Hochschild cohomology of Sullivan algebras and mapping spaces

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Abstract. Let \( f : X \to Y \) be a map between simply connected spaces having the homotopy of finite type CW-complexes, where \( H^\ast(Y, \mathbb{Q}) \) is finite dimensional and \( \phi : (\wedge V, d) \to (B, d) \) a Sullivan model of \( f \). We consider \((B, d)\) as a module over \( \wedge V \) via the mapping \( \phi \).

Let \( \text{map}(X, Y; f) \) denote the component of \( f \) in the space of mappings from \( X \) to \( Y \). In this paper we show that there is a canonical injection \( \pi_\ast(\Omega \text{map}(X, Y; f)) \otimes \mathbb{Q} \to H H^\ast(\wedge V; B) \).

Keywords: Hochschild cohomology; Mapping space; \( L_\infty \) algebra

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1. Introduction

We work in the rational homotopy setting for which the standard reference is [6]. In this section we fix notation and recall a few facts on the Hochschild cohomology of an algebra. All vector spaces and algebras are taken over a field \( k \) of characteristic 0.

Definition 1. A lower graded vector space \( V \) is a direct sum of vector spaces, that is, \( V = \oplus_i V_i \), where \( i \in \mathbb{Z} \). We say that element \( a \in V_i \) is homogeneous of degree \( i \) and we write \( |a| = i \) and \( V = V_\bullet \) is lower or homologically graded. If \( V = \oplus_{i \geq 0} V_i \), then \( V \) is said to be non negatively graded. In the same way \( V^\ast = \oplus_i V^i \) is called cohomologically.

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graded. We use the standard convention \( V^i := V_{-i} \). Hence if \( V = \oplus_{i \geq 0} V^i \), the dual space of \( V \) is denoted \( V^* = \prod_i \Hom(V^i, k) = \prod_i \Hom(V_{-i}, k) \) has a lower non positive grading.

**Definition 2.** A morphism of graded vector spaces \( f : V \to W \) of degree \( r \), is a family of linear maps \( f_n : V_n \to W_{n+r} \).

Let \( (M, d) \) be a differential \((A, d)\)-bimodule. The Hochschild cohomology of \( A \) with coefficients in \( M \) is defined as \( \Ext_{A^e}(A, M) \) where \( A \) is an \( A^e = A \otimes A^{op} \)-module under the action \((a_1 \otimes a_2)\alpha = (-1)^{|a_1||a_2|}a_1a_2\alpha \), where \( a, a_1, a_2 \in A \).

Let \((P, d_P) \to (A, d)\) be a semifree resolution of \( A \) as an \( A^e \)-module [5], and \((M, d_M)\) an \( A^e \)-differential module. Then \( HH^*(A; M) := \Ext_{A^e}(A, M) \) is the homology of the complex \((\Hom_{A^e}(P, M), D)\), where the differential is defined by

\[
(DF)(x) = d_M f(x) - (-1)^{|f|} f(d_P x).
\]

In the sequel we work in the category of commutative differential graded algebras (cdga’s for short). This implies that left (or right) modules have a natural bimodule structure. Let \( f : A \to B \) be a morphism of cdga’s. Then \( B \) is considered as an \( A \)-module by the action induced by \( f \).

Our aim is to study the structure of \( HH^*(A; B) \). Let \((\wedge V, d)\) be a Sullivan algebra, and \( m : (\wedge V \otimes \wedge V, d' = d \otimes 1 + 1 \otimes d) \to (\wedge V, d)\) the multiplication. Then there is a quasi isomorphism

\[
(\wedge V \otimes \wedge V \otimes \wedge \bar{V}, D) \to (\wedge V, d)
\]

making the following diagram commutative.

\[
\begin{array}{ccc}
(\wedge V \otimes \wedge V \otimes \wedge \bar{V}, D) & \xrightarrow{m} & (\wedge V, d) \\
\downarrow & & \downarrow \\
(\wedge V \otimes \wedge V \otimes \wedge \bar{V}, D).
\end{array}
\]

Moreover \( \bar{V}^n = V^{n+1} \) and the differential \( D \) is defined by

\[
D(v) = v \otimes 1 - 1 \otimes v + \alpha, \alpha \in \wedge V \otimes \wedge V \otimes \wedge \bar{V},
\]

and \( \iota \) is the canonical inclusion [6, §15]. The quasi isomorphism

\[
(\wedge V \otimes \wedge V \otimes \wedge \bar{V}, D) \xrightarrow{\partial} (\wedge V, d)
\]

is a semifree resolution of \((\wedge V, d)\) as a \( \wedge V \otimes \wedge V \)-module [5,10]. Therefore, for any \( \wedge V \)-module \( M \), \( HH^*(\wedge V; M) \) is the homology of the complex \((\Hom_{\wedge V \otimes \wedge V}(\wedge V \otimes \wedge V \otimes \wedge \bar{V}, M), D)\),

where the differential is defined by \((1)\).

We consider the cdga \((\wedge V \otimes \wedge V, \bar{D})\) where \( Dv = dv, \bar{D}(\bar{v}) = -S(dv) \) and \( S \) is the unique derivation on \( \wedge V \otimes \wedge \bar{V} \) defined by \( Sv = \bar{v} \) and \( S\bar{v} = 0 \). It is obtained as a push out in the diagram below.

\[
\begin{array}{ccc}
(\wedge V \otimes \wedge V, d') & \xrightarrow{\iota} & (\wedge V \otimes \wedge V \otimes \wedge \bar{V}, D) \\
\downarrow & & \downarrow \\
(\wedge V, d) & \xrightarrow{m} & (\wedge V \otimes \wedge \bar{V}, \bar{D}).
\end{array}
\]
Moreover, the composition with $m'$ yields an isomorphism of complexes

$$\text{Hom}_{\wedge V}(\wedge V \otimes \wedge \bar{V}, M) \overset{\sim}{\to} \text{Hom}_{\wedge V \otimes \wedge V}(\wedge V \otimes \wedge \bar{V}, M).$$

As $\tilde{D}(\wedge V \otimes \wedge^n \bar{V}) \subset \wedge V \otimes \wedge^n \bar{V}$, hence each $(\text{Hom}_{\wedge V}(\wedge V \otimes \wedge^n \bar{V}, M), \tilde{D})$ is a sub cochain complex [8]. This gives a Hodge type decomposition of the Hochschild cohomology

$$HH^*(\wedge V; M) = \bigoplus_{n \geq 0} HH^*_n(\wedge V; M)$$

for any $\wedge V$-differential module $(M, d)$ [11,7].

Let $f : X \to Y$ be a map between simply connected spaces having the homotopy of finite type CW-complexes and assume that $H^*(Y, \mathbb{Q})$ is finite dimensional. Let $\phi : (\wedge V, d) \to (B, d)$ be a cdga model of $f$. We consider $(B, d)$ as a module over $\wedge V$ via the mapping $\phi$. Denote by $\text{map}(X, Y; f)$ the component of $f$ in the space of mappings from $X$ to $Y$. In this paper we show the following result.

**Theorem 3.** There is a canonical injection

$$\pi_*(\Omega \text{ map}(X, Y; f)) \otimes \mathbb{k} \to HH^*(\wedge V; B).$$

Moreover $\pi_*(\text{map}(X, Y; f)) \otimes \mathbb{k} \cong HH^*_1(\wedge V; B)$.

The result is a generalization of the inclusion $\pi_*(\Omega \text{ map}(X, X; 1_X)) \otimes \mathbb{k} \to HH^*(\wedge V; \wedge V)$. See [7, Theorem 2] and [9, Theorem 1.1].

## 2. $L_\infty$-MODELS OF MAPPING SPACES

The notion of $L_\infty$ algebra was introduced by Lada [14] and $L_\infty$ models of mapping spaces were used by Félix et al. in [3,4]. We remind here their definition.

**Definition 4.** A permutation $\sigma \in S_k$ is called an $(i, k-i)$ shuffle if $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(k)$ where $i = 1, \ldots, n$. For graded objects $x_1, \ldots, x_k$, the Koszul sign $\epsilon(\sigma)$ is determined by

$$x_1 \wedge \cdots \wedge x_k = \epsilon(\sigma)x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k)}.$$

It depends not only of the permutation $\sigma$ but also on degrees of $x_1, \ldots, x_k$.

**Definition 5.** An $L_\infty$-algebra or a strongly homotopy Lie algebra is a graded vector space $L = \bigoplus_i L_i$ with maps $\ell_k := [\ldots, 1] : L^\otimes k \to L$ of degree $k - 2$ such that

1. $\ell_k$ is graded skew symmetric, that is, for a $k$-permutation $\sigma$

$$\ell_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \text{sgn}(\sigma)\epsilon(\sigma)\ell_k(x_1, \ldots, x_k),$$

where $\text{sgn}(\sigma)$ is the sign of $\sigma$,

2. There are generalized Jacobi identities

$$\sum_{i+j=k+1} \sum_\sigma \epsilon(\sigma)(-1)^{(i-k)} \ell_j(\ell_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(k)}) = 0,$$

where the second summation extends to all $(i, k-i)$ shuffles of the symmetric group $S_k$. 
In particular if $\ell_k = 0$ for $k \geq 3$, one recovers the notion of differential graded Lie algebra $(L, d)$ where $[x, y] := \ell_2(x, y)$ and $dx = \ell_1(x)$.

There is a 1-1 correspondence between $L_\infty$ structures on $L$ and codifferentials $d_n : \wedge^m(sL) \to \wedge^{m-n+1}(sL)$ of degree $-1$ on the coalgebra $\wedge sL$, such that $d^2 = 0$, where $d = d_1 + d_2 + \cdots + d_n + \ldots$ [14].

**Definition 6 ([12]).** Let $(A, \mu)$ be a commutative algebra and $D : A \to A$ an operator. Define multi-brackets on $A$ as follows.

$$
\begin{align*}
F_D^1(a) &= Da \\
F_D(a_1, \ldots, a_n) &= \mu((D \otimes 1)(a_1 \otimes 1 - 1 \otimes a_1) \cdots (a_n \otimes 1 - 1 \otimes a_n)).
\end{align*}
$$

Then $D$ is called an operator of order $n$ if $F_D^{n+1} = 0$.

There is a generalization of multi-brackets to non commutative algebras that is due to Akman [1].

**Definition 7.** A Gerstenhaber algebra is a graded commutative algebra $A = \oplus_i A_i$ together with a bracket

$$
A_i \otimes A_j \to A_{i+j+1}, \quad a \otimes b \mapsto \{a, b\},
$$

such that $sL$ is a graded Lie algebra and the bracket acts like a derivation of algebras. That is, for $a, b, c \in A$,

1. $\{a, b\} = -(-1)^{|a|+1}|b|\{b, a\}$,
2. $\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{|a|+1}|b|\{b, \{a, c\}\}$,
3. $\{a, bc\} = \{a, b\}c + (-1)^{|b|(|a|+1)}b\{a, c\}$.

**Definition 8.** A Batalin–Vilkovisky algebra (BV-algebra for short) is a graded commutative algebra $A$, together with an operator $\Delta : A_i \to A_{i+1}$ of order 2 and of square 0.

Any BV-algebra $(A, \Delta)$ is a Gerstenhaber algebra with the bracket defined by

$$
[a, b] = (-1)^{|a|}\Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b).
$$

**Definition 9 ([13,2]).** A commutative $BV_\infty$-algebra is a graded commutative algebra $A = \oplus_{i \in \mathbb{Z}} A_i$ together with an operator $D = \sum_{i \geq 1} D_i$ such that $D^2 = 0$ and each $D_n$ is an operator of order $n$ and of degree $2n - 3$.

From the relation $D^2 = 0$, one gets $D_1^2 = 0$, hence $D_1$ is a differential on the algebra $A$. Moreover $D_1D_2 + D_2D_1 = 0$, therefore $D_2$ induces an action on the homology $H_n(A, D_1)$ which induces a BV-algebra structure [13]. If $D_i = 0$ for all $i \geq 3$, then $(A, D_1 + D_2)$ is called a differential BV-algebra.

**Definition 10.** Let $\phi : (A, d) \to (B, d)$ be a morphism of cdga’s. A $\phi$-derivation of degree $k$ is a linear mapping $\theta : A^n \to B^{n-k}$ such that $\theta(ab) = \theta(a)\phi(b) + (-1)^{|a|}\phi(a)\theta(b)$. We denote by $\text{Der}_n(A, B; \phi)$ the vector space of $\phi$-derivations of degree $n$ and by $\text{Der}(A, B; \phi) = \oplus_n \text{Der}_n(A, B; \phi)$ the $\mathbb{Z}$-graded vector space of all $\phi$-derivations. The differential on $\text{Der}(A, B; \phi)$ is defined by $\delta \theta = d\theta - (-1)^k \theta d$. 
Theorem 13. Let $A = B$ and $\phi = 1_A$, then we get the Lie algebra of derivations $\text{Der} A$, where the Lie bracket is the commutator bracket. If $V$ is finite, then $\text{Der}(\wedge V) \cong \wedge V \otimes V^\#$. We have the following result for $\phi$-derivations.

**Proposition 11.** Let $\phi : (\wedge V, d) \to (B, d)$ be a surjective morphism between cdga’s where $V$ is finite dimensional and $I = \text{Ker} \phi$. Then $\text{Der}(\wedge V, B; \phi) \cong \wedge V/I \otimes V^\#$.

**Proof.** Let $\{v_1, \ldots, v_k\}$ be a basis of $V$. In $\text{Der}(\wedge V, B; \phi)$, we denote by $(v_i, 1)$ the $\phi$-derivation $\theta_i$ such that $\theta_i(v_i) = \delta_{ij}$. We observe that $v_i^\#$ corresponds to the derivation $\theta_i = (v_i, 1)$. Let $\theta$ be a $\phi$-derivation. Then $\theta(v_i) = b_i$, where $b_i \in B$. As $\phi$ is surjective, there exist $a_i \in \wedge V$ such that $\phi(a_i) = b_i$. Hence $\theta = \sum_i a_i \theta_i = a_i v_i^\#$. By the first isomorphism theorem $\text{Der}(\wedge V, B; \phi) \cong \wedge V/I \otimes V^\#$. $\square$

Define $\widetilde{\text{Der}}(A, B; \phi)$ as follows.

$$\widetilde{\text{Der}}(A, B; \phi)_i = \begin{cases} \text{Der}(A, B; \phi)_i, & i > 1, \\ \{ \theta \in \text{ Der}_1(A, B; \phi) : \delta \theta = 0 \}, & i = 1. \end{cases}$$

Let $A = \wedge V$ and $\theta_1, \ldots, \theta_k \in \widetilde{\text{Der}}(\wedge V, B; \phi)$ be $\phi$-derivations of respective degrees $n_1, \ldots, n_k$, define

$$[\theta_1, \ldots, \theta_k](v) = (-1)^{\eta(k)} \sum_{i_1, \ldots, i_k} \epsilon \phi(v_1 \ldots \hat{v}_{i_1} \ldots \hat{v}_{i_k} \ldots v_m) \theta_1(v_{i_1}) \ldots \theta_k(v_{i_k}),$$

where $d v = \sum v_1 \ldots v_m$, $\eta(j) = n_1 + \cdots + n_k - 1$, and $\epsilon$ is the corresponding Koszul sign of the permutation

$$(v_1, \ldots, v_m) \to (v_1, \ldots, \hat{v}_{i_1}, \ldots, \hat{v}_{i_k}, \ldots, v_m, v_{i_1}, \ldots, v_{i_k}).$$

We note that $[\theta_1, \ldots, \theta_k]$ is of degree $n_1 + \cdots + n_k - 1$. Now define linear maps $\ell_k$ of degree $k - 2$ on $s^{-1}\text{Der}(\wedge V, B; \phi)$ by

$$\ell_1(s^{-1}\theta) = -s^{-1}\delta \theta, \quad \ell_k(s^{-1}\theta_1, \ldots, s^{-1}\theta_k) = (-1)^k s^{-1}[\theta_1, \ldots, \theta_k],$$

where $\epsilon_k = \frac{k(k-1)}{2} + \sum_{i=1}^{k-1} (k-i)|\theta_i|\ [4]$.

**Proposition 12 (Lemma 3.3,[4]).** If $\phi : \wedge V \to B$ is a Sullivan model of a mapping $f : X \to Y$ between simply connected spaces and $V$ is finite dimensional, then $(s^{-1}\text{Der}(\wedge V, B; \phi), \ell_k)$ is an $L_\infty$ model of map$(X, Y; f)$.

**Theorem 13.** Let $(\wedge V, d) \to (B, d)$ be a cdga model of map $f : X \to Y$ between 1-connected spaces of finite type where $Y$ is finite dimensional.

1. Then there is a natural isomorphism

$$\Gamma : \pi_*(\Omega \text{ map}(X, Y; f)) \otimes \mathbb{Q} \to H H^*_{(1)}(\wedge V; B),$$

2. Moreover the following diagram commutes:

$$\begin{array}{c}
\pi_*(\text{aut}_1 Y) \otimes \mathbb{Q} \longrightarrow \pi_*(\text{map}(X, Y; f)) \otimes \mathbb{Q} \\
\downarrow \quad \downarrow \\
H H^*(\wedge V; \wedge V) \longrightarrow H H^*(\wedge V; B).
\end{array}$$
Proof of the theorem. Before we prove the theorem, we need a generalization of derivations.

Definition 14. Let $A$ be a commutative cochain algebra and $M$ a differential $A$-module (considered here as an $A$-bimodule). A derivation $\theta$ from $A$ to $M$ of degree $k$ is a linear map $\theta : A^* \to M^{*-k}$ such that $\theta(ab) = \theta(a)b + (-1)^{|a|}a\theta(b)$.

It is easily seen that if $\theta : A \to M$ is derivation and $f : M \to N$ a morphism of $A$-bimodules, then the composition $f \circ \theta : A \to N$ is a derivation.

Let $(\wedge V, d)$ be a Sullivan model of a simply connected space. Define $\tilde{V} = sV$, that is, $\tilde{V}^n = V^{n+1}$. A Sullivan model of the loop space map $(\wedge(V \oplus \tilde{V}), D)$, the cdga defined in Section 1. For recall, $Dv = dv, \tilde{D}v = -S(dv)$ where $S$ is the unique derivation defined by $Su = \tilde{u}$ and $\tilde{S}u = 0$ [6].

Consider the linear map $S : (\wedge V, d) \to (\wedge V \otimes \tilde{V}, D)$ defined $Sv = \tilde{v}$ and extended $S$ as a derivation in the sense of Definition 14. As $S(dv) = -D(Sv)$, then $Sd + DS = 0$, then $S$ is a morphism of differential modules of upper degree $-1$.

We define a map

$$\Phi : \text{Hom}_{\wedge V}(\wedge V \otimes \tilde{V}, B) \to \text{Der}(\wedge V, B; \phi)$$

such that $\Phi(f)$ is the following composition mapping

$$\wedge V \xrightarrow{S} \wedge V \otimes \tilde{V} \overset{f}{\longrightarrow} B,$$

that is, $\Phi(f)(v) = f(\tilde{v})$.

Lemma 15. The map $\Phi$ commutes with differentials.

Proof. Let $f \in \text{Hom}_{\wedge V}(\wedge V \otimes \tilde{V}, \wedge V)$.

$$(Df)(\tilde{v}) = d(f(\tilde{v})) - (-1)^{|f|}f(D(\tilde{v})) = d(f(\tilde{v})) + (-1)^{|f|}(f(sdv)),$$

hence $(\Phi(Df))(v) = d(f(\tilde{v})) + (-1)^{|f|}(f(sdv))$.

On the other hand

$$(D\Phi(f))(v) = d(\Phi(f)(v)) - (-1)^{|\Phi(f)|}\Phi(f)(dv) = d(f(sv)) + (-1)^{|f|}f(sdv).$$

Hence $\Phi$ is a morphism of chain complexes. \qed

Moreover, there are isomorphisms of vector spaces $\text{Hom}_{\wedge V}(\wedge V \otimes \tilde{V}, B) \cong \text{Hom}(\tilde{V}, B) \cong \text{Der}(\wedge V, B)$. Hence $\Phi$ is bijective. Therefore

$$H_n(s^{-1}\text{Der}(\wedge V, B)) \cong HH_{(1)}^{*}(\wedge V, B) \hookrightarrow HH^{*}(\wedge V; B).$$

Remark 16. It was shown that if $L$ is an $L_\infty$-algebra, then $\wedge s^{-1}L$ is a $BV_\infty$ algebra [2]. It would be interesting to find a link between the $BV_\infty$-algebra $\wedge s^{-1}L$ and $HH^{*}(\wedge V; B)$.

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