X'_k)$. If $a_k = u_1$ then $b_k \notin \{u_2, \ldots, u_{2t_k}\}$, since f does not involve the indeterminate x_{a_k,b_k}. By expanding f along the u_1-th row and column one has

$$f = \sum_{h=1}^{2t_k} \pm[u_1, u_h][u_2, \ldots, \check{u}_h, \ldots, u_{2t_k}].$$

The notation $[..., \check{u}_i, ...]$ means that the index u_i is not involved in the pfaffian. Since $[u_2, \ldots, \check{u}_h, \ldots, u_{2t_k}] \in I_{2(t_k-1)}(X'_k)$, one has $f \in I_{2(t_k-1)}(X'_k)$. Similarly, if $u_{2t_k} = b_k$ then

$a_k \notin \{u_1, \ldots, u_{2t_k-1}\}$, and expanding f along the u_{2t_k}-th row and column the conclusion follows.

The variety W is reduced, irreducible, and arithmetically Cohen-Macaulay by Proposition 1.9. In particular it is generically Gorenstein. Therefore we can regard V and V' as generalized divisors on W (see [12] about the theory of generalized divisors). Then V and V' are G-bilinked on W if and only if $V \sim V' + mH$ for some $m \in \mathbb{Z}$, where H is the hyperplane section divisor on W. This is in turn equivalent to

$$\mathcal{I}_{V|W} \cong \mathcal{I}_{V'|W}(-m) \tag{2}$$

as subsheaves of the sheaf of total quotient rings of W. In order to construct an isomorphism as (2), we prove that

$$\frac{[a_k, u_2, \ldots, u_{2t_k-1}, b_k]}{[u_2, \ldots, u_{2t_k-1}]} = \frac{[a_k, v_2, \ldots, v_{2t_k-1}, b_k]}{[v_2, \ldots, v_{2t_k-1}]}$$

modulo I_W, for any choice of u_i, v_i such that $a_k < u_2 < \ldots < u_{2t_k-1} < b_k$ and $a_k < v_2 < \ldots < v_{2t_k-1} < b_k$. Then multiplication by $\frac{[a_k, u_2, \ldots, u_{2t_k-1}, b_k]}{[u_2, \ldots, u_{2t_k-1}]}$ for a fixed choice of $a_k < u_2 < \ldots < u_{2t_k-1} < b_k$ yields an isomorphism as (2). By Lemma 2.1 one has

$$\begin{aligned}
&[v_2, \ldots, v_{2t_k-1}][a_k, u_2, \ldots, u_{2t_k-1}, b_k] = \\
&\sum_{h=2}^{2t_k-1} [v_2, \ldots, v_{h-1}, a_k, v_{h+1}, , \ldots, v_{2t_k-1}][v_h, u_2, \ldots, u_{2t_k-1}, b_k] + \\
&+ \sum_{l=2}^{2t_k-1} (-1)^{l-1} [u_l, a_k, v_2, \ldots, v_{2t_k-1}][u_2, \ldots, \check{u}_l, \ldots, u_{2t_k-1}, b_k] + \\
&(-1)^{2t_k-1} [b_k, a_k, v_2, \ldots, v_{2t_k-1}][u_2, \ldots, u_{2t_k-1}].
\end{aligned}$$

Since the pfaffians $[v_h, u_2, \ldots, u_{2t_k-1}, b_k], [u_l, a_k, v_2, \ldots, v_{2t_k-1}]$ are in I_W for every h and l, one has that $[v_2, \ldots, v_{2t_k-1}][a_k, u_2, \ldots, u_{2t_k-1}, b_k] = -[b_k, a_k, v_2, \ldots, v_{2t_k-1}][u_2, \ldots, u_{2t_k-1}] = [a_k, v_2, \ldots, v_{2t_k-1}, b_k][u_2, \ldots, u_{2t_k-1}]$ modulo I_W. Therefore the isomorphism (2) holds, and V and V' are G-bilinked on W. Repeating this procedure, one eventually gets to the pfaffians of order 2 of the ladder $\mathcal{L}$ with upper corners $\{(a_k + t_k - 1, b_k - t_k + 1) \mid k = 1, \ldots, s\}$. Clearly $I_2(L) = (x_{ij} \mid (i,j) \in \mathcal{L}, i < j)$ defines a linear variety. □

Kleppe, Migliore, Miró-Roig, Nagel, and Peterson proved in [15] that a G-biliaison on an arithmetically Cohen-Macaulay, G_1 scheme can be realized by two Gorenstein links. In [12] Hartshorne generalized their result to a G-biliaison on an arithmetically Cohen-Macaulay, generically Gorenstein scheme. Therefore we get the following corollary.

Corollary 2.4. *Every ladder pfaffian variety V can be G-linked in $2(t_1 + \ldots + t_s - s)$ steps to a linear variety of the same codimension. In particular, ladder pfaffian varieties are glicci.*

Remark 2.5. The varieties cut out by pfaffians of fixed size of a skew-symmetric matrix (i.e. those for which $Y = X$ and $t_1 = \ldots = t_s$) are known to be arithmetically Gorenstein (see [1] and [14]). In [7] ideals generated by pfaffians of fixed size in a ladder are considered, and a characterization is given of the ones defining arithmetically Gorenstein varieties.

The results in [4] play a central role in the argument. It turns out that the arithmetically Gorenstein varieties are essentially only those cut out by pfaffians of fixed size of a skew-symmetric matrix, and a few more cases that are directly connected to those. Notice that combining Proposition 1.8 and the results in [7], one easily obtains a characterization of the arithmetically Gorenstein ladder pfaffian varieties for pfaffians of mixed size, in terms of the upper outside corners of the ladder and of the vector t. The technique of Theorem 6.3.1 in [8] applies to this situation.

Arithmetically Gorenstein schemes are known to be glicci (see Theorem 7.1 of [3]). Notice however that only very special ladder pfaffian varieties are arithmetically Gorenstein. Moreover, the question of whether every glicci scheme belongs to the G-biliaison class of a complete intersection remains open.

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