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One-loop β-functions in 4-derivative gauge theory in 6 dimensions

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Abstract

A classically scale-invariant 6d analog of the 4d Yang-Mills theory is the 4-derivative $(\nabla F)^2 + F^3$ gauge theory with two independent couplings. Motivated by a search for a perturbatively conformal but possibly non-unitary 6d models we compute the one-loop β-functions in this theory. A systematic way of doing this using the background field method requires the (previously unknown) expression for the b_6 Seeley-DeWitt coefficient for a generic 4-derivative operator; we derive it here. As an application, we also compute the one-loop β-function in the (1,0) supersymmetric $(\nabla F)^2$ 6d gauge theory constructed in hep-th/0505082.

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Contents

1 Introduction

Like Einstein theory in 4 dimensions, the 6d Yang-Mills theory with the standard F^2 action has dimensional coupling and is not power-counting renormalizable. A 6d analog of the classically scale invariant and renormalizable $R^2 + C^2$ 4d gravity is the 4-derivative $(\nabla F)^2 + F^3$ gauge theory. Such 4-derivative terms are induced as counterterms when considering the standard scalars, fermions or YM vectors coupled to a background gauge field in 6d [1]. While non-unitary, this model may serve as a building block of possible higher-derivative (super)conformal theories in 6 dimensions.[1] Similar 4-derivative 6d gauge theories were discussed, e.g., in [5, 6, 7, 8, 9, 10, 11, 12, 13].

The aim of the present paper is to compute the one-loop β-functions in the Euclidean 6d theory with the action[2]

$$\begin{aligned} S &= -\frac{1}{g^2}\int d^6x\ \mathrm{Tr}\Big[(\nabla_m F_{mn})^2 + 2\gamma F_{mn}F_{nk}F_{km}\Big] \\ &= \frac{1}{2g^2}\int d^6x\Big[(\nabla_m F^a_{mn})^2 + \gamma f^{abc}F^a_{mn}F^b_{nk}F^c_{km}\Big]\ . \end{aligned} \tag{1.1}$$

Here g and γ are the two independent dimensionless coupling parameters.[3]

In general, the UV logarithmically divergent part of the 6d one-loop effective action Γ_1 in a gauge field background may be written as[4]

$$\Gamma_{1\infty} = -\frac{\log\Lambda}{(4\pi)^3}\int d^6x\ \mathrm{tr}\Big[-\frac{1}{60}\beta_2\,(\nabla_m F_{mn})^2 + \frac{1}{90}\beta_3 F_{mn}F_{nk}F_{km}\Big]\ , \tag{1.2}$$

[1] In 4 dimensions the $F^2 + (\nabla F)^2 + F^3$ theory was studied in [2] and later in [3]. The result of [2] for the one-loop divergences in this 4d theory was corrected in [4] making it in agreement with that of [3].

[2] We use $m, n, k, ... = 1, ..., 6$ for coordinate indices and flat Euclidean 6d metric so that the position of contracted indices is irrelevant. The gauge group generators are normalized as $\mathrm{tr}(t^a t^b) = -T_R\delta^{ab}$, $[t^a, t^b] = f^{abc}t^c$, where $T_R = \frac{1}{2}$ in the fundamental representation of $SU(N)$ (we denote the trace in this case as Tr) and $T_R = C_2 = N$ in the adjoint representation.

[3] Two other possible 4-derivative $\nabla F \nabla F$ invariants are related to the above two by the Bianchi identity, e.g., $F_{mn}\nabla^2 F_{mn} = -2\,(\nabla_m F_{mn})^2 + 4F_{mn}F_{nk}F_{km} +$ total derivative.

[4] Here tr is the trace over the matrix indices of a particular representation to which the quantum field belongs; for example, in the gauge theory case it is in the adjoint representation $A^{ab}_m = f^{acb}A^c_m$, $f_{acd}f_{bcd} = C_2\delta_{ab}$.

where the 1-loop β-function coefficients β_2, β_3 depend on the field content of the theory. As we shall find below, their values in the case of the 4-derivative theory (1.1) are given by the following functions of the coupling γ (1-loop coefficients do not depend on the overall g^2 coupling)

$$\beta_{2A} = 249 \ , \qquad\qquad \beta_{3A} = 9 - 900\gamma + \frac{405}{2}\gamma^3 \ . \tag{1.3}$$

Somewhat surprisingly, the coefficient β_{2A} of the $(\nabla F)^2$ divergence turns out to be independent of the coupling γ.

The total values of β_2, β_3 in a 6d renormalizable model containing the gauge theory (1.1) minimally coupled to the ordinary-derivative "matter" fields – N_0 real scalars, $N_{\frac{1}{2}}$ Weyl fermions, N_1 YM vectors and N_T self-dual tensors (interacting with A_m as in [12]) are then [1, 12][5]

$$\begin{aligned} \beta_2 &= \beta_{2A} \ - 27\, N_T - 36\, N_1 + N_0 + 16\, N_{\frac{1}{2}} \ , \\ \beta_3 &= \beta_{3A} \ - 57\, N_T + 4\, N_1 + N_0 - 4\, N_{\frac{1}{2}} \ . \end{aligned} \tag{1.4}$$

Note that for the ordinary spin 0, 1/2, 1 fields their contributions to β_3 are proportional to the number of dynamical degrees of freedom. The same is true also for the 4-derivative gauge theory (1.1) with $\gamma = 0$: $\beta_{3A} = 9$ is the number of d.o.f. of a 4-derivative gauge vector in 6d.[6] As a consequence one should get $\beta_3 = 0$ in a supersymmetric theory; this is consistent with the non-existence of a super-invariant containing $\mathrm{tr}(F_{mn}F_{nk}F_{km})$. Indeed, for the standard 2-derivative 6d (1,0) SYM theory ($N_1 = 1,\ N_{\frac{1}{2}} = 1$) and for the scalar (hyper) multiplet ($N_0 = 4,\ N_{\frac{1}{2}} = 1$) one finds

$$\beta_{2\,(1,0)\,\mathrm{SYM}} = -20 \ , \qquad \beta_{2\,\mathrm{scal}} = 20 \ , \qquad \beta_{3\,(1,0)\,\mathrm{SYM}} = \beta_{3\,\mathrm{scal}} = 0 \ . \tag{1.5}$$

Since $\nabla_m F_{mn} = 0$ on the standard YM equations of motion the (1,0) SYM theory is 1-loop finite on shell. The sum of the contributions of the two multiplets in (1.5) corresponds to the (1,1) SYM theory in 6d (and thus to $N = 4$ SYM in 4d) which is 1-loop finite even off-shell [1]

$$\beta_{2\,(1,1)\,\mathrm{SYM}} = \beta_{3\,(1,1)\,\mathrm{SYM}} = 0 \ . \tag{1.6}$$

In the $(1,0)$ supersymmetric 4-derivative gauge theory with the action given by the super-extension [5] of $\mathrm{tr}\,(\nabla_m F_{mn})^2$ (containing also interacting $\nabla\!\!\!/^3$ Weyl fermion and three ∇^2 scalars) we will find below that

$$\beta_{2\,(1,0)} = 220 \ , \qquad \beta_{3\,(1,0)} = 0 \, . \tag{1.7}$$

This result is in agreement (modulo notation change) with the one given in the recently revised version of [5]. This theory is non-unitary and is also formally inconsistent having a chiral anomaly [6] (the same as in the (1,0) 6d SYM theory containing Weyl fermion). One may still hope to cancel all of its anomalies by adding some higher derivative 6d "matter" multiplets (cf. [14, 15, 16]).

The calculation of the β-functions (1.3) is most straightforward in the background field method and using the heat kernel expansion to extract the log divergences of the determinants. This requires the knowledge of the corresponding b_6 Seeley-DeWitt coefficient for the

[5] Here all the fields are taken for simplicity in the adjoint representation; in the case of other representations one is to rescale the numbers N_s by T_R/C_2. We corrected misprints in [1] mentioned in [12]. Note that the vector N_1 terms here are formal: they indicate the 6d YM contribution in the absence of higher-derivative terms in (1.1). In the combined $F^2 + (\nabla F)^2 + F^3$ theory discussed below in Appendix B the values of β_2 and β_3 are the same as in the theory (1.1) without the YM F^2 term.

[6] While the 2-derivative YM vector in d dimensions has $\frac{1}{2}2(d-1) - 1 = d - 2$ dynamical d.o.f., for the 4-derivative gauge vector in (1.1) one finds $\frac{1}{2}4(d-1) - 1 = 2d - 3$, i.e. 5 in $d = 4$ and 9 in $d = 6$.

4-derivative operator $\Delta_4 = \nabla^4 + ...$ in a gauge field background. While b_6 is available for the 2-derivative Δ_2 operators [17], its expression for Δ_4 was not known so far. The main new technical result of this paper is the computation of $b_6(\Delta_4)$. We shall use the same strategy as employed previously in [2] to obtain $b_4(\Delta_4)$ from the known expression for $b_4(\Delta_2)$ by considering special factorized cases of the operator Δ_4.

The rest of the paper is organized as follows. In section 2 we present the general form of the one-loop effective action of the theory (1.1). In section 3 the result for the heat kernel coefficient b_6 that controls the logarithmic divergence of the determinant of a generic 4-derivative operator is given. In section 4 this expression is applied to compute the one-loop divergences in the bosonic gauge theory (1.1) and its (1,0) supersymmetric extension (with $\gamma = 0$). Details of the derivation of $b_6(\Delta_4)$ are described in Appendix A. In Appendix B we discuss divergences of the combined 2- and 4-derivative $\frac{1}{g^2}[\kappa^2 F^2 + (\nabla F)^2 + \gamma F^3]$ gauge theory and its (1,0) supersymmetric version: adding F^2 does not change the β-functions (1.3) for g and γ but leads to the γ-dependent β-function for κ.

2 One-loop effective action

The derivation of the one-loop effective action in the 4-derivative theory (1.1) in 6d follows the same steps as in the 4d case discussed in Appendix C of [2] (for a review, see also [4]). Expanding the invariants in (1.1) near a classical background $A^a_m \to A^a_m + \tilde{A}^a_m$ we get

$$\begin{aligned}\mathrm{Tr}\,(\nabla_m F_{mn})^2 \to -\frac{1}{2}\tilde{A}^a_m \Big[\delta_{mn}\nabla^4 + 4F_{mn}\nabla^2 - 2\Big(\nabla_k F_{km}\,\delta_{nr} + 2\nabla_k F_{k[n}\,\delta_{r]m}\Big)\nabla_r \\ - 2\nabla_n\nabla_k F_{km} + 4F_{mk}F_{kn}\Big]^{ab}\tilde{A}^b_n - \frac{1}{2}(\nabla_m\tilde{A}^a_m)\nabla^2(\nabla_n\tilde{A}^a_n)\ , \quad (2.1)\\ \mathrm{Tr}(F_{mn}F_{nk}F_{km}) \to \tilde{A}^a_m \Big[\Big(\frac{3}{2}F_{[m}{}^{(r}\delta^{k)}_{n]} - \frac{3}{4}F_{mn}\delta^{rk}\Big)\nabla_r\nabla_k + 3\nabla_k F_{[m}{}^{[r}\delta^{n]}_{k]}\nabla_r \\ - \Big(\frac{3}{4}[F_{mk},F_{kn}] + \frac{3}{4}F_{r(m}F_{n)r} + \frac{3}{8}F_{rk}F_{rk}\delta_{mn}\Big)\Big]^{ab}\tilde{A}^b_n\ , \quad (2.2)\end{aligned}$$

where F_{mn} and ∇_m depend on the background A_m and a, b are indices in the adjoint representation. Then the quadratic part of the fluctuation Lagrangian in (1.1) may be written as

$$\mathcal{L}^{(2)} = \frac{1}{2g^2}\tilde{A}^a_m\,(\Delta_{4A})^{ab}_{mn}\,\tilde{A}^b_n + \frac{1}{2g^2}(\nabla_m\tilde{A}^a_m)(-\nabla^2)(\nabla_n\tilde{A}^a_n)\ . \quad (2.3)$$

The second term here can be cancelled by adding a gauge-fixing ($\nabla_m\tilde{A}_m = f(x)$) term averaged with the operator $-\nabla^2$. The 4-derivative operator Δ_{4A} acting on $\tilde{A}^a_m$ can be written in the following "symmetric" form

$$\Delta_4 = \nabla^4 + \nabla_r\hat{V}_{rk}\nabla_k + \hat{N}_k\nabla_k + \nabla_k\hat{N}_k + \hat{U}, \qquad \hat{V}_{rk} = \hat{V}_{kr}\ , \quad (2.4)$$

where $\hat{V}_{rk}$, $\hat{N}_k$, $\hat{U}$ are local covariant matrices in the internal $(a,m),(b,n)$ indices reading

$$\begin{aligned}(\hat{V}_{rk})_{mn} &= (4+3\gamma)F_{mn}\delta^{rk} - 6\gamma F_{[m}{}^{(r}\delta^{k)}_{n]}\ ,\\ (\hat{N}_k)_{mn} &= \frac{1}{2}(2+3\gamma)\nabla_r F_{rk}\delta_{mn} - \frac{1}{2}(4+3\gamma)\nabla_r F_{r(m}\delta_{n)k} - \frac{3}{2}\gamma\nabla_{(m}F_{n)k}\ , \quad (2.5)\\ (\hat{U})_{mn} &= -\frac{1}{2}(4+3\gamma)F_{kn}F_{mk} + \frac{3}{2}(4+3\gamma)F_{km}F_{nk} + \frac{3}{2}\gamma F_{rk}F_{rk}\delta_{mn} + 3\nabla^2 F_{mn}\ .\end{aligned}$$

The operator that appears in the effective action after path-integral is performed (i.e. Δ_{4A} in (2.3)) should be self-adjoint and this is so for (2.4) with (2.5).[7]

The 1-loop effective action is then given by

$$\Gamma_1 = \frac{1}{2}\log\frac{\det\Delta_{4A}}{(\det\Delta_{\rm gh})^2\,\det H} = \frac{1}{2}\log\det\Delta_{4A} - \frac{3}{2}\log\det\Delta_{2,0}\ , \qquad \Delta_{2,0} = -\nabla^2\ , \tag{2.6}$$

where $\Delta_{\rm gh} = -\nabla^2$ is the ghost operator and $H = -\nabla^2$ is the gauge-condition averaging operator required to cancel the last term in (2.3). Using the proper-time cutoff, the log divergent part of a determinant can be expressed (in general dimension d) in terms of the corresponding Seeley-DeWitt coefficient B_d[8]

$$\Gamma_{1\infty}(\Delta) = \frac{1}{2}(\log\det\Delta)_\infty = -\frac{\log\Lambda}{(4\pi)^{d/2}}\,B_d(\Delta)\ , \qquad B_d = \int d^dx\, b_d(\Delta)\ . \tag{2.7}$$

The values of b_p for 2-derivative Laplacian Δ_2 (in general curved space and gauge field background) are known up to $p = 10$ (see, e.g., [17, 18]) while for the 4-derivative operator Δ_4 only b_2 and b_4 were found so far [19, 2, 20, 21]. Thus to compute the divergent part of (2.6) we need first to determine the coefficient b_6 for Δ_4 in (2.4). This will be the subject of the next section and Appendix A.

3 Heat kernel coefficient $b_6(\Delta_4)$

In general, given an elliptic differential operator Δ_ℓ of an even order ℓ in d dimensions one has

$$\log\det\Delta_\ell = -\int d^dx \int_\varepsilon^\infty \frac{dt}{t}\ {\rm tr}\,\langle x|e^{-t\Delta_\ell}|x\rangle\ , \tag{3.1}$$

where tr is the trace over internal indices of the operator. The heat kernel has an asymptotic expansion for $t \to 0$ so that (see, e.g., [20, 18, 21])

$${\rm tr}\,\langle x|e^{-t\Delta_\ell}|x\rangle \equiv {\rm tr}\,K(t;x,x;\Delta_\ell) \simeq \sum_{p\geq 0}\frac{2}{(4\pi)^{d/2}\,\ell}\ t^{(p-d)/\ell}\ b_p(\Delta_\ell)\ . \tag{3.2}$$

The Seeley-DeWitt coefficients b_p are local invariant expressions of dimension p constructed out of the background metric and gauge field (below we shall consider them up to total derivative terms). Using the proper-time cutoff $\varepsilon = \Lambda^{-\ell}$ we obtain for the divergent part of (3.1)[9]

$$\begin{aligned}(\log\det\Delta_\ell)_\infty &= -\frac{2}{(4\pi)^{d/2}}\Big[\sum_{p=0}^{d-1}\frac{B_p(\Delta_\ell)}{d-p}\Lambda^{d-p} + B_d(\Delta_\ell)\log\frac{\Lambda}{\mu}\Big],\\ B_p(\Delta_\ell) &= \int d^dx\ b_p(\Delta_\ell)\ .\end{aligned} \tag{3.3}$$

The renormalization scale μ in log will be sometimes left implicit below. For example, for the 2-derivative operator defined on a vector bundle with the covariant derivative ∇_m and the

[7] Note that (2.4) is a completely general form for a fourth-order elliptic differential operator without the three-derivative term. The self-adjointness can be imposed via the following additional conditions on the coefficients $\hat V^\dagger_{mn} = \hat V_{mn}$, $\hat N^\dagger_m = -\hat N_m$, $\hat U^\dagger = \hat U$ where † is transposition if the field is real, and hermitian conjugation if the field is complex.

[8] Here we ignore boundary terms. Note also that in the dimensional regularization one is to replace $\log\Lambda \to -\frac{1}{{\rm d}-d}$ where d is integer and ${\rm d} < d$ is its analytic continuation.

[9] Note that the form of (3.3) is universal for any order ℓ of the differential operator – that is the reason for the above normalization of the Seeley-DeWitt coefficients.

curvature $F_{mn} = [\nabla_m, \nabla_n]$ one has[10]

$$\Delta_2 = -\nabla^2 + X, \tag{3.4}$$

$$b_6(\Delta_2) = \mathrm{tr}\Big[- \frac{1}{60}\,(\nabla_m F_{mn})^2 + \frac{1}{90} F_{mn}F_{nk}F_{km} - \frac{1}{12} X F_{mn}F_{mn} + \frac{1}{12} X\nabla^2 X - \frac{1}{6} X^3\Big]. \tag{3.5}$$

To find $b_6(\Delta_4)$ for the operator in (2.4) we will use the same idea as in [2] and consider several special cases of factorized operators Δ_4 for which

$$\Delta_4 = \Delta_2\,\Delta_2'\ , \qquad \det \Delta_4 = \det \Delta_2\ \det \Delta_2'\ , \qquad b_p(\Delta_2\,\Delta_2') = b_p(\Delta_2) + b_p(\Delta_2')\ . \tag{3.6}$$

The 4-derivative operator that we are interested in is given in (2.4). As explained in Appendix A, a general expression for its b_6 coefficient is ($\hat V \equiv \hat V_{mm}$)

$$\begin{aligned} b_6(\Delta_4) = \mathrm{tr}\Big[&\hat k_1 (\nabla_m F_{mn})^2 + \hat k_2 F_{mn}F_{nk}F_{km} \\ &+ \hat k_3 \hat V_{mn}\hat V_{nk}\hat V_{km} + \hat k_4 \hat V_{mn}\hat V_{mn}\hat V + \hat k_5 \hat V\hat V\hat V + \hat k_6 \hat V_{mn}\nabla_{(n}\nabla_{k)}\hat V_{km} \\ &+ \hat k_7 \hat V_{mn}\nabla^2\hat V_{mn} + \hat k_8 \hat V_{mn}\nabla_m\nabla_n\hat V + \hat k_9 \hat V\nabla^2\hat V + \hat k_{10}\hat V_{mn}\hat V_{nk}F_{mk} \\ &+ \hat k_{11} F_{mn}\nabla_{(m}\nabla_{k)}\hat V_{kn} + \hat k_{12}\hat V F_{mn}F_{mn} + \hat k_{13}\hat V_{mn}F_{mk}F_{nk} \\ &+ \hat k_{14} F_{mn}\nabla_m \hat N_n + \hat k_{15}\hat V_{mn}\nabla_m\hat N_n + \hat k_{16}\hat V\nabla_m\hat N_m + \hat k_{17}\hat N_m\hat N_m + \hat k_{18}\hat U\hat V\Big]\ . \end{aligned} \tag{3.7}$$

In contrast to what happens in the case of Δ_2 in (3.5), some of the coefficients in (3.7) in general depend on the number of dimensions d. In the case of $d = 6$ we are interested in here one finds

$$\begin{aligned} &\hat k_1 = -\frac{1}{30}, \quad \hat k_2 = \frac{1}{45}, \quad \hat k_3 = \frac{1}{360}, \quad \hat k_4 = \frac{1}{480}, \quad \hat k_5 = \frac{1}{2880}, \quad \hat k_6 = -\frac{1}{120}, \\ &\hat k_7 = \frac{1}{120}, \quad \hat k_8 = \frac{1}{60}, \quad \hat k_9 = \frac{1}{240}, \quad \hat k_{10} = -\frac{1}{24}, \quad \hat k_{11} = 0, \quad \hat k_{12} = \frac{1}{24}, \\ &\hat k_{13} = -\frac{1}{6}, \quad \hat k_{14} = -\frac{1}{3}, \quad \hat k_{15} = 0, \quad \hat k_{16} = 0, \quad \hat k_{17} = -\frac{1}{6}, \quad \hat k_{18} = -\frac{1}{12}. \end{aligned} \tag{3.8}$$

4 Divergences of 4-derivative 6d gauge theories

Let us now apply the above general expression (3.7), (3.8) for $b_6(\Delta_4)$ to the gauge theories of interest.

4.1 Bosonic theory

Starting with the explicit form of the coefficient functions (2.4), (2.5) in the operator Δ_{4A} and applying (3.7), (3.8) as well as (3.5), we can compute the coefficient b_6 in the divergent part of the effective action (2.6), (2.7) of the 4-derivative bosonic 6d gauge theory (1.1)[11]

$$b_6 = b_6(\Delta_{4A}) - 3 b_6(\Delta_{2,0})\ , \tag{4.1}$$

$$b_6(\Delta_{4A}) = \mathrm{tr}\Big[- \frac{21}{5}\,(\nabla_m F_{mn})^2 + \Big(\frac{2}{15} - 10\gamma + \frac{9}{4}\gamma^3\Big) F_{mn}F_{nk}F_{km}\Big]\ , \tag{4.2}$$

[10] Here we will somewhat abuse the notation and adopt the same labels for the connection, covariant derivative and its curvature of the vector bundle as in the gauge theory (A_m, ∇_m, F_{mn}) with an implicit understanding that the connection in the differential operators Δ_ℓ may take more general values that in a particular representation of a gauge group.

[11] In applying (3.7) to the gauge field case, the trace there is acting on the full internal index structure of the operator Δ_{4A}, i.e. involving both spacetime and gauge indices (cf. footnote 10).

$$b_6(\Delta_{2,0}) = \mathrm{tr}\Big[-\frac{1}{60}(\nabla_m F_{mn})^2 + \frac{1}{90} F_{mn}F_{nk}F_{km}\Big] \ . \tag{4.3}$$

Thus finally

$$b_6 = \mathrm{tr}\Big[-\frac{83}{20}(\nabla_m F_{mn})^2 + \Big(\frac{1}{10} - 10\gamma + \frac{9}{4}\gamma^3\Big) F_{mn}F_{nk}F_{km}\Big] \ . \tag{4.4}$$

Comparing to (1.2) we end up with the values of the one-loop β-function coefficients β_{2A}, β_{3A} quoted in (1.3). It is remarkable that the divergence proportional to $(\nabla F)^2$ turned out to be independent of the parameter γ: various terms in b_6 in (3.7) generically do give γ-dependent $(\nabla F)^2$ contributions and they cancel out only when combined together weighted with the $\hat{k}_i$ coefficients in (3.8).

The corresponding RG equations for the renormalized couplings $g(\mu)$ and $\gamma(\mu)$ in (1.1) may be written as ($t = \frac{1}{(4\pi)^3} \log \mu^2$, $C_2(\mathrm{SU}(N)) = N$)

$$\frac{dg^{-2}}{dt} = \beta_{2A} C \ , \qquad \frac{d\gamma}{dt} = \beta_\gamma C g^2 \ , \qquad C \equiv \frac{1}{60} C_2 \ , \tag{4.5}$$

$$\beta_{2A} = 249 \ , \qquad \beta_\gamma = -\gamma\beta_{2A} - \frac{1}{3}\beta_{3A} = \frac{3}{2}(-2 + 34\gamma - 45\gamma^3) \ . \tag{4.6}$$

The flow of g is independent of the parameter γ and the sign of β_{2A} corresponds to asymptotic freedom. The fixed points of the flow of γ are the solutions of $\beta_\gamma = 0$, i.e. $\gamma_1 \simeq -0.897$, $\gamma_2 \simeq 0.059$, $\gamma_3 \simeq 0.838$. Since $\beta_\gamma > 0$ for $\gamma < \gamma_1$ or $\gamma_2 < \gamma < \gamma_3$, we have that γ_1 and γ_3 are attractive fixed points of the flow. As the sign of the F^3 term in (1.1) is not a priori constrained by the requirement of positivity of the Euclidean action we formally define a second coupling $h^2 = \gamma^{-1} g^2$ that may assume positive as well as negative values. Then near the fixed points h^2 also goes to zero in the UV, i.e. like g^2 the second coupling is also asymptotically free.

In Appendix B we shall present also the one-loop β-functions for the combined YM plus 4-derivative gauge theory with $\mathcal{L} = \frac{1}{g^2}\big[\kappa^2 F^2 + (\nabla F)^2 + \gamma F^3\big]$.

4.2 (1,0) supersymmetric theory

Let us now consider the 6d supersymmetric version of the theory (1.1) constructed in [5]. In this case $\gamma = 0$ since, in general, there is no supersymmetric extension of the F^3 term.[12] The field content includes the 4-derivative gauge field A_m, the 3-derivative 6d Weyl spinor Ψ, and the three 2-derivative real scalars Φ_I ($I = 1, 2, 3$).[13] In total, one has $9 + 3$ bosonic and 3×4 fermionic on-shell degrees of freedom (for each value of the internal index).

Using an off-shell harmonic superspace formulation ref. [5] found the following (1,0) supersymmetric 6d action[14]

$$\begin{aligned} S_{(1,0)} = -\frac{1}{g^2}\int d^6x \ \mathrm{Tr}\Big[&(\nabla_m F_{mn})^2 - i\bar\Psi \nabla\!\!\!\!/ \nabla^2 \Psi - (\nabla_m \Phi_I)^2 \\ &- \frac{i}{2}\bar\Psi \Gamma_k \Gamma_{mn} \nabla_k [F_{mn}, \Psi] + 2i \nabla_m F_{mn} \bar\Psi \Gamma_n \Psi + \mathcal{O}(\Phi\Psi^2, \Phi^3)\Big] \ . \end{aligned} \tag{4.7}$$

[12] This can be easily understood using, e.g., the standard $N = 1$ 4d superspace formulation: the YM field strength F_{mn} is part of the spinor superfield strength W_α and thus constructing an invariant cubic in W_α is not possible.

[13] In the case of the standard (1,0) SYM theory (corresponding to $N = 2$ SYM theory in 4d) the latter correspond to the auxiliary scalars.

[14] Our notation differ significantly from that of [5] (where, e.g., the scalar kinetic term is defined using ϵ^{ij} to raise the indices and thus implicitly is negative definite). Here, the Dirac matrices Γ_m are 8×8 hermitian complex matrices satisfying $\Gamma_{(m}\Gamma_{n)} = \frac{1}{2}\{\Gamma_m, \Gamma_n\} = \delta_{mn}$ and $\Gamma_{mn} \equiv \Gamma_{[m}\Gamma_{n]}$.

We suppressed interactions that are more than second order in the scalars and fermions, as they will not contribute to the one-loop divergences in a gauge-field background. Note that with our definition of the coupling constant g (i.e. the choice of the overall sign of the action) the gauge field term in (4.7) is positive definite (cf. (1.1)) but the scalar term is not, and this is one indication of the non-unitarity of the theory.[15]

The 4-derivative operator for the fluctuations of the gauge field is given by (2.4), (2.5) with $\gamma = 0$, i.e. it is $\Delta^{(0)}_{4A} \equiv \Delta_{4A}\big|_{\gamma=0}$, while the 3-derivative fermion and the 2-derivative scalar operators in gauge field background may be written as[16]

$$\begin{aligned} &\Delta_{3\Psi} = i\nabla\!\!\!/ \nabla^2 + \frac{i}{2}\nabla\!\!\!/ \Gamma_{mn}F_{mn} + i\Gamma_n(\nabla_m F_{mn}) = i\nabla\!\!\!/^3 + i\Gamma_n(\nabla_m F_{mn}) \ , \\ &\Delta_{2\Phi} = -\nabla^2 = \Delta_{2,0} \ . \end{aligned} \tag{4.8}$$

Here $i\nabla\!\!\!/^3$ is the cube of the Dirac operator $\Delta_{1\Psi} = -i\nabla\!\!\!/ = -i\Gamma^m\nabla_m$ whose square is

$$\Delta_{2\Psi} = -\nabla\!\!\!/^2 = -\nabla^2 - \frac{1}{2}\Gamma_{mn}F_{mn} \ . \tag{4.9}$$

As a result, the one-loop effective action of the supersymmetric theory (4.7) is the following generalization of the bosonic case (2.6)

$$\Gamma_{1\,(1,0)} = \frac{1}{2}\log\frac{\det\Delta^{(0)}_{4A}\,\big[\det\Delta_{2\Phi}\big]^3}{\big[\det\Delta_{2,0}\big]^3\,\det\Delta_{3\Psi}} = \frac{1}{2}\log\det\Delta^{(0)}_{4A} - \frac{1}{2}\log\det\Delta_{3\Psi} \ . \tag{4.10}$$

Here the contributions of the ghost and gauge-averaging operators in (2.6) got canceled against the contribution of the three scalars Φ_I. We also used that $\det\Delta_\Psi$ is defined for the Dirac 6d spinors so that the factor $\frac{1}{2}$ accounts for the fact that the fermion Ψ is a Weyl spinor. As a result, the coefficient of the log divergent part of the effective action (2.7) is given by (cf. (4.1))

$$b_{6\,(1,0)} = b_6(\Delta^{(0)}_{4A}) - b_6(\Delta_{3\Psi}) \ . \tag{4.11}$$

Setting $\gamma = 0$ in (4.2) gives

$$b_6(\Delta^{(0)}_{4A}) = \mathrm{tr}\Big[-\frac{21}{5}\,(\nabla_m F_{mn})^2 + \frac{2}{15}F_{mn}F_{nk}F_{km}\Big] \ . \tag{4.12}$$

To compute the fermionic contribution, let us first construct a 4-derivative operator by taking the product of $\Delta_{3\Psi}$ in (4.8) with the standard Dirac operator $\Delta_{1\Psi} = -i\nabla\!\!\!/$

$$\Delta_{4\Psi} \equiv \Delta_{1\Psi}\,\Delta_{3\Psi} = \nabla\!\!\!/^4 + \nabla\!\!\!/\Gamma_n(\nabla_m F_{mn}) \ , \qquad b_6(\Delta_{3\Psi}) = b_6(\Delta_{4\Psi}) - b_6(\Delta_{1\Psi}) \ . \tag{4.13}$$

$\Delta_{4\Psi}$ is then a 4-order operator of the form (2.4) with the coefficients[17]

$$\begin{aligned} &\hat V_{rk} = \Gamma_{mn}F_{mn}\,\delta_{rk} \ , \qquad\qquad \hat N_k = \frac{1}{2}\Gamma_k\Gamma_n\nabla_m F_{mn} \ , \\ &\hat U = \frac{1}{2}\Gamma_{mn}\nabla^2 F_{mn} + \frac{1}{4}\Gamma_{mn}\Gamma_{rk}F_{mn}F_{rk} + \frac{1}{2}\Gamma_k\Gamma_n\nabla_k\nabla_m F_{mn} \ . \end{aligned} \tag{4.14}$$

[15] In [5] the opposite overall sign was chosen so that their coupling is related to ours by $g^2 \to -g^2$. This translates into the opposite sign of the β-function for g in (4.17). Note that here there is thus no "preferred" choice of the sign of the action (redefining the scalars $\Phi_I \to i\Phi_I$ leads to imaginary Φ^3 interaction, i.e. to non-hermiticity of the action). For a review of related issues in higher-derivative theories see [22].

[16] In the first form of $\Delta_{3\Psi}$ the derivative in the second term acts all the way to the right while the third term term it acts only on F_{mn}.

[17] Notice that this operator is not self-adjoint, i.e. the symmetry requirements in footnote 7 are not satisfied.

Applying the general expression for $b_6(\Delta_4)$ that we found in (3.7), (3.8) (where now the connection and its curvature are understood to include also the internal spinor indices, see footnote 10) and also using that squaring $\Delta_{1\Psi}$ one obtains (4.9), for which b_6 can then found from (3.5), we end up with

$$b_6(\Delta_{3\Psi}) = b_6(\Delta_{4\Psi}) - \frac{1}{2}b_6(\Delta_{2\Psi}) = \mathrm{tr}\Big[-\frac{8}{15}(\nabla_m F_{mn})^2 + \frac{2}{15}F_{mn}F_{nk}F_{km}\Big]. \tag{4.15}$$

Combining the bosonic (4.12) and the fermionic (4.15) contributions to (4.11) we conclude that the F^3 terms cancel as expected and finally

$$b_{6\,(1,0)} = -\frac{11}{3}\,\mathrm{tr}\,(\nabla_m F_{mn})^2 \ . \tag{4.16}$$

This is the same result as quoted in (1.2), (1.7). The resulting renormalized coupling in (4.7) is thus (cf. (2.7), (4.7))

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\Lambda)} - \frac{22}{3}\frac{C_2}{(4\pi)^3}\log\frac{\Lambda}{\mu} \ , \tag{4.17}$$

corresponding to an asymptotically free behaviour. This agrees with the (recently revised) result of [5] (cf. footnote 15). Note that the computation of the β-function in [5] was done in the scalar field Φ_I background while here we used the gauge field background, thus providing an independent check of the result.

For comparison, let us recall the result [1] of a similar computation in the ordinary-derivative (1,0) 6d SYM theory

$$S_{(1,0)\,\mathrm{SYM}} = -\frac{\kappa^2}{g^2}\int d^6x\ \mathrm{Tr}\Big(\frac{1}{2}F_{mn}F_{mn} + i\overline{\Psi}\nabla\!\!\!\!/\,\Psi - \Phi_I\Phi_I\Big) \ , \tag{4.18}$$

where Ψ is a Weyl spinor, Φ_I are 3 auxiliary fields (cf. (4.7)) and κ is a mass scale. The analog of the one-loop effective action in a gauge field background (4.10) here is

$$\Gamma_{1\,(1,0)\,\mathrm{SYM}} = \frac{1}{2}\log\frac{\det\Delta_{2A}}{\big[\det\Delta_{2,0}\big]^2\,\det\Delta_{1\Psi}} \ , \qquad (\Delta_{2A})_{mn} = -\delta_{mn}\nabla^2 - 2F_{mn} \ . \tag{4.19}$$

Using (3.5) we get

$$\begin{aligned} b_6(\Delta_{2A}) &= \mathrm{tr}\Big[\frac{17}{30}\,(\nabla_m F_{mn})^2 + \frac{1}{15}F_{mn}F_{nk}F_{km}\Big] \ , \\ b_6(\Delta_{2,0}) &= \mathrm{tr}\Big[-\frac{1}{60}\,(\nabla_m F_{mn})^2 + \frac{1}{90}F_{mn}F_{nc}F_{cm}\Big] \ , \\ b_6(\Delta_{1\Psi}) &= \frac{1}{2}b_6(\Delta_{2\Psi}) = \mathrm{tr}\Big[\frac{4}{15}(\nabla_m F_{mn})^2 + \frac{2}{45}F_{mn}F_{nc}F_{cm}\Big] \ . \end{aligned} \tag{4.20}$$

As a result, the one-loop logarithmic divergence is given by (2.7) with

$$b_{6\,(1,0)\,\mathrm{SYM}} = b_6(\Delta_{2A}) - 2b_6(\Delta_{2,0}) - b_6(\Delta_{1\Psi}) = \frac{1}{3}\,\mathrm{tr}\,(\nabla_m F_{mn})^2 \ . \tag{4.21}$$

Once again, the F^3 divergence cancels, and (4.21) implies the value of $\beta_2 = -20$ in (1.2), (1.5). Since here $\nabla_m F_{mn} = 0$ is an equation of motion, the divergence (4.21) vanishes on-shell, i.e. the (1,0) 6d SYM theory is finite on-shell[18] though is not renormalizable off-shell. The (1,1) 6d SYM found by combining the (1,0) SYM with a scalar multiplet (cf. (1.5)) is one-loop finite even off-shell [1] (cf. also [24]).

[18] The coefficient in (4.21) here is, in fact, gauge-dependent, see also [23].

Let us also note that it is easy to check the cancellation of F^3 divergences in the (1,0) supersymmetric gauge theory (4.7) by restricting the background to satisfy $\nabla_m F_{mn} = 0$ (which is a special on-shell background also in this theory). Then $\Delta_{3\Psi}$ in (4.8) becomes simply $(\Delta_{1\Psi})^3 = i\not{\nabla}^3$ and also the vector field operator in (2.4), (2.5) (with $\gamma = 0$) becomes a square of the standard YM operator in (4.19), i.e. $\Delta_{4A} = (\Delta_{2A})^2$. As a result, the effective action (4.10) reduces to

$$\begin{aligned}\Gamma_{1\,(1,0)} &= \frac{1}{2}\log\det(\Delta_{2A})^2 - \frac{1}{2}\log\det(\Delta_{1\Psi})^3 \\ &= 2\cdot\frac{1}{2}\Big[\log\det\Delta_{2A} - 2\log\det\Delta_{2,0} - \det\Delta_{1\Psi}\Big] + \frac{1}{2}\Big[4\log\det\Delta_{2,0} - \det\Delta_{1\Psi}\Big] \\ &= 2\,\Gamma_{1\,(1,0)\,\mathrm{SYM}} + \Gamma_{1\,\mathrm{scal}}\ , \end{aligned} \tag{4.22}$$

i.e. equal to the sum of twice the effective action of the standard (1,0) SYM in (4.19) with the effective action of the scalar (hyper) multiplet (containing 4 real scalars and one Weyl fermion). Each of these do not contribute to the F^3 divergent terms according to (1.5).

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Note added

After this paper was submitted to the arXiv we learned about the earlier work [25] (see also [26]) in which a diagrammatic computation of the two-loop β-functions in the 6d gauge theory (1.1) coupled to standard fermions was performed.[19] After correcting a mistake in the original version of this paper we found that our result (1.3), (1.4) for the β-functions of the theory (1.1) coupled to fermions is in full agreement with the one-loop β-functions in [25].[20]

A Derivation of the expression for $b_6(\Delta_4)$

The operator that we shall consider is

$$\Delta_4 = \nabla^4 + V_{mn}\nabla_m\nabla_n + 2N_m\nabla_m + U, \qquad V_{mn} = V_{nm}\ , \tag{A.1}$$

which is the most general fourth-order elliptic differential operator operator without 3-derivative term. It is related to the "symmetrized" operator in (2.4) by

$$V_{mn} = \hat{V}_{mn}\ , \qquad N_m = \hat{N}_m + \frac{1}{2}\nabla_m\hat{V}_{mn}\ , \qquad U = \tilde{U} + \nabla_m\hat{N}_m\ . \tag{A.2}$$

[19] We are grateful to I. Klebanov for drawing our attention to this paper.

[20] The translation between the notation in [25] and ours is as follows. Instead of $(\nabla_m F_{mn})^2$ in (1.1) the action in [25] contained $(\nabla_k F_{mn})^2$ with the two invariants related as in footnote 3. As a result, the couplings g_1 and g_2 in [25] are related to ours as $g_1 = g,\ \ g_2 = 3g(1+\gamma)$ (using also that $g_2 \to -g_2$ due to apparent sign difference in notation for F_{mn}). For the gauge theory (1.1) coupled to Weyl fermions in generic representation our result (1.3), (1.4) for the β-functions reads (cf. (4.5), (4.6)): $\beta_g \equiv \frac{dg}{dt} = -\frac{1}{120}C_2\beta_2,\ \ \beta_\gamma \equiv \frac{d\gamma}{dt} = -\frac{1}{120}C_2(2\gamma\beta_2 + \frac{2}{3}\beta_3)g^2$, $\beta_2 = 249 + N_{\frac{1}{2}},\ \beta_3 = 9 - 900\gamma + \frac{405}{2}\gamma^3 - 4N_{\frac{1}{2}},\ \ N_{\frac{1}{2}} = \frac{T_R}{C_2}N_f$. Then the β-functions for the above g_1 and g_2, i.e. $\beta_{g_1} = \frac{dg_1}{dt} = \beta_g,\ \ \beta_{g_2} = \frac{dg_2}{dt} = 3\beta_g(1+\gamma) + 3g\beta_\gamma$ match the expressions in [25].

The general expression for its coefficient b_6 including only independent invariants may be written as ($V \equiv V_{mm}$)

$$\begin{aligned} b_6(\Delta_4) = \mathrm{tr}\Big[& k_1\,(\nabla_m F_{mn})^2 + k_2 F_{mn}F_{nk}F_{km} \\ & + k_3 V_{mn}V_{nk}V_{km} + k_4 V_{mn}V_{mn}V + k_5 VVV + k_6 V_{mn}\nabla_{(n}\nabla_{k)}V_{km} \\ & + k_7 V_{mn}\nabla^2 V_{mn} + k_8 V_{mn}\nabla_m\nabla_n V + k_9 V\nabla^2 V + k_{10}V_{mn}V_{nk}F_{mk} \\ & + k_{11}F_{mn}\nabla_{(m}\nabla_{k)}V_{kn} + k_{12}VF_{mn}F_{mn} + k_{13}V_{mn}F_{mk}F_{nk} \\ & + k_{14}F_{mn}\nabla_m N_n + k_{15}V_{mn}\nabla_m N_n + k_{16}V\nabla_m N_m + k_{17}N_m N_m + k_{18}UV\Big], \end{aligned} \tag{A.3}$$

where the trace is over internal indices and k_i are real coefficients.[21] Their values in $d = 6$ found below are

$$\begin{aligned} &k_1 = -\frac{1}{30}, && k_2 = \frac{1}{45}, && k_3 = \frac{1}{360}, && k_4 = \frac{1}{480}, && k_5 = \frac{1}{2880}, && k_6 = \frac{1}{30}, \\ &k_7 = \frac{1}{120}, && k_8 = -\frac{1}{40}, && k_9 = \frac{1}{240}, && k_{10} = -\frac{1}{12}, && k_{11} = \frac{1}{6}, && k_{12} = \frac{1}{24}, \\ &k_{13} = -\frac{1}{6}, && k_{14} = -\frac{1}{3}, && k_{15} = -\frac{1}{6}, && k_{16} = \frac{1}{12}, && k_{17} = -\frac{1}{6}, && k_{18} = -\frac{1}{12}. \end{aligned} \tag{A.4}$$

To determine k_i we shall exploit the factorization property (3.6), i.e.

$$b_6(\Delta_4) = b_6(\Delta_2) + b_6(\Delta_2') \ , \qquad \Delta_4 = \Delta_2\Delta_2' \ , \tag{A.5}$$

where $b_6(\Delta_2)$ is given by (3.5). One needs to identify enough special cases and consistency conditions to fix all k_i. When comparing the two sides of the b_6-relation in (A.5) it is important to take into account (i) that they are defined up to total derivatives (which we drop in discussing UV divergences), (ii) that the terms can be cyclically permuted because they appear under an overall trace, and (iii) relations between the invariants (implied, e.g., by the Bianchi identity).

Considering $\Delta_2 = -\nabla^2 + X$ and $\Delta_2' = -\nabla^2 + X'$ their product is given by (A.1) with

$$V_{mn} = -\delta_{mn}(X + X'), \quad N_m = -\nabla_m X', \quad U = XX' - \nabla^2 X', \quad V = -6(X + X') \ . \tag{A.6}$$

Using (3.5) and comparing with (A.3) gives

$$\begin{aligned} &k_1 = -\frac{1}{30}, && k_3 + 6k_4 + 36k_5 = \frac{1}{36}, && k_{13} + 6k_{12} = \frac{1}{12}, && k_{17} = -\frac{1}{6}, \\ &k_2 = \frac{1}{45}, && k_6 + 6k_7 + 6k_8 + 36k_9 = \frac{1}{12}, && k_{15} + 6k_{16} = \frac{1}{3}, && k_{18} = -\frac{1}{12}. \end{aligned} \tag{A.7}$$

Next, let us assume that

$$\Delta_4 = \Delta_+\Delta_-, \qquad \nabla_m^\pm \equiv \nabla_m \pm K_m \ , \tag{A.8}$$

$$\Delta_\pm = -(\nabla_m^\pm)^2 = -\nabla^2 \mp 2K_m\nabla_m \mp (\nabla_m K_m) - K_m K_m \ . \tag{A.9}$$

Here $\nabla_m K_n = \partial_m K_n + [A_m, K_n]$ (K_m is in the adjoint representation of the gauge group). The coefficient functions in the corresponding operator $\Delta_4 = \Delta_+\Delta_-$ in (A.1) read

$$\begin{aligned} V_{mn} &= -4\nabla_{(m}K_{n)} + 2K^2\delta_{mn} - 4K_{(m}K_{n)}, \qquad V = -4\nabla_n K_n + 8K^2 \ , \\ N_m &= -\nabla^2 K_m - \nabla_m\nabla_n K_n + \nabla_m K^2 \\ &\quad + K_m K^2 - K^2 K_m - 2K_n\nabla_n K_m - K_m\nabla_n K_n + 2K_n F_{nm}, \\ U &= -\nabla^2\nabla_n K_n + \nabla^2 K^2 - 2K_m\nabla_m\nabla_n K_n + 2K_m\nabla_m K^2 - (\nabla_n K_n)^2 + K^4 \\ &\quad + (\nabla_n K_n)K^2 - K^2\nabla_n K_n - 2\nabla_m K_n F_{mn} - 2K_m K_n F_{mn} + 2K_m\nabla_n F_{mn}. \end{aligned} \tag{A.10}$$

[21] The relations between the k_i and $\hat{k}_i$ in (3.8) are, using (A.2), $\hat{k}_6 = k_6 + \frac{1}{2}k_{15} - \frac{1}{4}k_{17}$, $\hat{k}_8 = k_8 + \frac{1}{2}k_{16}$, $\hat{k}_{10} = k_{10} - \frac{1}{2}k_{15} + \frac{1}{4}k_{17}$, $\hat{k}_{11} = k_{11} + \frac{1}{2}k_{14}$, $\hat{k}_{15} = k_{15} - k_{17}$, $\hat{k}_{16} = k_{16} + k_{18}$ with $\hat{k}_i = k_i$ otherwise.

Using (3.5) and the relations

$$F_{mn}^{\pm} \equiv [\nabla_m^{\pm}, \nabla_n^{\pm}] = F_{mn} + [K_m, K_n] \pm (\nabla_m K_n - \nabla_n K_m) \ , \tag{A.11}$$

$$\begin{aligned} \nabla_m^{\pm} F_{mn}^{\pm} = \nabla_m \Big[F_{mn} + [K_m, K_n] \pm (\nabla_m K_n - \nabla_n K_m) \Big] \\ \pm \Big[K_m, F_{mn} + [K_m, K_n] \pm (\nabla_m K_n - \nabla_n K_m) \Big] \ , \end{aligned} \tag{A.12}$$

one can compute $b_6(\Delta_\pm)$ and then compare to $b_6(\Delta_4)$ in (A.5).

It is enough to consider the following special cases:

1. Abelian gauge group, $\nabla_n K_m = \partial_n K_m$, $[F_{mn}, K_k] = 0$. In (A.5) we consider the terms with $\partial^r K_m$, $r = 0, 1, 4$ that can always be uniquely cast into the form

$$K^6, \qquad K^4 \partial_m K_m \ , \qquad (\partial_m K_m) \partial^2 (\partial_n K_n) \ , \qquad K_m \partial^4 K_m \ . \tag{A.13}$$

   Then comparing also the coefficients of $F_{nm} K^2 \partial_n K_m$ and $F_{nm} K_m \partial_n K^2$ (the latter does not actually appear) one obtains

$$\begin{aligned} &32k_3 + 16 \cdot 12k_4 + 512k_5 + 8k_{18} = 0 \ , \qquad k_{15} + 4k_{16} - k_{17} + 4k_{18} = 0 \ , \\ &4k_6 + 8k_7 + 2k_{15} - k_{17} = \frac{1}{30} \ , \qquad 12k_{18} + 256 \cdot 3k_5 + 16 \cdot 18k_4 + 48k_3 = 0 \ , \\ &12k_6 + 8k_7 + 16k_8 + 16k_9 + 6k_{15} + 8k_{16} - 3k_{17} + 4k_{18} = -\frac{1}{30} \ . \end{aligned} \tag{A.14}$$

2. K_n constrained by $\nabla_m K_n = 0$, implying $2\nabla_{[k}\nabla_{m]} K_n = [F_{km}, K_n] = 0$. This leads to a number of nontrivial relations, e.g., $\mathrm{tr}([K_m, K_n] F_{nk} F_{km}) = 0$. All the remaining invariants can be uniquely written as a combination of

$$\begin{aligned} &K^6 \ , \quad K_m K_n K_k K_m K_n K_k \ , \quad K^2 K_m K^2 K_m \ , \quad K_m K_n K_m K_k K_n K_k \ , \\ &K^2 F_{mn} F_{mn} \ , \quad K^2 K_m K_n K_m K_n \ , \quad K_m K_n F_{mk} F_{nk} \ , \quad F_{mn} K_m K_n K^2 \ . \end{aligned} \tag{A.15}$$

   Their coefficients can then be compared to get ($(K_m K_n K_p)^2$ and $(K_m K_p K_m)^2$ give the same equation)

$$\begin{aligned} &24k_3 = \frac{1}{15} \ , \qquad -8k_3 + 64k_4 + 512k_5 + 8k_{18} - 2k_{17} = -\frac{2}{45} \ , \\ &k_{13} - k_{17} = 0 \ , \qquad 64k_4 + 48k_3 + 2k_{17} = -\frac{1}{15} \ , \qquad 24k_3 + 64k_4 = \frac{1}{5} \ , \\ &- k_{17} + 2k_{18} = 0 \ , \qquad 8k_{12} + 2k_{13} = 0 \ . \end{aligned} \tag{A.16}$$

3. General unconstrained K_n, comparing the terms with one K_m or two of them contracted together. A basis of such tensors contains

$$\begin{aligned} &K_m \nabla^2 \nabla_n F_{mn}, \qquad K_m F_{nk} \nabla_n F_{km}, \qquad K_m F_{mn} \nabla_k F_{kn}, \\ &K_m \nabla_n F_{km} F_{nk}, \qquad K_m \nabla_k F_{kn} F_{mn}, \qquad K_m \nabla^4 K_m, \\ &K_m \nabla_k F_{kn} \nabla_n K_m, \qquad K^2 F_{kn} F_{kn}, \qquad K_m F_{kn} K_m F_{kn}. \end{aligned} \tag{A.17}$$

   In this case we obtain (the two $KKFF$ terms give the same equation)

$$\begin{aligned} &2k_{11} + k_{14} = 0 \ , \qquad 8k_{12} + 2k_{13} = 0 \ , \qquad 2k_{11} - 2k_{13} + 2k_{14} = 0 \ , \\ &4k_6 + 8k_7 + 2k_{15} - k_{17} = \frac{1}{30} \ , \qquad 2k_6 + 16k_7 + 2k_{10} + 8k_{12} + 2k_{13} = \frac{1}{30} \ , \\ &4k_6 + 16k_7 + 4k_{10} = -\frac{1}{15} \ , \qquad 2k_6 + 16k_7 + 2k_{10} = \frac{1}{30} \ , \qquad k_{11} + k_{13} = 0 \ . \end{aligned} \tag{A.18}$$

The final system of equations is given by (A.7), (A.14), (A.16) and (A.18). This system is over-determined, with the unique solution for k_i given by (A.4). That some of the equations are actually redundant gives a non-trivial consistency check of the calculation. We also checked some of the coefficients k_i by explicit diagrammatic calculations of the corresponding UV divergences.

B One-loop divergences in $F^2 + (\nabla F)^2 + F^3$ theory

It is straightforward to generalize the expression for the effective action (2.6) to the case when one adds to the action (1.1) the standard YM term, i.e. the first term in (4.18)

$$\Gamma_1 = \frac{1}{2}\log\frac{\det\Delta'_{4A}}{\left[\,\det(-\nabla^2)\,\right]^2\det(-\nabla^2+\kappa^2)}\ , \qquad \Delta'_{4A} = \Delta_{4A} + \kappa^2\Delta_{2A}\ . \tag{B.1}$$

Here Δ_{2A} is given in (4.19). The quadratic and logarithmic divergences of (B.1) are determined by the total b_4 and b_6 coefficients (cf. (3.3))

$$\Gamma_{1\infty} = -\frac{1}{(4\pi)^3}\Big(\frac{1}{2}B_4\Lambda^2 + B_6\log\Lambda\Big)\ , \tag{B.2}$$

$$B_p = \int d^6x\ b_p\,, \qquad b_p = b_p(\Delta'_{4A}) - 2\,b_p(-\nabla^2) - b_p(-\nabla^2+\kappa^2)\ . \tag{B.3}$$

The expression for b_4 is known for both for Δ_2 (3.4) and Δ_4 (A.1) operators[22]

$$b_4(\Delta_2) = \mathrm{tr}\Big[\frac{1}{12}F_{mn}F_{mn} + \frac{1}{2}X^2\Big]\ , \tag{B.4}$$

$$b_4(\Delta_4) = \mathrm{tr}\Big[\frac{1}{6}F_{mn}F_{mn} + p_1V_{mn}V_{mn} + p_2VV - U\Big]\ . \tag{B.5}$$

While the coefficients in (B.4) are universal, i.e. the same in any dimension d [17], the coefficients p_1 and p_2 in (B.5) are dimension-dependent. In $d=4$ their values are [2] $p_1=\frac{1}{24}$, $p_2=\frac{1}{48}$ while for general d we found

$$p_1 = -\frac{8-8d+d^2}{16\,d\,(d-1)}\ , \qquad p_2 = \frac{1}{16\,(d-1)}\ , \qquad p_1\Big|_{d=6} = \frac{1}{120}\ , \quad p_2\Big|_{d=6} = \frac{1}{80}\ . \tag{B.6}$$

The coefficient b_4 controls the logarithmic divergences in the corresponding 4d theory where their computation was done in [2] (see also [4]). For the operators in (B.1) we get in $d=6$ (here tr is in the adjoint representation and F_{mn} is the gauge field strength)[23]

$$b_4(-\nabla^2+\kappa^2) = \frac{1}{12}\,\mathrm{tr}\,F_{mn}F_{mn} + \frac{1}{2}\,\kappa^4\,C_2\ , \tag{B.7}$$

$$b_4(\Delta'_{4A}) = -\Big(3 + 14\gamma + \frac{12}{5}\gamma^2\Big)\,\mathrm{tr}\,F_{mn}F_{mn} + 3\,\kappa^4\,C_2\ . \tag{B.8}$$

Similarly, using (3.5) and (3.7), (3.8) we find

$$b_6(\Delta'_{4A}) = -\frac{21}{5}\,\mathrm{tr}(\nabla_mF_{mn})^2 + \Big(\frac{1}{10} - 10\gamma + \frac{9}{4}\gamma^3\Big)\,\mathrm{tr}\,F_{mn}F_{nk}F_{km} + \Big(\frac{3}{2} + 9\gamma + 3\gamma^2\Big)\kappa^2\,\mathrm{tr}\,F_{mn}F_{mn} - \kappa^6C_2\ , \tag{B.9}$$

[22] Here tr and F_{mn} are the general trace and the curvature on the bundle, cf. footnote 10.

[23] $b_4(\Delta'_{4A})$ has two sources of dependence on space-time dimension d: the operator itself and the coefficients in b_4 in (B.5),(B.6). The gauge fixing contributions are independent of d. As a result, in 4d theory the coefficient β_{1A} in (B.13) below is given by $\beta_{1A} = -2(43 + 108\gamma + 27\gamma^2)$ (cf. [2, 4]).

$$b_6(-\nabla^2+\kappa^2) = -\frac{1}{60}\,\mathrm{tr}(\nabla_m F_{mn})^2 + \frac{1}{90}\,\mathrm{tr}\, F_{mn}F_{nk}F_{km} - \frac{1}{12}\kappa^2\,\mathrm{tr}\, F_{mn}F_{mn} - \frac{1}{6}\kappa^6 C_2\ . \quad \text{(B.10)}$$

As a result, the total values of the coefficients of the quadratic and logarithmic divergences in (B.2) in $d=6$ are (omitting field-independent terms)

$$b_4 = \frac{1}{12}\beta_1\,\mathrm{tr}\, F_{mn}F_{mn}\ , \quad \text{(B.11)}$$

$$b_6 = \kappa^2\beta_\kappa\,\mathrm{tr}\, F_{mn}F_{mn} - \frac{1}{60}\beta_{2A}\,\mathrm{tr}(\nabla_m F_{mn})^2 + \frac{1}{90}\beta_{3A}\,\mathrm{tr}\, F_{mn}F_{nk}F_{km}\ , \quad \text{(B.12)}$$

$$\beta_{1A} = -39 - 168\gamma - \frac{144}{5}\gamma^2\ , \qquad \beta_{\kappa,A} = \frac{19}{12} + 9\gamma + 3\gamma^2\ , \quad \text{(B.13)}$$

where β_{2A} and β_{3A} in (B.12) are the same as in (1.3). Ignoring non-universal quadratic divergence (absent in dimensional regularization), the logarithmic renormalization of κ is controlled by $\beta_{\kappa,A}$ with the RG equation (cf. (4.5), (4.6))[24]

$$\frac{d\kappa^2}{dt} = -\Big(\beta_{\kappa,A} + \frac{1}{60}\beta_{2A}\Big)2\kappa^2 g^2 C_2 = \Big(\frac{175}{12} - 18\gamma - 6\gamma^2\Big)\kappa^2 g^2 C_2\ . \quad \text{(B.14)}$$

Near both the attractive fixed points $\gamma_1 \simeq -0.897$ and $\gamma_3 \simeq 0.838$ of β_γ in (4.6), the r.h.s of (B.14) is negative and thus $\kappa^2 \to 0$ in the UV.

Let us now consider the log divergence in the (1,0) supersymmetric extension of this bosonic model, i.e. the $(1,0)$ SYM combined with the (1,0) theory (4.7). Here the operators in the 1-loop effective action (4.10) get κ-dependent terms as in (B.1) (with $\gamma = 0$)

$$\Delta'^{(0)}_{4A} = \Delta^{(0)}_{4A} + \kappa^2\Delta_{2A}\ , \qquad \Delta'_{3\Psi} = \Delta_{3\Psi} + \kappa^2\Delta_{1\Psi}\ , \qquad \Delta'_{2\Phi} = \Delta_{2\Phi} + \kappa^2\ , \quad \text{(B.15)}$$

where $\Delta_{1\Psi} = -i\nabla\!\!\!\!/$ and $\Delta_{2\Phi} = -\nabla^2$. Explicitly, we get (cf. (4.10), (B.1))

$$\begin{aligned}\Gamma'_{1(1,0)} &= \frac{1}{2}\log\Big[\frac{\det \Delta'^{(0)}_{4A}}{[\det(-\nabla^2)]^2\det(-\nabla^2+\kappa^2)}\,\frac{[\det\Delta'_{2\Phi}]^3}{\det\Delta'_{3\Psi}}\Big] \\ &= \frac{1}{2}\log\Big[\frac{\det\Delta'^{(0)}_{4A}}{\det\Delta'_{3\Psi}}\,\frac{[\det(-\nabla^2+\kappa^2)]^2}{[\det(-\nabla^2)]^2}\Big]\ .\end{aligned} \quad \text{(B.16)}$$

For the gauge field and scalar determinants the expressions for b_4 and b_6 are given by (B.7)–(B.10) with $\gamma = 0$ while for the fermion contribution we get as in (4.13),

$$b_6(\Delta'_{3\Psi}) = b_6(\Delta_{1\Psi}\Delta'_{3\Psi}) - b_6(\Delta_{1\Psi}) = b_6(\Delta_{3\Psi}) + \frac{14}{3}\kappa^2\,\mathrm{tr}\, F_{mn}F_{mn} - \frac{4}{3}\kappa^6 C_2. \quad \text{(B.17)}$$

As a result, the analog of (B.13) is

$$\begin{aligned} b_6 &= \kappa^2\beta_{\kappa\,(1,0)}\,\mathrm{tr}\, F_{mn}F_{mn} - \frac{1}{60}\beta_{2\,(1,0)}\,\mathrm{tr}\,(\nabla_m F_{mn})^2\ , \\ \beta_{\kappa\,(1,0)} &= -\frac{29}{6}\ , \qquad \beta_{2\,(1,0)} = 220\,. \end{aligned} \quad \text{(B.18)}$$

where $\beta_{2\,(1,0)}$ is the same as in (4.16), (1.7). Since the combination $\beta_{\kappa\,(1,0)} + \frac{1}{60}\beta_{2A}$ is negative, as a result of (B.14) we do not have asymptotic freedom in the supersymmetric case.

Let us note also that on $\nabla_m F_{mn} = 0$ background (B.16) becomes the following generalization of (4.22)

$$\Gamma'_{1\,(1,0)} = \frac{1}{2}\Big[\log\det\Delta_{2A} - 2\log\det\Delta_{2,0} - \det\Delta_{1\Psi}\Big]$$

[24] Recall that the coefficient of the YM term is chosen as $\frac{\kappa^2}{g^2}$, cf. (4.18).

$$+\frac{1}{2}\Big[\log\det(\Delta_{2A}+\kappa^2)-2\log\det(\Delta_{2,0}+\kappa^2)-\det(\Delta_{1\Psi}+\kappa)\Big]$$
$$+\frac{1}{2}\Big[4\log\det(\Delta_{2,0}+\kappa^2)-\det(\Delta_{1\Psi}+\kappa)\Big]\ , \qquad (B.19)$$

i.e. the sum of contributions of massless (1,0) SYM, its massive analog, and massive analog of scalar multiplet. From (B.19) it is easy to compute the quadratic divergence coefficient b_4 (B.4) or β_1 coefficient in (B.11) in the effective action (B.16) or (B.19)[25]

$$\beta_{1\,(1,0)\,\mathrm{SYM}}=-12,\qquad \beta_{1\,(1,0)\,\mathrm{scal}}=12\ ,\qquad \beta_{1\,(1,0)}=\beta'_{1\,(1,0)}=-12-12+12=-12\ . \qquad (B.20)$$

The coefficient of the field independent κ^4 quadratic divergence is proportional to the number of degrees of freedom and thus vanishes in supersymmetric cases.

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[25] Explicitly, one finds from (B.4) in general dimension d [1]: $\beta_1(\Delta_{2,0})=1$, $\beta_1(\Delta_{2A})=d-24$, $\Delta_{1\Psi}=-\nu$, $\nu=2^{[d/2]}$, so that in $d=6$ we get $\beta_{1\,(1,0)\,\mathrm{SYM}}=-12$, $\beta_{1\,(1,0)\,\mathrm{scal}}=12$, $\beta_{1\,(1,1)\,\mathrm{SYM}}=0$.

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