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# $L$-Series, Modular Imbeddings, and Signatures*

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## Introduction

For each cusp singularity of the Hilbert modular variety of a totally real algebraic number field there is associated an $L$-series which determines the corresponding "parabolic contribution" to the version of Selberg's trace formula worked out by Shimizu.

For a real quadratic field Meyer [9, 10] has given an elementary algorithm by Dedekind sums for the computation of the parabolic contribution (see also Siegel [14]) from which a simple formula for the parabolic contribution in terms of the continued fraction associated with the cyclic resolution of the corresponding cusp singularity can be derived ([5, 7, 8]). In fact, the parabolic contribution equals $\frac{1}{12}(3r-b_0-b_1-\cdots-b_{r-1})$ where $b_0, \ldots, b_{r-1}$ are the characteristic numbers of the dual normal bundles of the rational curves which form the exceptional fibre of the cyclic resolution.

The first purpose of this note is to give a simple formula for the total parabolic contribution, i.e., the sum of the parabolic contributions for the different cusps. This formula is deduced from a general identity satisfied by certain $L$-series in any real algebraic number field. In the quadratic case one obtains an expression for the total parabolic contribution in terms of the class numbers of imaginary quadratic fields. Further, in the quadratic case the total parabolic contribution vanishes if and only if the discriminant of the field is the sum of two squares and is negative otherwise.

The total parabolic contribution is of additional interest in the quadratic case for two reasons: first, it is one-fourth the signature (adjusted for defects coming from quotient singularities in certain special cases) of the open rational homology 4-manifold obtained as the quotient of the product of two upper-half planes by the non-symmetric Hilbert modular group. This result, announced to some extent in [5], will be proved here. Thus, from [2] the vanishing of this (adjusted) signature is a necessary and sufficient condition for the existence of modular imbeddings. Second, it will be shown that this signature is twice the difference of the arithmetic genera for the ordinary and mixed Hilbert modular groups.

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## 1. A Relation between *L*-Series

Let $K$ be a real algebraic number field, i.e., an extension of finite degree of the field of rational numbers which is isomorphic to a subfield of the real field. Let $G$ stand for the group of ideal classes of $K$, and let $Gn$ denote the *norm class group* of $K$: the quotient of the group of fractional ideals by the subgroup of principal ideals generated by elements of positive norm. A norm class character (being any character of $Gn$) is said to be of *norm-signature type* if it induces a non-trivial character on the multiplicative group of $K$. Such characters of $Gn$ exist if and only if $K$ has no unit of norm $-1$. If norm-signature characters exist, then the kernel $Pn$ of the canonical homomorphism $Gn \to G$ is a group of order 2 since $K$ contains elements of negative norm (which follows, as Igusa has pointed out to the authors, from the "approximation theorem" for a finite number of inequivalent valuations – in this case the archimedean valuations of the field $K$), and conversely. In this case let $\theta$ denote the unique non-trivial element of the group $Pn$; then norm-signature characters are those norm class characters satisfying $\chi(\theta) = -1$. Following traditional notation, for any norm class character $\chi$ the symbol $L(s, \chi)$ denotes the $L$-series

$$\Sigma\, \chi(a)\,(Nma)^{-s} \tag{$*$}$$

in which the summation is taken over all non-zero integral ideals $a$ of $K$. Further, for a class $A$ in $G$ the symbol $L(s, \chi, A)$ denotes the series of type ($*$) in which the summation is taken over the integral ideals in $A$. Note that the square of an element of $G$ is a well-defined element of $Gn$; if $A$ is in $G$, $A^{(2)}$ will denote its square in $Gn$. For any character $\chi$ the complex-conjugate character will be denoted by $\chi^*$.

**Lemma 1.1.** *For any ideal class $A$ the expression $\chi^*(A^{(2)})\, L(s, \chi, A^2)$ does not depend on the choice of the norm-signature character $\chi$.*

The proof is obvious; the statement is recorded only for ease of reference.

**Theorem 1.2.** *Let $\chi$ be any character of norm-signature type, and let $\chi_1, \ldots, \chi_r$ denote the real characters of norm-signature type. Then*

$$\sum_{A \in G} \chi^*(A^{(2)})\, L(s, \chi, A^2) = \sum_{j=1}^{r} L(s, \chi_j)\,.$$

*When $r = 0$, the right-hand side is understood to be zero.*

*Proof.* The argument is given in two cases according to whether or not $r = 0$. First suppose $r = 0$. Then $\chi(\theta) = 1$ for every real character $\chi$ of $Gn$. For any finite abelian group the real characters are precisely the characters of order 2. Hence, by the duality theory of finite abelian groups $\theta$ must be an element of the group $Sqn$ of squares of elements of $Gn$. Letting $Sq$ denote the group of squares in $G$, the left-hand side of the formula to be proved may be written

$$\sum_{B\in Sq} L(s,\chi,B) \sum_{AA=B} \chi^*(A^{(2)}) .$$

The inner sum may be re-written as

$$\chi^*(A_0^{(2)}) \sum_{C} \chi^*(C^{(2)}) ,$$

where $A_0$ is a fixed element of $G$ whose square is $B$ and where the summation is taken over the group of elements $C$ of order 2 in $G$. This latter group is mapped surjectively to $Pn$ by the homomorphism $C \mapsto C^{(2)}$ since $\theta$ is in $Sqn$. Hence, the sum is zero, as required.

Next suppose that $r$ is positive. Then one may take $\chi$ real in the left-hand side by the lemma, and the left-hand side becomes

$$r \sum_{B\in Sq} L(s,\chi,B)$$

since $r$ is easily seen to be the order of the group of elements of order 2 in $G$. The right-hand side of the formula to be proved is

$$\sum_{A\in G} \sum_{j=1}^{r} L(s,\chi_j,A),$$

and this sum may be split according to whether or not $A$ lies in the group $Sq$ of squares in $G$. If $A$ is a square, then the fact that $\chi_j$ is real yields $L(s,\chi_j,A)=L(s,\chi,A)$ by the lemma, so that the part of the latter sum involving $A$ in $Sq$ agrees with the left-hand side. It remains to be seen that

$$\sum_{j=1}^{r} L(s,\chi_j,A)=0$$

when $A$ is not a square. This follows from the assertion that

$$\sum_{j=1}^{r} \chi_j(a)=0$$

for every ideal $a$ in $A$. To see this last fact let $\chi_{r+1},\ldots,\chi_{2r}$ be the remaining real characters of $Gn$. Then these are the real characters of the group $G$, and, therefore, the sum of $\chi_j(a)$ as $j$ varies from $r+1$ to $2r$ is zero since $A$ is not a square. The proof is completed by the similar observation that the sum of $\chi_j(a)$ as $j$ varies from 1 to $2r$ is also zero since the norm class of $a$ is also not a square.

*Supplement 1.3.* The number $r$ of real norm-signature characters vanishes if and only if the non-trivial element $\theta$ of the group $Pn$ is the square of an element of $Gn$.

## 2. The Total Parabolic Contribution

In the case where $K$ is a totally real number field of degree $n$ with discriminant $d$ the ideal classes are in bijective correspondence with the cusp singularities of the Hilbert modular variety, and the parabolic contribution, according to Shimizu [13], for a given cusp has the value

$$w_A = (-1)^{n/2} \pi^{-n} d^{\frac{1}{2}} \chi^*(A^{(2)}) \, L(1, \chi, A^2)$$

whenever norm-signature characters $\chi$ exist (and the value zero otherwise). Consequently, in view of Theorem 1, the total parabolic contribution is given by

$$w = (-1)^{n/2} \pi^{-n} d^{\frac{1}{2}} \sum_{j=1}^{r} L(1, \chi_j),$$

which generalizes the formula given by Shimizu for the case where the ideal class group $G$ has odd order (hence $r=1$).

In the quadratic case the real norm-signature characters are given in a very explicit form (see Siegel [14]). Precisely, the real norm-signature characters $\chi$ correspond bijectively to factorizations $d = d_1 d_2$ of the discriminant $d$ into the product of two *negative discriminants* $d_1, d_2$. For such $\chi$

$$L(1, \chi) = 4\pi^2 d^{-\frac{1}{2}} h(d_1)\, h(d_2)\, u(d_1)^{-1} u(d_2)^{-1},$$

where $h(m)$ denotes the ideal class number and $u(m)$ the order of the unit group for the quadratic field of discriminant $m$. Thus,

**Theorem 2.1.** *The total parabolic contribution for the real quadratic field of discriminant $d$ is given by the formula:*

$$w = -4 \Sigma\, h(d_1)\, h(d_2)\, u(d_1)^{-1} u(d_2)^{-1}$$

*with the summation taken over all decompositions $d = d_1 d_2$ in which $d_1$, $d_2$ are the discriminants of imaginary quadratic fields. (The decomposition $d = d_2 d_1$ is identified with $d = d_1 d_2$.)*

**Corollary 2.2.** *The total parabolic contribution for the real quadratic field of discriminant $d$ vanishes if and only if $d$ is the sum of two squares, i.e., $d$ contains no prime* $\equiv 3 \bmod 4$. *Otherwise it is negative.*

*Proof.* In the first place, the norm class group $Gn$ coincides with the *narrow class group*, which is defined for any totally real field as the quotient of the group of fractional ideals by the subgroup of principal ideals generated by totally positive elements. Secondly the different of $K$, which is the principal ideal generated by $d^{\frac{1}{2}}$, defines the non-trivial norm ideal class $\theta$. Thus, from the supplement (1.3), the number $r$ of real norm-signature characters vanishes if and only if the narrow ideal class of the different is a square. By a result of Igusa (see [2]) this is equivalent to the condition that the discriminant $d$ should be the sum of two squares. For the convenience of the reader a special case available from Shimizu [13] and Siegel [14] is stated:

**Corollary 2.3.** *For the field* $Q(\sqrt{p})$, *p prime* $p \equiv 3 \pmod 4$ *one has:*

$$w = \begin{cases} -1/6, & p=3. \\ -\frac{1}{2}h(-p), & p>3. \end{cases}$$

## 3. Signatures

In earlier articles [3, 6] the authors have obtained two expressions for the arithmetic genus $\chi$ of the minimal desingularization of the normal projective surface which compactifies the quotient $H \times H/\Gamma$ of the product $H \times H$ of two copies of the upper-half plane by the Hilbert modular group $\Gamma$ of a real quadratic field $K$. Combining these two expressions enables one to prove quickly the following theorem:

**Theorem 3.1.** *Let d be the square-free part of the discriminant of the real quadratic field K. Then the signature of the rational homology* 4*-manifold* $H \times H/\Gamma$ *is given by*

$$\operatorname{sgn}(H \times H/\Gamma) = \begin{cases} 0, & d<6, \\ 4w - \frac{2}{3}h(-d/3), & d \geqq 6, \quad d \equiv 0 \pmod 3, \\ 4w, & d \geqq 6, \quad d \not\equiv 0 \pmod 3, \end{cases}$$

*where w is the total parabolic contribution and* $h(-d/3)$ *denotes the ideal class number of the field* $Q(\sqrt{-d/3})$.

*Proof.* Denoting by $e(\ )$ the Euler number, one has from [6]

$$\operatorname{sgn}(H \times H/\Gamma) = 4\chi - e(H \times H/\Gamma),$$

where

$$e(H \times H/\Gamma) = 2\zeta_K(-1) + \sum_{(r,q)} a(r,q)\frac{r-1}{r},$$

with $a(r,q)$ denoting the number of quotient singularities of type $(r,q)$, i.e., arising from the cyclic group action on the complex plane with eigenvalues $\lambda$, $\lambda^q$, $(\lambda^r = 1)$. From [3] (see also [1])

$$4\chi = 2\zeta_K(-1) + 4\sum_{(r,q)} a(r,q)\,c(r,q) + 4w,$$

where $c(r,q)$ is the contribution to Shimizu's formula for a quotient singularity of type $(r,q)$. From Shimizu [13] one has

$$c(r,q) = r^{-1}\,\Sigma\, g_q(\lambda),$$

where the summation is extended over the non-trivial $r^{\text{th}}$ roots of unity $\lambda$: $\lambda^r = 1$, $\lambda \neq 1$, and $g_q$ denotes the rational function of one variable

$$g_q(z) = (1-z)^{-1}(1-z^q)^{-1}.$$

The $c(r, q)$ are very close to Dedekind sums and to the signature defects $\operatorname{def}(r; q, 1)$ introduced in [4], formula (27). We have

$$\operatorname{def}(r; q, 1) = \Sigma(1+\lambda)(1+\lambda^q)\, g_q(\lambda)$$

where the summation is extended as before. The following trivial formula holds:

$$4\Sigma\, g_q(\lambda) = r - 1 + \operatorname{def}(r; q, 1).$$

Therefore we get

**Theorem 3.2.** $\operatorname{sgn}(H \times H/\Gamma) = 4w + \sum_{(r,q)} a(r, q)\, r^{-1} \operatorname{def}(r; q, 1)$.

The calculation of signature defects can be done by the following classical formulas ([12], compare also [4], p. 16–17): For any real $x$ let $[x]$ denote the greatest integer not larger than $x$, and let $((x))$ be defined by

$$((x)) = \begin{cases} 0, & \text{if } x \text{ is an integer} \\ x - [x] - \frac{1}{2}, & \text{otherwise}. \end{cases}$$

For $r, q$ coprime integers with $r$ positive let

$$s(q, r) = \sum_{k=1}^{r-1} \left(\left(\frac{k}{r}\right)\right)\left(\left(\frac{kq}{r}\right)\right); \tag{3.1.1}$$

then

$$\operatorname{def}(r; q, 1) = -4r\, s(q, r). \tag{3.1.2}$$

To finish the proof of (3.1) it is sufficient to apply the formulae (3.1.1) and (3.1.2) together with the results of Prestel [11] on the computation of the $a(r, q)$. When $d < 6$, one must also compute $w$ using, for example, (2.1). (It vanishes for $d = 2$ and $d = 5$.) For $d \geqq 6$ the only values of $r$ are 2 and 3. One may forget the case $r = 2$ since $s(1, 2) = 0$, while for $r = 3$ one has $s(1, 3) = -s(2, 3) = 1/18$. From Prestel

$$a(3, 1) - a(3, 2) = \begin{cases} 3h(-d/3), & d \equiv 0 \pmod 3. \\ 0, & d \not\equiv 0 \pmod 3. \end{cases}$$

As a consequence of (3.1) and (2.3) one has immediately:

**Corollary 3.3.** *If $d = p$, a prime, with $p \equiv 3 \pmod 4$ and $p > 3$, then*

$$\operatorname{sgn}(H \times H/\Gamma) = -2h(-p).$$

## 4. Application to the Mixed Hilbert Modular Group

The Hilbert modular group $\Gamma$ of a real quadratic field $K$ acts also on the product $H \times H'$ of the upper-half plane and the lower-half plane.

The rational homology manifolds $H \times H'/\Gamma$ and $H \times H/\Gamma$ are homeomorphic with opposite orientation, and, therefore,

$$e(H \times H'/\Gamma) = e(H \times H/\Gamma)$$

and

$$\operatorname{sgn}(H \times H'/\Gamma) = -\operatorname{sgn}(H \times H/\Gamma).$$

Denoting by $\chi'$ the arithmetic genus of a smooth model for the field of modular functions associated to $H \times H'/\Gamma$, one has as in Section 3:

$$4\chi' = e(H \times H'/\Gamma) + \operatorname{sgn}(H \times H'/\Gamma).$$

If $n$ and $n'$ denote the dimensions of the spaces of cusp forms of lowest weight, then one knows from Freitag [1] that $n = \chi - 1$ and $n' = \chi' - 1$. Thus,

**Theorem 4.1.** *The difference of the arithmetic genera or, equivalently, the difference of the dimensions of the spaces of cusp forms is given by the formula:*

$$\chi' - \chi = n' - n = -\tfrac{1}{2}\operatorname{sgn}(H \times H/\Gamma) \geqq 0.$$

*Moreover, this difference vanishes if and only if $d = 3$ or $d$ is not divisible by a prime $\equiv 3 \bmod 4$.*

Finally, one has the special case:

**Corollary 4.2.** *If $d = p$, a prime, with $p \equiv 3 \pmod 4$ and $p > 3$, then*

$$\chi' - \chi = n' - n = h(-p).$$

## References


1. Freitag, E.: Lokale und globale Invarianten der Hilbertschen Modulgruppe. Invent. Math. **17**, 106—134 (1972).
2. Hammond, W. F.: The modular groups of Hilbert and Siegel. Amer. J. Math. **88**, 497—516 (1966).
3. Hammond, W. F.: The Hilbert modular surface of a real quadratic field. Math. Ann. **200**, 25—45 (1973).
4. Hirzebruch, F.: The signature theorem: reminiscences and recreation. Prospects in Mathematics, Ann. Math. Stud., no. 70, Princeton, 1971.
5. Hirzebruch, F.: The Hilbert modular group, resolution of the singularities at the cusps and related problems. Séminaire Bourbaki, exp. 396 (1971).
6. Hirzebruch, F.: The Hilbert modular group and some algebraic surfaces, International Symposium in Number Theory, Moscow, 1971 (to appear).
7. Hirzebruch, F.: Hilbert modular surfaces. IMU-lectures, Tokyo, 1972 (to appear in L'Enseignement Mathématique).
8. Hirzebruch, F., Zagier, D.: Class numbers, continued fractions and the Hilbert modular group (in preparation).
9. Meyer, C.: Die Berechnung der Klassenzahl abelscher Körper über quadratischen Zahlkörpern. Berlin 1957.
10. Meyer, C.: Über die Bildung von elementar-arithmetischen Klasseninvarianten in reell-quadratischen Zahlkörpern. Algebraische Zahlentheorie, Oberwolfach, pp. 165—215. Mannheim: Bibliogr. Institut 1966.

11. Prestel, A.: Die elliptischen Fixpunkte der Hilbertschen Modulgruppen. Math. Ann. **177**, 181—209 (1968).
12. Rademacher, H., Grosswald, E.: Dedekind Sums. Math. Assoc. of America, 1972.
13. Shimizu, H.: On discontinuous groups operating on the product of the upper half planes. Ann. of Math. **77**, 33—71 (1963).
14. Siegel, C. L.: Lectures on advanced analytic number theory. Tata Inst., Bombay, 1961 (re-issued 1965).

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