On some variational problems in geometry

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Summary

In this thesis we consider several variational problems in geometry that have a connection to the spectrum of the Laplacian acting on functions.

In the first part, we study a quantity called the analytic systole, which was defined in recent joint work of the author with Werner Ballmann and Sugata Mondal. The motivation to study the analytic systole is its connection to small eigenvalues on Riemannian surfaces of finite topological type. We prove qualitative bounds comparing the analytic systole to the classical systole in the presence of curvature bounds. Moreover, we prove that in many situations the analytic systole is bounded strictly from below by the bottom of the spectrum of the universal covering, thereby further refining bounds on small eigenvalues. These results are published in *Geometric and Functional Analysis* [**BMM17a**].

The next few chapters deal with extremal metrics for Laplace eigenalues. Extremal metrics somewhat resemble the notion of critical points for the non-smooth functionals given by the eigenvalues of the Laplacian up to normalization. We study these functionals either on the space of all metrics with normalized volume or on a fixed conformal class. More precisely, we are interested in questions related to existence and regularity of extremal metrics.

Firstly, we give an existence result for maximizers for the first eigenvalue on non-orientable surface relying on two spectral gap assumptions that prevent degenerations of a carefully chosen maximizing sequence in the moduli space. A similar spectral gap assumption occurs in a recent result of Petrides dealing with the orientable case. Slightly more general than actually required, these spectral gap assumptions ask, whether it is possible to strictly increase the quantity $\lambda_1 \cdot$ area by attaching a handle or a cross cap to a given closed Riemannian surface. We establish this under some extra assumptions. Unfortunately, our assumptions are too restrictive to establish the existence of maximizers at this point. However, there are some examples to which our techniques apply. In particular, we obtain the existence of a maximizing metric for the first eigenvalue on the surface of genus three. Large parts of these results are contained in a preprint jointly with Anna Siffert, where we claim a much stronger result, [**MS17**]. The arguments in [**MS17**] contain a significant gap, which we do not know how to fix at the moment, leading to the weaker results presented in this thesis.

Next, we consider extremal metrics for eigenvalues in a conformal class. Exploiting a connection of extremal metrics and n-harmonic maps we give an existence result for extremal metrics in perturbed conformal classes on products. A shortened version of this already appeared in the preprint [**Mat17**]. In a similar way the connection to n-harmonic maps is used to prove a regularity result for extremal metrics.

Finally, we exhibit a natural class of metrics with an integral scalar curvature bound in which one can maximize the first eigenvalue. More abstractly, we prove a regularity result for the Yamabe equation under an L^p-scalar curvature bound, provided the first eigenvalue is sufficiently large. By standard compactness results, this then easily implies the existence of a sharp eigenvalue bound in this class of metrics. The results of this chapter are published in *Annales de la Faculté des Sciences de Toulouse. Mathématiques. Série 6*, [**Mat16**].

In the last part, motivated by recent results on sharp eigenvalue bounds for the first eigenvalue on closed surfaces, we study the geometry of embedded, minimal surfaces of unbounded genus in ambient three-manifolds. Our main result here states that the systole of such a sequence tends to zero if the ambient manifold has positive Ricci curvature. The contents of this chapter are joint work with Anna Siffert and are contained in the preprint [**MS18**]

Contents

CHAPTER 1

Introduction

This thesis is concerned with variational problems in geometry that have some connection to the eigenvalues or, more generally, the spectrum of the Laplace operator acting on functions. Before we discuss the contents, we briefly present some general background material on the Laplacian.

1.1. The Laplacian

Given a Riemannian manifold (M, g) the *Laplacian* is the second order differential operator given by

$$\Delta u = -\operatorname{div}(\nabla u),$$

for any smooth function $u \in C^\infty(M)$. The minus sign gives the geometer's sign convention that makes the operator positive, i.e.

$$\int_M v\Delta u = \int_M \nabla u \cdot \nabla v$$

for any two smooth, compactly supported functions u and v. The Laplacian is an elliptic differential opearator of second order, as it is given in local coordinates by

$$\Delta u = -\frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j u\right),$$

where we make use of the summation convention. As usual, (g^{ij}) denotes the inverse of (g_{ij}) and $\sqrt{g}$ is shorthand for $\sqrt{\det(g_{ij})}$.

If the manifold M is complete and with (possible empty) smooth, compact boundary, it can be shown, that Δ acting on $C_c^\infty(M)$ (the smooth functions on M having compact support in $M\setminus\partial M$) extends to a self-adjoint operator with domain $\operatorname{dom}(\Delta) = W^{2,2}(M)\cap W_0^{1,2}(M)$. If the boundary of M is non-empty, we will also refer to this operator as the *Dirichlet Laplacian.* Its spectrum is given by

$$\operatorname{spec}(\Delta) = \{\lambda \in \mathbb{C} \ : \ (\Delta - \lambda \operatorname{id})\colon \operatorname{dom}(\Delta) \to L^2(M) \text{ is not invertible}\}.$$

If M is compact, the embedding $W^{1,2}(M) \hookrightarrow L^2(M)$ is compact and, combined with standard elliptic estimates, this easily implies that $\operatorname{spec}(\Delta)$ consists of eigenvalues only. We call $\lambda \in \operatorname{spec}(\Delta)$ an *eigenvalue* if the following three properties hold:

- There is a non-trivial solution $u \in C^\infty(M) \cap \operatorname{dom}(\Delta)$ to

$$\Delta u = \lambda u, \tag{1.1.1}$$

- the space of these solutions is finite dimensional, and
- λ is isolated in $\operatorname{spec}(\Delta)$.

In this case any non-trivial solution u to (1.1.1) is called an *eigenfunction*. Note that this definition makes sense for compact as well as non-compact manifolds. However, if M is non-compact, $\operatorname{spec}(\Delta)$ might not consist of eigenvalues only. In fact, it can even happen that there are no eigenvalues at all. The most basic example for this is Euclidean space $\mathbb{R}^n$, in which case $\operatorname{spec}(\Delta) = [0, \infty)$ but no eigenvalues exist at all. It can happen that there is more than one linearly independent solution to (1.1.1), but by definition there can be at most finitely many of these. The number of linearly independent solutions is called the *multiplicity* of the eigenvalue. Since Δ is self-adjoint and non-negative, we also have that

$$\operatorname{spec}(\Delta) \subset [0, \infty).$$

Thus, if M is compact, we can list the eigenvalues as

$$0 = \lambda_0(M, g) < \lambda_1(M, g) \leq \lambda_2(M, g) \leq \cdots \to +\infty,$$

where each eigenvalue is repeated as often as its multiplicity requires.

The eigenvalues also admit a variational characterization. In fact, we have

$$\lambda_k(M, g) = \inf_V \sup_{u \in V \setminus \{0\}} \frac{\int_M |\nabla u|^2}{\int_M u^2}, \tag{1.1.2}$$

where the infimum is taken over all $(k+1)$-dimensional subspaces of $C_c^\infty(M)$. The fraction on the right hand side is called the *Rayleigh quotient* of u. The characterization (1.1.2) has various useful consequences. For instance, (1.1.2) makes sense even if the metric is not smooth but much less regular. This can be used to extend the notion of eigenvalues to more singular metrics. Moreover, a standard method to give estimates for λ_k from above is to construct a $(k+1)$-dimensional subspace of $C_c^\infty(M)$, such that the Rayleigh quotient of any function in this space can be controlled from above.

Another observation is that the right hand side of (1.1.2) also makes sense if M is non-compact. In particular, for any manifold, compact or non-compact, we can define

$$\lambda_0(M, g) = \inf_{u \in C_c^\infty(M) \setminus \{0\}} \frac{\int_M |\nabla u|^2}{\int_M u^2}.$$

For any manifold, λ_0, called the *bottom of the spectrum*, has the useful property that one always has

$$\inf \operatorname{spec}(\Delta) = \lambda_0(M, g).$$

It is one of the most classical subjects in geometry and analysis to understand the interaction of the geometric and the analytic aspects related to the Laplace operator. More precisely, the geometric aspects refer to quantities such as curvature, diameter, volume, or topology, whereas analytic aspects are bounds on the eigenvalues or their asymptotic properties. We are also concerned with this subject, mostly in one way or another in terms of eigenvalue bounds.

1.2. The analytic systole of a Riemannian surface

It was known for a long time that the first $2\gamma - 3$ eigenvalues on a closed hyperbolic surface, i.e. a closed, orientable surface endowed with a metric of constant curvature -1, of genus γ behave fundamentally different from the other eigenvalues. This led to a conjecture attributed to Buser and Schmutz, stating that, for any closed hyperbolic surface S of genus γ the eigenvalue bound

$$\lambda_{2\gamma-2}(S) > 1/4 \tag{1.2.1}$$

should hold. The right hand side is exactly the bottom of the spectrum of the hyperbolic plane, which is the universal covering of any hyperbolic surface.

The conjectured bound (1.2.1) was recently proved by Otal and Rosas [**OR09**] in much greater generality. Otal and Rosas showed that if (S, g) is closed, orientable, negatively curved, and g is analytic, then

$$\lambda_{-\chi(S)}(S) > \lambda_0(\tilde{S}).$$

In joint work with Werner Ballmann and Sugata Mondal, we extended this result to all complete Riemannian surfaces of finite topological type and sharpened the lower bound in terms of a quantity, which has an analytic as well as a geometric flavour, [**BMM16, BMM17b**]. In case of non-empty boundary, we assume the boundary to be smooth and compact and our results refer to the Dirichlet Laplacian as introduced in Section 1.1. If S is non-compact, the spectrum might not consist of eigenvalues only. This forces us to chose a slightly different formulation. For any complete Riemannian surface of finite topological type with $\chi(S) < 0$, we showed that

$$\#\{\lambda \le \Lambda(S) \;:\; \lambda \text{ is an eigenvalue}\} \le -\chi(S), \tag{1.2.2}$$

where $\Lambda(S)$ is the so-called *analytic systole* of S. This is defined by

$$\Lambda(S) = \inf_{\Omega} \lambda_0(\Omega),$$

with the infimum taken over over all smooth subdomains $\Omega \subset S$ diffeomorphic to a disk, an annulus, or a Möbius strip. By a result of Brooks one always has

$$\Lambda(S) \ge \lambda_0(\tilde{S}). \tag{1.2.3}$$

Another quantity related to the analytic systole is $\lambda_{\text{ess}}(S)$, the *bottom of the essential spectrum.* The *essential spectrum* of S is given by

$$\operatorname{spec}_{\text{ess}}(S) = \{\lambda \in \operatorname{spec}(\Delta) \;:\; (\Delta - \lambda\,\mathrm{id}) \text{ is } \textit{not} \text{ a Fredholm operator}\}.$$

In analogy with the bottom of the spectrum, we define the *bottom of the essential spectrum* to be

$$\lambda_{\text{ess}}(S) = \inf \operatorname{spec}_{\text{ess}}(S).$$

If S is of finite type, any end is topologically an annulus. Therefore, using Weyl sequences, it is not hard to see that

$$\Lambda(S) \le \lambda_{\text{ess}}(S). \tag{1.2.4}$$

The eigenvalue bound (1.2.2) as well as the relation to the spectral quantities in (1.2.3) and (1.2.4) motivate a closer inspection of the analytic systole. The corresponding results are contained in Chapter 2. This is joint work with Werner Ballmann and Sugata Mondal and published in [**BMM17a**].

Our main quantitative result characterizes precisely when the strict inequality

$$\Lambda(S) > \lambda_0(\tilde{S}) \tag{1.2.5}$$

holds. By (1.2.3) and (1.2.4), an obious obstruction to this is that we need to have $\lambda_{\mathrm{ess}}(S) > \lambda_0(\tilde{S})$. Moreover, up to removing sets of vanishing capacity, S itself should not be a competitor for $\Lambda(S)$. For example, if S is a sphere $S \setminus B(p,\varepsilon)$ is a disk, but $\lambda_0(S \setminus B(p,\varepsilon)) \to 0 = \lambda_0(\tilde{S})$ as $\varepsilon \to 0$.

Theorem 1.2.6 (Theorem 2.1.7)**.** *If S is a complete and connected Riemannian surface of finite type whose fundamental group is not cyclic, then $\Lambda(S) > \lambda_0(\tilde{S})$ if and only if $\lambda_{\mathrm{ess}}(S) > \lambda_0(\tilde{S})$.*

In particular, if S is compact, it follows that $\mathrm{spec}_{\mathrm{ess}}(S) = \emptyset$ and thus the strict inequality (1.2.5) holds on any compact surface with $\chi(S) \leq 0$.

This qualitative bound is related to an interesting new compactness property of the ground states of competing domains for the analytic systole. The *ground state* of a smooth, compact domain $\Omega \subset S$ is the unique non-negative, L^2-normalized solution to

$$\Delta u = \lambda_0(\Omega)u,$$

which vanishes along $\partial\Omega$. If Ω is a disk, we can lift it to a disk $\tilde{\Omega} \subset \tilde{S}$ and can consider the lift $\tilde{u}$ of u to $\tilde{\Omega}$. The trivial bound $\|\tilde{u}\|_{W^{1,2}(\tilde{S})} \leq 1+\lambda_0(\Omega)$ is not very helpful, since the non-compactness of $\tilde{S}$ prevents the embedding $W^{1,2}(\tilde{S}) \hookrightarrow L^2(\tilde{S})$ from being compact. However, it turns out that the two conditions $\lambda_0(\Omega) \leq \theta\lambda_{\mathrm{ess}}(S)$ for some $\theta < 1$ and $\chi(\Omega) \geq 0$ imply restrictions on the geometry of the superlevel sets $\Omega_t = \{x \in \Omega \;:\; u \geq t\}$. Clearly, these conitinue to hold for the superlevel sets of $\tilde{u}$. These restrictions play a pivotal role in gaining the needed compactness to obtain the qualtitative bound (1.2.5).

We also prove quantitative bounds on the analytic systole that compare it to the classical systole in the presence of curvature bounds. Recall that the *systole* of a closed surface Σ is the length of a shortest non-contractible curve,

$$\mathrm{sys}(\Sigma) = \inf\{\mathrm{length}(c) \;:\; c\colon S^1 \to \Sigma \text{ non-contractible}\}.$$

The lower bound we prove is the following.

Theorem 1.2.7 (Theorem 2.1.11)**.** *For a closed Riemannian surface S with curvature $K \leq \kappa \leq 0$, we have*

$$\Lambda(S) \geq -\frac{\kappa}{4} + \frac{\mathrm{sys}(S)^2}{|S|^2}.$$

This is accompanied by an upper bound, which requires a lower curvature bound, that we normalize to be -1. We call a geodesic a *systolic geodesic* if it realizes the systole.

Theorem 1.2.8 (Theorem 2.1.13)**.** *If S is a closed Riemannian surface with $\chi(S) < 0$ and curvature $K \geq -1$, then*

$$\Lambda(S) \leq \frac{1}{4} + \frac{4\pi^2}{w^2},$$

where

$$w = w(\mathrm{sys}(S)) = \begin{cases} \mathrm{arsinh}(1/\sinh(\mathrm{sys}(S)/2))) \\ \mathrm{arsinh}(1/\sinh(\mathrm{sys}(S))) \end{cases}$$

if S has a two-sided systolic geodesic or if all systolic geodesics of S are one-sided, respectively.

Finally, we investigate how generic the strict inequlaity $\lambda_{\text{ess}}(S) > \lambda_0(\tilde{S})$, and therefore also the strict inequality (1.2.5), is among all smooth metrics on a non-compact surface. It turns out that this is in fact the case precisely when $\lambda_0(\tilde{S}) > 0$.

1.3. Extremal metrics for eigenvalue functionals

A rather new subject related to Laplace eigenvalues are extremal metrics. To the author's knowledge, the ideas around this notion appeared first in [**Nad96**] and were then systematically developed in [**ESI00, ESI03, ESI08**]. Given a closed manifold M, one can consider the k-th eigenvalue as functionals

$$\lambda_k\colon \mathcal{R} \to \mathbb{R} \tag{1.3.1}$$

and

$$\lambda_k\colon [g] \to \mathbb{R}, \tag{1.3.2}$$

where

$$\mathcal{R} = \{g \ : \ g \text{ is a smooth Riemannian metric on } M \text{ with } \operatorname{vol}(M,g) = 1\}$$

and

$$[g] = \{\phi g \ : \ \phi \in C^\infty(M), \phi > 0, \operatorname{vol}(M, \phi g) = 1\}.$$

Both of these functionals are in general not smooth but only Lipschitz. The notion of an extremal metric resembles somewhat the notion of a critical point. We can not use the notion of critical points themselves here, since the eigenvalues do not have the required regularity properties. Two illustrating examples that are among the class of extremal metrics are local minima and maxima. To be precise, we call a metric g *extremal* for problem (1.3.1) if

$$\frac{d}{dt}\lambda_k(M, g_t)\bigg|_{t=0-} \frac{d}{dt}\lambda_k(M, g_t)\bigg|_{t=0+} \leq 0 \tag{1.3.3}$$

for any analytic, volume preserving deformation $(g_t) \subset \mathcal{R}$ with $g_0 = g$. It should be remarked here that the left and right derivative in (1.3.3) always exist. Extremal metrics for (1.3.2) are defined completely analogously, but require the deformation to stay inside $[g]$.

One reason why extremal metrics are interesting is that they have a connection to more classical objects from differential geometry.

The corresponding object for (1.3.1) are minimal immersions into spheres. More generally, a (possibly branched) immersion $\Phi\colon M \to N$ is called *minimal* if it is a critical point of the area functional among all compactly supported variations. Equivalently, the mean curvature of the (locally embedded) image vanishes at any unbranched point. If a metric g is extremal for (1.3.1), there is a family $(u_1, \dots, u_{\ell+1})$ of λ_k-eigenfunctions such that $u = (u_1, \dots, u_{\ell+1})\colon (M, g) \to S^\ell$ defines a branched immersion. Since each component function is an eigenfunction corresponding to the same eigenvalue, u is then automatically minimal.

Similarly, (1.3.2) is related to harmonic maps. A map $u\colon M \to N$ is called *harmonic* if it is a critical point of the energy functional

$$\int_M |du|^2 dV_g$$

with respect to all deformations in N. We want to point out that the latter point is related to some delicate regularity issues. If also variations in the domain are allowed one obtains so-called *stationary harmonic maps* which posses better regularity properties than harmonic maps.

If $N = S^\ell$, the differential equation satisfied by harmonic maps is given by

$$\Delta u = |\nabla u|^2 u.$$

Harmonic maps arising in the context of extremal metrics for (1.3.2) have the additional property of having constant energydensity, i.e. $|\nabla u|^2$ is constant. Clearly, this implies that there is a family of λ_k-eigenfunctions $(u_1, \dots, u_{\ell+1})$ such that we obtain a harmonic map $u = (u_1, \dots, u_{\ell+1})\colon (M, g) \to S^\ell$. Therefore, we also call a constant density harmonic map an *eigenmap*. In contrast to extremal metrics for (1.3.1) this might not be an immersion. It will be important for us that the existence of a constant density harmonic map also implies extremality for (1.3.2).

Extremal metrics also play a role in connection with sharp eigenvalue bounds. In terms of the functionals (1.3.1) and (1.3.2) the question on the existence of sharp eigenvalue bounds simply asks:

Given a Riemannian manifold (M, g), do (1.3.1) and (1.3.2) attain global maxima.

In modern approaches to sharp eigenvalue bounds, the connection to harmonic maps and minimal surfaces plays an important role to gain compactness for carefully chosen maximizing sequences. An example for this is given in Section 1.4.

1.4. Maximizing the first eigenvalue on non-orientable surfaces

As an important example for extremal metrics we mentioned global maxima. It turns out that the functional (1.3.2) is always bounded from above, [**Kor93**]. Therefore, one can ask whether (1.3.2) attains a global maximum on any closed manifold M endowed with a conformal class $[g]$. This is completely open if the dimension of M is at least three. We discuss two related problems in Section 1.7 and Section 1.8. In dimension two, maximizers always exist as proved recently by Petrides [**Pet14**]. The reason for the dimensional restriction of this result is twofold. On the one hand, the Laplace operator is conformally covariant in dimension two, which simplifies the analysis significantly. On the other hand, in order to exploit the connection of extremal metrics and harmonic maps, it is very helpful to fix the dimension of the codomain, i.e. to consider only harmonic maps $\Phi\colon (\Sigma, g) \to S^N$ for *fixed* N. In dimension two, this is automatic since the multiplicity of the k-th eigenvalue can be bounded purely in terms of the topology of Σ and k. It is known, that such a bound does not hold in higher dimensions without additional assumptions on the geometry.

In contrast, the functional (1.3.1) is never bounded from above if the dimension of M is at least three. The situation is very different for surfaces. In fact, by [**Kor93**] the functionals (1.3.1) are bounded from above on any closed surface.

We write

$$\Lambda_1(\Sigma) = \sup_g \lambda_1(\Sigma, g) \operatorname{area}(\Sigma_g)$$

where Σ is a closed surface and g runs over all smooth metrics on Σ.

If Σ is a closed, orientable surfaces of genus γ, we write $\Lambda_1(\gamma) = \Lambda_1(\Sigma)$. Similarly, if Σ is a closed, non-orientable surface of non-orientable genus δ, we write $\Lambda_1^K(\delta) = \Lambda_1(\Sigma)$.

Petrides proved the following result regarding the existence of maximizers for the first eigenvalue on orientable surfaces.

Theorem 1.4.1 ([**Pet14**]). *If $\Lambda_1(\gamma - 1) < \Lambda_1(\gamma)$, there is a metric, smooth away from at most finitely many conical singularities, achieving $\Lambda_1(\gamma)$.*

In Chapter 3, we extend this result to non-orientable surfaces. This has been obtained in joint work with Anna Siffert.

Theorem 1.4.2 (Theorem 3.1.2). *If $\Lambda_1^K(\delta - 1) < \Lambda_1^K(\delta)$ and $\Lambda_1(\lfloor(\delta - 1)/2\rfloor) < \Lambda_1^K(\delta)$, there is a metric smooth away from at most finitely many conical singularities achieving $\Lambda_1^K(\delta)$.*

The reason why we need two compared to only a single spectral assumption in Theorem 1.4.1 stems from the possible degenrations in the moduli space of non-orientable surfaces. As for orientable surfaces, non-orientable surfaces can degenerate via neck-pinches. However, there are more options for these degenerations if the initial surface is non-orientable. It can degenerate to a non-orientable surface with lower non-orientable genus. But it can also degenerate to an orientable surface.

As advertised earlier, the proof uses the connection of extremal metrics and harmonic maps. Instead of taking any maximizing sequence, we can take a sequence of metrics (g_k) on Σ with unit area, such that g_k maximizes the first eigenvalue in its own conformal class $[g_k]$. Such a choice is possible thanks to a result from [**Pet14**]. In particular, there are associated harmonic maps $\Phi_k\colon (\Sigma, h_k) \to S^N$, where $h_k \in [g_k]$ is the unique hyperbolic metric. Moreover, Φ_k and h_k determine g_k. We are then concerned with the possible degenerations of the hyperbolic metrics h_k and the harmonic maps Φ_k. It is helpful to work with the hyperbolic metrics here, since their possible degenerations can be described very explicitly. This is in turn useful in the analysis of the corresponding harmonic maps.

1.5. Attaching handles and cross caps and the first eigenvalue

Given Theorem 1.4.1 and Theorem 1.4.2, it is natural to aks when the assumptions in these results actually hold. More generally, we want to ask the following question: Given a closed Riemannian surface (Σ, g), is it possible to construct a closed Riemannian surface (Σ', g'), which is topologically obtained from Σ by adding a handle or a cross cap, such that

$$\lambda_1(\Sigma', g')\,\mathrm{area}(\Sigma', g') > \lambda_1(\Sigma, g)\,\mathrm{area}(\Sigma, g)?$$

In Chapter 4, we present joint work with Anna Siffert, in which we exhibit two situations in which this is possible. The first one applies to both, handles and cross caps.

Theorem 1.5.1 (Theorem 4.1.1). *Let (Σ, g) be a closed Riemannian surface and assume that there is a point $x \in \Sigma$ such that $u(x) = 0$ for any $\lambda_1(\Sigma, g)$-eigenfunction u. Let Σ' be the surface obtained from Σ by attaching a handle or a cross cap. Then there is a metric g' on Σ' such that*

$$\lambda_1(\Sigma', g')\,\mathrm{area}(\Sigma', g') > \lambda_1(\Sigma, g)\,\mathrm{area}(\Sigma, g). \tag{1.5.2}$$

Unfortunately, the assumptions from the first result are never satisfied by maximizers. The second result is more general, but deals only with attaching handles.

Theorem 1.5.3 (Theorem 4.1.3)**.** *Let (Σ, g) be a closed Riemannian surface and assume that there are points $x, y \in \Sigma$ such that $u(x) = -u(y)$ for any $\lambda_1(\Sigma, g)$-eigenfunction u. Let Σ' be the surface obtained from Σ by attaching a handle. Then there is a metric g' on Σ' such that*

$$\lambda_1(\Sigma', g')\,\mathrm{area}(\Sigma', g') > \lambda_1(\Sigma, g)\,\mathrm{area}(\Sigma, g). \tag{1.5.4}$$

The proofs of both these results follow similar ideas. We remove balls of radius ε^k for sufficiently large k and then attach a flat cylinder or flat cross cap of radius ε along its boundary. As ε tends to zero, these surfaces have area approximately $\mathrm{area}(\Sigma) + 2\pi\varepsilon h$ up to terms of higher order, where h denotes the height of the cross cap and cylinder, respectively. The goal is then to show that the first eigenvalue is not smaller than $\lambda_1(\Sigma) - o(1)\varepsilon$, which will then imply the results.

Very recently, an explicit maximizing metric on the surface of genus two was constructed, [**NS17b**]. This metric has lots of symmetries. In particular, it satisfies the assumptions from Theorem 1.5.3.

Corollary 1.5.5. *There is a metric, smooth away from at most finitely many conical singularities, achieving $\Lambda_1(3)$.*

1.6. Existence of extremal metrics in perturbed conformal classes on products

As we have indicated already, it is a rather difficult problem to find extremal metrics for (1.3.1) or (1.3.2). In dimensions $n \geq 3$ all previously known examples were homogeneous spaces. Using a perturbativion approach we show in Chapter 5 that one can find extremal metrics for (1.3.2) on manifolds M endowed with a conformal class $[g]$ if g is sufficiently close to a metric g_0 such that either

(i) (M, g_0) admits an eigenmap to S^1,
or
(ii) $M = N \times S^\ell$ and $g_0 = g_N + g_{st.}$, where $g_{st.}$ is the standard round metric.

Note that *(i)* holds in particular for all fibrations $M \to S^1$ with totally geodesic fibres. A shortened version of **??** is contained in the preprint [**Mat17**].

To make this more precise, we introduce the space

$$\mathcal{C} = C^\infty_+(M) \backslash \mathcal{R},$$

where

$$C^\infty_+(M) = \{\phi \in C^\infty(M) \ : \ \phi > 0\}$$

acts via normalized pointwise multiplication on $\mathcal{R}$,

$$\phi.g = \mathrm{vol}(M, \phi g)^{2/n} \phi g.$$

Our approach is based on the characterization of extremal metrics in terms of eigenmaps. This implies that in order to find extremal metrics it suffices to obtain the following result.

Theorem 1.6.1 (Theorem 5.1.6). *Let (M, g) be a closed Riemannian manifold of dimension* $\dim(M) \geq 3$*, and assume that*

(i) there is a a non-constant eigenmap $u\colon (M, g) \to S^1$*,*
or

(ii) $(M, g) = (N \times S^\ell, g_N + g_{st.})$*, where* $g_{st.}$ *denotes the round metric of curvature* 1 *on* S^ℓ*.*

Then there is a neighbourhood U of $[g]$ in $\mathcal{C}$ such that for any $c \in U$ there is a representative $h \in c$ such that (M, h) admits a non-constant eigenmap to S^1 or S^ℓ, respectively.

In order to find eigenmaps in nearby conformal classes we exploit a connection of eigenmaps to n-harmonic maps with nowhere vanishing density. An *n-harmonic map* $u \in W^{1,n}(M, S^\ell)$ is a weak solution to

$$-\operatorname{div}(|\nabla u|^{n-2}\nabla u) = |\nabla u|^n u.$$

Its energy density is given by $|\nabla u|^n$. If $|\nabla u|^n$ is everywhere non-vanishing, one can define a new metric g' on M such that $u\colon (M, g') \to S^\ell$ is an eigenmap. It is easier to perturb n-harmonic maps with nowhere vanishing density instead of eigenmaps mainly for one reason. The condition $|\nabla u| \neq 0$ is easier to preserve than $|\nabla u|^2 = const.$, let alone doing this while u also solves a differential equation.

Our main tools are a compactness result of B. White for n-harmonic maps, [**Whi88**], and some regularity results for n-harmonic maps. White's result is based on a beautiful application of Fubini's theorem that is also used in [**SY79**]. For $p = n$, there is no embedding $W^{1,p} \hookrightarrow C^0$, but the restriction to a generic k-skeleton $X^{(k)}$ for $k < n$ is well-defined and lies in $W^{1,p}(X^{(k)}) \hookrightarrow C^{0,\alpha}(X^{(k)})$, where we ignore the technical problems arising because $X^{(k)}$ is not smooth. We can use White's compactness theorem to obtain non-smooth n-harmonic maps in perturbed conformal classes. In a next step we use regularity estimates for these maps to obtain that they actually have nowhere vanishing gradient. Finally, this allows us to show that these maps are smooth, because the equation is no longer degenerate elliptic.

1.7. Regularity of extremal metrics

Besides the existence of extremal metrics, another natural question are their regularity properties. It is well known, that regularity and compactness properties are typically closely tied, so that it is desirable to obtain regularity results also to tackle the question of existence. More precisely, a modern approach in the calculus of variations and partial differential equations to find minimizers for functionals or solutions to equations consists of the following two steps. One first tries to find a generalized solution in a very large space of functions, that has good compactness properties, so that it is easier to show existence. Starting with this generalized solution one then tries to show that it actually has to be a classical solution. This second step is classically referred to as regularity theory. Unfortunately, we are not able to carry out the first step of this program for the first eigenvalue in a conformal class. We think that it is nevertheless of some interest to give an approach to the second step. This will be carried out in Chapter 6.

As we briefly indicated in Section 1.1, it is possible to extend the eigenvalue functional to a much larger class than smooth metrics. For various reasons, we only consider the case $n = \dim(M) \geq 3$. For $\dim(M) = 2$ the eigenvalues can be extended to an even larger class of metrics as described below. This has been treated in great detail in [**Kok14**].

We will be working in the conformal setting so that it is convenient to work with the following class of conformal factors. We fix a smooth metric g and consider

$$L^{n/2}_{\geq 0}(M,g) := \{\phi \in L^{n/2}(M,g) \ : \ \phi \geq 0 \ a.e.\}.$$

If we take $0 \neq \phi \in L^{n/2}_{\geq 0}$, the metric ϕg has a well defined volume measure given by

$$\mathrm{vol}(\Omega) = \int_\Omega \phi^{n/2} dV_g,$$

for any measurable subset $\Omega \subset M$. If ϕ is smooth, the Dirichlet energy with respect to ϕg is computed as

$$\int_M |du|_g^2 \phi^{n/2-1} dV_g,$$

which is clearly also well defined if $u \in C^1(M)$ and $\phi \in L^{n/2}_{\geq 0}$. Thanks to the variational characterization of eigenvalues (1.1.2), this allows us to extend the functionals λ_k from $[g]$ to $L^{n/2}(M,g)$. We prove a regularity result for extremal metrics if the conformal factor is *a priori* only assumed to lie in $L^{n/2}_{\geq 0}$ assuming some initial regularity. More precisely, we assume that

$$W^{1,2}(M,\phi g) \hookrightarrow L^2(M,\phi g) \tag{1.7.1}$$

is compact.

Theorem 1.7.2 (Theorem 6.1.5)**.** *Let ϕg be extremal for one of the functionals λ_k on $[g]$. Assume that the embedding* (1.7.1) *is compact and that $\lambda_k(\phi g) > 0$. Then ϕ is smooth in the interior if its support.*

The proof exploits the connection to n-harmonic maps in a very similar way as the proof of Theorem 1.6.1. We first compute the left and right derivatives of the eigenvalues for metrics with low regularity. Using this we can show that extremal metrics give rise to eigenmaps also in the singular setting and that the conformal factor can be obtained from this harmonic map. Finally, we use that the eigenmap for the singular metric ϕg is an n-harmonic map with respect to the smooth metric g. This allows us to use regularity results for n-harmonic maps, which then imply the regularity of ϕ.

1.8. Regularity of conformal metrics with large first eigenvalue

In Chapter 7 we give an approach to find a maximizer for the first eigenvalue in a conformal class under an additional curvature bound. Differently stated, we exhibit a natural class of metrics in a conformal class which admits a sharp bound for the first eigenvalue The curvature bound that we assume is an L^p scalar curvature bound. Therefore, we have to work with the Yamabe equation (1.8.1) describing the change of scalar curvature under a conformal change of the metric. The corresponding results are published in [**Mat16**].

More precisely, given a fixed background metric g on a closed manifold M of dimension $n \geq 3$, we consider conformal metrics of the form $u^{4/n-2}g$, where u is a smooth, positive function. It is convenient to write the conformal factor in this way, because the scalar curvature then transforms according to

$$4\frac{n-1}{n-2}\Delta_g u + R_g u = R_{u^{4/(n-2)}g} u^{2^\star-1}, \tag{1.8.1}$$

where $2^\star = 2n/(n-2)$ is the critical exponent for the Sobolev embedding $W^{1,2} \hookrightarrow L^p$, meaning that this embedding is compact precisely for $p < 2^\star$. In view of the elliptic equation (1.8.1), the scalar curvature is very natural to work with, if a conformal class is fixed. The scalar curvature bound that we assume is that there is a constant $A > 0$ such that

$$\int_M |R_{u^{4/(n-2)}g}|^p dV_{u^{4/(n-2)}g} \leq A \tag{1.8.2}$$

for some $p > n/2$. The restriction on the exponent p has its origin in the analytic properties of (1.8.1).

Since we have an application for the first eigenvalue in mind, it is very natural to look for a helpful assumption that involves only the first eigenvalue. Our assumption for the first eigenvalue is that we can find a constant $B > n((n+1)\omega_{n+1})^{2/n}$ such that

$$\lambda_1(M, u^{4/(n-2)}g)\,\mathrm{vol}(M, u^{4/(n-2)}g)^{2/n} \geq B. \tag{1.8.3}$$

The geometric significance of B is that $n((n+1)\omega_{n+1})^{2/n} = \lambda_1(S^n)\,\mathrm{vol}(S^n)^{2/n}$. Assumption (1.8.3) might appear rather restrictive. However, by a result of Petrides [**Pet15**] any conformal class not conformally equivalent to that of the round metric on S^n admits a metric for which (1.8.3) holds.

Assumption (1.8.3) allows us to a rule out bubbling phenomena that can in general occur for solutions of (1.8.1). By this we mean that, given a sequence of normalized solutions (u_j) to (1.8.1), it might happen that

$$\inf_{\delta>0} \sup_j \int_{B_g(x,r)} u_j^{2^\star}\, dV_g > 0.$$

The non-bubbling result in combination with (1.8.1) can then be used to obtain the following bound.

Theorem 1.8.4 (Theorem 7.1.2)**.** *Let (M,g) be a closed Riemannian manifold of dimension $n \geq 3$ and u a smooth positive function. Consider the conformal metric $\tilde g = u^{4/(n-2)}g$ and denote by $\tilde R$ its scalar curvature. Assume that*

(i) $\mathrm{vol}(M, \tilde g) = 1$,

(ii) $\int_M |\tilde R|^p u^{2^\star} dV_g \leq A$ *for some* $n/2 < p < \infty$,

(iii) $\lambda_1(M, \tilde g) \geq B > n\left((n+1)\omega_{n+1}\right)^{2/n}$.

Then there exist constants $C_1, C_2, C_3 > 0$, depending on (M,g) and A, B, such that $C_1 \leq u \leq C_2$ and $\|u\|_{W^{2,p}(M,g)} \leq C_3$.

Once the pointwise upper bound in this result is established, standard elliptic estimates applied to (1.8.1) give the $W^{2,p}$-bound but also uniform bounds in Hölder spaces,

$$\|u\|_{C^{0,\alpha}(M,g)} \leq C_4.$$

In combination with the pointwise lower bound, we can then use this to obtain a Hölder continous maximizing metric for the first eigenvalue in the class of conformal metrics having the bound (1.8.2). Unfortunately, we do not know whether this approach can be used to obtain extremal metrics. The problem is that there could be close by metrics with slightly larger first eigenvalue for which the scalar curvature bound (1.8.2) no longer holds.

In a very similar fashion, we also obtain a new compactness result for isospectral metrics.

1.9. The systole of large genus minimal surfaces

The last part is not directly concerned with the spectrum of the Laplacian but draws some motivation from recent results on the first eigenvalue, more precisely from Theorem 1.4.1. Let us write Σ_γ for a closed, orientable surface of genus γ. By [**Pet14**], there is a family of branched minimal immersions $\Sigma_\gamma \to S^{N_\gamma}$ that correspond to maximizers for the first eigenvalue for infinitely many values of γ. Clearly one always has $N_\gamma \geq 2$. Let us assume that any of these immersions is full, i.e. its image does not lie in any great sphere. Besides these minimal immersions, there are quite a few families of minimal surfaces in spheres known, see e.g. [**Bry82, Law70**]. It is an interesting question to understand how the family arising as maximizers for eigenvalues fits into this picture. One question is if necessarily $N_\gamma \to \infty$ as $\gamma \to \infty$. This seems to be very ambitious and insteadn one can try to rule out the possibility of $N_\gamma = 2$ or $N_\gamma = 3$ for γ sufficiently large. We will focus on the latter case under the additional assumption of embeddedness. A way to approach this problem is to understand geometric properties of minimal surfaces in S^3 when the genus is large. For instance, if one could show that the area of such a sequence of embedded minimal surfaces in S^3 grows sublinearly in the genus, it would follow that the maximizers can not be of this type. Although we can not answer this question, we are able to give some information on the intrinsic geometry of minimal surfaces in S^3 if the genus is very large. To our knowledge, this is the first result of this type.

In order to give some background and describe our results it is convenient to set up some notation. Given a closed three-manifold (M, g), we write

$$\mathcal{M} = \{\Sigma \subset M \ : \ \Sigma \ \text{ is a closed, embedded, minimal surface}\},$$

and similarly,

$$\mathcal{M}_\gamma = \{\Sigma \in \mathcal{M} \ : \ \mathrm{genus}(\Sigma) = \gamma\}.$$

A by-now classical result of Choi and Schoen asserts that $\mathcal{M}_\gamma$ is smoothly compact if (M, g) has positive Ricci curvature. For so-called bumpy metrics, which are generic among all metrics thanks to work of B. White, with positive Ricci curvature, an argument of Colding and Minicozzi even implies that $\mathcal{M}_\gamma$ is finite. On the other hand, recent work by Marques and Neves gives that $\mathcal{M}$ is infinite for any ambient three-manifold of positive Ricci curvature. Given these results it appears very natural to ask for properties of minimal surfaces in ambient three-manifolds of positive Ricci curvature if the genus is very large.

In a little more detail, we assume that M is a closed three manifold endowed with a metric of positive Ricci curvature. We want to study properties of a sequence $\Sigma_j \in \mathcal{M}$ of closed, embedded, minimal surfaces with $\mathrm{genus}(\Sigma_j) \to \infty$. One class of such results concerns how Σ_j lies in the ambient space M for j very large. An example is the *index* of Σ, i.e. the largest number of linearly independent deformations that do not increase area up to second order. If the genus becomes unbounded, the index has to do so as well. Another interesting phenomenon is that for generic metrics there is a family Σ_j with $\text{genus}(\Sigma_j) \to \infty$ that equidistributes, i.e.

$$\lim_{k\to\infty} \frac{1}{\sum_{j=1}^k \text{area}(\Sigma_j)} \sum_{j=1}^k \int_{\Sigma_j} f d\mathcal{H}^2 \to \frac{1}{\text{vol}(M)} \int_M f,$$

for any smooth function $f \in C^\infty(M)$, see [**IMN18, LMN18, MNS17**].

Both of these results concern the extrinsic geometry of Σ_j, i.e. how Σ_j sits in M. In contrast, we are able to give a first result on the intrinsic geometry of Σ_j for j large if M has positive Ricci curvature.

In Chapter 8 we show that for Σ_j as above, the systole tends to zero,

$$\text{sys}(\Sigma_j) \to 0. \tag{1.9.1}$$

In fact, we are even able to find a non-separating short curve if M is simply connected. To that end let us define the *homology systole* of a closed surface Σ as

$$\text{hsys}(\Sigma) = \inf\{\text{length}(c) \ : \ 0 \neq [c] \in H_1(\Sigma, \mathbb{Z})\}.$$

Our precise result, which is joint work with Anna Siffert, can now be stated as follows.

Theorem 1.9.2 (Theorem 8.1.1)**.** *Assume that (M, g) is a simply connected three-manifold with positive Ricci curvature. Let $\Sigma_j \subset M$ be a sequence of closed, embedded minimal surfaces with* $\text{genus}(\Sigma_j) \to \infty$ *as $j \to \infty$. Then we have for the homology systole that*

$$\text{hsys}(\Sigma_j) \to 0,$$

as $j \to \infty$.

By a covering space argument this easily implies the same type of result for the systole of such a sequence if M is not assumed to be simply connected.

Our argument relies on groundbreaking work by Colding and Minicozzi on the structure of minimal surfaces in the presence of only a bound on the genus, but the absence of a curvature bound. It is *a priori* not clear at all that their techniques apply to our setting. This stems from the global nature of the systole. In general, a curve could be non-contractible in a small ball but contractible in a much larger one. Therefore, any information on the systole might give only little information about the structure in small extrinsic balls. Crucially, for minimal surfaces, the systole has a pseudo locality property:

There is $R_0 > 0$ depending only on the ambient manifold M such that if $c \subset \Sigma \cap B(x_0, r_0)$ is contractible in $\Sigma \cap B(x, R_0)$ then it is already contractible in $\Sigma \cap B(x_0, r_0)$.

In other words, one can either locally decide if a curve is contractible or the contraction has to leave a large ball. It is exactly this phenomenon that allows us to show that the techniques of Colding–Minicozzi can be applied to obtain (1.9.1). More precisely, we assume that we can find a sequence Σ_j with unbounded genus but $\text{sys}(\Sigma_j) \geq l_0 > 0$. We then show that this prevents any topology from concentrating on a fixed small scale and study a limit lamination $\mathcal{L}$ of a subsequence of Σ_j. *A priori*, the lamination $\mathcal{L}$ is defined only outside a closed set $\mathcal{S} \subset M$. We then show that $\mathcal{L}$ has at least one leaf which extends to a closed, stable, two-sided minimal surface. Since M has positive Ricci curvature, such a surface can not exist.

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CHAPTER 2

On the analytic systole of Riemannian surfaces

2.1. Introduction

Small eigenvalues of Riemannian surfaces, in particular of hyperbolic surfaces, have been of interest in different mathematical fields for a long time. Buser and Schmutz conjectured that a hyperbolic metric on the closed surface $S = S_g$ of genus $g \geq 2$ has at most $2g-2$ eigenvalues below $1/4$ [**Bus77, Sch91**]. In [**OR09**], Otal and Rosas proved a generalized version of this conjecture. They showed that a real analytic Riemannian metric on S_g with negative curvature has at most $2g-2$ eigenvalues $\leq \lambda_0(\tilde{S})$, where $\tilde{S}$ denotes the universal covering surface of S, endowed with the lifted Riemannian metric, and where $\lambda_0(\tilde{S})$ denotes the bottom of the spectrum of $\tilde{S}$. Recall here that, for a Riemannian surface F (possibly not complete) with piecewise smooth boundary ∂F (possibly empty), the *bottom of the spectrum of* F is defined to be

$$\lambda_0 = \lambda_0(F) = \inf R(\varphi), \tag{2.1.1}$$

where φ runs over all non-vanishing smooth functions on F with compact support in the interior $\mathring{F} = F \setminus \partial F$ of F and where $R(\varphi)$ denotes the Rayleigh quotient of φ. It is well known that the bottom of the spectrum of the Euclidean and hyperbolic plane is 0 and $1/4$, respectively. When F is closed, $\lambda_0(F) = 0$, when F is compact and connected with non-empty boundary, $\lambda_0(F)$ is the first Dirichlet eigenvalue of F. In the latter case, $\lambda_0(F)$-eigenfunctions of F do not have zeros in $\mathring{F}$ and, therefore, the multiplicity of $\lambda_0(F)$ as an eigenvalue of F is one. We then call the corresponding positive eigenfunction of F with L^2-norm one the *ground state* of F.

For a Riemannian surface S, with or without boundary, we define the *analytic systole* to be the quantity

$$\Lambda(S) = \inf_F \lambda_0(F), \tag{2.1.2}$$

where the infimum is taken over all subsurfaces F in $\mathring{S}$ with smooth boundary which are diffeomorphic to a closed disc, annulus, or cross cap. (A cross cap is frequently also called a Möbius strip.) Note that the fundamental groups of disc, annulus, and cross cap are cyclic, hence amenable. By the work of Brooks, we therefore have

$$\Lambda(S) \geq \lambda_0(\tilde{S}) \tag{2.1.3}$$

for all complete and connected Riemannian surfaces S, see [**Bro85**, Theorem 1] and also Theorem 2.8.9 below. The strictness of this inequalilty and other estimates of $\Lambda(S)$ are the topics of this chapter.

To clarify our terminology, a surface is a smooth manifold of dimension two. A Riemann surface is a surface together with a conformal structure. They are not the topic of this chapter. We study Riemannian surfaces, that is, surfaces together with a Riemannian metric.

We say that a surface S is of *finite type* if its Euler characteristic $\chi(S)$ is finite and its boundary is compact (possibly empty). It is well known that a connected surface S is of finite type if and only if S can be obtained from a closed surface by deleting a finite number of pairwise disjoint points and open discs.

After first extensions of the results of Otal and Rosas in [**Mat13**] and [**Mon14**], we showed in [**BMM16**] and [**BMM17b**] that any complete Riemannian metric on a connected surface S of finite type with $\chi(S) < 0$ has at most $-\chi(S)$ eigenvalues $\leq \Lambda(S)$, where the eigenvalues are understood to be Dirichlet eigenvalues if $\partial S \neq \emptyset$. This result explains the significance of the analytic systole and the interest in establishing strictness in (2.1.3).

2.1.1. Statement of main results. In our first three results, we discuss the strictness of (2.1.3).

Theorem 2.1.4. *If S is a compact and connected Riemannian surface whose fundamental group is not cyclic, then $\Lambda(S) > \lambda_0(\tilde{S})$.*

Note that the compact and connected surfaces with cyclic fundamental group are precisely sphere, projective plane, closed disc, closed annulus, and closed cross cap. For these, we always have equality in (2.1.3) as we will see in Proposition 2.1.10.

Recall that the spectrum of S is discrete if S is compact. In general, the spectrum of S is the disjoint union of its discrete and essential parts, where $\lambda \in \mathbb{R}$ belongs to the essential spectrum of S if $\Delta - \lambda$ is not a Fredholm operator. The *bottom of the essential spectrum* is given by

$$\lambda_{\text{ess}}(S) = \lim_{K} \lambda_0(S \setminus K), \tag{2.1.5}$$

where K runs over the compact subsets of S, ordered by inclusion, see Proposition 2.8.1. By domain monotonicity, the limit is monotone with respect to the ordering of the compact subsets of S. If S is compact, then $\lambda_{\text{ess}}(S) = \infty$. If S is a complete and connected Riemannian surface of finite type, then

$$\lambda_{\text{ess}}(S) \geq \Lambda(S) \tag{2.1.6}$$

since the the ends of S admit neighborhoods in S whose connected components are diffeomorphic to open annuli. Such neighborhoods of the ends of S will be called *cylindrical.*

Theorem 2.1.4 extends in the following way to surfaces of finite type, be they compact or non-compact.

Theorem 2.1.7. *If S is a complete and connected Riemannian surface of finite type whose fundamental group is not cyclic, then $\Lambda(S) > \lambda_0(\tilde{S})$ if and only if $\lambda_{\text{ess}}(S) > \lambda_0(\tilde{S})$.*

For complete and connected Riemannian surfaces of finite type, $\Lambda(S)$ is always between $\lambda_0(\tilde{S})$ and $\lambda_{\text{ess}}(S)$, by (2.1.3) and (2.1.6). Hence the condition $\lambda_{\text{ess}}(S) > \lambda_0(\tilde{S})$ in Theorem 2.1.7 is obviously necessary to have the strict inequality $\Lambda(S) > \lambda_0(\tilde{S})$. The hard part of the proof of Theorem 2.1.7 is to show that the condition is also sufficient.

Remark 2.1.8. Another way of stating the condition $\lambda_{\text{ess}}(S) > \lambda_0(\tilde{S})$ in Theorem 2.1.7 is to require that there is a compact subset K of S such that $\lambda_0(S \setminus K) > \lambda_0(\tilde{S})$. By domain monotonicity, this condition is then also satisfied for any compact subset K' of S containing K.

Example 2.1.9. Let S be a non-compact connected surface of finite type whose fundamental group is not cyclic. Using a decomposition of S into pairs of pants, it becomes obvious that S carries complete hyperbolic metrics with (possibly empty) geodesic boundary. For any such metric, we have

$$\lambda_{\text{ess}}(S) = \Lambda(S) = \lambda_0(\tilde{S}) = 1/4$$

since the ends of S are then hyperbolic cusps or funnels.

Tempted by this equality we investigate how *generic* this equality is among all smooth complete metrics on a non-compact surface of finite type. Our next result is that it is in fact rare. Some form of rigidity in the case of equality would of course be very interesting; compare with Section 2.8.6.

Proposition 2.1.10. *Let S be a connected surface of finite type.*
1) If the fundamental group of S is cyclic, then $\Lambda(S) = \lambda_0(\tilde{S})$ for any complete Riemannian metric on S.
2) If S is non-compact and the fundamental group of S is not cyclic, then S carries complete Riemannian metrics such that $\lambda_{\text{ess}}(S) > \Lambda(S)$. Moreover, if $\chi(S) < 0$, then such metrics may be chosen to have curvature $K \leq -1$ and finite or infinite area.
3) If $\lambda_0(\tilde{S}, \tilde{g}) > 0$ for some Riemannian metric g on S, then a generic complete Riemannian metric g' on S in any neighborhood of g in the uniform C^∞ topology satisfies the strict inequality $\lambda_{\text{ess}}(S, g') > \Lambda(S, g')$.

By Theorem 2.1.7, $\lambda_{\text{ess}}(S) > \Lambda(S)$ implies $\Lambda(S) > \lambda_0(\tilde{S})$.

In our next result, we generalize the main result of [**Mon14**], which asserts that a hyperbolic metric on the closed surface S_g of genus $g \geq 2$ has at most $2g-2$ eigenvalues $\leq 1/4 + \delta$, where

$$\delta = \min\{\pi/|S|, \operatorname{sys}(S)^2/|S|^2\}.$$

Here $|S|$ denotes the area of S and $\operatorname{sys}(S)$, the *systole* of S, is defined to be the minimal possible length of an essential closed curve in S.

Theorem 2.1.11. *For a closed Riemannian surface S with curvature $K \leq \kappa \leq 0$, we have*

$$\Lambda(S) \geq -\frac{\kappa}{4} + \frac{\operatorname{sys}(S)^2}{|S|^2}.$$

Remarks 2.1.12. 1) For closed Riemannian surfaces S with curvature $K \leq \kappa < 0$, we know in general only that $\lambda_0(\tilde{S}) \geq -\kappa/4$. Therefore Theorem 2.1.11 may not imply the strict inequality $\Lambda(S) > \lambda_0(\tilde{S})$ for such S. In fact, the relation between $\lambda_0(\tilde{S})$ and the right hand side in Theorem 2.1.11 is not clear. Our method of proof, involving isoperimetric inequalities and Cheeger's inequality, does not seem to be sophisticated enough to capture the difference between them.

2) The proof of Theorem 2.1.11 also applies to non-compact surfaces of finite type. In this case one needs to define the systole as the infimum over *all* homotopically non-trivial

curves, not only the essential (not homotopic to a boundary component or a puncture) ones. For this reason, the corresponding statement is not really interesting anymore. If $|S| < \infty$, then $\mathrm{sys}(S) = 0$ (by a refinement of the isosystolic inequality), and if $|S| = \infty$, then $\mathrm{sys}(S)/|S| = 0$.

3) The difference $\Lambda(S) - \lambda_0(\tilde{S})$ can not be estimated from below by a positive constant, which only depends on the topology and the area of S. In fact, given any $\varepsilon > 0$ and natural number n, if the metric on S is hyperbolic with sufficiently small systole, then $\lambda_n(S) < 1/4 + \varepsilon$, by [**Bus77**, Satz 2] or the proof of Theorem 8.1.2 in [**Bus10**].

One may view Theorem 2.1.11 also as an upper bound on the systole in terms of a curvature bound and $\Lambda(S)$. Together with our next result, this explains the name *analytic systole*.

For a closed Riemannian surface S, we say that a closed geodesic c of S is a *systolic geodesic* if it is essential with length $L(c) = \mathrm{sys}(S)$. Clearly, systolic geodesics are simple.

Theorem 2.1.13. *If S is a closed Riemannian surface with $\chi(S) < 0$ and curvature $K \geq -1$, then*

$$\Lambda(S) \leq \frac{1}{4} + \frac{4\pi^2}{w^2},$$

where

$$w = w(\mathrm{sys}(S)) = \begin{cases} \operatorname{arsinh}(1/\sinh(\mathrm{sys}(S)/2))) \\ \operatorname{arsinh}(1/\sinh(\mathrm{sys}(S))) \end{cases}$$

if S has a two-sided systolic geodesic or if all systolic geodesics of S are one-sided, respectively.

Here we say that a simple closed curve in S is two-sided or one-sided if it has a tubular neigborhood which is diffeomorphic to an annulus or a cross cap, respectively.

Combining Theorem 2.1.11 and Theorem 2.1.13, we get that, for hyperbolic metrics, $\Lambda(S)$ is squeezed between two functions of the systole.

Corollary 2.1.14. *For closed hyperbolic surfaces, we have*

$$\frac{1}{4} + \frac{\mathrm{sys}(S)^2}{4\pi^2\chi(S)^2} \leq \Lambda(S) \leq \frac{1}{4} + \frac{4\pi^2}{w^2}$$

with $w = w(\mathrm{sys}(S))$ as in Theorem 2.1.13.

Recall that $\operatorname{arsinh} x = \ln(x + \sqrt{x^2+1})$. In particular, we have

$$w(\mathrm{sys}(S)) \sim -\ln(\mathrm{sys}(S)) \to \infty \quad \text{as } \mathrm{sys}(S) \to 0.$$

We conclude that the analytic systole of hyperbolic metrics on closed surfaces tends to $1/4$ if and only if their systole tends to 0.

2.1.2. Main problems and arguments. The only surfaces S in Theorem 2.1.4 and Theorem 2.1.7 with Euler characteristic $\chi(S) \geq 0$ are torus and Klein bottle. For these, the proof of the inequality $\Lambda(S) > \lambda_0(\tilde{S})$ is quite elementary. The proof of the hard direction of Theorem 2.1.7, namely establishing the strict inequality $\Lambda(S) > \lambda_0(\tilde{S})$ under the condition $\lambda_{\mathrm{ess}}(S) > \lambda_0(\tilde{S})$, is rather involved in the case $\chi(S) < 0$.

The domain monotonicity of the first Dirichlet eigenvalue implies that $\Lambda(S)$ can not be realized by any compact subsurface $F \subseteq S$ diffeomorphic to a disc, an annulus or a cross cap. Keeping this in mind, our general strategy for the proof of Theorem 2.1.7 is to show that the equality $\Lambda(S) = \lambda_0(\tilde{S})$ would imply the existence of a non-trivial $\lambda_0(\tilde{S})$-eigenfunction $\tilde{\varphi}$ on $\tilde{S}$ or an appropriate cyclic quotient $\hat{S}$ of $\tilde{S}$ that vanishes on an open set.

The condition $\lambda_{\text{ess}}(S) > \lambda_0(\tilde{S})$ forces a subsurface F with $\lambda_0(F)$ close to $\lambda_0(\tilde{S})$ to stay almost completely in a large compact set in a weighted sense, the weight being the ground state. One then works essentially within a fixed compact subsurface of S. Two main problems that we still have to overcome in establishing the existence of $\tilde{\varphi}$ as above are

1) a priori non-existence of a fixed quotient of $\tilde{S}$ along a sequence of subsurfaces approximating $\lambda_0(\tilde{S})$ and
2) the absence of the compact Sobolev embedding $H^1 \hookrightarrow L^2$ on these covering spaces.

As for the first problem, the case of cross caps can be reduced to the case of annuli by considering the two-sheeted orientation covering of the original surface. The case of annuli is tackled by showing that only finitely many isotopy types of annuli have the bottom of their spectrum close to $\lambda_0(\tilde{S})$. A keystone of the argument is Lemma 2.5.3 which relates the bottom of the spectrum of compact surfaces F with the sum of the lengths of shortest curves in the free homotopy classes of the boundary circles of F.

To tackle the second problem, we establish, in Lemma 2.6.8, an inradius estimate for superlevel sets of suitably truncated ground states of a sequence of subsurfaces F_n approximating $\lambda_0(\tilde{S})$. The inradius estimate is proved by means of isoperimetric inequalities, extending arguments from the proof of the Cheeger inequality.

2.1.3. Structure of the chapter. In Section 2.2, we collect the relevant facts about isoperimetric inequalities on Riemannian surfaces. In Section 2.3, we extend Osserman's refined version of the Cheeger inequality [**Oss77**, Lemma 1] for plane domains to compact Riemannian surfaces with boundary. We also recall Osserman's elegant proof since we will need consequences and extensions of his arguments. The isoperimetric inequalities from Corollary 2.2.3 and the Cheeger inequality are then used in Section 2.4 to obtain a generalized version of Theorem 2.1.11. The arguments here are very much in the spirit of Osserman [**Oss77**] and Croke [**Cro81**]. As an application of our discussion, we obtain Theorem 2.1.7 for the case where S is a torus or a Klein bottle. This section closes with the proof of Theorem 2.1.13, which involves methods which are different from those of the rest of the chapter. Sections Section 2.5 and Section 2.6 are concerned with properties of the ground states of compact Riemannian surfaces with boundary. The main objectives are Lemma 2.5.3 on the relation of the bottom of the spectrum to other geometric quantities and Lemma 2.6.8 on the inradius of superlevel sets of ground states. In Section 2.7, we complete the proof of Theorem 2.1.7. Section 2.8 contains the proof of Proposition 2.1.10 and some remarks and questions. In particular, we draw attention to problems in optimal design which are related to optimal estimates of the analytic systol. In Section 2.8.6, we discuss an extension of the result of Brooks quoted in connection with (2.1.3).

2.2. Isoperimetric inequalities

The content of the present section is related to and extends Lemma 1 of [**Oss77**] in the way we will need it.

Let F be a compact and connected surface with piecewise smooth boundary $\partial F \neq \emptyset$ and interior $\mathring{F} = F \setminus \partial F$. The components of ∂F are piecewise smooth circles. Denote by $\chi = \chi(F)$ the Euler characteristic of F.

Assume that F is endowed with a Riemannian metric and denote by K the Gauss curvature of F. Let $|F|$ and $|\partial F|$ be the area of F and the length of ∂F, respectively, and

$$\rho = \rho_F = \max\{d(x, \partial F) \mid x \in F\} \tag{2.2.1}$$

be the *inradius* of F.

For a function $f\colon F \to \mathbb{R}$, write $f^+ = \max(f, 0)$ for its positive part. We recall the following isoperimetric inequalities.

Theorem 2.2.2. *For any F as above and $\kappa \in \mathbb{R}$, we have*

$$|\partial F|^2 \geq -\kappa |F|^2 + 2\left(2\pi\chi - \int_F (K-\kappa)^+ dx\right)|F|. \tag{1}$$

If $\kappa \leq 0$, then

$$|\partial F| \geq |F| \operatorname{ct}_\kappa \rho + \left(2\pi\chi - \int_F (K-\kappa)^+ dx\right) \operatorname{tn}_\kappa \frac{\rho}{2}. \tag{2}$$

If F is not a disc and $\kappa < 0$, then

$$(|\partial F|^2 - \ell^2)^{1/2} \geq \sqrt{-\kappa}|F| + \frac{1}{\sqrt{-\kappa}}\left(2\pi\chi - \int_F (K-\kappa)^+ dx\right). \tag{3}$$

where ℓ denotes the sum of the lengths of the shortest loops in the free homotopy classes (in F) of the boundary circles of F.

In the second inequality, $\operatorname{tn}_\kappa = \operatorname{sn}_\kappa / \operatorname{cs}_\kappa$ and $\operatorname{ct}_\kappa = \operatorname{cs}_\kappa / \operatorname{sn}_\kappa$, where sn_κ and cs_κ are the solutions of the differential equation $\ddot{u} + \kappa u = 0$ with respective initial conditions

$$\operatorname{sn}_\kappa(0) = 0,\ \operatorname{sn}_\kappa'(0) = 1 \quad \text{and} \quad \operatorname{cs}_\kappa(0) = 1,\ \operatorname{cs}_\kappa'(0) = 0.$$

The first inequality of Theorem 2.2.2 corresponds to [**BZ88**, Theorem 2.2.1], the third to the (outer) inequality in (20) of [**BZ88**, p. 15]. We added "in F" in parentheses in the statement since we will use Theorem 2.2.2 in the case where F is a domain in a surface S. Then the length of a shortest loop in the free homotopy class in S of a boundary circle c of F might be smaller than the length of a shortest loop in the free homotopy class of c in F.

Proof of Theorem 2.2.2 (Eq. (2)). We apply [**BZ88**, Theorem 2.4.2] in the case $t = \rho$. The function $f = f(t)$ of [**BZ88**] measures the area of the collar of width t about ∂F and, therefore, we have $f(\rho) = |F|$ by the definition of ρ. The function $a = a(t)$ of [**BZ88**] satisfies

$$a(\rho) = \kappa|F| - 2\pi\chi + \int_F (K-\kappa)^+ dx.$$

In our notation, the function ψ of [**BZ88**] is given by

$$\psi(t) = a(t)\frac{1-\mathrm{cs}_\kappa t}{\kappa} + |\partial F| \,\mathrm{sn}_\kappa t,$$

where we set $(1-\mathrm{cs}_\kappa t)/\kappa = t^2/2$ for $\kappa = 0$. Now Theorem 2.4.2 of [**BZ88**], in the case $\kappa \le 0$ and $t = \rho$, asserts that $f(\rho) \le \psi(\rho)$, that is, that

$$|F| \le |F|(1-\mathrm{cs}_\kappa \rho) - \left(2\pi\chi - \int_F (K-\kappa)^+ dx\right)\frac{1-\mathrm{cs}_\kappa \rho}{\kappa} + |\partial F|\,\mathrm{sn}_\kappa \rho.$$

Therefore we get

$$|\partial F|\,\mathrm{sn}_\kappa \rho \ge |F|\,\mathrm{cs}_\kappa \rho + \left(2\pi\chi - \int_F (K-\kappa)^+ dx\right)\frac{1-\mathrm{cs}_\kappa \rho}{\kappa}.$$

This implies (2) since $(1-\mathrm{cs}_\kappa t)/\kappa\,\mathrm{sn}_\kappa t = \mathrm{tn}_\kappa(t/2)$. □

Corollary 2.2.3. *If $K \le \kappa$, then we have:*
1) If F is a disc, then $|\partial F|^2 \ge -\kappa|F|^2 + 4\pi|F|$.
2) If $\chi \ge 0$ and $\kappa \le 0$, then $|\partial F| \ge |F|\,\mathrm{ct}_\kappa \rho$.
3) If $\chi = 0$ and $\kappa \le 0$, then $|\partial F|^2 \ge -\kappa|F|^2 + \ell^2$.

Note that we always have $|\partial F|^2 \ge \ell^2$, by the definition of ℓ.

2.3. Cheeger inequality revisited

In Lemma 2 of [**Oss77**], Osserman discusses a refinement of the Cheeger inequality for compact planar domains, endowed with Riemannian metrics. We will need an extension of Osserman's Lemma 2.

As above, we let F be a compact and connected Riemannian surface F with piecewise smooth boundary $\partial F \ne \emptyset$. The *Cheeger constant* of F is defined to be the number

$$h = h(F) = \inf |\partial F'|/|F'|,$$

where the infimum is taken over all compact subsurfaces F' of $\mathring{F}$ with smooth boundary. Note that closed surfaces cannot occur as subsurfaces F' of F since F is connected with non-empty boundary.

Lemma 2.3.1. *The Cheeger constant is given by*

$$h = \inf |\partial F'|/|F'|,$$

where the infimum is taken over all compact and connected subsurfaces F' of $\mathring{F}$ with smooth boundary such that the boundary of each component of $F \setminus F'$ has at least one boundary circle in $\mathring{F}$ and contains at least one boundary circle of F.

Proof. Let F' be a compact subsurface of $\mathring{F}$ with smooth boundary and denote by $F'_1, \dots, F'_k$ the components of F'. Then

$$|F'| = |F'_1| + \cdots + |F'_k| \quad \text{and} \quad |\partial F'| = |\partial F'_1| + \cdots + |\partial F'_k|$$

and hence

$$\inf \frac{|\partial F'_i|}{|F'_i|} \le \frac{|\partial F'_1| + \cdots + |\partial F'_k|}{|F'_1| + \cdots + |F'_k|} = \frac{|\partial F'|}{|F'|}.$$

This shows that the infimum h can be taken over compact and connected subsurfaces F' of $\mathring{F}$ with smooth boundary.

Let now C be a component of $F \setminus F'$. Suppose first that the boundary of C does not contain a boundary circle of F. Then $F'' = F' \cup C$ is a compact and connected subsurface of $\mathring{F}$ with area $|F''| > |F'|$ and length of boundary $|\partial F''| < |\partial F'|$. It follows that the infimum h is attained by compact and connected subsurfaces F' of $\mathring{F}$ with smooth boundary such that the boundary of each component of $F \setminus F'$ contains a boundary circle of F.

If the boundary of C would not have a boundary circle in $\mathring{F}$, then C would have to coincide with F since F is connected. But then F' would be empty, a contradiction. □

By a slight variation of the standard terminology, we say that a subsurface S of a surface T is *incompressible in* T if the induced maps of fundamental groups are injective, for all connected component C of S. In particular, embedded discs are always incompressible.

Proposition 2.3.2. *The Cheeger constant is given by*

$$h = \inf |\partial F'|/|F'|,$$

where the infimum is taken over all incompressible compact and connected subsurfaces F' of $\mathring{F}$ with smooth boundary such that no component of $F \setminus F'$ is a disc or a cross cap. Any such F' satisfies $\chi(F') \geq \chi(F)$ with equality if and only if $F \setminus F'$ is a collared neighborhood of ∂F, consisting of annuli about the boundary circles of F.

For example, if F is an annulus, then we only need to consider discs and incompressible annuli F' in F; if F is a cross cap, then only discs and incompressible annuli and cross caps F'.

Proof of Proposition 2.3.2. By Lemma 2.3.1, the Cheeger constant h is realized by compact and connected subsurfaces F' of $\mathring{F}$ with smooth boundary such that each component of $F \setminus F'$ has at least two boundary circles. This excludes discs and cross caps as components of $F \setminus F'$. We have

$$\chi(F) = \chi(F') + \chi(F \setminus \mathring{F}') \leq \chi(F')$$

since the intersection of F' with $F \setminus \mathring{F}'$ consists of circles and since no component of $F \setminus \mathring{F}'$ is a disc. Furthermore, equality can only occur if $\chi(F \setminus \mathring{F}') = 0$. By what we already know, this can only happen if the components of $F \setminus \mathring{F}'$ are annuli. By Lemma 2.3.1 and since $F' \subseteq \mathring{F}$, they constitute a collared neighborhood of ∂F.

It remains to show the incompressibility of F'. If this would not hold, F' would contain a Jordan loop c which is not contractible in F', but is contractible in $\mathring{F}$. Then c would be the boundary of an embedded disc D in $\mathring{F}$ which is not contained in F'. Since $\partial D \subseteq F'$, $D \setminus F'$ would consist of components of $F \setminus F'$. Their boundary would be in $D \subseteq \mathring{F}$ in contradiction to Lemma 2.3.1. □

Recall the classical Cheeger inequality.

Theorem 2.3.3 (Cheeger inequality)**.** *We have $\lambda_0(F) \geq h^2/4$.*

In the proofs of Lemma 2.5.3 and Lemma 2.6.8, we will need arguments and consequences of the proof of Theorem 2.3.3 and, therefore, recall the elegant arguments from the proof of the corresponding Lemma 2 in [**Oss77**].

Recalling the proof of the Cheeger inequality. Since F is compact with piecewise smooth boundary, $\lambda_0 = \lambda_0(F)$ is the first Dirichlet eigenvalue of F. Let φ be the corresponding ground state and set $\psi = \varphi^2$. By the Schwarz inequality, we have

$$\begin{aligned}\int_F |\nabla\psi| &= \int_F 2|\varphi||\nabla\varphi| \le 2\left(\int_F |\nabla\varphi|^2\right)^{1/2}\left(\int_F \varphi^2\right)^{1/2} \\ &= 2\sqrt{\lambda_0}\int_F \varphi^2 = 2\sqrt{\lambda_0}\int_F \psi.\end{aligned} \tag{2.3.4}$$

This implies

$$\sqrt{\lambda_0} \ge \frac{1}{2}\frac{\int_F |\nabla\psi|}{\int_F \psi}. \tag{2.3.5}$$

For regular values $t > 0$ of ψ, let $F_t = \{\psi \ge t\}$ and denote by $A(t)$ and $L(t)$ the area and the length of F_t and $\partial F_t = \{\psi = t\}$, respectively. For the null set of singular values of ψ, set $A(t) = L(t) = 0$. The coarea formula gives

$$\int_F |\nabla\psi| = \int_0^\infty L(t)dt. \tag{2.3.6}$$

On the other hand, since $\int_F \psi$ computes the volume of the domain

$$\{(x, y) \in F \times \mathbb{R} \mid 0 \le y \le \psi(x)\},$$

Cavalieri's principle gives

$$\int_F \psi = \int_0^\infty A(t)dt. \tag{2.3.7}$$

By the definition of $h = h(F)$, we have

$$\int_F |\nabla\psi| = \int_0^\infty L(t)dt \ge h\int_0^\infty A(t)dt = h\int_F \psi. \tag{2.3.8}$$

Combining (2.3.5) and (2.3.8), we get $\lambda_0 \ge h^2/4$ as asserted. □

2.4. Quantitative estimates of $\Lambda(S)$

We start with a version of Theorem 2.1.11 for surfaces with (possibly empty) boundary.

Theorem 2.4.1. *Let S be a compact and connected Riemannian surface, with or without boundary, with infinite fundamental group and curvature $K \le \kappa$, where κ is a constant. Then we have:*
1) If $\kappa \le 0$, then

$$\Lambda(S) \ge -\frac{\kappa}{4} + \frac{1}{|S|}\min\left\{\pi, \frac{\mathrm{sys}(S)^2}{|S|}\right\}.$$

2) If $\kappa > 0$ and S is orientable, then

$$\Lambda(S) \geq \min\left\{\frac{\pi}{|S|} - \frac{\kappa}{4}, \frac{\mathrm{sys}(S)^2}{|S|^2}\right\}.$$

3) If $\kappa > 0$ and S is non-orientable, then

$$\Lambda(S) \geq \min\left\{\frac{\pi}{|S|} - \frac{\kappa}{4}, \frac{\mathrm{sys}(S)^2}{4|S|^2}\right\}.$$

PROOF. For a closed disc D in S, Corollary 2.2.3.Item 1 implies that

$$\frac{|\partial D|^2}{|D|^2} \geq -\kappa + \frac{4\pi}{|D|} \geq -\kappa + \frac{4}{|S|}\pi. \tag{2.4.2}$$

Suppose now that A is a closed annulus in S. Suppose first that the boundary circles of A are null-homotopic in S. Then by the Schoenflies theorem (see also [**BMM16**, Appendix A]) there is a disc D in $S \setminus \mathring{A}$ such that $F' = A \cup D$ is a disc. Then $|\partial F'| \leq |\partial A|$ and $|F'| \geq |A|$. Using Corollary 2.2.3.Item 1 again, we get that (2.4.2) also holds for A in place of D.

Assume now that the boundary circles of A are not null-homotopic in S. By Corollary 2.2.3.Item 3 and the statement after it, we have

$$|\partial A|^2 \geq -\min(\kappa, 0)|A|^2 + 4l(A)^2,$$

where $l(A)$ denotes the length of a shortest curve in the free homotopy class in A of the two boundary circles of A. Since the boundary circles of A are not homotopic to zero in S, we have $l(A) \geq \mathrm{sys}(S)$. Hence

$$\frac{|\partial A|^2}{|A|^2} \geq -\min(\kappa, 0) + 4\frac{\mathrm{sys}(S)^2}{|A|^2} \geq -\min(\kappa, 0) + \frac{4}{|S|}\frac{\mathrm{sys}(S)^2}{|S|}. \tag{2.4.3}$$

If C is a cross cap in S, then S is not orientable. Now the soul of C is not homotopic to zero in S and the fundamental group of S is torsion free. Since the boundary circle ∂C of C is freely homotopic to the soul of C, run twice, we get that ∂C is not homotopic to zero in S. In particular, we always have $|\partial C| \geq \mathrm{sys}(S)$. If $\kappa \leq 0$, then a shortest curve in S in the free homotopy class of the soul of C, run twice, is a shortest curve in S in the free homotopy class of the boundary circle of C. Hence $|\partial C| \geq 2\,\mathrm{sys}(S)$ if $\kappa \leq 0$. We conclude that (2.4.3) also holds for C in place of A if $\kappa \leq 0$. In the general case, $|\partial C| \geq \mathrm{sys}(S)$ implies a modified version of (2.4.3) with C in place of A, where the factor 4 on the right hand side is replaced by 1.

Now the assertions of Theorem 2.4.1 follows from the Cheeger inequality (Theorem 2.3.3) in combination with Proposition 2.3.2, (2.4.2), and (2.4.3) or the modified version of (2.4.3), respectively. □

PROOF OF THEOREM 2.1.11. It remains to show that $\mathrm{sys}(S)^2/|S| \leq \pi$ if S is closed with curvature $K \leq 0$. In fact, in that case, the injectivity radius of S is $\mathrm{sys}(S)/2$. Then the exponential map $\exp_p$ at any point $p \in S$ is a diffeomorphism from the disc of radius $\mathrm{sys}(S)/2$ in T_pS to its image, the metric ball $B = B(p, \mathrm{sys}(S)/2)$ about p in S. By comparison with the flat case, we get $|B| \geq \pi\,\mathrm{sys}(S)^2/4$ and therefore

$$\mathrm{sys}(S)^2/|S| < \mathrm{sys}(S)^2/|B| \leq 4/\pi.$$

□

Remarks 2.4.4. 1) If S is a compact and connected surface with non-empty boundary, then S contains a finite graph G in its interior which is a deformation retract of S. Given a Riemannian metric on S, a sufficiently small tubular neighborhood T of G in S is a Riemannian surface diffeomorphic to S with $\mathrm{sys}(T) \geq \mathrm{sys}(S)$ and with arbitrarily small area. Moreover, any upper bound on the curvature persists. In other words, we cannot expect to remove the minimum on the right hand side of the estimates in Theorem 2.4.1.

Note also that the right hand side of the inequalities in Item 2) and Item 3) of Theorem 2.4.1 is positive if and only if $|S| < 4\pi/\kappa$, that is, if and only if $|S|$ is smaller than the area of the sphere of constant curvature $\kappa > 0$.

2) In [**Gro83**, Corollary 5.2.B], Gromov shows that $\mathrm{sys}(S)^2/|S|^2 \leq 4/3$ for any closed Riemannian surface. The point is, of course, that his estimate is curvature free. His work in [**Gro83**] also implies that

$$\mathrm{sys}(S_g)^2/|S_g|^2 \leq C_g(\ln g)^2/g$$

with $\limsup C_g \leq 1/\pi$ as $g \to \infty$; see Section 11.3 in [**Kat07**].

Proof of Theorem 2.1.7 in the case $\chi(S) \geq 0$. In view of Proposition 2.1.10, it remains to show that $\Lambda(S) > \lambda_0(\tilde{S})$ in the case where S is a torus or a Klein bottle. Then S admits a flat background metric h which is conformal to the given metric g of S. Theorem 2.1.11 applies to h and shows that

$$\Lambda(S,h) \geq \mathrm{sys}(S,h)^2/|(S,h)|^2, \tag{2.4.5}$$

where (S,h) denotes S, endowed with the metric h. Furthermore, since we are in the case of surfaces, the Dirichlet integral of smooth functions is invariant under conformal changes; that is, we have

$$\int |\nabla\varphi|^2\, da = \int |\nabla\varphi|_h^2\, da_h. \tag{2.4.6}$$

Since S is compact, there is a constant $\alpha \geq 1$ such that

$$\alpha^{-1}|v| \leq |v|_h \leq \alpha|v|$$

for all tangent vectors v of S. Using (2.4.5) and (2.4.6), we obtain

$$\Lambda(S) \geq \alpha^{-2}\Lambda(S,h) \geq \alpha^{-8}\,\mathrm{sys}(S)^2/|S|^2 > 0.$$

On the other hand, the fundamental group of S is amenable and hence $\lambda_0(\tilde{S}) = \lambda_0(S)$ by [**Bro85**, Theorem 1]. Now S is a torus or a Klein bottle, hence $\lambda_0(\tilde{S}) = 0$. $\square$

Proof of Theorem 2.1.13. Suppose first that S is orientable, that is, that $S = S_g$ for some $g \geq 2$, and let c be a systolic geodesic on S. Then by [**Bus10**, Theorem 4.3.2], the tubular neighborhood T of c of width

$$w_2 = \mathrm{arsinh}(1/\sinh(\mathrm{sys}(S)/2)))$$

is an open annulus. Since c is essential, T is incompressible. Note also that T can be exhausted by incompressible compact annuli with smooth boundary. In particular, for any $r < w_2$, the closed metric ball $\bar{B}(p,r)$ of radius r about a point p on c is contained in an incompressible compact annulus $A_r \subseteq T$ with smooth boundary. Since $B(p,r) \subseteq A_r$, we may use Theorem 1.1 and the first displayed formula on page 294 of [**Che75**] to conclude that

$$\lambda_0(A_r) \le \lambda_0(\bar{B}(p,r)) \le -\frac{\kappa}{4} + \frac{4\pi^2}{r^2}.$$

By the definition of $\Lambda(S)$, we have $\Lambda(S) \le \lambda_0(A_r)$ for any r as above. Hence the claim of Theorem 2.1.13 follows in the case $S = S_g$.

Suppose now that S is not orientable. Let $\mathrm{Or}(S) \to S$ be the orientation covering of S and c be a systolic geodesic on S. There are two cases:

1) If c is one-sided, then the lift $\tilde{c}$ of c to $\mathrm{Or}(S)$ is simple of length $2L$ and is invariant under the non-trivial covering transformation f of $\mathrm{Or}(S)$. Again by [**Bus10**, Theorem 4.3.2], the tubular neighborhood T of $\tilde{c}$ of width

$$w_1 = \operatorname{arsinh}(1/\sinh(\operatorname{sys}(S)))$$

is an open annulus. Since f leaves $\tilde{c}$ invariant, it also leaves T invariant and T/f is an open cross cap with soul c and width w_1 about c. Hence for any $r < w_1$, the closed metric ball $\bar{B}(p,r)$ of radius r about a point p on c is contained in a compact cross cap $C_r \subseteq T/f$ with smooth boundary.

2) If c is two-sided, then c has two lifts c_1 and c_2 to $\mathrm{Or}(S)$ and both are simple of length L. Moreover, by [**Bus10**, Theorem 4.3.2], the tubular neighborhoods T_1 of c_1 and T_2 of c_2 of width w_2 are open annuli and do not intersect. Now f permutes c_1 and c_2, therefore also T_1 and T_2, and hence the tubular neighborhood T of c of width w_2 is an open annulus.

In both cases, Item 1) and Item 2), we can now conclude the proof of the claim of Theorem 2.1.13 as in the case $S = S_g$. □

Remark 2.4.7. The arguments in the proof of Theorem 2.1.13 also show that $\operatorname{diam} S \ge w$ with $w = w_2$ and $w = w_1$, respectively. Hence we get

$$\lambda_{-\chi(S)} \le -\frac{\kappa}{4} + \chi(S)^2 \frac{16\pi^2}{w^2}$$

from Corollary 2.3 of [**Che75**]. In view of $\Lambda(S) < \lambda_{-\chi(S)}$, this gives another, but weaker upper bound for $\Lambda(S)$.

2.5. On the ground state

Throughout this section, we let F be a compact Riemannian surface with smooth boundary $\partial F \neq \emptyset$ and φ be the ground state of F. We also set $\psi = \varphi^2$ and let $F_t = \{\psi \ge t\}$. Note that $\int \psi = 1$.

By the Hopf boundary lemma [**GT83**, Lemma 3.4], φ does not have critical points on ∂F. Moreover, since $\varphi > 0$ in the interior $\mathring{F}$ of F, a point in $\mathring{F}$ is critical for ψ if and only if it is critical for φ. All points of ∂F are critical for ψ.

In our first result, we elaborate on the argument from the middle of page 549 in [**Oss77**].

Lemma 2.5.1. *Let $0 < t < \max \psi$ be a regular value of ψ. Then F_t is a compact subsurface of $\mathring{F}$ such that the boundary of each component of $F \setminus F_t$ has at least one boundary circle in $\mathring{F}$ and contains at least one boundary circle of F.*

Proof. Since t is a regular value of ψ with $0 < t < \max \psi$, F_t is a compact subsurface of $\mathring{F}$ with smooth boundary. If the boundary of a component C of $F \setminus F_t$ would not contain a boundary circle of F, then φ would be a non-constant superharmonic function on C which attains its maximum $\sqrt{t}$ along ∂F_t, a contradiction. Clearly, the boundary of C must have at least one boundary circle in $\partial F_t \subseteq \mathring{F}$. $\square$

Proposition 2.5.2. *Let $0 < t < \max \psi$ be a regular value of ψ. Then any connected component C of F_t is incompressible in F and no component of $F \setminus C$ is a disc or a cross cap. Furthermore, $\chi(C) \geq \chi(F)$ with equality if and only if $F \setminus C$ is a collared neighborhood of ∂F, consisting of annuli about the boundary circles of F.*

Proof. Suppose that a component D of $F \setminus C$ would be a disc or a cross cap. Then $D \setminus F_t$ would consist of components of $F \setminus F_t$ with boundary in $\mathring{F}$, a contradiction to Lemma 2.5.1. Substituting C for F', the rest of the proof of Proposition 2.5.2 is now more or less the same as that of Proposition 2.3.2. $\square$

As in Theorem 2.2.2, we denote by ℓ the sum of the lengths of the shortest loops in the free homotopy classes (in F) of the boundary circles of F. Furthermore, we let $\Lambda'(F) = \inf \lambda_0(F')$, where the infimum is taken over all incompressible compact and connected subsurfaces F' of $\mathring{F}$ with smooth boundary and Euler characteristic $\chi(F') > \chi(F)$.

Lemma 2.5.3. *If $\chi(F) \leq 0$, then*

$$\lambda_0(F) \geq \big\{1 - \delta + 2\big(1 - \frac{1}{\delta}\big)\frac{|F|}{\ell}\sqrt{\lambda_0(F)}\big\}\Lambda'(F)$$

for all $0 < \delta < 1/2$.

Proof. Since the quantities involved in the lemma vary continuously with respect to variations of the metric (in the C^0-topology), we may assume, by Theorem 8 in [**Uhl76**], that φ is a Morse function. Then the critical points of ψ in $\mathring{F}$ are non-degenerate. Moreover, since φ does not have critical points on ∂F, $F \setminus F_t$ is a collared neighborhood of ∂F, consisting of annuli about the boundary circles of F, for all sufficiently small $t > 0$. On the other hand, for $t < \max \psi$ suffciently close to $\max \psi$, F_t is a union of embedded discs, one for each maximum point of ψ. Hence the topology of F_t undergoes changes as t increases from 0 to $\max \psi$.

Since φ is a Morse function, ψ has only finitely many critical points in $\mathring{F}$. By Lemma 2.5.1, ψ does not have local minima in $\mathring{F}$. Hence critical points of ψ in $\mathring{F}$ are saddle points and local maxima.

Let $0 = \beta_0 < \cdots < \beta_m = \max \psi$ be the finite sequence of critical values of ψ and choose $\varepsilon > 0$ with $\varepsilon < \min\{\beta_{i+1} - \beta_i\}$. In a first step, we select now a critical value $\beta = \beta_i$ according to specific requirements.

By Proposition 2.5.2, each component C of $F_{\beta_1+\varepsilon}$ has Euler characteristic $\chi(C) \geq \chi(F)$. Therefore there are two cases. Either each component C of $F_{\beta_1+\varepsilon}$ has Euler characteristic $\chi(C) > \chi(F)$. Then we set $\beta = \beta_1$. Or else there is a component C with $\chi(C) = \chi(F)$. Then $F \setminus C$ is a collared neighborhood of ∂F, consisting of annuli about the boundary circles of F, by Proposition 2.5.2. In that case, by Lemma 2.5.1, the other components of $F_{\beta_1+\varepsilon}$ are discs contained in these annuli.

We assume that we are in the second case and consider the second critical value β_2. By Proposition 2.5.2, there are again two cases. Either each component C of $F_{\beta_2+\varepsilon}$ has Euler characteristic $\chi(C) > \chi(F)$; then we set $\beta = \beta_2$. Or else there is a component C of $F_{\beta_2+\varepsilon}$ with $\chi(C) = \chi(F)$. Then $F \setminus C$ is a collared neighborhood of ∂F consisting of annuli about the boundary circles of F. In the latter case, we pass on to the next critical value β_3. Since $\chi(F) \le 0$, we will eventually arrive at a first critical value $\beta = \beta_i$ with the property that the complement of a component of $F_{\beta-\varepsilon}$ is a collared neighborhood of ∂F consisting of annuli about the boundary circles of F and such that each component C of $F_{\beta+\varepsilon}$ has Euler characteristic $\chi(C) > \chi(F)$. Note that this property then holds for all sufficiently small $\varepsilon > 0$ since β is the only critical value of φ in $(\beta_{i-1}, \beta_{i+1})$. It follows that for any regular value $0 < t < \beta$ of ψ, F_t has a component C such that $F \setminus C$ is a collared neighborhood of ∂F consisting of annuli about the boundary circles of F. In particular, $|\partial F_t| \ge \ell$ for all such t. Using (2.3.5) and (2.3.6), we obtain

$$\beta\ell \le \int_0^\infty L(t)dt \le 2\sqrt{\lambda_0(F)}, \tag{2.5.4}$$

where $L(t)$ denotes the length of ∂F_t.

For $\varepsilon > 0$ as above, the smooth function $\varphi_\varepsilon = \varphi - \sqrt{\beta+\varepsilon}$ is smooth on $F_{\beta+\varepsilon}$, vanishes on $\partial F_{\beta+\varepsilon}$ and satisfies

$$\begin{aligned} \int_{F_{\beta+\varepsilon}} \varphi_\varepsilon^2 &= \int_{F_{\beta+\varepsilon}} \big(\varphi - \sqrt{\beta+\varepsilon}\big)^2 \\ &= \int_{F_{\beta+\varepsilon}} \varphi^2 - 2\sqrt{\beta+\varepsilon}\int_{F_{\beta+\varepsilon}} \varphi + (\beta+\varepsilon)|F_{\beta+\varepsilon}|. \end{aligned} \tag{2.5.5}$$

Now the first term on the right hand side of (2.5.5) satisfies

$$\int_{F_{\beta+\varepsilon}} \varphi^2 \ge \int_F \varphi^2 - (\beta+\varepsilon)(|F| - |F_{\beta+\varepsilon}|) \tag{2.5.6}$$

since $\varphi^2 \le \beta+\varepsilon$ on $F \setminus F_{\beta+\varepsilon}$. For the second term on the right hand side of (2.5.5), we have

$$\begin{aligned} 2\sqrt{\beta+\varepsilon}\int_{F_{\beta+\varepsilon}} \varphi &\le 2\sqrt{\beta+\varepsilon}|F_{\beta+\varepsilon}|^{1/2}\big(\int_{F_{\beta+\varepsilon}} \varphi^2\big)^{1/2} \\ &\le \frac{1}{\delta}(\beta+\varepsilon)|F_{\beta+\varepsilon}| + \delta\int_{F_{\beta+\varepsilon}} \varphi^2 \\ &\le \frac{1}{\delta}(\beta+\varepsilon)|F_{\beta+\varepsilon}| + \delta\int_F \varphi^2 \end{aligned} \tag{2.5.7}$$

by the Schwarz inequality and the Peter and Paul principle. Combining (2.5.5), (2.5.6), and (2.5.7) and using that the L^2-norm of φ is one and that $2 - 1/\delta < 0$, we obtain

$$\begin{aligned} \int_{F_{\beta+\varepsilon}} \varphi_\varepsilon^2 &\ge (1-\delta)\int_F \varphi^2 - (\beta+\varepsilon)|F| + \big(2 - \frac{1}{\delta}\big)(\beta+\varepsilon)|F_{\beta+\varepsilon}| \\ &\ge 1 - \delta + \big(1 - \frac{1}{\delta}\big)(\beta+\varepsilon)|F|. \end{aligned}$$

For the Rayleigh quotient of φ_ε, we get

$$\begin{aligned} R(\varphi_\varepsilon)(1-\delta+\big(1-\frac{1}{\delta}\big)(\beta+\varepsilon)|F|) &\le R(\varphi_\varepsilon)\int_{F_{\beta+\varepsilon}} \varphi_\varepsilon^2 \\ &= \int_{F_{\beta+\varepsilon}} |\nabla\varphi_\varepsilon|^2 = \int_{F_{\beta+\varepsilon}} |\nabla\varphi|^2 \le \int_F |\nabla\varphi|^2 = \lambda_0(F). \end{aligned}$$

Since $F_{\beta+\varepsilon}$ is a disjoint union of incompressible compact and connected subsurfaces F' with smooth boundary and $\chi(F') > \chi(F)$, we also have $R(\varphi_\varepsilon) \ge \inf\Lambda'(F)$. Letting ε tend to 0, we finally obtain

$$\Lambda'(F)(1-\delta+\big(1-\frac{1}{\delta}\big)\beta|F|) \le \lambda_0(F). \tag{2.5.8}$$

Combining (2.5.4) and (2.5.8), we arrive at Lemma 2.5.3. □

2.6. On the ground state (continued)

Let S be a complete and connected Riemannian surface of finite type with $\chi(S) < 0$ and $\lambda_0(\tilde S) < \lambda_{\mathrm{ess}}(S)$.

Since $\chi(S) < 0$, S carries complete hyperbolic metrics. Using a decomposition of S into a finite number of pairs of pants, it is clear that we may choose such a metric h such that the connected components of a neighborhood of the ends of S is a finite union of *hyperbolic funnels*, that is, cylinders of the form $(-1,\infty)\times\mathbb{R}/\mathbb{Z}$ with metric

$$dr^2 + \cosh(r)^2 d\vartheta^2.$$

Then the curves $r = 0$ are closed h-geodesics of length 1. The original metric of S will be denoted by g.

We fix a smooth and proper function from S to $[0,\infty)$, which agrees outside a compact set with the coordinates r in each of the ends. By abuse of notation, we denote this function by r. Choose an increasing sequence $0 < r_0 < r_1 < r_2 < \cdots \to \infty$. Then the subsurfaces

$$K_i = \{r \le r_i\} \tag{2.6.1}$$

of S are compact with smooth boundary $\partial K_i = \{r = r_i\}$ such that $S\setminus K_i$ is a cylindrical neighbourhood of infinity. Furthermore,

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots \tag{2.6.2}$$

is an exhaustion of S. By choosing the sequence of r_i suitably, we may assume that
a) there exist cutoff functions $\eta_i\colon S \to [0,1]$ with $\eta_i = 1$ on K_i, $\eta_i = 0$ outside K_{i+1}, and $|\nabla\eta_i|^2 \le 1/i$,
b) $\lambda_0(\tilde S) < \lambda_0(S\setminus K_0)$,
where we note that $\lambda_0(S\setminus K_i) < \lambda_0(S\setminus K_{i+1})\cdots \to \lambda_{\mathrm{ess}}(S)$.

In the case where S is compact, we have $K_i = S$ for all i and part of the following discussion becomes trivial.

We now let F be a compact subsurface of S with smooth boundary $\partial F \ne \emptyset$. As in Section 2.5, we denote by φ the ground state of F and let $F_t = \{\varphi^2 \ge t\}$.

Lemma 2.6.3. *For a subset $\mathcal{R} \subseteq (0,\max\varphi^2)$ of full measure, F_t is a smooth subsurface of $\mathring{F}$ such that $\partial F_t = \{\varphi = t\}$ and ∂K_i intersect transversally for all i.*

Proof. Since φ is smooth up to the boundary of F and has no critical points on ∂F, there is a smooth extension $\tilde{\varphi}$ of φ to S such that $\tilde{\varphi}$ is strictly negative on $S \setminus F$. The restriction $\tilde{\varphi}_0$ of $\tilde{\varphi}$ to the union of the curves $\{r = r_i\}$ is then smooth, and hence there is a set $\mathcal{R}_0 \subseteq \mathbb{R}$ of full measure such that any $t \in \mathcal{R}_0$ is a regular value of $\tilde{\varphi}_0$. Note that $\nabla\tilde{\varphi}$ is not perpendicular to the curve $\{r = r_i\}$ at points $p \in \{r = r_i\}$ with $\tilde{\varphi}(p) \in \mathcal{R}_0$. On the other hand, $\nabla\tilde{\varphi}$ is perpendicular to ∂F_t for any regular value t of φ^2 in $(0, \max\varphi^2)$. Therefore the intersection $\mathcal{R}$ of $\mathcal{R}_0$ with the set of regular values of φ^2 in $(0, \max\varphi^2)$ satisfies the required assertions. $\square$

Lemma 2.6.4. *For any $t \in \mathcal{R}$, the intersection $F_t \cap K_i$ is a subsurface of F with piecewise smooth boundary and any connected component C of $F_t \cap K_i$ is incompressible in F. In particular, we have $\chi(C) \geq \chi(F)$.*

Proof. For any $t \in \mathcal{R}$, $\partial F_t = \{\varphi = t\}$ and ∂K_i intersect transversally for all i, and then $F_t \cap K_i$ is a subsurface of F with piecewise smooth boundary. Since $S \setminus K_i$ is a cylindrical neighborhood of the ends of S, a disc in S has to be contained in K_i if its boundary is in K_i. Hence the components of $F_t \cap K_i$ are incompressible in F_t. By Proposition 2.5.2, F_t is incompressible in F. Therefore $F_t \cap K_i$ is incompressible in F. $\square$

Lemma 2.6.5. *Assume that $\lambda_0(F) \leq \theta\lambda_0(S \setminus K_0)$ for some $0 < \theta < 1$, and let $\varepsilon > 0$. Then there is an integer $i_0 = i_0(\theta, \varepsilon) \geq 0$ such that*

$$\int_{F \cap K_i} \varphi^2 \geq 1 - \varepsilon \quad \textit{for all } i \geq i_0.$$

Proof. Since $(1 - \eta_i)\varphi$ has support in $F \setminus K_i \subseteq S \setminus K_0$, we have

$$\begin{aligned}
\lambda_0(S \setminus K_0) \int_F (1 - \eta_i)^2\varphi^2 &\leq \int_F |\nabla((1 - \eta_i)\varphi)|^2 \\
&= \int_F \nabla((1 - \eta_i)^2\varphi) \cdot \nabla\varphi + \int_F \varphi^2|\nabla(1 - \eta_i)|^2 \\
&= \int_F ((1 - \eta_i)^2\varphi) \cdot \Delta\varphi + \int_F \varphi^2|\nabla(1 - \eta_i)|^2 \\
&\leq \lambda_0(F) \int_F (1 - \eta_i)^2\varphi^2 + \frac{1}{i}\int_F \varphi^2 \\
&\leq \theta\lambda_0(S \setminus K_0) \int_F (1 - \eta_i)^2\varphi^2 + \frac{1}{i}.
\end{aligned}$$

Since $0 < \theta < 1$, we conclude that

$$\int_F (1 - \eta_i)^2\varphi^2 \leq \frac{1}{(1 - \theta)\lambda_0(S \setminus K_0)i}.$$

Now for i_0 sufficiently large, the right hand side is smaller than ε for all $i \geq i_0 - 1$. For any $i \geq i_0$, we then have

$$\begin{aligned}\int_{F\cap K_i} \varphi^2 &= 1 - \int_{F\setminus K_i} \varphi^2 \\ &= 1 - \int_{F\setminus K_i} (1-\eta_{i-1})^2\varphi^2 \\ &\geq 1 - \int_F (1-\eta_{i-1})^2\varphi^2 \\ &\geq 1-\varepsilon. \end{aligned} \qquad \square$$

There is a sequence of constants $1 \leq \alpha_0 \leq \alpha_1 \leq \cdots$ such that

$$\alpha_i^{-1}|v| \leq |v|_h \leq \alpha_i|v| \tag{2.6.6}$$

for all tangent vectors v of S with foot point in K_i, where no index and index h indicate measurement with respect to g and h, respectively. Over K_i, the area elements da of g and da_h of h are then estimated by

$$\alpha_i^{-2}da \leq da_h \leq \alpha_i^2 da \tag{2.6.7}$$

with corresponding inequalities for the areas of measurable subsets and for integrals of non-negative measurable functions.

Let now again φ be the ground state of F, $t \in \mathcal{R}$, and $F_t = \{\varphi^2 \geq t\}$. In our next result, we estimate the inradius of $F_t \cap K_i$ for sufficiently large i.

Lemma 2.6.8. *Let F be a disc, an annulus, or a cross cap. Assume that $\lambda_0(F) \leq \theta\lambda_0(S\setminus K_0)$ for some $0<\theta<1$ and let $\delta>0$. Then there is an integer $i_1 = i_1(\theta,\delta) \geq 0$ such that the inradius $\rho(t)$ of $F_t \cap K_i$ satifies*

$$\coth(\alpha_{i+1}\rho(t)) \leq \frac{2\alpha_{i+1}^3\sqrt{\lambda_0(F)+\delta}}{1-\delta-t|F\cap K_i|}$$

for all $0 \leq t < (1-\delta)/|F\cap K_i|$ and $i \geq i_1$.

PROOF. In a first step, we estimate the Rayleigh quotient of $\eta_i\varphi$. Computing as in the proof of Lemma 2.6.5, we have

$$\begin{aligned}\int_F |\nabla(\eta_i\varphi)|^2 &= \int_F \nabla(\eta_i^2\varphi)\nabla\varphi + \int_F |\nabla\eta_i|^2\varphi^2 \\ &= \int_F \eta_i^2\varphi\Delta\varphi + \int_F|\nabla\eta_i|^2\varphi^2 \\ &\leq \lambda_0(F)\int_F \eta_i^2\varphi^2 + 1/i.\end{aligned}$$

Since $\eta_i = 1$ on K_i, we get $R(\eta_i\varphi) \leq \lambda_0(F) + 2/i$ for all $i \geq i_0(\theta, 1/2)$, where i_0 is taken from Lemma 2.6.5. Therefore

$$R(\eta_i\varphi) \leq \lambda_0(F) + \delta \quad \text{for all } i \geq i_1(\theta,\delta), \tag{2.6.9}$$

where we may assume that $i_1(\theta,\delta) \ge i_0(\theta,\delta)$. In a second step, we follow the proof of Cheeger's inequality, Theorem 2.3.3. Computing as in (2.3.4), we get

$$\int_F |\nabla(\eta_i^2\varphi^2)| \le 2\sqrt{R(\eta_i\varphi)} \int_F \eta_i^2\varphi^2 \le 2\sqrt{R(\eta_i\varphi)}.$$

By the coarea formula and (2.6.6) and since $\operatorname{supp}\eta_i \subseteq K_{i+1}$ and $\eta_i = 1$ on K_i, we have

$$\begin{aligned} \alpha_{i+1}\int_F |\nabla(\eta_i^2\varphi^2)| &= \alpha_{i+1}\int_0^\infty |\{\eta_i^2\varphi^2 = s\}|\,ds \\ &\ge \int_t^\infty |\{\eta_i^2\varphi^2 = s\}|_h\,ds \\ &= \int_t^\infty |\{\varphi^2 = s\}\cap K_i|_h\,ds \\ &\qquad + \int_t^\infty |\{\eta_i^2\varphi^2 = s\}\cap(F\setminus K_i)|_h\,ds \end{aligned} \tag{2.6.10}$$

Here we note that, for the integration, it suffices to consider $s \in \mathcal{R}$ which are also regular for $\eta_i^2\varphi^2$. Then $\{\varphi^2 = s\}$ meets ∂K_i transversally and, therefore, $\{\eta_i^2\varphi^2 = s\}\cap(F\setminus K_i)$ consists of arcs a_j connecting their corresponding end points on $\{\varphi^2 = s\}\cap\partial K_i$. Replacing the a_j by the corresponding segments b_j on ∂K_i, we obtain the boundary of the subsurface $F_s\cap K_i$. Now $|b_j|_h \le |a_j|_h$ by the choice of the hyperbolic metric h on S and since the numbers r_i defining the K_i are positive. Hence we have

$$\begin{aligned} |\{\varphi^2 = s\}\cap K_i|_h &+ |\{\eta_i^2\varphi^2 = s\}\cap F\setminus K_i|_h \\ &= |\{\varphi^2 = s\}\cap K_i|_h + \sum |a_j|_h \\ &\ge |\{\varphi^2 = s\}\cap K_i|_h + \sum |b_j|_h \\ &= |\partial(\{\varphi^2 \ge s\}\cap K_i)|_h, \end{aligned} \tag{2.6.11}$$

for any $s \in \mathcal{R}$. This implies

$$\begin{aligned} \int_t^\infty |\{\varphi^2 = s\}\cap K_i|_h\,ds &+ \int_t^\infty |\{\eta_i^2\varphi^2 = s\}\cap(F\setminus K_i)|_h\,ds \\ &\ge \int_t^\infty |\partial(\{\varphi^2 \ge s\}\cap K_i)|_h\,ds. \end{aligned} \tag{2.6.12}$$

Furthermore, by Lemma 2.6.4, the connected components of $F_s\cap K_i$ are subsurfaces of F with piecewise smooth boundary and have non-negative Euler characteristic for any $s \in \mathcal{R}$. Therefore we get

$$\begin{aligned} \int_t^\infty &|\partial(\{\varphi^2 \ge s\}\cap K_i)|_h\,ds \\ &\ge \int_t^\infty |\{\varphi^2 \ge s\}\cap K_i|_h \coth(\rho_h(s))\,ds \\ &\ge \coth(\rho_h(t))\int_t^\infty |\{\varphi^2 \ge s\}\cap K_i|_h\,ds \\ &\ge \alpha_i^{-2}\coth(\alpha_i\rho(t))\int_t^\infty |\{\varphi^2 \ge s\}\cap K_i|\,ds, \end{aligned} \tag{2.6.13}$$

by Corollary 2.2.3.Item 2, (2.6.6), and (2.6.7). Finally, since $i_1(\theta,\delta) \geq i_0(\theta,\delta)$,

$$
\begin{aligned}
\int_t^\infty |\{\varphi^2 \geq s\} \cap K_i|\, ds &\geq \int_0^\infty |\{\varphi^2 \geq s\} \cap K_i|\, ds - t|F \cap K_i| \\
&= \int_{F\cap K_i} \varphi^2 - t|F \cap K_i| \geq 1 - \delta - t|F \cap K_i|.
\end{aligned}
\tag{2.6.14}
$$

Lemma 2.6.8 follows now from combining (2.6.9) – (2.6.14). □

Lemmas Lemma 2.5.3 and Lemma 2.6.8 will lead to the apriori estimates in Lemma 2.7.2 and in the proof of Lemma 2.7.9, which are essential in the proof of Theorem 2.1.4.

2.7. Qualititative estimates of $\Lambda(S)$

In this section, we prove Theorem 2.1.7 in the case $\chi(S) < 0$. Throughout, we let S be a complete and connected Riemannian surface of finite type and set

$$
\Lambda_D(S) = \inf_D \lambda_0(D), \quad \Lambda_A(S) = \inf_D \lambda_0(A), \quad \Lambda_C(S) = \inf_D \lambda_0(C),
\tag{2.7.1}
$$

where the infimum is taken over all embedded closed discs D, incompressible annuli A, and cross caps C in $\mathring{S}$ with smooth boundary, respectively. As we will explain in Section 2.8.1, we have

$$
\Lambda_D(S) \geq \Lambda_A(S) \quad \text{and} \quad \Lambda(S) = \inf\{\Lambda_A(S), \Lambda_C(S)\}
$$

if the fundamental group of S is infinite. Nevertheless, since the case of discs reveals an essential idea of the proof and since we will need the estimate anyway, we include the discussion of $\Lambda_D(S)$.

We fix a hyperbolic metric h on S as in Section 2.6 and denote by g the original Riemannian metric of S. If not otherwise mentioned, statements refer to g and not to h.

We will use the setup and notation from the previous section. The following assertion is an immediate consequence of Lemma 2.6.8.

Lemma 2.7.2. *Let F be a closed disc, annulus, or cross cap in S and φ be the ground state of F. Assume that $\lambda_0(F) \leq \theta\lambda_0(S\setminus K_0)$ for some $0 < \theta < 1$. Then the inradius $\rho(\varepsilon)$ of $\{\varphi^2 \geq \varepsilon\} \cap K_i$ satisfies $\rho(\varepsilon) \geq \rho > 0$ for all $0 < \varepsilon < 1/4|K_{i_1}|$ and $i \geq i_1 = i_1(\theta, 1/2)$.*

We now discuss the cases of discs and annuli separately.

Theorem 2.7.3. *If S is a complete and connected Riemannian surface of finite type with $\chi(S) < 0$ and $\lambda_{\mathrm{ess}}(S) > \lambda_0(\tilde{S})$, then $\Lambda_D(S) > \lambda_0(\tilde{S})$.*

Proof. Suppose that there is a sequence of discs D_n in S with smooth boundary such that $\lambda_0(D_n) \to \lambda_0(\tilde{S})$. Let $\varphi_n \colon D_n \to \mathbb{R}$ be the positive $\lambda_0(D_n)$-Dirichlet eigenfunction with $||\varphi_n||_2 = 1$. By passing to a subsequence if necessary, we may assume that $\lambda_0(D_n) \leq \theta\lambda_0(S \setminus K_0)$ for some $0 < \theta < 1$. By Lemma 2.7.2 and up to passing to a subsequence, there are positive constants ε_0 and ρ_0 and a point $x_0 \in S$ such that $B(x_0, 2\rho_0)$ is contained in $\{\varphi_n^2 \geq \varepsilon_0\} \cap K$ for all n, where $K = K_{i_1(\theta,1/2)}$.

Fix a point $\tilde{x}_0 \in \tilde{S}$ above x_0. Then there is a unique lift $\tilde{D}_n$ of D_n to $\tilde{S}$ containing $\tilde{x}_0$ such that $\tilde{D}_n \to D_n$ is a diffeomorphism (including the boundary). Thus we may also lift φ_n to $\tilde{\varphi}_n$ on $\tilde{D}_n$ and extend $\tilde{\varphi}_n$ to a function $\tilde{\varphi}_n$ on $\tilde{S}_n$ by setting $\tilde{\varphi}_n = 0$ on

$\tilde{S} \setminus \tilde{D}_n$. Since the boundary of $\tilde{D}_n$ is smooth and $\tilde{\varphi}_n$ is smooth on $\tilde{D}_n$, it follows that $\tilde{\varphi}_n \in H_0^1(\tilde{S})$ with H^1-norm

$$||\tilde{\varphi}_n||_{H^1} = ||\varphi_n||_{H^1}^2 = \lambda_0(D_n) + 1,$$

where we use Green's formula for the second equality. In particular, up to extracting a subsequence, we have weak convergence

$$\tilde{\varphi}_n \rightharpoonup \tilde{\varphi} \in H_0^1(\tilde{S}) \quad \text{with} \quad ||\tilde{\varphi}||_{H^1} \leq \liminf ||\varphi_n||_{H^1}.$$

Up to extracting a further subsequence, the sequence of $\tilde{\varphi}_n$ converges uniformly in any C^k-norm in $B(\tilde{x}_0, \rho_0)$, by Theorem 8.10 in [**GT83**]. In particular $\tilde{\varphi}^2 \geq \varepsilon_0$ on $B(\tilde{x}_0, \rho_0)$.

By Theorem 1 of [**AFLMR07**], we may approximate the distance function d_0 to $\tilde{x}_0$ in $\tilde{S}$ by a smooth function u on $\tilde{S}$ such that $|u - d_0| \leq 1$ and $|\nabla u| \leq 2$. Then the sublevels $B(r) = \{u \leq r\}$ form an exhaustion of $\tilde{S}$ by compact subsets. Clearly,

$$R(\tilde{\varphi}) = \lim_{r \to \infty} R(\tilde{\varphi}|_{B(r)}).$$

Furthermore, up to passing to a subsequence, we have weak convergence

$$\tilde{\varphi}_n|_{B(r)} \rightharpoonup \tilde{\varphi}|_{B(r)} \quad \text{in } H^1(B(r))$$

and strong convergence

$$\tilde{\varphi}_n|_{B(r)} \to \tilde{\varphi}|_{B(r)} \quad \text{in } L^2(B(r)).$$

Hence

$$R(\tilde{\varphi}|_{B(r)}) \leq \liminf R(\tilde{\varphi}_n|_{B(r)}).$$

For any regular value r of u such that $\partial B(r)$ intersects $\partial \tilde{D}_n$ transversally we have

$$\int_{B(r)} |\nabla \tilde{\varphi}_n|^2 = \lambda_0(D_n) \int_{B(r)} |\tilde{\varphi}_n|^2 + \int_{\partial B(r)} \tilde{\varphi}_n \langle \nabla \tilde{\varphi}_n, \nu \rangle,$$

where $\nu = \nabla u / |\nabla u|$ is the outward unit vector field along $\partial B(r)$. Clearly, the second term on the right satisfies

$$\int_{\partial B(r)} \tilde{\varphi}_n \langle \nabla \tilde{\varphi}_n, \nu \rangle \leq \int_{\partial B(r)} (|\tilde{\varphi}_n|^2 + |\nabla \tilde{\varphi}_n|^2).$$

Let now $\varepsilon > 0$ be given. Then there is a sequence of integers $k_m \to \infty$ such that, for each m, there is a subsequence of $n \to \infty$ with

$$\int_{K(k_m)} (|\tilde{\varphi}_n|^2 + |\nabla \tilde{\varphi}_n|^2) < \varepsilon, \tag{2.7.4}$$

where $K(r) = B(r) \setminus B(r-1)$. If this would not be the case, there would be some $\varepsilon > 0$ and a positive integer m such that for all integers $k \geq m$ there is an integer l such that, for all integers $n \geq l$,

$$\int_{K(k)} (|\tilde{\varphi}_n|^2 + |\nabla \tilde{\varphi}_n|^2) \geq \varepsilon. \tag{2.7.5}$$

We thus find, for any fixed M, an index N such that (2.7.5) holds for $k = m, \dots, m + M$ and $n \geq N$. If we choose M such that $M\varepsilon > \lambda_0(\tilde{S}) + 1$, we get a contradiction.

By the coarea formula, we have

$$\int_{K(k_m)} (|\tilde{\varphi}_n|^2 + |\nabla \tilde{\varphi}_n|^2)|\nabla u| = \int_{k_m-1}^{k_m} \int_{\partial B(r)} (|\tilde{\varphi}_n|^2 + |\nabla \tilde{\varphi}_n|^2).$$

Since $|\nabla u| \leq 2$, we then get, for n as in (2.7.4), that there is a regular value $r_n \in (k_m - 1, k_m)$ of u such that

$$\int_{\partial B(r_n)} (|\tilde{\varphi}_n|^2 + |\nabla \tilde{\varphi}_n|^2) < 2\varepsilon,$$

where we may also assume that $\partial B(r_n)$ intersects $\partial \tilde{D}_n$ transversally. We obtain

$$\begin{aligned} R(\tilde{\varphi}_n|_{B(k_m)}) &\leq \frac{\int_{B(r_n)} |\nabla \tilde{\varphi}_n|^2}{\int_{B(k_m)} |\tilde{\varphi}_n|^2} + \frac{\int_{K(k_m)} |\nabla \tilde{\varphi}_n|^2}{\int_{B(k_m)} |\tilde{\varphi}_n|^2} \\ &\leq \frac{\int_{B(r_n)} |\nabla \tilde{\varphi}_n|^2}{\int_{B(r_n)} |\tilde{\varphi}_n|^2} + \frac{\varepsilon}{\int_{B(k_m)} |\tilde{\varphi}_n|^2} \\ &\leq \lambda_0(D_n) + \frac{\int_{\partial B(r_n)} \tilde{\varphi}_n \langle \nabla \tilde{\varphi}_n, \nu \rangle}{\int_{B(r_n)} |\tilde{\varphi}_n|^2} + \frac{\varepsilon}{\int_{B(k_m)} |\tilde{\varphi}_n|^2} \\ &\leq \lambda_0(D_n) + \frac{2\varepsilon}{\int_{B(r_n)} |\tilde{\varphi}_n|^2} + \frac{\varepsilon}{\int_{B(k_m)} |\tilde{\varphi}_n|^2}. \end{aligned}$$

It follows that

$$R(\tilde{\varphi}|_{B(k_m)}) \leq \lambda_0(\tilde{S}) + \frac{3\varepsilon}{\int_{B(\tilde{x}_0, \rho_0)} |\tilde{\varphi}|^2}$$

for all sufficiently large m. In conclusion,

$$R(\tilde{\varphi}) \leq \lambda_0(\tilde{S}).$$

Since $\tilde{\varphi} \in H_0^1(\tilde{S})$, this implies that $\tilde{\varphi}$ is an eigenfunction of the Laplacian with eigenvalue $\lambda_0(\tilde{S})$.

Now $\tilde{\varphi}$ is non-zero on $B(\tilde{x}_0, \rho_0)$. On the other hand, by the definition of the lifts $\tilde{\varphi}_n$, $\tilde{\varphi}$ vanishes on any other preimages $B(\tilde{x}, \rho_0)$ of $B(x_0, \rho_0)$ under the covering projection $\tilde{S} \to S$. Now the fundamental group of S is not trivial, hence there are such preimages $\tilde{x} \in \tilde{S}$. Thus we arrive at a contradiction to the unique continuation property for eigenfunctions of Laplacians [**Aro57**]. □

Theorem 2.7.6. *If S is a complete and connected Riemannian surface of finite type with $\chi(S) < 0$ and $\lambda_{\text{ess}}(S) > \lambda_0(\tilde{S})$, then $\Lambda_A(S) > \lambda_0(\tilde{S})$.*

To prove Theorem 2.7.6, we assume the contrary and let (A_n) be a sequence of incompressible annuli in S with smooth boundary such that $\lambda_0(A_n) \to \lambda_0(\tilde{S})$. We may assume that

$$\lambda_0(A_n) + 4\delta < \min(\Lambda_D(S), \theta\lambda_0(S \setminus K_0)) \tag{2.7.7}$$

for all n and some fixed constants $\delta, \theta \in (0, 1)$, by invoking Theorem 2.7.3 and that $\lambda_0(\tilde{S}) < \lambda_0(S \setminus K_0)$ by the choice of K_0 in (2.6.2). By deforming the A_n (slightly), we may also assume that

$$(2.7.8) \qquad \partial A_n \text{ and } \partial K_i \text{ intersect transversally}$$

for all n and i. Then the intersections $A_n \cap K_i$ and $A_n \cap (S \setminus \mathring{K}_i)$ are incompressible subsurfaces of S with piecewise smooth boundary.

Recall the constant $i_1(\theta, \delta)$ from Lemma 2.6.8.

Lemma 2.7.9. *By passing to a subsequence, we may assume that*
1) all A_n are isotopic in S and, for each $i > i_1 = i_1(\theta, \delta)$, exactly one component A_n' of $A_n \cap K_i$ is an annulus (topologically) isotopic to A_n;
2) there is a constant $\ell_0 > 0$ such that the free homotopy classes of the boundary curves of A_n in A_n contain curves of length at most ℓ_0 with respect to g and h;
3) there are $x_0 \in S$ and $\rho, \varepsilon > 0$ such that $B(x_0, \rho) \subseteq A_n$ and such that the ground states φ_n of A_n satisfy $\varphi_n \geq \varepsilon$ on $B(x_0, \rho)$.

Proof. The connected components of $A_n \cap \partial K_i$ consist of embedded segments connecting two boundary points of A_n and of embedded circles in the interior of A_n. By the Schoenflies theorem and the topology of S, there are the following two possibilities:
a) All connected components of $A_n \cap K_i$ are discs.
b) The connected components of $A_n \cap K_i$ consist of one annulus A_n' (topologically) isotopic to A_n and discs.
Now let $i > i_1 = i_1(\theta, \delta)$. Then $\eta_{i-1}\varphi_n$ is a non-zero smooth function on A_n with compact support in $A_n \cap K_i$ and Rayleigh quotient

$$R(\eta_{i-1}\varphi_n) < \lambda_0(A_n) + \delta < \Lambda_D(S),$$

by (2.6.9) and (2.7.7). Hence, if all the connected components of $A_n \cap K_i$ were discs, we would have $R(\eta_{i-1}\varphi_n) \geq \Lambda_D(S)$. Therefore only case Item b) can occur. Then the Rayleigh quotients of $\eta_{i-1}\varphi_n$ on the discs of $A_n \cap K_i$ (on which $\eta_{i-1}\varphi_n$ does not vanish) must be at least $\Lambda_D(S)$. Hence the Rayleigh quotient of $\eta_{i-1}\varphi_n$ on A_n' must be less than $\lambda_0(A_n) + \delta$. In particular, $\lambda_0(A_n') \leq \lambda_0(A_n) + \delta$.

Note that one of the boundary circles of A_n' may only be piecewise smooth. Therefore A_n' may only be topologically isotopic to A_n.

Now we let $i = i_1 + 1$. Since $A_n' \subseteq K_i$, we have the uniform area bound $|A_n'| \leq |K_i|$. This together with the above estimate on $\lambda_0(A_n')$ and Lemma 2.5.3 implies that the length of shortest curves in A_n', which are freely homotopic to the boundary circles of A_n' in A_n', is uniformly bounded with respect to g. Then their length is also uniformly bounded with respect to h, by (2.6.6). In particular, there are only finitely many isotopy types of A_n' and, therefore also, of A_n. Therefore we may pass to a subsequence so that all of them are isotopic.

By Lemma 2.7.2 and since K_i is compact, we may pass to a further subsequence so that all $A_n \cap K_i$, and hence also all A_n, contain a geodesic ball $B(x_0, \rho)$ such that $\varphi_n \geq \varepsilon$ on $B(x_0, \rho)$ as claimed. $\square$

Proof of Theorem 2.7.6. By passing to a subsequence, we may assume that the sequence of A_n satisfies all the properties from (2.7.7), (2.7.8), and Lemma 2.7.9.

Choose a shortest (with respect to h) closed h-geodesic c in the free homotopy class in S of a generator of the fundamental group of A_n. This is possible since the ends of S

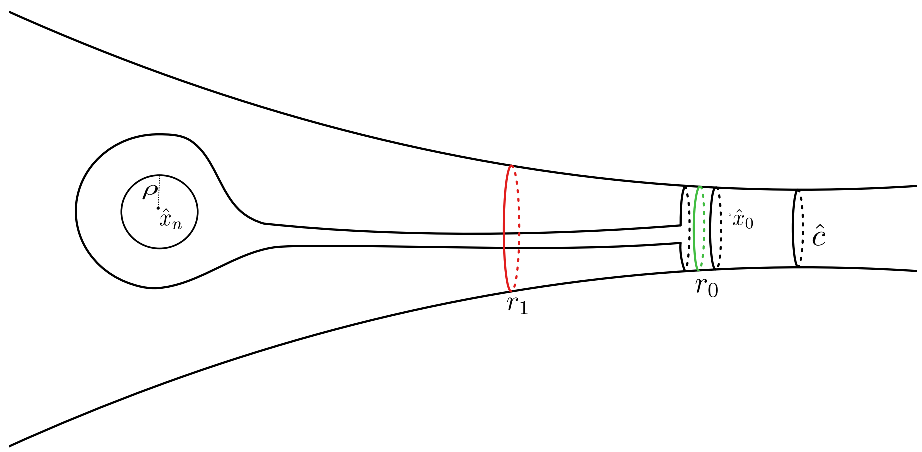
are hyperbolic funnels with respect to h. Note that c does not depend on n since all A_n are isotopic. We let $\hat{S}$ be the cyclic subcover of $\tilde{S}$ to which c lifts as a closed $\hat{h}$-geodesic $\hat{c}$, where $\hat{h}$ denotes the lift of h to $\hat{S}$. Note that all annuli A_n are isotopic to a small tubular neighbourhood of c. Lifting the corresponding isotopies, we get lifts $\hat{A}_n \subseteq \hat{S}$ of the annuli A_n. Note that $\hat{A}_n$ is the unique compact component of $\pi^{-1}(A_n)$ and that $\pi\colon \hat{A}_n \to A_n$ is a diffeomorphism which is isometric with respect to g and h and their respective lifts $\hat{g}$ and $\hat{h}$.

Denote by $\hat{x}_n$ the lift of x_0 which is contained in $\hat{A}_n$. If $\hat{x}_n$ stays at bounded distance to $\hat{c}$, the arguments for the case of discs in the proof of Theorem 2.7.3 apply again and lead to a contradiction since the fundamental group of S is not cyclic and $\lambda_0(\hat{S}) = \lambda_0(\tilde{S})$.

Suppose now that $\hat{x}_n \to \infty$ in $\hat{S}$. Let $(r,\theta)\colon \hat{S} \to \mathbb{R} \times (\mathbb{R}/\ell\mathbb{Z})$ be Fermi coordinates about $\hat{c}$, where ℓ denotes the h-length of c and $\hat{c}$, such that $\hat{c} = \{r = 0\}$. Then we have

$$\hat{h} = dr^2 + \cosh^2(r) d\theta^2.$$

Since the h-length of shortest curves, c_n, in the free homotopy class of the boundary curves of A_n in A_n is bounded by ℓ_0, there is a constant $r_0 > 0$ such that the lifts $\hat{c}_n$ of c_n to $\hat{A}_n$ are contained in the region $\{|r| \leq r_0\}$ of $\hat{S}$.



The case $\hat{x}_n \to \infty$

Let $\hat{\varphi}_n$ be the lift of φ_n to $\hat{A}_n$. Then $\hat{\varphi}_n$ is the ground state of $\hat{A}_n$ with respect to $\hat{g}$ and we have

$$\int_{B(\hat{x}_n,\rho)} \hat{\varphi}_n^2 = \int_{B(x_0,\rho)} \varphi_n^2 \geq \varepsilon^2 \operatorname{vol} B(x_0,\rho), \tag{2.7.10}$$

by Lemma 2.7.9.Item 3. Choose $j > i_1 = i_1(\theta,\delta)$ such that

$$|\nabla \eta_j|^2 < \delta\varepsilon^2 \operatorname{vol} B(x_0,\rho). \tag{2.7.11}$$

Since $\operatorname{supp} \eta_j \subseteq K_{j+1}$, the h-area of $\operatorname{supp} \eta_j$ is bounded by the h-area $|K_{j+1}|_h$ of K_{j+1}. Now choose an $r_1 > r_0$ such that the $\hat{h}$-area of either of the regions $-r_1 \leq r \leq -r_0$ and $r_0 \leq r \leq r_1$ in $\hat{S}$ is larger than $|K_{j+1}|_h$. Finally, choose a cut off function χ on $\hat{S}$ such that $\chi = 0$ on $\{|r| \le r_1\}$, $\chi = 1$ on $\{|r| \ge r_2\}$ for some $r_2 > r_1$, and such that

$$|\nabla\chi|^2 < \delta\varepsilon^2 \operatorname{vol} B(x_0, \rho). \tag{2.7.12}$$

Computing as in the proofs of Lemma 2.6.5 and Lemma 2.6.8 and with $\eta = \eta_j \circ \pi$, we get

$$\begin{aligned} \int_{\hat{A}_n} |\nabla(\chi\eta\hat{\varphi}_n)|^2 &= \int_{\hat{A}_n} \nabla(\chi^2\eta^2\hat{\varphi}_n)\nabla\hat{\varphi}_n + \int_{\hat{A}_n} \hat{\varphi}_n^2 |\nabla(\chi\eta)|^2 \\ &= \lambda_0(A_n) \int_{\hat{A}_n} \chi^2\eta^2\hat{\varphi}_n^2 + \int_{\hat{A}_n} \hat{\varphi}_n^2 |\nabla(\chi\eta)|^2 \\ &\le \lambda_0(A_n) \int_{\hat{A}_n} \chi^2\eta^2\hat{\varphi}_n^2 + 4\delta\varepsilon^2 \operatorname{vol} B(x_0, \rho), \end{aligned} \tag{2.7.13}$$

where we use (2.7.11) and (2.7.12) for the passage to the last line.

Since $\hat{x}_n \to \infty$ and $B(x_0, \rho) \subseteq K_j$ by the choice of j, $\chi = \eta = 1$ on $B(\hat{x}_n, \rho)$ for all sufficiently large n. Combining (2.7.10) and (2.7.13), we then get

$$R(\chi\eta\hat{\varphi}_n) \le \lambda_0(A_n) + 4\delta < \Lambda_D(S). \tag{2.7.14}$$

On the other hand, $\operatorname{supp}(\chi\eta\hat{\varphi}_n)$ is contained in the lift B_n of $A_n \cap K_{j+1}$ to $\hat{A}_n$, intersected with $\{|r| \ge r_1\}$. Now the h-area of B_n is bounded by $|K_{j+1}|_h$ and B_n contains c_n. Hence B_n does not contain loops freely homotopic to c_n in the region $\{|r| \ge r_1\}$ of $\hat{S}$ since otherwise, by the uinqueness of A'_n, it would contain one of the regions $-r_1 \le r \le -r_0$ or $r_0 \le r \le r_1$. Hence $B_n \cap \{|r| \ge r_1\}$ is a union of discs and, therefore, the Rayleigh quotient of $\chi\eta\hat{\varphi}_n$ has to be at least $\Lambda_D(S)$, a contradiction to (2.7.14). It follows that the sequence of $\hat{x}_n$ is bounded. □

Proof of Theorem 2.1.7 in the case $\chi(S) < 0$. By Theorems Theorem 2.7.3 and Theorem 2.7.6, we have

$$\min\big(\Lambda_D(S), \Lambda_A(S)\big) > \lambda_0(\tilde{S})$$

for any complete and connected Riemannian surface of finite type with $\chi(S) < 0$. This implies the assertion of Theorem 2.1.7 in the case where S is orientable since then S does not contain cross caps.

Assume now that S is not orientable and let $\operatorname{Or}(S)$ be the orientation covering space of S. Let C be a cross cap in S. Then the lift of C to $\operatorname{Or}(S)$ is an annulus A in $\operatorname{Or}(S)$ with

$$\lambda_0(C) \ge \lambda_0(A) \ge \Lambda_A(\operatorname{Or}(S)).$$

We conclude that $\Lambda_C(S) \ge \Lambda_A(\operatorname{Or}(S)) > \lambda_0(\tilde{S})$ as asserted. □

2.8. Remarks, examples, and questions

In this section, we collect some loose ends. We start with a comment which gives another argument for calling Λ the analytic systole.

2.8.1. On the definition of Λ. For complete and connected surfaces S with infinite fundamental group, an equivalent definition of the analytic systole is $\Lambda(S) = \inf_F \lambda_0(F)$, where the infimum is taken over incompressible annuli and cross caps F with smooth boundary in S:
a) For any disc D with smooth boundary and any free homotopy class $[c]$ of closed curves in S, there is an annulus A with smooth boundary in S containing D whose soul belongs to $[c]$. If $[c]$ is non-trivial, then A is incompressible. Moreover, $\lambda_0(A) \le \lambda_0(D)$ by the domain monotonicity of λ_0.
b) For any compressible annulus A in S with smooth boundary, there is a disc D in S whose boundary ∂D is one of the boundary circles of A such that $A \cup D$ is a disc in S with smooth boundary, by the Schoenflies theorem, and then a) applies.
c) Cross caps only occur in the case where S is not orientable. The soul of a cross cap C in S is not homotopic to zero in S. Since the fundamental group of S is torsion free, we get that C is incompressible.

However, in view of our previous articles [**BMM16, BMM17b**], it is more natural to include discs into the definition. Moreover, it is important in our analysis to handle the case of discs separately.

2.8.2. On the essential spectrum. The following result, Proposition 3.6 from [**BMM17b**] formulated for surfaces, is probably folklore. It shows that the essential spectrum of the Laplacian only depends on the geometry of the underlying surface S at infinity and that the essential spectrum of the Laplacian is empty if S is compact.

Proposition 2.8.1. *For a complete Riemannian surface S with compact boundary (possibly empty), $\lambda \in \mathbb{R}$ belongs to the essential spectrum of Δ if and only if there is a* Weyl sequence *for λ, that is, a sequence φ_n of smooth functions on S with compact support such that*
1) for any compact $K \subseteq S$, $\operatorname{supp} \varphi_n \cap K = \emptyset$ for all sufficiently large n;
2) $\limsup_{n\to\infty} \|\varphi_n\|_2 > 0$ and $\lim_{n\to\infty} \|\Delta\varphi_n - \lambda\varphi_n\|_2 = 0$.

In work of Arne Persson and of Richard Froese and Peter Hislop, the bottom of the essential spectrum of Laplacians and more general operators has been characterized in the sense of Proposition 2.8.1 or, more specifically, in the sense of (2.1.5); compare with [**HS96**, Section 14.4].

Corollary 2.8.2. *For a complete Riemannian surface S, we have*

$$\lambda_{\mathrm{ess}}(S) = \lim_K \lambda_0(S \setminus K),$$

where K runs over the compact subsets of S, ordered by inclusion.

Corollary 2.8.3. *If S is a compact Riemannian surface, then the spectrum of S is discrete; that is, $\lambda_{\mathrm{ess}}(S) = \infty$.*

2.8.3. Surfaces with cyclic fundamental group. In the (unnumbered) lemma on page 551 of [**Oss77**], Osserman establishes the following result in the special case of domains in the Euclidean plane.

Lemma 2.8.4. *Let S be a complete Riemannian surface with boundary (possibly empty) and p be a point in the interior of S. For sufficiently small $\varepsilon > 0$, let $S_\varepsilon(p) = S \setminus B_\varepsilon(p)$.*

Then

$$\lambda_0(S_\varepsilon(p)) \to \lambda_0(S) \quad \textit{as} \quad \varepsilon \to 0.$$

The arguments in [**Oss77**] also apply to the more general situation of Lemma 2.8.4 and therefore we skip its proof.

Proposition 2.8.5. *If S is a complete Riemannian surface diffeomorphic to sphere or projective plane, then $\Lambda(S) = 0$.*

Proof. For $p \in S$ and $\varepsilon > 0$, $S_\varepsilon(p) = S \setminus B_\varepsilon(p)$ is a closed disc or cross cap, respectively, and hence

$$0 \le \Lambda(S) \le \inf_{p,\varepsilon} \lambda_0(S_\varepsilon(p)) = \lambda_0(S) = 0,$$

where we use Lemma 2.8.4 for the penultimate equality. □

Proposition 2.8.6. *If S is a complete Riemannian surface diffeomorphic to disc (open or closed), annulus (open, half-open, or closed), or cross cap (open or closed), then $\lambda_0(S) = \Lambda(S)$.*

Proof. In each case, there exists an increasing sequence of closed discs, annuli, or cross caps F_n, respectively, which exhausts the interior $\mathring{S}$ of S. Hence

$$\Lambda(S) = \lim \lambda_0(F_n) = \lambda_0(S),$$

by domain monotonicity and the definitions of $\Lambda(S)$ and $\lambda_0(S)$. □

2.8.4. Examples. It follows from the constructions in [**BMM17b**, Example 3.7] that any non-compact and connected surface S of finite type carries complete Riemannian metrics of finite or infinite area with discrete spectrum, that is, with $\lambda_{\mathrm{ess}}(S) = \infty$. If the fundamental group of S is not cyclic, then $\Lambda(S) > \lambda_0(\tilde{S})$ for any such metric, by Theorem 2.1.7. In the following, we extend some constructions from [**BMM17b**] slightly.

Let $F = \{(x, y) \mid x \ge 0,\, y \in \mathbb{R}/L\mathbb{Z}\}$ be a funnel with the expanding hyperbolic metric $dx^2 + \cosh(x)^2 dy^2$. Let $\kappa\colon \mathbb{R} \to \mathbb{R}$ be a monotonic smooth function with $\kappa(x) = -1$ for $x \le 1$ and $\kappa(x) \to \kappa_\infty \in [-1, -\infty]$ as $x \to \infty$. Let $j\colon \mathbb{R} \to \mathbb{R}$ solve $j'' + \kappa j = 0$ with initial condition $j(0) = 1$ and $j'(0) = 0$. Then $j(x) \ge \cosh x$. The funnel F with Riemannian metric $g = dx^2 + j(x)^2 dy^2$ has curvature $K(x, y) = \kappa(x) \le -1$ and infinite area. By comparison, the Rayleigh quotient with respect to g of any smooth function φ with compact support in the part $\{x \ge x_0\}$ of the funnel is at least $-\kappa(x_0)/4$.

Let S be a non-compact surface of finite type. Endow S with a hyperbolic metric which is expanding along its funnels as above. Replace the hyperbolic metric on the funnels by the above Riemannian metric g. Then the new Riemannian metric on S is complete with curvature $K \le -1$ and infinite area. By Proposition 2.8.1 and by what we said above about the Rayleigh quotients, the essential spectrum of the new Riemannian metric is contained in $[\kappa_\infty, \infty)$. Choosing κ such that κ_∞ is larger than the first Dirichlet eigenvalue $\lambda_0(D)$ of some smooth closed disc D inside the surface yields the estimate

$$\lambda_{\mathrm{ess}}(S) > \lambda_0(D) > \lambda_0(\tilde{S}).$$

As a variation, let j solve $j'' + \kappa j = 0$ with initial condition $j(0) = 1$ and $j(\infty) = 0$. Then $j'(0) \le -1$ and $j(x) \le \exp(-x)$. The funnel F with Riemannian metric $g =$

$dx^2 + j(x)^2 dy^2$ has curvature $K(x,y) = \kappa(x)$ and finite area. Again by comparison, the Rayleigh quotient with respect to g of any smooth function φ with compact support in the part $\{x \geq x_0\}$ of the funnel is at least $-\kappa(x_0)/4$.

Let S be a non-compact surface of finite type, and choose $r > 0$ such that $\coth(r) = -j'(0)$. Now S minus the parts $\{x \geq r\}$ of its funnels carries hyperbolic metrics which are equal to $dx^2 + j_0(x)^2 dy^2$ along the parts $\{x < r\}$ of its funnels, where $j_0(x) = \sinh(r-x)/\sinh(r)$. Then $j_0(x) = j(x)$ for $x < \min\{1, r\}$. Hence any such hyperbolic metric, restricted to S minus the parts $\{x \geq \min\{1, r\}\}$ of its funnels, when combined with g along the funnels, defines a smooth and complete Riemannian metric on S which has curvature $K \leq -1$ and finite area. Choosing κ and D as in the first case, we again have $\lambda_{\mathrm{ess}}(S) > \lambda_0(\tilde{S})$.

2.8.5. Generic metrics. In view of Section 2.8.3 and Section 2.8.4 we are now prepared for the proof of Proposition 2.1.10.

Proof of Proposition 2.1.10. For the first part observe that for any complete Riemannian surface S of finite type, we have

$$\lambda_0(S) \leq \lambda_0(\tilde{S}) \leq \Lambda(S),$$

by (2.8.8) and (2.1.3), respectively. We conclude that

$$\lambda_0(S) = \lambda_0(\tilde{S}) = \Lambda(S)$$

for the surfaces considered in Propositions Proposition 2.8.5 and Proposition 2.8.6. These surfaces are precisely the ones with cyclic fundamental group.

The second part follows immediately from Section 2.8.4, so we are only left with the proof of the third part.

Let S be a non-compact surface of finite type and g be a complete Riemannian metric on S with

$$\lambda_0(\tilde{S}, \tilde{g}) = \lambda_{\mathrm{ess}}(S, g).$$

By Theorem Theorem 2.1.4 we then have

$$\lambda_0(\tilde{S}, \tilde{g}) = \Lambda(S, g) = \lambda_{\mathrm{ess}}(S, g).$$

Now assume that $\lambda_{\mathrm{ess}}(S, g) > 0$. For $n \geq 1$, let $F_n \subseteq S$ be a smooth closed disc, annulus or cross cap with

$$\lambda_0(F_n, g) < e^{1/n+1}\Lambda(S, g) = e^{1/n+1}\lambda_{\mathrm{ess}}(S, g).$$

Choose exhaustions of S by compact subsets K_n and L_n and smooth functions h_n such that, for all $n \geq 1$,

$$F_n \subseteq \mathring{K}_n \subseteq K_n \subseteq \mathring{L}_n$$

and

$$e^{-1/n} \leq h_n \leq 1, \quad h_n = 1 \text{ on } K_n, \quad h_n = e^{-1/n} \text{ on } S \setminus L_n.$$

There exists a smooth function $f = f_t = f(t,x)$ on $(0,1] \times S$ with $f_{1/n} = h_n$ such that, for all $0 < t \leq 1/n$,

$$f_t = 1 \text{ on } K_n \text{ and } f_t = e^{-t} \text{ on } S \setminus L_n$$

and such that f is monotonically decreasing in t. Since $f_t = 1$ on K_n for $t \leq 1/n$ and the K_n exhaust S, f can be smoothly extended to $[0,1] \times S$ by setting $f_0 = 1$.

Let $g_t = f_t g$. Then g_t is a smooth family of conformal metrics on S and is a continuous curve of metrics with respect to the uniform distance. For $t \le 1/n$, we have

$$\Lambda(S, g_t) \le \lambda_0(F_n, g_t) = \lambda_0(F_n, g) < e^{1/n+1}\lambda_{\mathrm{ess}}(S, g).$$

Since the Dirichlet integral is invariant under conformal change in dimension two, we obtain, for $1/n + 1 \le t \le 1/n$,

$$\Lambda(S, g_t) < e^{1/n+1}\lambda_{\mathrm{ess}}(S, g) \le e^t \lambda_{\mathrm{ess}}(S, g) = \lambda_{\mathrm{ess}}(S, g_t).$$

Invoking Theorem Theorem 2.1.4 we conclude that for all $t > 0$ one has the inequality:

$$\lambda_0(\tilde{S}, \tilde{g}_t) < \Lambda(S, g_t) < \lambda_{\mathrm{ess}}(S, g_t).$$

It remains to show that the set of metrics g on S that satisfy the strict inequality

$$\Lambda(S, g) > \lambda_0(\tilde{S}, \tilde{g})$$

is an open set in the uniform C^∞ topology. This follows from the fact that two metrics that are close to each other in the uniform C^∞ topology are quasi-isometric by a quasi-isometry with quasi-isometry constant close to 1. □

Remarks 2.8.7. 1) Any two complete Riemannian metrics g_0, g_1 on S of finite uniform distance are quasi-isometric. This implies, that there is a constant $C > 0$ depending on the distance of g_0 to g_1 such that

$$C^{-1}\lambda_{\mathrm{ess}}(S, g_0) \le \lambda_{\mathrm{ess}}(S, g_1) \le C\lambda_{\mathrm{ess}}(S, g_0).$$

In particular, we have that

(i) $\lambda_{\mathrm{ess}}(S, g_0)$ is finite iff $\lambda_{\mathrm{ess}}(S, g_1)$ is finite,
(ii) $\lambda_{\mathrm{ess}}(S, g_0) = 0$ iff $\lambda_{\mathrm{ess}}(S, g_1) = 0$.

2) The above construction can be extended to get metrics with

$$\lambda_{\mathrm{ess}}(S, g_t) = e^t \lambda_{\mathrm{ess}}(S, g)$$

for all $t \ge 0$.
3) If S is non-compact, any complete hyperbolic metric on S satisfies $\lambda_0(\tilde{S}, \tilde{g}) = \lambda_{\mathrm{ess}}(S, g) = 1/4$.
4) If S is non-compact, any complete Riemannian metric on S, which is in zeroth order asymptotic to a flat cylinder $\mathbb{R}/L\mathbb{Z} \times [0, \infty)$, has $\lambda_0(\tilde{S}, g) = \lambda_{\mathrm{ess}}(S, g) = 0$.

2.8.6. Problems and questions. We collect some questions naturally arising in view of our results. Let S be a compact and connected Riemannian surface with negative Euler characteristic.
1) (Optimal design) For a given compact subsurface T of $\mathring{S}$ with smooth boundary $\partial T \ne \emptyset$, we may consider the constant

$$\Lambda_T(S) = \inf_F \lambda_0(F),$$

where F runs over all subsurfaces of S isotopic to T. The analytic systole is an infimum over such constants. It is interesting to ask for estimates of $\Lambda_T(S)$. The infimum is probably achieved by degenerate F, where ∂F is mapped onto a graph Γ in S such that $S \setminus \Gamma$ is diffeomorphic to the interior of T. In fact, for any F isotopic to T, there is a graph Γ in S such that $F \subseteq S \setminus \Gamma$ and such that $S \setminus \Gamma$ is isotopic to the interior of F.

Hence, by domain monotonicity, $\Lambda_T(S)$ is the infimum over all $\lambda_0(S \setminus \Gamma)$, where Γ runs through such graphs. What are the optimal graphs? This circle of problems is related to the work of Helffer, Hoffman-Ostenhof, and Terracini [**HHOT09**].

2) (RIGIDITY) The inequality $\lambda_{-\chi(S)}(S) > \Lambda(S)$, mentioned in the introduction, raises the question whether there is another natural geometric constant $\Lambda'(S) > \Lambda(S)$, where we only have the weak inequality $\lambda_{-\chi(S)} \geq \Lambda'(S)$ and where equality occurs only for a distinguished class of Riemannian metrics.

3) (ANOTHER RIGIDITY) The last part of Proposition 2.1.10 suggests that hyperbolic metrics on non-compact surfaces of finite type are among a small collection of metrics that satisfy

$$\Lambda(S) = \lambda_0(\tilde{S}) = \lambda_{ess}(S).$$

It would be interesting to see what other implications this equality has on the metric. If we rescale the metric by a function $f\colon S \to (0,1]$ which is 1 outside a compact set, then $\lambda_0(\tilde{S})$ can only increase, while $\lambda_{\mathrm{ess}}(S)$ remains unaffected. Using our main theorem we can see that the new metric also satisfy the above equality. Hence one can not have a rigidity among all smooth metrics. Also, observe that points 1) and 4) of Remarks 2.8.7 imply that there is no such rigidity for metrics with $\lambda_{\mathrm{ess}}(S) = 0$.

4) (HIGHER DIMENSIONS) All our definitions extend in a natural way to higher dimensional manifolds. For instance, we may define the analytic systole of an n-dimensional manifold M by $\Lambda(M) = \inf_\Omega \lambda_0(\Omega)$, where Ω runs over all tubular neighborhoods about essential simple loops in M. By (2.8.8), we have $\Lambda(M) \geq \lambda_0(\tilde{M})$. One may ask whether the strict inequality holds true under reasonable assumptions on M. Our methods seem to be too weak to adress this question.

Appendix: On λ_0 under coverings

In [**Bro85**], Brooks states that, for a Riemannian covering $\pi\colon \hat{M} \to M$ of complete Riemannian manifolds without boundary, the bottom of the spectrum remains unchanged provided the covering is normal with amenable covering group and that M has finite topological type, that is, that M is the union of finitely many simplices. We use the corresponding result in the case where the covering is normal with cyclic fundamental group, but where the boundaries of $\hat{M}$ and M may not be empty. In fact, in [**BMM17b**] we also claim that the results there remain true for Schrödinger operators $\Delta + V$ with non-negative potential V.

By the proof of [**Sul87**, Theorem (2.1)] or [**CY75**, Theorem 7], the bottom $\lambda_0(M, V)$ of the spectrum of a Schrödinger operator $\Delta + V$ on a complete and connected Riemannian manifold M with boundary (possibly empty) with non-negative potential V. is the top of the positive spectrum of $\Delta + V$. Now for a Riemannian covering $\pi\colon \hat{M} \to M$ of complete and connected Riemannian manifolds with boundary (possibly empty) and non-negative potentials V and $\hat{V} = V \circ \pi$, the lift of a positive λ-eigenfunction of $\Delta + V$ on M to $\hat{M}$ is a positive λ-eigenfunction of $\Delta + \hat{V}$. Therefore

$$\lambda_0(M, V) \leq \lambda_0(\hat{M}, \hat{V}) \tag{2.8.8}$$

in this situation. Since the lift of a square integrable function on M to $\hat{M}$ is square integrable if the covering is finite, the reverse inequality holds for such coverings, but does not hold in general.

Theorem 2.8.9. *Let $\pi\colon \hat{M} \to M$ be a normal Riemannian covering of complete and connected Riemannian manifolds with boundary (possibly empty) with infinite cyclic covering group. Let $V\colon M \to \mathbb{R}$ be a smooth non-negative function and set $\hat{V} = V \circ \pi$. Then $\lambda_0(M, V) = \lambda_0(\hat{M}, \hat{V})$.*

The case of the standard Laplacian corresponds to the case $V = 0$. Note that we do not need to assume that M has finite topological type in the sense of Brooks.

Proof of Theorem 2.8.9. By (2.8.8), we have $\lambda_0(M, V) \le \lambda_0(\hat{M}, \hat{V})$. To show the reverse inequality, let $\varepsilon > 0$ and φ be a smooth function on M with compact support in the interior of M and Rayleigh quotient

$$R(\varphi) = \int_M \{|\nabla\varphi|^2 + V\varphi^2\} \Big/ \int_M \varphi^2 < \lambda_0(M, V) + \varepsilon.$$

The strategy is now to cut off the lift $\hat{\varphi} = \varphi \circ \pi$ of φ to $\hat{M}$ conveniently so that the Rayleigh quotient of the new function is bounded by $R(\varphi) + \varepsilon$.

We note first that the covering $\pi_0\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is universal. Hence the covering π is the pull back of π_0 by a smooth map $f\colon M \to \mathbb{R}/\mathbb{Z}$. Without loss of generality, we may assume that $[0] \in \mathbb{R}/\mathbb{Z}$ is a regular value of f. Then $f^{-1}([0])$ is a smooth hypersurface of M.

Up to covering transformation, there is a unique lift $\hat{f}\colon \hat{M} \to \mathbb{R}$ of f. Then $\pi^{-1}(f^{-1}([0])) \subseteq \hat{M}$ is the union of the smooth hypersurfaces $\hat{f}^{-1}(k)$, $k \in \mathbb{Z}$. Moreover, $\hat{f}^{-1}([k, k+1])$ is a smooth fundamental domain for the action of $\mathbb{Z}$ on $\hat{M}$, for all $k \in \mathbb{Z}$, and $\operatorname{supp}\hat{\varphi} \cap \hat{f}^{-1}([a, b])$ is compact, for all $a \le b$.

Let η_0 be a non-negative smooth function on $\hat{M}$ which is positive on $\hat{f}^{-1}([0, 1])$ and which has support in $\hat{f}^{-1}([-1, 2])$. Set $\eta_k = \eta_0 \circ \lambda_k$, where λ_k denotes the action of $k \in \mathbb{Z}$ on $\hat{M}$, and $\zeta_k = \eta_k / \sum_{j\in\mathbb{Z}} \eta_j$. Note that the sum in the denominator on the right is well defined since it is locally finite. Then (ζ_k) is a partition of unity on $\hat{M}$ such that $\zeta_k = \zeta_0 \circ \lambda_k$. In particular, since ζ_0 has support in $\hat{f}^{-1}([-1, 2])$ and $\operatorname{supp}\hat{\varphi} \cap \hat{f}^{-1}([-1, 2])$ is compact, there is a uniform bound $|\nabla\zeta_k| \le C$ on $\operatorname{supp}\hat{\varphi}$. Therefore

$$\chi_k = \sum_{-1 \le j \le k+1} \zeta_j$$

is a smooth cut-off function on $\hat{M}$ with values in $[0, 1]$ which is equal to 1 on $f^{-1}([0, k])$, has support in $f^{-1}([-2, k+2])$, and gradient bounded by $3C$ on $\operatorname{supp}\hat{\varphi}$. We conclude that

$$\int_{\hat{M}} (\chi_k\hat{\varphi})^2 \ge k \int_M \varphi^2, \quad \int_{\hat{M}} \hat{V}(\chi_k\hat{\varphi})^2 \le (k+4) \int_M V\varphi^2,$$

and, using Young's inequality,

$$\begin{aligned}\int_{\hat{M}} |\nabla(\chi_k\hat{\varphi})|^2 &\le (1+\delta) \int_{\hat{M}} |\nabla\varphi|^2 + \left(1 + \frac{1}{\delta}\right) \int_{\operatorname{supp}\nabla\chi_k} |\nabla\chi_k|^2 \hat{\varphi}^2 \\ &\le (1+\delta)(k+4) \int_M |\nabla\varphi|^2 + \left(1 + \frac{1}{\delta}\right) 36C^2 \int_M \varphi^2,\end{aligned}$$

since $\operatorname{supp} \nabla\chi_k \subseteq \hat{f}^{-1}([-2,0] \cup [k, k+2])$. Choosing δ small enough, we hence find that $\chi_k \hat{\varphi}$ is a smooth function on $\hat{M}$ with compact support such that $R(\chi_k\hat{\varphi}) < R(\varphi) + \varepsilon$ for all sufficiently large k. $\square$

CHAPTER 3

Maximizing the first eigenvalue on non-orientable surfaces

3.1. Introduction

For a closed Riemannian surface (Σ, g) the spectrum of the Laplace operator acting on smooth functions, is purely discrete and can be written as

$$0 = \lambda_0 < \lambda_1(\Sigma, g) \leq \lambda_2(\Sigma, g) \leq \lambda_3(\Sigma, g) \leq \cdots \to \infty,$$

where we repeat an eigenvalue as often as its multiplicity requires.

The pioneering work of Hersch [**Her70**] and Yang–Yau [**YY80**] raised the natural question, whether there are metrics g that maximize the quantities

$$\lambda_1(\Sigma, g)\,\mathrm{area}(\Sigma, g)$$

if Σ is a closed surface of fixed topological type (see also [**Kar16, LY82**] for the case of non-orientable surfaces). Such maximizers have remarkable properties. In fact, they always arise as immersed minimal surfaces (of possibly high codimension) in a sphere [**ESI00**] and are unique in their conformal class [**MR86**]. By a slight abuse of notation, we also call Σ, endowed with a maximizing metric, a 'maximizer'.

For the statement of our results and related work, we need to introduce some notation. We write Σ_γ for a closed orientable surface of genus γ. Similarly, Σ_δ^K denotes a closed non-orientable surface of non-orientable genus δ. We briefly elaborate on these notions in Section 3.3. Furthermore, we use the common notation

$$\Lambda_1(\gamma) = \sup_g \lambda_1(\Sigma_\gamma, g)\,\mathrm{area}(\Sigma_\gamma, g),$$

and similarly,

$$\Lambda_1^K(\delta) = \sup_g \lambda_1(\Sigma_\delta^K, g)\,\mathrm{area}(\Sigma_\delta^K, g),$$

with the supremum taken over all smooth metrics on Σ_γ, respectively Σ_δ^K. It is convenient to use the notation

$$\Lambda_1(\Sigma) = \sup_g \lambda_1(\Sigma, g)\,\mathrm{area}(\Sigma, g),$$

where Σ is a closed surface and the supremum is taken over all smooth metrics g on Σ. If Σ is orientable and has genus γ, then $\Lambda_1(\Sigma) = \Lambda_1(\gamma)$. If Σ is non-orientable and has non-orientable genus δ, then $\Lambda_1(\Sigma) = \Lambda_1^K(\delta)$.

Explicit values for $\Lambda_1(\gamma)$ or $\Lambda_1^K(\delta)$ are only known in very few cases. However, in all of these cases not only the values but also the maximizing metrics are known.

The case of the sphere is due to Hersch. We have $\Lambda_1(\mathbb{S}^2) = 8\pi$ with unique maximizer the round metric [**Her70**]. His arguments are very elegant and a cornerstone in the development of the subject.

For the real projective plane, we have $\Lambda_1(\mathbb{RP}^2) = 12\pi$ with unique maximizer the round metric [**LY82**]. The proof extends the ideas from [**Her70**] in a conceptually very nice way.

The first result for higher genus surfaces is due to Nadirashvili, namely $\Lambda_1(T^2) = 8\pi^2/\sqrt{3}$ with unique maximizer the flat equilateral torus [**Nad96**]. Nadirashvili's arguments are very different from the previously employed methods. The crucial step in his proof is to obtain the existence of a maximizer. Using [**MR86**] it follows that such a maximizer necessarily has to be flat. The sharp bound follows then from earlier work of Berger [**Ber73**].

For the Klein bottle, conjecturally, $\Lambda_1(K) = 12\pi E(2\sqrt{2}/3)$ with unique maximizer a metric of revolution [**ESGJ06, Nad96**]. Here E is the complete elliptic integral of the second kind.

There is also a conjecture concerning the sharp bound on genus 2 surfaces [**JLN$^+$05**], a proof of which has very recently been outlined in [**NS17b**].

Let us also mention that there are a quite some results concerning similar questions for higher order eigenvalues, see [**KNPP17, Nad02, NP16, NS15b, NS17a**].

Since Nadirashvili's paper [**Nad96**] there was growing interest in finding maximizers for eigenvalue functionals on surfaces. No doubt partly because of their connection to minimal surfaces. For the Steklov eigenvalue problem, there is a connection to free boundary minimal surfaces in Euclidean balls. Fraser and Schoen showed the existence of maximizers for the first Steklov eigenvalue on bordered surfaces of genus 0 [**FS16**]. Recently, Petrides used many of the ideas in [**FS16**] to prove the following beautiful result concerning metrics realizing $\Lambda_1(\gamma)$.

Theorem 3.1.1 (Theorem 2 in [**Pet14**])**.** *If $\Lambda_1(\gamma-1) < \Lambda_1(\gamma)$, there is a metric g on $\Sigma = \Sigma_\gamma$, which is smooth away from finitely many conical singularities, such that*

$$\lambda_1(\Sigma, g)\,\mathrm{area}(\Sigma, g) = \Lambda_1(\gamma).$$

We extend this to non-orientable surfaces. Since non-orientable surfaces can degenerate to non-orientable surfaces as well as orientable ones, we need to make two instead of only a single spectral assumption.

Theorem 3.1.2. *If $\Lambda_1^K(\delta-1) < \Lambda_1^K(\delta)$ and $\Lambda_1(\lfloor(\delta-1)/2\rfloor) < \Lambda_1^K(\delta)$, there there is a metric smooth away from at most finitely many conical singularities achieving $\Lambda_1^K(\delta)$.*

Our methods are very similar to those in [**Pet14**]. In addition to the cases already handle in there, we also need to take care of degenerating one-sided geodesics.

3.2. Compactness for non-orientable surfaces

The Mumford compactness criterion [**Mum71**] states that the set of orientable, hyperbolic surfaces with injectivity radius bounded below is a compact subset of Teichmüller space. In this section we show that this also holds for non-orientable surfaces. Probably, this is well-known, but for the sake of completeness and since we will use the arguments from our proof again, we include a proof below.

Given any Riemannian metric g_0 on $\Sigma = \Sigma_\delta^K$, the Poincaré Uniformization theorem asserts that we can find a new metric on Σ, which is pointwise conformal to g_0 and has constant curvature $+1, 0$, or -1, depending on the sign of $\chi(\Sigma)$. Assuming $\delta \geq 3$, these metrics have curvature -1. Let h_k be a sequence of such metrics on Σ with injectivity radius bounded uniformly from below, $\mathrm{inj}(\Sigma, h_k) \geq c > 0$. The goal is to prove that there exist diffeomorphisms σ_k of Σ and a hyperbolic metric h of Σ, such that $\sigma_k^* h_k$ converges smoothly to h as $k \to \infty$. Our strategy is to apply the Mumford compactness criterion to the orientation double covers of the surfaces (Σ, h_k).

So consider the orientation double cover $\hat{\Sigma} = \Sigma_{\delta-1}$ of Σ endowed with the pullback metrics of h_k, denoted by $\hat{h}_k$. Since $\delta \geq 3$, these are orientable hyperbolic surface of genus $\delta - 1$ and may thus be regarded as elements in Teichmüller space $\mathcal{T}_{\delta-1}$, which in addition admit fixed point free, isometric, orientation reversing involutions ι_k.

We have the following lemma.

Lemma 3.2.1. *Assume that* $\inf_k \mathrm{inj}(\Sigma_\delta^K, h_k) > 0$. *Then there are diffeomorphisms* $\tau_k \colon \Sigma_{\delta-1} \to \Sigma_{\delta-1}$, *such that, up to taking a subsequence,* $\tau_k^* \hat{h}_k \to \hat{h}$ *in* C^∞. *Moreover,* $(\Sigma_{\delta-1}, \hat{h})$ *admits a fixed point free, isometric, orientation reversing involution* ι, *which is obtained as the* C^0*-limit of the involutions* $\tau_k^{-1} \circ \iota_k \circ \tau_k$.

Proof. As above, we simply write Σ instead of Σ_δ^K, and $\hat{\Sigma}$ instead of $\Sigma_{\delta-1}$. It is elementary to see that $\mathrm{inj}(\hat{\Sigma}, \hat{h}_k) \geq \mathrm{inj}(\Sigma, h_k)$. Therefore, we can apply the Mumford compactness criterion [**Mum71**] and find diffeomorphisms τ_k and a limit metric $\hat{h}$ as asserted.

It remains to show that we can find the involution ι. Since $\tau_k^* \hat{h}_k \to \hat{h}$ in C^∞, we have the uniform Lipschitz bound

$$\begin{aligned} d_{\hat{h}}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(p), (\tau_k^{-1} \circ \iota_k \circ \tau_k)(q)) \\ &\leq C d_{\tau_k^* \hat{h}_k}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(p), (\tau_k^{-1} \circ \iota_k \circ \tau_k)(q)) \\ &= C d_{\tau_k^* \hat{h}_k}(p, q) \\ &\leq C d_{\hat{h}}(p, q). \end{aligned}$$

Since $\hat{\Sigma}$ is compact, it follows from Arzela–Ascoli, that, up to taking a subsequence, $\tau_k^{-1} \circ \iota_k \circ \tau_k \to \iota$ in $C^0(\hat{\Sigma}, \hat{h})$. We have

$$\begin{aligned} d_{\hat{h}}(\iota(p), \iota(q)) &\leq \lim_{k\to\infty} d_{\tau_k^* \hat{h}_k}(\iota(p), (\tau_k^{-1} \circ \iota_k \circ \tau_k)(p)) \\ &\quad + \lim_{k\to\infty} d_{\tau_k^* \hat{h}_k}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(p), (\tau_k^{-1} \circ \iota_k \circ \tau_k)(q)) \\ &\quad + \lim_{k\to\infty} d_{\tau_k^* \hat{h}_k}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(q), \iota(q)) \\ &\leq C \lim_{k\to\infty} d_{C^0(\hat{\Sigma}, \hat{h})}(\tau_k^{-1} \circ \iota_k \circ \tau_k, \iota) \\ &\quad + \lim_{k\to\infty} d_{\tau_k^* \hat{h}_k}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(p), (\tau_k^{-1} \circ \iota_k \circ \tau_k)(q)) \\ &= d_{\hat{h}}(p, q), \end{aligned} \tag{3.2.2}$$

using that $\tau_k^* \hat{h}_k \to \hat{h}$ in C^∞, and $\tau_k^{-1} \circ \iota_k \circ \tau_k \to \iota$ in $C^0(\hat{\Sigma}, \hat{h})$. Observe that ι is an involution again, hence (3.2.2) implies that actually

$$d_{\hat{h}}(\iota(p), \iota(q)) = d_{\hat{h}}(p, q).$$

By the Myers–Steenrod theorem it thus follows that ι is a smooth, isometric involution.

We need to show that ι does not have any fixed points. But this is a consequence of the general bound $d_{\tau_k^*\hat{h}_k}((\tau_k^{-1}\circ\iota_k\circ\tau_k)(p),p)\geq c>0$ for some uniform c. To prove this let $c>0$ be such that $B_{\hat{h}}(x,2c)\subset\hat{\Sigma}$ is strictly geodesically convex for any $x\in\hat{\Sigma}$. Then $B_{\tau_k^*\hat{h}_k}(x,c)$ is strictly geodesically convex for $k\geq K$ sufficiently large. Assume now that there is $k\geq K$, such that $d_{\hat{h}_k}((\tau_k^{-1}\circ\iota_k\circ\tau_k)(p),p)<c$. Let γ be the unique minimizing geodesic connecting p to $(\tau_k^{-1}\circ\iota_k\circ\tau_k)(p)$. Since $\tau_k^{-1}\circ\iota_k\circ\tau_k$ is an isometry, we need to have $\operatorname{im}(\tau_k^{-1}\circ\iota_k\circ\tau_k\circ\gamma)=\operatorname{im}\gamma$. Since ι_k is fixed point free, γ is non-constant. Therefore, $\tau_k^{-1}\circ\iota_k\circ\tau_k$ restricted to $\operatorname{im}\gamma$ induces an involution of the interval $[0,1]$ mapping 0 to 1 and vice versa. But such an involution needs to have a fixed point. It follows that ι_k has a fixed point for large k, which is a contradiction.

Finally, note that ι is orientation reversing by C^0-convergence. $\square$

It follows that the metric $\hat{h}$ on $\hat{\Sigma}$ is ι-invariant. Therefore, it induces a smooth hyperbolic metric h on Σ. Moreover, the hyperbolic metrics on Σ induced from $\tau_k^*h_k$ and $\tau_k^{-1}\circ\iota_k\circ\tau_k$ converge smoothly to h on Σ. Finally, observe that the diffeomorphisms τ_k induce diffeomorphisms σ_k of Σ, such that $\sigma_k^*h_k$ are the metrics described above and converge smoothly to h.

Thus we have proved the following proposition.

Proposition 3.2.3. *Let (h_k) be a sequence of hyperbolic metrics on Σ_δ^K such that the injectivity radius satisfies $\operatorname{inj}(\Sigma_\delta^K,h_k)\geq c>0$. Then there are diffeomorphisms σ_k of Σ_δ^K and a hyperbolic metric h, such that $\sigma_k^*h_k\to h$ smoothly.*

3.3. Maximizing the first eigenvalue

In this section we extend [**Pet14**, Theorem 2] to the non-orientable case. The strategy is the same as in [**Pet14**]. That is, we first use that we can maximize the first eigenvalue in each conformal class. We then pick a maximizing sequence, consisting of maximizers in their own conformal class. This has the advantage, that these metrics can be studied in terms of sphere valued harmonic maps. Using these harmonic maps it is possible to estimate the first eigenvalue along the maximizing sequence in case that the conformal class degenerates. To do so, we extend the results from [**Zhu10**] to non-orientable surfaces.

For fixed non-orientable genus $\delta\geq 3$, let c_k be a sequence of conformal classes on $\Sigma=\Sigma_\delta^K$ represented by hyperbolic metrics h_k, such that

$$\lim_{k\to\infty}\sup_{g\in c_k}\lambda_1(\Sigma,g)\operatorname{area}(\Sigma,g)=\Lambda_1^K(\delta).$$

We will now use the following result due to Nadirashvili–Sire (with an extra assumption not relevant for our purposes) and, independently, Petrides

Theorem 3.3.1 ([**NS15a**, Theorem 2.1] or [**Pet14**, Theorem 1])**.** *For each conformal class c_k as above, there is a metric g_k, which is smooth away from finitely many conical singularities such that*

$$\lambda_1(\Sigma,g_k)\operatorname{area}(\Sigma,g_k)=\sup_{g\in c_k}\lambda_1(\Sigma,g)\operatorname{area}(\Sigma,g).$$

From now on we assume that $g_k \in c_k$ is picked as in the preceding theorem. Moreover, we assume that they are normalized to have

$$\operatorname{area}(\Sigma, g_k) = 1.$$

Since these metrics are maximizers, there is a family of first eigenfunctions $u_1^k, \dots u_{\ell(k)+1}^k$, such that $\Phi_k = (u_1^k, \dots u_{\ell(k)+1}^k) \colon (\Sigma, h_k) \to \mathbb{S}^{\ell(k)}$ is a harmonic map [**ESI03**]. Since the multiplicity of λ_1 is uniformly bounded in terms of the topology of Σ [**Bes80, Che76**], we may pass to a subsequence, such that $\ell(k)$ is some constant number l. Moreover, in this situation the maximizing metrics can be recovered via

$$g_k = \frac{|\nabla \Phi_k|_{h_k}}{\lambda_1(\Sigma, g_k)} h_k.$$

In view of Proposition 3.2.3, we want to show the following proposition.

Proposition 3.3.2. *The injectivity radius of g_k is uniformly bounded from below, provided that $\Lambda_1^K(\delta) > \Lambda_1(\delta - 1)$, and $\Lambda_1^K(\delta) > \Lambda_1^K(\delta - 1)$.*

We will argue by contradiction and assume $\operatorname{inj}(\Sigma, g_k) \to 0$. The Margulis lemma implies that we can find closed geodesics $\gamma_1^k, \dots, \gamma_s^k$ in (Σ, h_k), such that their lengths go to zero, i.e. $l_{h_k}(\gamma_i^k) \to 0$, as $k \to \infty$. We assume that s is chosen maximal with this property.

Each of these geodesics is either one-sided or two-sided. If a such a geodesic is two-sided, tubular neighborhoods are just described by the classical collar lemma for hyperbolic surfaces [**Bus10**]. In the second case we may apply the collar lemma to the orientation double cover as follows.

Let c be a one-sided closed geodesic in Σ. We write $\hat{\Sigma}$ for the orientation double cover and τ for the non-trivial deck transformation. The lifts of c to $\hat{\Sigma}$ can not be closed, since in this case they would be disjoint and it would follow that c is two-sided. Thus the lifts c_1 and c_2 are geodesic segments with $\tau \circ c_1 = c_2$. Let $\mathcal{C}$ be a collar around the closed geodesic $c_2 * c_1$. It is not very difficult to see that the action of τ near $c_2 * c_1$ is just given by rotation about π and reflection at $c_2 * c_1$. Therefore, τ maps $\mathcal{C}$ to itself (by the explicit construction of $\mathcal{C}$), so that we can use $\mathcal{C}/\tau$ as a tubular neighborhood of c.

Our first goal is to prove that for the situation at hand the volume, measured with respect to g_k, either concentrates in the neighborhood of a pinching geodesic, or in one connected component of the complement of these neighborhoods. Before stating and proving this result we need to introduce some notation, which we borrow from Section 4 in [**Pet14**].

We write s_1 for the number of one-sided closed geodesics with length going to 0. Moreover, we denote by s_2 the number of such geodesics that are two-sided. Clearly, $s = s_1 + s_2$ and $0 \leq s_1, s_2 \leq s$. From now on we assume that the closed geodesics γ_k^i are ordered such that the first s_1 geodesics are one-sided.

For all $s_1 + 1 \leq i \leq s$ the collar theorem [**Bus10**] asserts the existence of an open neighborhood P_k^i of γ_k^i isometric to the following truncated hyperbolic cylinder

$$\mathcal{C}_k^i = \left\{ (t, \theta) \mid -w_k^i < t < w_k^i,\, 0 \leq \theta < 2\pi \right\}$$

with

$$w_k^i = \frac{\pi}{l_k^i}\left(\pi - 2\arctan\left(\sinh\frac{l_k^i}{2}\right)\right)$$

endowed with the metric

$$\left(\frac{l_k^i}{2\pi\cos\left(\frac{l_k^i}{2\pi}t\right)}\right)^2 (dt^2 + d\theta^2).$$

Below we identify $(\theta, t) = (0, t)$ with $(\theta, t) = (2\pi, t)$. Thus the closed geodesic γ_α^i corresponds to $\{t = 0\}$.

By the discussion above and the the collar theorem again, we get that for all $1 \leq i \leq s_1$, there exists an open neighborhood P_k^i of γ_α^i isometric to the following truncated Möbius strip

$$\mathcal{M}_k^i = \left\{(t,\theta) \mid -w_k^i < t < w_k^i,\, 0 \leq \theta < 2\pi\right\} / \sim$$

with

$$w_k^i = \frac{\pi}{2l_k^i}\left(\pi - 2\arctan\left(\sinh l_k^i\right)\right)$$

endowed with the metric

$$\left(\frac{2l_k^i}{2\pi\cos\left(\frac{2l_k^i}{2\pi}t\right)}\right)^2 \left(dt^2 + d\theta^2\right).$$

Moreover, the equivalence relation $\sim$ is given by identifying $(t, \theta,) \sim (-t, \theta + \pi)$, where $\theta + \pi \in \mathbb{R}/2\pi\mathbb{R}$. Hence, the closed geodesic γ_α^i corresponds to $\{t = 0\}$.

We denote by $\Sigma_k^1, \cdots, \Sigma_k^r$ the connected components of $\Sigma \setminus \bigcup_{i=1}^s P_k^i$. Consequently, Σ can be written as the disjoint union

$$\Sigma = \left(\bigcup_{i=1}^s P_k^i\right) \bigcup \left(\bigcup_{j=1}^r \Sigma_k^j\right).$$

For $s_1 + 1 \leq i \leq s$ and $0 < b < w_k^i$ we denote by $P_k^i(b)$ the truncated hyperbolic cylinder whose length, compared to P_k^i, is reduced by b, i.e.,

$$P_k^i(b) = \left\{(t,\theta),\, -w_k^i + b < t < w_k^i - b\right\}.$$

Analogously, for $1 \leq i \leq s_1$ and $0 < b < w_k^i$, we introduce

$$P_k^i(b) = \left\{(t,\theta),\, -w_k^i + b < t < w_k^i - b\right\} / \sim.$$

Finally, we denote by $\Sigma_k^j(b)$ the connected components of $\Sigma \setminus \displaystyle\bigcup_{i=1}^s P_k^i(b)$ which contains Σ_k^j.

We are now ready to prove the above mentioned result, namely, that the volume either concentrates in the neighborhood of a pinching geodesic P_k^i, or in one connected component Σ_k^j of the complement of these neighborhoods.

Lemma 3.3.3. *There exists $D > 0$ such that one of the two following assertions is true:*

(1) *There exists an* $i \in \{1, \dots, s\}$ *such that*

$$\operatorname{area}_{g_k}\left(P_k^i\left(a_k\right)\right) \geq 1 - \frac{D}{a_k}$$

for all sequences $a_k \to +\infty$ *with* $\frac{a_k}{w_k^i} \to 0$ *as* $k \to +\infty$ *for all* $1 \leq i \leq s$.

(2) *There exists a* $j \in \{1, \dots, r\}$ *such that*

$$\operatorname{area}_{g_k}\left(\Sigma_k^j\left(9a_k\right)\right) \geq 1 - \frac{D}{a_k}$$

for all sequences $a_k \to +\infty$ *with* $\frac{a_k}{w_k^i} \to 0$ *as* $k \to +\infty$ *for all* $1 \leq i \leq s$.

Proof. The proof of Claim 11 in [**Pet14**] can easily be adapted to the present situation. First recall the rough strategy of the proof: construct suitable test functions for $\lambda_1(\Sigma, g_k)$ in the P_k^i and the Σ_k^j's, apply the min-max formula for the first eigenvalue and prove the claim by contradiction. More precisely, on $\hat{\Sigma}$, the test functions are constructed with linear decay in the t variable in neck regions of the type $\hat{P}_k^i(2a_k) \setminus \hat{P}_k^i(3a_k)$ and $\hat{P}_k^i(1a_k) \setminus \hat{P}_k^i(2a_k)$, respectively, where the hat indicates that we consider the preimages under the covering map $\hat{\Sigma} \to \Sigma$. By conformal invariance, the Dirichlet energy of these can be estimated using the hyperbolic metric and decays like a_k^{-1}. From the construction it is clear that these functions are invariant under the relevant involutions. From this point on, one can just follow the arguments in [**Pet14**]. $\square$

Below we consider the two possible cases of the preceding lemma separately. The following lemma deals with the first case, i.e. when the volume concentrates in one of the P_k^i. We show that in this case we would have $\Lambda_1^K(\delta) \leq 8\pi$ if γ_k^i is 2-sided; and $\Lambda_1^K(\delta) \leq 12\pi$ if γ_k^i is 1-sided.

Lemma 3.3.4. *Suppose that there exists an* $i \in \{1, ..., s\}$ *such that*

$$\operatorname{area}_{g_k}(P_k^i(a_k)) \geq 1 - \tfrac{D}{a_k}$$

for all sequences $a_k \to \infty$ *which satisfy* $\lim_{k\to\infty} \frac{a_k}{w_k^i} = 0$ *for all* $1 \leq i \leq s$.

(1) *If* γ_k^i *is* 2*-sided, then* $\Lambda_1^K(\delta) \leq 8\pi$.
(2) *If* γ_k^i *is* 1*-sided, then* $\Lambda_1^K(\delta) \leq 12\pi$.

Proof. In [**Pet14**], Petrides proved the first statement by following ideas of Girouard [**Gir09**]. The proof of the second statement is carried out analogously.

By assumption, there exists an $i \in \{1, ..., s\}$, such that the volume concentrates on $P_k := P_k^i$. On P_k we have coordinates (t, θ) as above (on $\mathcal{M}_k$). By the assumptions on the volume and a_k, we can find cut-off functions η_k which are 1 on $P_k(a_k)$ and 0 outside P_k, and satisfy

$$\int_\Sigma |\nabla \eta_k|^2 dv_{g_k} \to 0.$$

We denote by $\mathcal{C} = (-\infty, \infty) \times \mathbb{S}^1$ the infinite cylinder with its canonical coordinates $(t, \theta) \in (-\infty, \infty) \times [0, 2\pi)$. Let $\phi : \mathbb{C} \to \mathbb{S}^2 \subset \mathbb{R}^3$ be given by

$$\phi(t, \theta) = \frac{1}{e^{2t} + 1}(2e^t \cos(\theta), 2e^t \sin(\theta), e^{2t} - 1).$$

Observe that this induces a map $\psi\colon \mathcal{M} \to \mathbb{RP}^2(\sqrt{3})$ if we divide by the $\mathbb{Z}/2$ actions that we have on both sides. More precisely, $\mathcal{M} = \mathcal{C}/\sim$, where $(t,\theta) \sim (-t, \theta+\pi)$ as above, and on $\mathbb{S}^2$ we simply take the antipodal map. If we denote by $v : \mathbb{RP}^2(\sqrt{3}) \to \mathbb{S}^4$ the Veronese map, the concatenation $v \circ \phi : \mathcal{M} \to \mathbb{S}^4$ is a conformal map [**Gir09**]. We may regard $\mathcal{M}_k \subset \mathcal{M}$ using Fermi coordinates as introduced above.

By a theorem of Hersch [**Her70**], there exists a conformal diffeomorphism τ_k of $\mathbb{S}^4$, such that

$$\int_{P_k} (\pi \circ \tau_k \circ v \circ \phi)\eta_k dv_{g_k} = 0,$$

where $\pi : \mathbb{S}^4 \hookrightarrow \mathbb{R}^5$ is the standard embedding. Set $u_k^i = (\pi_i \circ \tau_k \circ v \circ \phi)\eta_k$. By construction, we have

$$\sum_{i=1}^{5} \int_{\mathcal{M}_k} (u_k^i)^2 dv_{g_k} \geq 1 - \frac{D}{a_k},$$

since $\mathrm{area}_{g_k}(P_\alpha^i(a_k)) \geq 1 - \frac{D}{a_k}$. Using conformal invariance, one easily finds that

$$\int_\Sigma |\nabla u_k|^2_{g_k} dv_{g_k} \leq 12\pi + o(1).$$

For details we refer to [**Gir09**]. Consequently, there is $i = i(k) \in \{1, \dots, 5\}$, such that

$$\lambda_1(\Sigma, g_k) \leq \frac{\int_M |\nabla u_k^i|^2_{g_k} dv_{g_k}}{\int_M (u_k^i)^2 dv_{g_k}} \leq 12\pi + o(1).$$

This finally implies

$$\Lambda_1^K(\delta) \leq \limsup_{k\to\infty} \lambda_1(\Sigma, g_k) \leq 12\pi,$$

which establishes the claim. □

We are thus left with the case second case from Lemma 3.3.3. In this case, we have the following lemma, which concludes the proof of Proposition 3.3.2.

Lemma 3.3.5. *Suppose that the second alternative from Lemma 3.3.3 holds, then either*

(i) $\Lambda_1^K(\delta) \leq \Lambda_1^K(\delta - 1)$, *or*

(ii) $\Lambda_1^K(\delta) \leq \Lambda_1(\gamma)$,

where $\gamma = \lfloor (\delta-1)/2 \rfloor$.

Proof. Again, we apply the machinery from [**Pet14**] to the orientation cover. The essential point is to keep track of the geometry of the corresponding involutions. Denote by $(\hat{\Sigma}, \hat{h}_k)$ the orientation covers of (Σ, h_k), and by ι_k the corresponding deck transformations.

We can then identify the spectrum of the Laplacian for any metric g in $[h_k]$ with the spectrum of the Laplacian acting only on the even functions on $(\hat{\Sigma}, \hat{g})$. We consider the associated harmonic maps $\Phi_k\colon (\Sigma, g_k) \to \mathbb{S}^l$. By conformal invariance, we can also view these as harmonic maps from (Σ, h_k) to $\mathbb{S}^l$. In this situation, the metric can be recovered by

$$g_k = \frac{|\nabla \Phi_k|^2_{h_k}}{\lambda_1(\Sigma, g_k)} h_k,$$

see [**Pet14**, Proof of Theorem 1]. By pulling back the Φ_k's to $\hat{\Sigma}$, we obtain even harmonic maps $\hat{\Phi}_k \colon (\hat{\Sigma}, \hat{h}_k) \to \mathbb{S}^l$, such that

$$\hat{g}_k = \frac{|\nabla \hat{\Phi}_k|_{\hat{h}_k}}{\lambda_1(\Sigma, g_k)} \hat{h}_k.$$

With out loss of generality, we may assume that the volume concentrates in $\Sigma_k^1(9a_k)$. Denote by $\hat{\Sigma}_k^1(9a_k)$ its preimage under the covering projection. Note that this preimage might be disconnected. As in [**Pet14**, Sect. 4], there are a compact Riemann surface $\bar{\Sigma}$ and diffeomorphisms $\tau_k \colon \bar{\Sigma} \setminus \{p_1, \dots, p_r\} \to \hat{\Sigma}_k^1(9a_k)$. Moreover, the hyperbolic metrics $\bar{h}_k = \tau_k^* \hat{h}_k$ converge in $C^\infty_{loc}(\bar{\Sigma} \setminus \{p_1 \dots, p_r\})$ to a hyperbolic metric $\bar{h}$.

Observe, that we can restrict and pullback the involutions ι_k to get involutions $\bar{\iota}_k$ of $\bar{\Sigma} \setminus \{p_1, \dots p_r\}$. Clearly, these involutions are isometric with respect to the hyperbolic metrics $\bar{h}_k$.

In a next step, we construct a fixed point free limit involution on $\bar{\Sigma}$. For the compact subsets $\bar{\Sigma}_c := \{x \in \bar{\Sigma} \mid \mathrm{inj}_x(\bar{\Sigma}, \bar{h}) \geq c\}$, we can argue exactly as in the proof of Lemma 3.2.1 to get limit involutions $\bar{\sigma}_n$ on $\bar{\Sigma}_{1/n}$. Since any isometric involution must map $\bar{\Sigma}_c$ to itself, we may take subsequences, such that for $m \geq n$, we have $\bar{\sigma}_m|_{\bar{\Sigma}_{1/n}} = \bar{\sigma}_n$. Using a standard diagonal argument, we find a limit involution on $\bar{\Sigma} \setminus \{p_1, \dots, p_r\}$. Clearly, this involution extends to an involution $\bar{\iota}$ on all of $\bar{\Sigma}$. Moreover, $\bar{\iota}$ is fixed point free: Arguing again as in Lemma 3.2.1, we can not have fixed points different from the p_i's. If say p_1 is fixed under $\bar{\iota}$, the involution is just rotation by π in a disc centered at p_1. By C^0-convergence away from p_1, we see that the involutions $\hat{\iota}_k$ act just via rotation on the collars around the degenerating geodesic. But this is impossible, since this implies that $\hat{\iota}_k$ is orientation preserving.

By [**Zhu10**], the pullbacks $\bar{\Phi}_k$ of the harmonic maps $\hat{\Phi}_k$ along the diffeomorphisms τ_k are then harmonic maps that converge in $C^\infty_{loc}(\bar{\Sigma} \setminus \{p_1, \dots p_r, x_1, \dots, x_s\})$ to a limit harmonic map $\bar{\Phi}$. Clearly, $\bar{\Phi}$ is invariant under $\bar{\iota}$. Note, that no energy can be lost at the points x_i or p_i. By construction, no volume concentrates near the closed geodesics bounding $\hat{\Sigma}_k^1(9a_k) \subset \hat{\Sigma}$), which implies that no energy is lost at the points p_i. Observe next, that the points x_i always come in pairs by the invariance of the harmonic maps. Moreover, from the construction of the limit involution, it is clear, that two such points are bounded away from each other. Therefore, energy concentration of the harmonic maps in a point x_i implies that the volume with respect to the metric g_k concentrates at a point in Σ. But by [**Kok14**, Lemma 2.1 and 3.1] this implies

$$\Lambda_1^K(\delta) = \lim_{k \to \infty} (\Sigma, g_k) \leq 8\pi.$$

Let $\bar{h}_0$ be the hyperbolic metric in the conformal class of the cusp compactification of $(\bar{\Sigma} \setminus \{p_1, \dots p_r\}, [\bar{h}])$. Since $\bar{\Phi} \colon (\bar{\Sigma} \setminus \{p_1, \dots, p_s\}, [\bar{h}_0]) \to \mathbb{S}^l$ has finite energy, $\bar{\Phi}$ extends to a harmonic map $(\bar{\Sigma}, [\bar{h}_0]) \to \mathbb{S}^l$ [**SU81**, Theorem 3.6]. Moreover, this extension is certainly invariant under $\bar{\iota}$.

We consider the metric

$$\bar{g} = \frac{|\nabla \bar{\Phi}|^2_{\bar{h}_0}}{\Lambda_1^K(\delta)} \bar{h}_0$$

and observe that it is invariant under the involution $\bar{\iota}$, so that it descends to a metric g on $\bar{\Sigma}/\bar{\iota}$. Since there is no energy lost along the sequence $\bar{\Phi}_k$ of harmonic maps, we have

$$\operatorname{area}(\bar{\Sigma}/\bar{\iota}, g) = 1.$$

Using that the capacity of a point relative to any ball is 0 [**Maz11**, Chapter 2.2.4], it is easy to construct $\bar{\iota}_k$-invariant cut-offs $\eta_{\varepsilon,k}$ on $\bar{\Sigma}$ with the following two properties. For ε fixed, there are neighbourhoods $U_\varepsilon \subset V_\varepsilon$ of $\{p_1, \dots, p_r\}$ such that $\cap_{\varepsilon>0} V_\varepsilon = \{p_1, \dots p_r\}$, $\eta_{\varepsilon,k} = 0$ in U_ε, and $\eta_{\varepsilon,k} = 1$ outside V_ε. Moreover, $\int_{\bar{\Sigma}} |\nabla \eta_{\varepsilon,k}|^2 dv_{\bar{g}} \leq \varepsilon^2$.

We write $\bar{g}_k = \tau_k^*(\hat{g}_k)$. Let u be the lift of a first eigenfunction of $(\bar{\Sigma}/\bar{\iota}, g)$ to $\bar{\Sigma}$. Using $\eta_{\varepsilon,k} u$ as a test function on $\hat{\Sigma}_k(9a_k)$ for k large enough, we find with help of the dominated convergence theorem, that

$$\begin{aligned}
\Lambda_1^K(\delta) &= \lim_{k\to\infty} \lambda_1(\Sigma, g_k) \\
&\leq \limsup_{\varepsilon\to 0} \lim_{k\to\infty} \frac{\int_{\bar{\Sigma}} |\nabla(\eta_{\varepsilon,k} u)|^2 dv_{\hat{g}_k}}{\int_{\bar{\Sigma}} |\eta_{\varepsilon,k} u|^2 dv_{\bar{g}_k} - \left(\int_{\bar{\Sigma}} \eta_{\varepsilon,k} u dv_{\bar{g}_k}\right)^2} \\
&\leq \limsup_{\varepsilon\to 0} \frac{\int_{\bar{\Sigma}} |\nabla u|^2 dv_{\bar{g}} + C\varepsilon}{\int_{\bar{\Sigma}\setminus V_\varepsilon} |u|^2 dv_{\bar{g}} - \left(\int_{\bar{\Sigma}} u dv_{\bar{g}}\right)^2} \\
&\leq \frac{\int_{\hat{\Sigma}} |\nabla u|^2 dv_{\bar{g}}}{\int_{\bar{\Sigma}} |u|^2 dv_{\bar{g}}} \\
&\leq \lambda_1(\bar{\Sigma}/\bar{\iota}, g).
\end{aligned}$$

If $\bar{\Sigma}$ is disconnected, it has two connected components and the genus of each component is at most $\lfloor(\delta-1)/2\rfloor$. Therefore, the quotient $\bar{\Sigma}/\bar{\iota}$ is an orientable surface of genus at most $\lfloor(\delta-1)/2\rfloor$ in this case. In case $\bar{\Sigma}$ is connected, the quotient is non-orientable of non-orientable genus at most $\delta - 1$. □

Since $\Lambda_1^K(2) > 12\pi$, we can always rule out the first scenario from Lemma 3.3.3. The following theorem extends Theorem 3.1.1 to the non-orientable setting.

Proof of Theorem 3.1.2. By the assumptions, Proposition 3.3.2, and Proposition 3.2.3, we can take hyperbolic metrics $h_k \to h$ in C^∞, such that

$$\lim_{k\to\infty} \sup_{g\in[h_k]} \lambda_1(\Sigma, g) \operatorname{area}(\Sigma, g) = \Lambda_1^K(\delta).$$

As above, we take unit volume metrics $g_k \in [h_k]$, such that

$$\lambda_1(\Sigma, g_k) = \sup_{g\in[h_k]} \lambda_1(\Sigma, g) \operatorname{area}(\Sigma, g).$$

For the corresponding sequence of harmonic maps $\Phi_k \colon (\Sigma, h_k) \to \mathbb{S}^l$ no bubbling can occur since this would imply $\Lambda_1^K(\delta) \leq 8\pi$, by the same argument as above. Therefore, we can take a subsequence such that $\Phi_k \to \Phi$ in C^∞, which implies that $g_k \to g = \frac{|\nabla\Phi|_h^2}{\Lambda_1^K(\delta)} h$ in C^∞. In particular,

$$\lambda_1(\Sigma, g) \operatorname{area}(\Sigma, g) = \Lambda_1^K(\delta)$$

and g is smooth away from the branch points of Φ. The number of branch points is finite and the branch points correspond to conical singularities of g [**Sal85**]. □

Appendix: Topology of surfaces

For convenience of the reader and the authors, we review here the notion of non-orientable genus.

Recall the classification of closed surfaces. The classes of closed orientable and non-orientable surfaces are both uniquely described up to diffeomorphism by the Euler characteristic. More precisely, any closed orientable surface is diffeomorphic to a surface of the form

$$\Sigma_\gamma = \mathbb{S}^2 \# \underbrace{T^2 \# \ldots \# T^2}_{\gamma-\text{times}},$$

and any closed non-orientable surface is diffeomorphic to a surface of the form

$$\Sigma_\delta^K = \mathbb{S}^2 \# \underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{\delta-\text{times}}.$$

These two families provide – up to diffeomorphism – a complete list of all orientable respectively non-orientable surfaces. We call γ the *genus* of Σ_γ and δ the *non-orientable genus* of Σ_δ^K. Note that with this convention, the real projective plane has non-orientable genus 1. We have $\chi(\Sigma_\gamma) = 2 - 2\gamma$ and $\chi(\Sigma_\delta^K) = 2 - \delta$, so that the orientation cover of Σ_δ^K is given by $\Sigma_{\delta-1}$. Some authors prefer to refer to the genus of the orientation cover as the non-orientable genus. As explained above these two definitions differ. Moreover, recall that we have the relation

$$\mathbb{S}^2 \# \underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{\delta-\text{times}} \cong \mathbb{S}^2 \# \underbrace{T^2 \# \ldots \# T^2}_{k-\text{times}} \# \underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{(\delta-2k)-\text{times}},$$

if $2k < \delta$.

CHAPTER 4

Attaching handles and cross caps and the first eigenvalue

4.1. Introduction

In this chapter, we are interested in whether it is possible to strictly increase $\lambda_1 \cdot \mathrm{area}$ by adding a small handle or cross cap to a given closed Riemannian surface.

Clearly, this is motivated by the results Theorem 3.1.1 and Theorem 3.1.2. Recall, that we write $\Lambda_1(\gamma) = \sup_g \lambda_1(\Sigma_\gamma, g)\,\mathrm{area}(\Sigma_\gamma, g)$, where Σ_γ is closed, orientable and of genus γ, and the supremum is over all smooth metrics. Similarly, we write $\Lambda_1^K(\delta) = \sup_g \lambda_1(\Sigma_\delta^K, g)\,\mathrm{area}(\Sigma_\delta^K, g)$, where Σ_δ^K is closed, non-orientable and of non-orientable genus δ. Theorem 3.1.1 gives existence of metrics achieving $\Lambda_1(\gamma)$ provided that $\Lambda_1(\gamma) > \Lambda_1(\gamma-1)$. In the same direction, Theorem 3.1.2 gives existence of a metric achieving $\Lambda_1^K(\delta)$ if $\Lambda_1^K(\delta) > \Lambda_1^K(\delta-1)$ and $\Lambda_1^K(\delta) > \Lambda_1(\lfloor(\delta-1)/2\rfloor)$. Thus, in order to proof existence of maximizers by induction, we would like to show that $\lambda_1 \cdot \mathrm{area}$ can be strictly increased by attaching a handle or a cross cap.

Unfortunately, we are not able to show this in the required generality to perform this induction. However, we are able to exhibit two situations in which it is in fact possible to strictly increase $\lambda_1 \cdot \mathrm{area}$ be attaching a handle or a cross cap. The first result works for attaching a handle or a cross cap.

Theorem 4.1.1. *Let (Σ, g) be a closed Riemannian surface and assume that there is a point $x \in \Sigma$, such that $u(x) = 0$ for any $\lambda_1(\Sigma, g)$-eigenfunction u. Let Σ' be the surface obtained from Σ by attaching a handle or a cross cap. Then there is a metric g' on Σ', such that*

$$\lambda_1(\Sigma', g')\,\mathrm{area}(\Sigma', g') > \lambda_1(\Sigma, g)\,\mathrm{area}(\Sigma, g). \tag{4.1.2}$$

Since a metric achieving $\Lambda_1(\Sigma)$ is the induced metric from a branched minimal immersion into a sphere by first eigenfunctions, the assumptions of Theorem 4.1.1 can not be satisfied by any such metric. The second result is more general and actually applies to some maximizing metrics. It deals only with attaching handles.

Theorem 4.1.3. *Let (Σ, g) be a closed Riemannian surface and assume that there are points $x, y \in \Sigma$, such that $u(x) == -u(y)$ for any $\lambda_1(\Sigma, g)$-eigenfunction u. Let Σ' be the surface obtained from Σ by attaching a handle. Then there is a metric g' on Σ', such that*

$$\lambda_1(\Sigma', g')\,\mathrm{area}(\Sigma', g') > \lambda_1(\Sigma, g)\,\mathrm{area}(\Sigma, g). \tag{4.1.4}$$

Very recently, an explicit maximizing metric on the surface of genus two was constructed, [**NS17b**]. This metric has lots of symmetries. In particular, it satisfies the assumptions from Theorem 4.1.3.

Corollary 4.1.5. *There is a metric, smooth away from at most finitely many conical singularities, achieving* $\Lambda_1(3)$.

4.2. The construction and convergence of the spectrum

The surface Σ' will be obtained by attaching a very thin handle ore cross cap of radius ε We show that $\lambda_1 \cdot$ area can be strictly increased this way if ε is very small. In this section we explain how the spectrum behaves as the parameter ε degenerates to 0.

4.2.1. Attaching small cross caps and small handles. We first explain the construction for attaching cross caps in Theorem 4.1.1. Given a closed surface Σ, we glue a cross cap along its boundary. Write

$$M_{\varepsilon,h} = \mathbb{S}^1(\varepsilon) \times [0, 2h]/\sim,$$

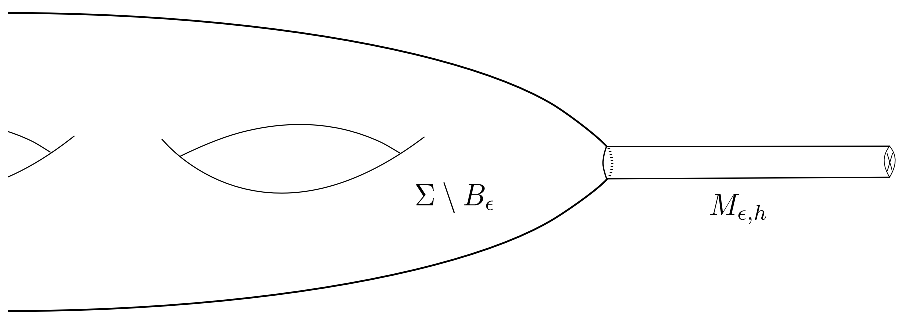
where $(\theta, t) \sim (\theta + \pi, 2h - t)$, and endow this with its canonical flat metric $f_{\varepsilon,h}$. Let $x_0 \in \Sigma$ and take a coordinate neighborhood U containing x_0, such that g is conformal to the Euclidean metric in U, that is $g = fg_e$ with f a smooth, positive function and g_e the Euclidean metric. We fix a natural number $k \geq 1$. Let $B_{\varepsilon^k} = B_{g_e}(x_0, \varepsilon^k)$ be a ball centered at x_0 with radius equals ε^k with respect to g_e. We then consider the surface

$$\Sigma_{\varepsilon,h} := (\Sigma \setminus B_{\varepsilon^k}) \cup_{\partial B_{\varepsilon^k}} M_{\varepsilon,h},$$

which we endow with the (non-smooth) metric $g_{\varepsilon,h}$ given by g on $\Sigma \setminus B_\varepsilon$ and by the flat metric $f_{\varepsilon,h}$ on $M_{\varepsilon,h}$. We will show below that we can find k, such that for ε small, there is a choice of $h \in [h_0, h_1] \subset (0, \infty)$ with

$$\lambda_1(\Sigma_{\varepsilon,h}) \operatorname{area}(\Sigma_{\varepsilon,h}) > \lambda_1(\Sigma) \operatorname{area}(\Sigma). \tag{4.2.1}$$

For ε and h such that the above holds, we can smooth the metric $g_{\varepsilon,h}$ in such a way, that we still have the strict inequality above.



A part of the surface $\Sigma_{\varepsilon,h}$

To show Theorem 4.1.3, we glue a flat cylinder along its two boundary components. More precisely, we take

$$C_{\varepsilon,h} = \mathbb{S}^1(\varepsilon) \times [0, h]$$

endowed with its canonical flat metric. For two points $x, y \in \Sigma$, such that $u(x) = -u(y)$ for any $\lambda_1(\Sigma)$-eigenfuncton u, we take neighbourhoods as above. We then consider for $k \geq 1$ the surface

$$\Sigma_{\varepsilon,h} = (\Sigma \setminus (B_{\varepsilon^k}(x) \cup B_{\varepsilon^k}(y))) \cup_{\partial B_{\varepsilon^k}(x) \cup \partial B_{\varepsilon^k}(y)} C_{\varepsilon,h},$$

where the balls B_{ε^k} are again with respect to the Euclidean metric. We will then show that we can find h and k such that for ε sufficiently small, such that

$$\lambda_1(\Sigma_{\varepsilon,h}) \operatorname{area}(\Sigma_{\varepsilon,h}) > \lambda_1(\Sigma) \operatorname{area}(\Sigma). \tag{4.2.2}$$

For a compact manifold with boundary, we denote by λ_0 its smallest Dirichlet eigenvalue and by μ_1 is smallest non-zero Neumann eigenvalue.

4.2.2. The limit spectrum. We mainly restrict our discussion in the following sections to the surfaces $\Sigma_{\varepsilon,h} = (\Sigma \setminus B_{\varepsilon^k}) \cup_{\partial B_{\varepsilon^k}} M_{\varepsilon,h}$. The discussion for glueing handles is similar or identical. We will indicate the necessary changes.

We will prove that the spectrum of $\Sigma_{\varepsilon,h}$ converges locally uniformly in the height h to the reordered union of the spectrum of Σ and the spectrum of the interval to which the handle respectively cross cap collapses to. In the case of attached handles and fixed height h, this is due to Anné [**Ann87**], see also [**Ann86, Pos03, Pos00**]. The arguments for the non-orientable case are essentially along the same lines.

For the precise statement of our result we first need to introduce some notation. Denote by $\operatorname{spec}_D^{\mathbb{Z}/2}([0,2h])$ the $\mathbb{Z}/2$-invariant Dirichlet spectrum of the interval $[0,2h]$, i.e. the spectrum of the Laplace operator acting on $\left(W_0^{1,2}([0,2h]) \cap W^{2,2}[0,2h]\right)^{\mathbb{Z}/2}$. The superscript indicates that we consider only those functions which are invariant under the involution $t \mapsto 2h - t$. For us the spectrum will always be a weakly increasing sequence, rather than just a set. (All operators we consider have purely discrete spectrum.) For fixed $h > 0$ denote by

$$0 = \nu_0^h < \nu_1^h \leq \nu_2^h \leq \dots$$

the reordered union of $\operatorname{spec}(\Sigma)$ and $\operatorname{spec}_D^{\mathbb{Z}/2}([0,2h])$.

The second thing we discuss is the convergence of the eigenfunctions on $\Sigma_{\varepsilon,h}$. The introduction of the following notation is convenient for this purpose. For $u \in W^{1,2}(\Sigma \setminus B_{\varepsilon^k})$, we write $\tilde{u} \in W^{1,2}(\Sigma)$ for the function which is given by u in $\Sigma \setminus B_{\varepsilon^k}$ and by the harmonic extension of $u|_{\partial B_{\varepsilon^k}}$ to B_{ε^k}.

We are now ready to state the above mentioned results.

Theorem 4.2.3. *The spectrum of $\Sigma_{\varepsilon,h}$ converges locally uniformly in h to $(\nu_i^h)_{i\in\mathbb{N}}$, i.e. for any a, b with $0 < a < b$, any $\delta > 0$ and $k \in \mathbb{N}$ there is $\varepsilon_0 > 0$ such that for any $h \in [a,b]$ and any $\varepsilon < \varepsilon_0$*

$$|\lambda_k(\Sigma_{\varepsilon,h}) - \nu_k^h| < \delta.$$

Let ε_l be a null sequence and $h_l \to h$. For any sequence u_l, of normalized eigenfunctions with bounded eigenvalues on $\Sigma_{\varepsilon_l,h_l}$, we have subsequential convergence in the following ways

(1) $r_l := u_l|_{\Sigma \setminus B_{\varepsilon_l}}$ satisfies $\tilde{r}_l \to u$ in $L^2(\Sigma)$, where u is a eigenfunction on Σ; or

(2) $\int_{M_{\varepsilon_l,h_l}} |(u_l - v_l) - \varepsilon_l^{-1/2} u_0|^2 \to 0$, *where* v_l *denotes the harmonic extension of* $u_l|_{\partial M_{\varepsilon_l,h-l}}$ *and* u_0 *is a eigenfunction corresponding to an eigenvalue contained in* $\sigma_D^{\mathbb{Z}/2}([0,2h])$.

Moreover, for a sequence u_l *such that we have convergence of both types as above, we have* $\|u_0\|_{L^2([0,2h])} + \|u\|_{L^2(\Sigma)} = 1$.

Remark 4.2.4. If we attach a collapsing handle as described in Section 4.2.1 instead of a cross cap, the analogous statement for the spectrum and eigenfunctions holds.

For the sake of completeness we give a proof of Theorem 4.2.3. Most of this material is contained in [**Ann87, Pos03**], but they deal only with handles. Moreover, in these papers balls of radius ε are removed. It will be important for us, that the construction works with balls of radius ε^k removed as well. Moreover, we give some more effective estimates during the course of our proof. We carry out the proof for the case of cross caps below.

We need to provide some preliminary estimates. In the first one we show that the Neumann spectrum of $\Sigma \setminus B_{\varepsilon^k}$ converges to the spectrum of Σ.

Lemma 4.2.5. *The spectrum of* $\Sigma \setminus B_{\varepsilon^k}$ *with Neumann boundary conditions converges to the spectrum of* Σ. *Moreover, for any sequence* $\varepsilon_l \to 0$ *and orthonormal eigenfunctions* $u_1^{\varepsilon_l}, \dots u_k^{\varepsilon_l}$ *on* $\Sigma \setminus B_{\varepsilon_l}$, *with uniformly bounded eigenvalues, we have subsequential convergence* $\tilde{u}_i^{\varepsilon_l} \to u_i$ *in* $L^2(\Sigma)$, *where* $u_1, \dots, u_k$ *are orthonormal eigenfunctions on* Σ.

Proof. First, note that a simple cut-off argument using that the capacity of $\{x_0\}$ in any ball is 0 yields

$$\lim_{\varepsilon \to 0} \mu_k(\Sigma \setminus B_{\varepsilon^k}) \leq \lambda_k(\Sigma). \tag{4.2.6}$$

To obtain the revers bound, let u_ε be a normalized μ_k-eigenfunction and let $\tilde{u}_\varepsilon$ be the function, that is obtained by extending u_ε harmonically to B_{ε^k}. By (4.2.6), u_ε is bounded in $W^{1,2}(\Sigma \setminus B_{\varepsilon^k})$, thus $\tilde{u}_\varepsilon$ is bounded in $W^{1,2}(\Sigma)$ and we may extract a subsequence $\varepsilon_l \to 0$, such that for $u_l = u_{\varepsilon_l}$ we have $u_l \rightharpoonup u$ in $W^{1,2}(\Sigma)$. By the compact Sobolev embedding we thus get $u_l \to u$ in $L^2(\Sigma)$. Hence, from standard elliptic estimates, we obtain $u_l \to u$ in $C^\infty_{loc}(\Sigma \setminus \{x_0\})$. If $\phi \in C^\infty_c(\Sigma \setminus \{x_0\})$, we find $\rho > 0$, such that $\operatorname{supp} \phi \subset \Sigma \setminus B_\rho$. By extracting a further subsequence if necessary, we may assume $\mu_1(\Sigma \setminus B_{\varepsilon_l}) \to \lambda$, using (4.2.6) another time. Then we have

$$\begin{aligned}
\int_\Sigma \nabla u \cdot \nabla \phi &= \lim_{l \to \infty} \int_{\Sigma \setminus B_\rho} \nabla u_l \cdot \nabla \phi \\
&= \lim_{l \to \infty} \mu_k(\Sigma \setminus B_{\varepsilon_l}) \int_{\Sigma \setminus B_\rho} u_l \phi \\
&= \lambda \int_\Sigma u \phi.
\end{aligned}$$

Since $C^\infty_c(\Sigma \setminus \{x_0\}) \subset W^{1,2}(\Sigma)$ is dense, it follows that u is an eigenfunction on Σ with eigenvalue λ. Thus we have that all accumulations of points of $(\mu_1(\Sigma \setminus B_{\varepsilon_l}))_l$ are contained in the spectrum of Σ. Moreover, we also have convergence of the eigenfunctions as claimed.

A simple argument using the ordering of the eigenvalues implies then that we actually have convergence $\mu_k(\Sigma \setminus B_{\varepsilon^k}) \to \lambda_k(\Sigma)$ (and not only subsequential convergence).

The assertion concerning the convergence of the eigenfunctions follows from the arguments above, combined with Lemma 4.2.12 and the maximum principle. $\square$

Remark 4.2.7. The same arguments as above give the same result if we remove a larger number of balls instead of just a single one.

Remark 4.2.8. For possible improvements of our result it seems interesting to have more precise bounds on the convergence rate in Lemma 4.2.5. We provide such an estimate in the appendix.

Next we prove that the Dirichlet spectrum of $M_{\varepsilon,h}$ converges to the spectrum of the interval to which $M_{\varepsilon,h}$ collapses to.

Lemma 4.2.9. *The Dirichlet spectrum of $M_{\varepsilon,h}$ converges locally uniformly in $h > 0$ to $\sigma_D^{\mathbb{Z}/2}([0, 2h])$. Moreover, any sequence of eigenfunctions u^{ε_l} for $\varepsilon_l \to 0$ with uniformly bounded eigenvalue consists of horizontal functions for ε_l sufficiently small.*

Proof. This is obvious since $M_{\varepsilon,h}$ is covered by a product, one of whose factors shrinks at rate ε. $\square$

For the proof of Theorem 4.2.3, we need a result relating the spectra of quadratic forms on different Hilbert spaces in the presence of a so-called coupling map. This result generalizes the 'Main Lemma' in [**Pos03**], since we have to take care of the additional parameter h.

Suppose we are given separable Hilbert spaces $\mathcal{H}_{\varepsilon,h}$ and $\mathcal{H}'_{\varepsilon,h}$, equipped with quadratic forms $q_{\varepsilon,h}$ and $q'_{\varepsilon,h}$, respectively. We assume that these quadratic forms are non-negative and closed. Then there is a unique self-adjoint operator associated to $q_{\varepsilon,h}$ which will henceforth be referred to as $Q_{\varepsilon,h}$, similarly we have $Q'_{\varepsilon,h}$ associated to $q'_{\varepsilon,h}$. Note, that the spectrum of $Q_{\varepsilon,h}$ and $Q_{\varepsilon,h'}$ is purely discrete.

The k-th eigenvalues of $q_{\varepsilon,h}$ and $q'_{\varepsilon,h}$ are henceforth denoted by $\lambda_k(\varepsilon, h)$ and $\lambda_k(\varepsilon, h)'$, respectively. Let $L_k(\varepsilon, h)$ denote the direct sum of the eigenspaces of $Q_{\varepsilon,h}$ corresponding to the first $(k + 1)$-eigenvalues. Finally, we denote by $\mathrm{dom}(q_{\varepsilon,h})$ the domain of $q_{\varepsilon,h}$.

Lemma 4.2.10. *For each $\varepsilon, h > 0$ let $\Phi_{\varepsilon,h} \colon \mathrm{dom}(q_{\varepsilon,h}) \to \mathrm{dom}(q'_{\varepsilon,h})$ be a linear map such that all $u_\varepsilon \in L_k(\varepsilon, h)$ with $\sup_\varepsilon(\|u_\varepsilon\|_{\mathcal{H}_{\varepsilon,h}} + q_{\varepsilon,h}(u_\varepsilon)) < \infty$ satisfy the following two conditions.*

(1) $\lim_{\varepsilon\to 0}(\|\Phi_{\varepsilon,h}u_\varepsilon\|_{\mathcal{H}'_{\varepsilon,h}} - \|u_\varepsilon\|_{\mathcal{H}_{\varepsilon,h}}) = 0$, *locally uniformly in h,*

(2) $q'_{\varepsilon,h}(\Phi_{\varepsilon,h}u_\varepsilon) \le q_{\varepsilon,h}(u_\varepsilon)$.

Moreover, assume that $\lambda_k(\varepsilon, h) \le C$ for any $\varepsilon > 0$, fixed k, and $h \in [h_0, h_1] \subset (0, \infty)$. Then we have

$$\lambda'_k(\varepsilon) \le \lambda_k(\varepsilon) + o(1),$$

where the $o(1)$ term is locally uniform in k and $h \in (0, \infty)$.

Proof. We just repeat the proof from [**Pos03**], where the result is proved without the additional parameter h.

Denote by $\phi^i_{\varepsilon,h}$ orthonormal bases of $\mathcal{H}_{\varepsilon,h}$ consisting of eigenfunctions of $Q_{\varepsilon,h}$. Given any $u \in L_k(\varepsilon,h)$, we can expand this as $u = \sum_{i=0}^k \alpha_i^{\varepsilon,h}\phi^i_{\varepsilon,h}$. Then, suppressing the indices ε and h whenever it is clear what they are, we get

$$\begin{aligned}\|u\|^2 - \|\Phi_{\varepsilon,h}u\|^2 &= \sum_{i,j=0}^k \alpha_i\alpha_j(\delta_{ij} - \langle \Phi_{\varepsilon,h}\phi^i_{\varepsilon,h}, \Phi_{\varepsilon,h}\phi^j_{\varepsilon,h}\rangle)\\ &\leq \delta'_k(\varepsilon,h)\sum_{j=1}^k |\alpha_j|^2 = \delta'_k(\varepsilon,h)\|u\|^2,\end{aligned}$$

where $\delta'_k(\varepsilon,h) = k\max_{i,j\leq k}|\delta_{ij} - \langle \Phi_{\varepsilon,h}\phi^i_{\varepsilon,h}, \Phi_{\varepsilon,h}\phi^j_{\varepsilon,h}\rangle|$. Assumption (1) combined with polarization implies that $\delta'_k(\varepsilon,h) \to 0$ locally uniformly in h. In particular, we find that

$$\|\Phi_{\varepsilon,h}u\|^2 \geq (1-\delta'_k(\varepsilon,h))\|u\|^2, \tag{4.2.11}$$

which also implies that $\Phi_{\varepsilon,h}$ is injective on $L_k(\varepsilon,h)$ for ε small enough. An easy computation then leads to

$$\frac{q'_{\varepsilon,h}(\Phi_{\varepsilon,h}u)}{\|\Phi_{\varepsilon,h}u\|^2} - \frac{q_{\varepsilon,h}(u)}{\|u\|^2} \leq \frac{C\delta'_k(\varepsilon,h)}{1-\delta'_k(\varepsilon,h)}.$$

Applying the min-max characterization of eigenvalues to the above estimate establishes the claim. ☐

Lemma 4.2.12. *Let u_ε be an L^2-normalized eigenfunction on $\Sigma_{\varepsilon,h}$ with eigenvalue $\lambda \leq \Lambda$. There is a constant C depending on Λ and k (from the construction of $\Sigma_{\varepsilon,h}$), such that the following holds. If we use Euclidean polar coordinates (r,θ) centered at x we have the uniform pointwise bounds*

$$|u_\varepsilon|(r,\theta) \leq C\log\left(\frac{1}{r}\right), \tag{4.2.13}$$

for $\varepsilon^k \leq r \leq 1/2$. and

$$|\nabla u_\varepsilon|(r,\theta) \leq \frac{C}{r} \tag{4.2.14}$$

for $2\varepsilon^k \leq r \leq 1/2$.

Proof. Recall that we have identified a conformally flat neighborhood of x_0 with $B_1 = B(0,1) \subset \mathbb{R}^2$, such that $x_0 = 0$. First, observe that, up to radius $2\varepsilon^k$ (4.2.13) is a direct consequence of (4.2.14). In fact, by the standard elliptic estimates [**Tay11a**, Chapter 5.1], the functions u_ε are uniformly bounded in C^∞ within compact subsets of $\Sigma \setminus \{x_0\}$. Given this, we can integrate the bound (4.2.14) from $\partial B_{1/2}$ to ∂B_r and find (4.2.13).

The bound (4.2.14) follows from standard elliptic estimates after rescaling the scale r to a fixed scale. More precisely, we consider the rescaled functions $w_r(z) := u_\varepsilon(rz)$. On $B_1 \setminus B_{\varepsilon^k}$ the metric of Σ is uniformly bounded from above and below by the Euclidean metric. Hence we can perform all computations in the Euclidean metric. We have

$$\int_{B_3\setminus B_{1/2}} |\nabla w_r|^2 = \int_{B_{3r}\setminus B_{r/2}} |\nabla u_\varepsilon|^2, \tag{4.2.15}$$

since the Dirichlet energy is conformally invariant in dimension two.

Since the Laplace operator is conformally covariant in dimension two, w_r solves the equation

$$\Delta_e w_r = r^2 f_r \lambda_\varepsilon w_r, \tag{4.2.16}$$

with $f_r(z) = f(rz)$ a smooth function and Δ_e the Euclidean Laplacian. Since $f \in C^\infty$, we have uniform C^∞-bounds on f_r for $r \leq 1$. Taking derivatives, we find that

$$\Delta_e \nabla w_r = r^2 \lambda_\varepsilon \nabla(f_r w_r), \tag{4.2.17}$$

where also the gradient is taken with respect to the Euclidean metric. The bound (4.2.15) implies that the right hand side of this equation is bounded by Cr^2 in $L^2(B_3 \setminus B_{1/2})$. Therefore, by elliptic estimates [**Tay11a**, Chapter 5.1] we have

$$\sup_{\{1 \leq s \leq 2\}} |\nabla w_r|(s,\theta) \leq Cr^2 + C|\nabla w_r|_{L^2(B_3 \setminus B_{1/2})} \leq C,$$

which scales to

$$\sup_{\{r \leq s \leq 2r\}} |\nabla u_\varepsilon|(s,\theta) \leq \frac{C}{s},$$

with C independent of r. This proves the estimate for $r \geq 2\varepsilon^k$.

To get the estimate (4.2.13) for the remaining radii we invoke the De Giorgi–Nash–Moser estimate. We fix $x_0 \in \partial B(x, \varepsilon^k)$ and consider the neighbourhood

$$U_\varepsilon(x_0, \alpha) = \{z \in \Sigma \setminus B_{\varepsilon^k} \ : \ d_e(x,z) \leq \alpha \varepsilon^k\} \cup \{z \in M_{\varepsilon,h} \ : \ d(x,z) \leq \alpha\varepsilon\}.$$

We rescale the metric on $U_\varepsilon(x_0, 4)$ by the singular conformal factor

$$f_\varepsilon = \begin{cases} \varepsilon^{-k} & \text{in } U_\varepsilon(x_0,4) \cap (\Sigma \setminus B(x,\varepsilon^k)) \\ \varepsilon^{-1} & \text{in } U_\varepsilon(x_0,4) \cap M_{\varepsilon,h}. \end{cases}$$

More precisely, we consider the metric $l_\varepsilon = f_\varepsilon g_{\varepsilon,h}$, where $g_{\varepsilon,h}$ is the metric on $\Sigma_{\varepsilon,h}$. We suppress the index h for the metric l since our analysis is independent of h for ε sufficiently small. Consider the function $w_\varepsilon = u_\varepsilon - (u_\varepsilon)_{U_\varepsilon(x_0,4)}$, where $(u_\varepsilon)_{U_\varepsilon(x_0,4)}$ denotes the mean value of u_ε on $U_\varepsilon(x_0, 4)$ with respect to the rescaled metric l_ε. By the conformal invariance of the Dirichlet energy, we find that w_ε has gradient bounded in L^2 with respect to the rescaled metric,

$$\int_{U_\varepsilon(x_0,4)} |\nabla w_\varepsilon|^2 dA_{l_\varepsilon} \leq C. \tag{4.2.18}$$

It is easy to see that the rescaled metric l_ε on $U_\varepsilon(x_0, 4)$ is uniformly bounded from above and below almost everywhere by a fixed metric. In fact, on $M_\varepsilon, h \cap U_\varepsilon(x_0, 4)$ the metric l_ε is the metric of a fixed flat cylinder, and on $\Sigma \setminus B(x, \varepsilon^k) \cap U_\varepsilon(x_0, 4)$ the metric l_ε is close to the standard flat metric on (a subset of) the unit disk. Therefore there is a constant C independent of ε and x_0 such that

$$\int_{U_\varepsilon(x_0,4)} |w_\varepsilon|^2 dA_{l_\varepsilon} \leq C \int_{U_\varepsilon(x_0,4)} |\nabla w_\varepsilon|^2 dA_{l_\varepsilon}. \tag{4.2.19}$$

Now observe that w_ε is a weak solution to the equation

$$\Delta_{l_\varepsilon} w_\varepsilon = \frac{1}{f_\varepsilon} \Delta_{g_{\varepsilon,h}} u_\varepsilon = \frac{1}{f_\varepsilon} \lambda_\varepsilon u_\varepsilon, \tag{4.2.20}$$

thanks to the conformal covariance of the Laplacian in dimension two, which is easily checked to hold also in the singular context required for the above equation. Finally, note that the right hand side of (4.2.20) is bounded in $L^2(U_\varepsilon(x_0,4), dA_{l_\varepsilon})$. Thanks to this, (4.2.18), (4.2.19), and (4.2.20) we can apply the inhomogeneous De Giorgi–Nash–Moser estimates (see e.g. [**Tay11b**, Chapter 14.9]) to obtain

$$\sup_{p\in U_\varepsilon(x_0,2), q\in U_\varepsilon(x_0,2)} |w_\varepsilon(p) - w_\varepsilon(q)| \leq C.$$

Since this is scale invariant, independent of ε and x_0 this implies (4.2.13). □

Combining the previous lemmata we can now prove Theorem 4.2.3.

PROOF OF THEOREM 4.2.3. Clearly, we have an upper bound

$$\lambda_k(\Sigma_{\varepsilon,h}) \leq \nu_k^h + o(1), \tag{4.2.21}$$

using extensions of Dirichlet eigenfunctions of $\Sigma \setminus B_\varepsilon$ and $M_{\varepsilon,h}$ as test functions, and the fact that the Dirichlet spectrum of $\Sigma \setminus B_\varepsilon$ converges to the spectrum of Σ (this is similar to, but easier than Lemma 4.2.5 above). In particular, we see that the $o(1)$ term is independent of h (but of course might depend on k).

For the lower bound and the assertion concerning the behavior of the eigenfunctions we use Lemma 4.2.10. Our first family of Hilbert spaces is $\mathcal{H}_{\varepsilon,h} = L^2(\Sigma_{\varepsilon,h})$ with quadratic forms $q_\varepsilon(u) = \int_{\Sigma_{\varepsilon,h}} |\nabla u|^2$. The second family is given by $\mathcal{H}'_{\varepsilon,h} = L^2(\Sigma \setminus B_{\varepsilon^k}) \oplus L^2(M_{\varepsilon,h})$, with quadratic forms $q'_{\varepsilon,h}(u) = \int_{\Sigma\setminus B_{\varepsilon^k}} |\nabla u_1|^2 + \int_{M_{\varepsilon,h}} |\nabla u_2|^2$. Here the first summand is subject to Neumann boundary conditions and the second one to Dirichlet boundary conditions. The coupling map $\Phi_{\varepsilon,h}\colon \mathcal{H}_{\varepsilon,h} \to \mathcal{H}'_{\varepsilon,h}$ is defined as follows

$$\Phi_{\varepsilon,h}(u) = u|_{\Sigma\setminus B_{\varepsilon^k}} \oplus (u|_{M_{\varepsilon,h}} - v_{\varepsilon,h}),$$

where $v_{\varepsilon,h} \in L^2(M_{\varepsilon,h})$ is the harmonic extension of $u|_{\partial M_{\varepsilon,h}}$ to $M_{\varepsilon,h}$. Next, we verify assumptions (1) and (2) from Lemma 4.2.10.

To check the first condition, we need to show $v_{\varepsilon,h} \to 0$ in L^2, meaning that

$$\int_{M_{\varepsilon,h}} |v_{\varepsilon,h}|^2 \to 0.$$

(Note that the measure depends on ε as well.) It suffices to check this in the case that u is an eigenfunction. The general case follows since the harmonic extension operator is linear. If u_ε is an eigenfunction, it follows from the maximum principle and Lemma 4.2.12, that

$$\sup_{M_{\varepsilon,h}} |v_{\varepsilon,h}| \leq \sup_{\partial M_{\varepsilon,h}} |v_{\varepsilon,h}| = \sup_{\partial B(x,\varepsilon^k)} |u_\varepsilon| \leq C|\log(\varepsilon^k)|.$$

which implies

$$\int_{M_{\varepsilon,h}} |v_\varepsilon|^2 \leq Ck\varepsilon h |\log(\varepsilon)| \to 0,$$

as $\varepsilon \to 0$.

In order to prove that the second condition is satisfied, observe that $u|_{M_{\varepsilon,h}} - v_{\varepsilon,h} \in W_0^{1,2}(M_{\varepsilon,h})$. Consequently, we have

$$\int_{M_{\varepsilon,h}} \nabla(u - v_{\varepsilon,h}) \cdot \nabla v_{\varepsilon,h} = 0.$$

This is turn implies that

$$\int_{M_{\varepsilon,h}} |\nabla(u - v_{\varepsilon,h})|^2 = \int_{M_{\varepsilon,h}} |\nabla u|^2 - \int_{M_{\varepsilon,h}} |\nabla v_{\varepsilon,h}|^2 \leq \int_{M_{\varepsilon,h}} |\nabla u|^2$$

so that $q'_{\varepsilon,h}(\Phi_{\varepsilon,h} u) \leq q_{\varepsilon,h}(u)$.

Trivially, the convergence of the Neumann spectrum of $\Sigma \setminus B_\varepsilon$ to the spectrum of Σ is uniform in h. Therefore it follows from (4.2.21) and Lemma 4.2.10 that the converge is locally uniform in h and k as claimed.

The assertion concerning the convergence of the eigenfunctions follows from the fact that the quantity $\delta'_k(\varepsilon)$ in the proof of Lemma 4.2.10 converges to zero. Indeed, let u_l be a normalized sequence of eigenfunctions corresponding to the eigenvalue $\lambda_k(\Sigma_{\varepsilon_l,h})$. From the bound (4.2.11), we can infer that we can extract a subsequence, such that either $\|u_l\|_{L^2(\Sigma \setminus B_\varepsilon)}$ or $\|u_l - v_l\|_{L^2(M_{\varepsilon,h})}$ is bounded away from zero. In the first case, we find that the sequence of harmonic extension $\tilde{u}_l$ is bounded in $W^{1,2}(\Sigma)$ and by the arguments from the proof of Lemma 4.2.5 we have subsequential convergence to a non-trivial eigenfunction on Σ in $L^2$and $C^\infty_{loc}(\Sigma \setminus \{x_0\})$. In the second case, we use that we know the Dirichlet spectrum and eigenfunctions of $M_{\varepsilon,h}$ explicitly. If one expands $u_{\varepsilon,h} - v_{\varepsilon,h}$ in the eigenfunctions, it is easily checked, that it becomes more and more horizontal, since the energy of the vertical eigenmodes explodes. Given this, the assertion follows easily by an argument similar to that of the first case. $\square$

4.3. Construction of quasimodes and conclusion I

Let $\eta\colon [1,2] \to [0,1]$ be a function with $\eta(1) = 0$ and $\eta(2) = 1$. We then define a cut-off function $\eta_\varepsilon\colon \Sigma_{\varepsilon,h} \to [0,1]$ by

$$\eta_\varepsilon = \begin{cases} 1 & \text{in } \Sigma \setminus B_{2\varepsilon^k} \\ \eta(\varepsilon^k r) & \text{in } B_{2\varepsilon^k} \setminus B_{\varepsilon^k} \\ 0 & \text{on } M_{\varepsilon,h}, \end{cases}$$

where we use (Euclidean) radial coordinates (θ, r) in $B_{2\varepsilon^k}$.

For u an L^2-normalized $\lambda_1(\Sigma)$-eigenfunction, we define a new function

$$v_\varepsilon = \eta_\varepsilon u.$$

Recall that we assume $u(x) = 0$. Therefore, the function v_ε turns out to be a good quasimode. Before we can actually prove this, we need to recall the following observation.

Lemma 4.3.1. *Let $1 < p < \infty$, then there is C_p independent of ε and k, such that*

$$\|\phi\|_{L^p(\Sigma \setminus B_{\varepsilon^k})} \leq C_p \|\phi\|_{W^{1,2}(\Sigma \setminus B_{\varepsilon^k})}. \tag{4.3.2}$$

Proof. This follows since the harmonic extension operator $W^{1,2}(\Sigma\backslash B_{\varepsilon^k}) \to W^{1,2}(\Sigma)$ is uniformly bounded. See e.g. [**RT75**], where this is proved by a scaling argument. The conclusion then follows by combining this with the Sobolev embedding $W^{1,2}(\Sigma) \hookrightarrow L^p(\Sigma)$. $\square$

Lemma 4.3.3. *For the function v_ε defined above and any $\phi \in W^{1,2}(\Sigma_{\varepsilon,h})$, we have that*

$$\left| \int_{\Sigma_{\varepsilon,h}} \nabla v_\varepsilon \cdot \nabla \phi - \lambda_1(\Sigma) \int_{\Sigma_{\varepsilon,h}} v_\varepsilon \phi \right| \leq C\varepsilon^{k/2} \|\phi\|_{W^{1,2}(\Sigma_{\varepsilon,h})}. \tag{4.3.4}$$

Proof. We compute

$$\begin{aligned} \int_{\Sigma_{\varepsilon,h}} \nabla v_\varepsilon \cdot \nabla \phi &= \int_{\Sigma_{\varepsilon,h}} \eta_\varepsilon \nabla u \cdot \nabla \phi + \int_{\Sigma_{\varepsilon,h}} u \nabla \eta_\varepsilon \cdot \nabla \phi \\ &= \int_{\Sigma_{\varepsilon,h}} \nabla u \cdot \nabla(\eta_\varepsilon \phi) - \int_{\Sigma_{\varepsilon,h}} \phi \nabla u \cdot \nabla \eta_\varepsilon + \int_{\Sigma_{\varepsilon,h}} u \nabla \eta_\varepsilon \cdot \nabla \phi \\ &= \lambda_1(\Sigma) \int_{\Sigma_{\varepsilon,h}} u \eta_\varepsilon \phi - \int_{\Sigma_{\varepsilon,h}} \phi \nabla u \cdot \nabla \eta_\varepsilon + \int_{\Sigma_{\varepsilon,h}} u \nabla \eta_\varepsilon \cdot \nabla \phi, \end{aligned} \tag{4.3.5}$$

since $\eta_\varepsilon \phi \in W^{1,2}(\Sigma)$. Let us estimate the two last terms separately. For the middle term, we use Lemma 4.3.1. Since u is smooth, there is a constant C such that $|\nabla u| \leq C$. Therefore, we can invoke Hölder's inequality, the scaling invariance of the Dirichlet energy, and Lemma 4.3.1 to find

$$\begin{aligned} \left| \int_{\Sigma_{\varepsilon,h}} \phi \nabla u \cdot \nabla \eta_\varepsilon \right| &\leq C \left(\int_{\Sigma_{\varepsilon,h}} |\phi|^p \right)^{1/p} \operatorname{area}(B_{2\varepsilon^k})^{1/q} \left(\int_{\Sigma_{\varepsilon,h}} |\nabla \eta_\varepsilon|^2 \right)^{1/2} \\ &\leq C_p \varepsilon^{2k/q} \|\phi\|_{W^{1.2}(\Sigma_{\varepsilon,h})}, \end{aligned} \tag{4.3.6}$$

where we used that it suffices to integrate over $\operatorname{supp} \nabla \eta_\varepsilon \subset B_{2\varepsilon^k}$ and $1/2 + 1/p + 1/q = 1$.

We now estimate the last term from (4.3.5). Since u is smooth and vanishes at x, there is a constant C, such that

$$|u| \leq C\varepsilon^k \tag{4.3.7}$$

in $B_{2\varepsilon^k}$. Since $\operatorname{supp} \nabla \eta_\varepsilon \subset B_{2\varepsilon^k}$, this implies

$$\begin{aligned} \left| \int_{\Sigma_{\varepsilon,h}} u \nabla \eta_\varepsilon \cdot \nabla \phi \right| &\leq C\varepsilon^k \left(\int_{\Sigma_{\varepsilon,h}} |\nabla \eta_\varepsilon|^2 \right)^{1/2} \left(\int_{\Sigma_{\varepsilon,h}} |\nabla \phi|^2 \right)^{1/2} \\ &\leq C\varepsilon^k \|\phi\|_{W^{1,2}(\Sigma_{\varepsilon,h})}. \end{aligned} \tag{4.3.8}$$

by Hölder's inequality and the scaling invariance of the Dirichlet energy. If we specify to $p = q = 4$ in (4.3.6) and combine this with (4.3.5) and (4.3.8), the assertion follows. $\square$

Proof of Theorem 4.1.1. We take $h_0 > 0$ such that $\lambda_0^{\mathbb{Z}/2}([0, h_0]) \leq 2\lambda_1(\Sigma)$ and $k \geq 4$ It follows from Theorem 4.2.3 that for ε sufficiently small there are exactly $\operatorname{mult}(\lambda_1(\Sigma))$ eigenvalues in $[\lambda_1(\Sigma) - \delta, \lambda_1(\Sigma) + \delta]$ and the only eigenvalue below $\lambda_1(\Sigma) - 2\delta$

is 0. On the other hand, we find from Lemma 4.3.3 combined with [**Ann90**, Proposition 2] that there are at least $\operatorname{mult}(\lambda_1(\Sigma))$ eigenvalues in $[\lambda_1(\Sigma)-C\varepsilon^2,\lambda_1(\Sigma)+C\varepsilon^2]$, since

$$\left|\int_{\Sigma_{\varepsilon,h}} v_\varepsilon w_\varepsilon\right| \leq \int_{B_{\varepsilon^k}} |vw| \leq C\varepsilon^{2k},$$

if v_ε and w_ε are quasimodes constructed as in the previous lemma starting from two orthonormal eigenfunctions v, w. In particular, the first eigenvalue satisfies

$$\lambda_1(\Sigma_{\varepsilon,h_0}) \geq \lambda_1(\Sigma) - C\varepsilon^2.$$

Since the gain in area is linear in ε, we find

$$\begin{aligned}\lambda_1(\Sigma_{\varepsilon,h})\operatorname{area}(\Sigma_{\varepsilon,h}) &\geq (\lambda_1(\Sigma)-C\varepsilon^2)(\operatorname{area}(\Sigma)+2\pi h\varepsilon - O(\varepsilon^8))\\ &\geq \lambda_1(\Sigma)\operatorname{area}(\Sigma) + 2\pi\varepsilon h\lambda_1(\Sigma) - O(\varepsilon^2).\end{aligned}$$

which implies the assertion for ε sufficiently small after smoothing the metric on $\Sigma_{\varepsilon,h}$. $\square$

4.4. Construction of quasimodes and conclusion II

We will need to following observation. For the flat cylinder $C_{\varepsilon,h}$, the eigenvalues $\lambda_0(C_{\varepsilon,h})$ with Dirichlet boundary conditions and $\mu_1(C_{\varepsilon,h})$ with Neumann boundary conditions agree if ε is sufficiently small. Moreover, for such ε, any $\mu_1(C_{\varepsilon,h})$-eigenfunctions is antisymmetric. Thus, we can hope that if we interpolate from $u(x)$ to $u(y)$ by a $\mu_1(C_{\varepsilon,h})$-eigenfunction on $C_{\varepsilon,h}$ to find a good quasimode.

To make this precise, we define a function $v_\varepsilon \in W^{1,2}(\Sigma_{\varepsilon,h})$ as follows. For u an L^2-normalized $\lambda_1(\Sigma)$-eigenfunction, we define a new function v_ε by

$$v_\varepsilon = \begin{cases} u & \text{in } \Sigma\setminus B_{2\varepsilon^k}(x)\cup B_{2\varepsilon^k}(y)\\ \eta(\varepsilon^k r)u + (1-\eta(\varepsilon^k r)u(x)) & \text{in } B_{2\varepsilon^k}\setminus B_{\varepsilon^k}(x)\\ \eta(\varepsilon^k r)u + (1-\eta(\varepsilon^k r)u(y)) & \text{in } B_{2\varepsilon^k}\setminus B_{\varepsilon^k}(y)\\ \varphi & \text{on } C_{\varepsilon,h},\end{cases} \tag{4.4.1}$$

where $\varphi\colon C_{\varepsilon,h}\to\mathbb{R}$ is a $\mu_1(C_{\varepsilon,h})$-eigenfunction that is equal to $u(x)$ respectively $u(y)$ on the boundary components of $C_{\varepsilon,h}$. Note that such ϕ exists since $u(x) = -u(y)$.

Similarly as in the previous section, v_ε provides a good quasimode as well. For the proof of Theorem 4.1.3 it is necessary to carefully keep track of the dependence of the estimate on the parameter h.

Lemma 4.4.2. *For the function v_ε defined above and any $\phi\in W^{1,2}(\Sigma_{\varepsilon,h})$, we have that*

$$\left|\int_{\Sigma_{\varepsilon,h}} \nabla v_\varepsilon\cdot\nabla\phi - \lambda_1(\Sigma)\int_{\Sigma_{\varepsilon,h}} v_\varepsilon\phi\right| \leq C\left(\left|\frac{1}{h^2}-\frac{1}{h_*^2}\right|\varepsilon^{1/2}+\varepsilon^{k/2}\right)\|\phi\|_{W^{1,2}(\Sigma_{\varepsilon,h})}$$

Proof. The estimate in $\Sigma\setminus(B(x,\varepsilon^k)\cup B(y,\varepsilon^k))$ carries over mutatis mutandis from the proof of Lemma 4.3.3 and implies

$$\left|\int_{\Sigma\setminus(B(x,\varepsilon^k)\cup B(y,\varepsilon^k))} \nabla v_\varepsilon\cdot\nabla\phi - \lambda_1(\Sigma)\int_{\Sigma\setminus(B(x,\varepsilon^k)\cup B(y,\varepsilon^k))} v_\varepsilon\phi\right| \leq C_p\varepsilon^{2k/q}\|\phi\|_{W^{1,2}(\Sigma_{\varepsilon,h})}$$

where $1/p+1/q = 1/2$.

Define $h_* > 0$ by $\mu_1(C_{\varepsilon,h_*}) = \pi^2/h_*^2 = \lambda_1(\Sigma)$. Then we have

$$\begin{aligned}\left|\int_{C_{\varepsilon,h}} \nabla v_\varepsilon \cdot \nabla \phi - \lambda_1(\Sigma) \int_{C_{\varepsilon,h}} v_\varepsilon \phi\right| &= \left|\int_{C_{\varepsilon,h}} \nabla \varphi \cdot \nabla \phi - \lambda_1(\Sigma) \int_{C_{\varepsilon,h}} v_\varepsilon \phi\right| \\ &\le |\mu_1(C_{\varepsilon,h}) - \lambda_1(\Sigma)| \int_{C_{\varepsilon,h}} |\varphi\phi| \\ &\le C \left|\frac{1}{h^2} - \frac{1}{h_*^2}\right| \int_{C_{\varepsilon,h}} |\phi| \\ &\le C \left|\frac{1}{h^2} - \frac{1}{h_*^2}\right| \varepsilon^{1/2} \|\phi\|_{W^{1,2}(\Sigma_{\varepsilon,h})}.\end{aligned}$$

Combining the above two estimates and specifying to $p = q = 2$ once again implies the assertion. ☐

Proof of Theorem 4.1.3. We define $h_\varepsilon > 0$ by requiring that $h_\varepsilon < h_*$ and

$$\left|\frac{\pi^2}{h_\varepsilon^2} - \frac{\pi^2}{h_*^2}\right| = \varepsilon^{3/4}. \tag{4.4.3}$$

Let $k = 4$ and define $\Sigma_\varepsilon = \Sigma_{\varepsilon,h_\varepsilon}$. If we take a basis $(u_1, \dots, u_k)$ of $\lambda_1(\Sigma)$ eigenfunctions with the property that $u_1(x) = \cdots = u_{k-1}(x) = 0$ it is easy to see that if $(v_i)_\varepsilon$ denotes the quasimode constructed starting with u_i in Lemma 4.4.2, then we have

$$\left|\int_{\Sigma_{\varepsilon,h}} (v_i)_\varepsilon (v_j)_\varepsilon\right| \le C\varepsilon^{2k}\delta_{ij}.$$

Therefore it follows from [**Ann90**] and Lemma 4.4.2 that there are at least $\operatorname{mult}(\lambda_1(\Sigma))$ eigenvalues in $[\lambda_1(\Sigma) - C\varepsilon^{5/4}, \lambda_1(\Sigma) + C\varepsilon^{5/4}]$. Moreover, [**Ann90**] implies that there is an eigenvalue λ_ε converging to $\lambda_1(\Sigma)$ with

$$\lambda_\varepsilon \ge \frac{\pi^2}{h_\varepsilon^2} - C\varepsilon \ge \lambda_1(\Sigma) + \left|\frac{\pi^2}{h_\varepsilon^2} - \frac{\pi^2}{h_*^2}\right| - C\varepsilon \ge \lambda_1(\Sigma) + \varepsilon^{3/4}/2. \tag{4.4.4}$$

for ε sufficiently small. In particular, the first eigenvalue is bounded from below by

$$\lambda_1(\Sigma_\varepsilon) \ge \lambda_1(\Sigma) - C\varepsilon^{5/4}.$$

The assertion follows now exactly as in the proof of Theorem 4.1.1. ☐

Remark 4.4.5. Note that the results from [**Ann90**] apply here as well, since we have the bound (4.2.13). The linear term in ε comes from the radius of the cylinder and it is not clear that it should improve for h fixed when removing balls of size ε^k instead of ε.

Appendix: Improved convergence of the Neumann spectrum

In this appendix we prove that the convergence of $\mu_1(\Sigma \setminus B_\varepsilon)$ to $\lambda_1(\Sigma)$ happens at a good rate at least from below. We use some ideas that are contained in [**FS16**], where a monotonicity result for the first Steklov eigenvalue under adding a boundary component is proved.

Theorem 4.4.6. *We have the following lower bound*

$$\mu_1(\Sigma \setminus B_\varepsilon) \geq \lambda_1(\Sigma) - O(\varepsilon^2 |\log(\varepsilon)|),$$

as $\varepsilon \to 0$.

The first ingredient we need is a version of Lemma 4.2.12 for Neumann eigenfunctions.

Lemma 4.4.7. *Let* u_ε *be an* L^2*-normalized* $\mu_1(\Sigma \setminus B_\varepsilon)$*-eigenfunction. If we use Euclidean polar coordinates* (r, θ) *centered at* x *we have the uniform pointwise bounds*

$$|u_\varepsilon|(r,\theta) \leq C \log\left(\frac{1}{r}\right), \tag{4.4.8}$$

and

$$|\nabla u_\varepsilon|(r,\theta) \leq \frac{C}{r} \tag{4.4.9}$$

for any $\varepsilon \leq r \leq 1/2$.

Proof. Up to radius 2ε the estimate follows from the proof of Lemma 4.2.12. For the remaining radii, we use the same argument but apply elliptic boundary estimates [**Tay11a**, Chapter 5.7]. □

The second ingredient we need is a good bound on the L^2-norm of the tangential derivative of a Neumann eigenfunction along ∂B_ε.

Lemma 4.4.10. *Let* u_ε *be an* L^2*-normalized* $\mu_1(\Sigma \setminus B_\varepsilon)$*-eigenfunction. Then we have*

$$\int_{\partial B_\varepsilon} |\partial_T u_\varepsilon|^2 d\mathcal{H}^1 \leq C\varepsilon. \tag{4.4.11}$$

Proof. As above, we denote by $\tilde{u}_\varepsilon$ the function obtained by extending u_ε harmonically to B_ε, where u_ε denotes a normalized $\mu_1(\Sigma \setminus B_\varepsilon)$-eigenfunction. By a scaling argument, $\tilde{u}_\varepsilon$ is uniformly bounded in $W^{1,2}(\Sigma)$ in terms of the $W^{1,2}$-norm of u_ε [**RT75**, p. 40].

Let w_ε be the unique weak solution to

$$\begin{cases} \Delta w_\varepsilon = \mu_1(\Sigma \setminus B_\varepsilon)\tilde{u}_\varepsilon & \text{in } B_1 \\ w_\varepsilon = 0 & \text{on } \partial B_1. \end{cases}$$

By elliptic estimates, w_ε is bounded in $W^{3,2}(B_{1/2})$, which embeds into $C^{1,\alpha}(B_{1/2})$ for any $\alpha < 1$. We can then write

$$u_\varepsilon = w_\varepsilon + v_\varepsilon,$$

with $v_\varepsilon \in W^{1,2}(B_1 \setminus B_\varepsilon)$ a harmonic function.

Note that the bound (4.4.11) clearly holds for w_ε, so it suffices to consider v_ε. If we denote by ν the inward pointing normal of B_ε, we have

$$|\partial_\nu v_\varepsilon| = |\partial_\nu u_\varepsilon - \partial_\nu w_\varepsilon| = |\partial_\nu w_\varepsilon| \leq C \tag{4.4.12}$$

along ∂B_ε, since w_ε is bounded in $C^{1,\alpha}(B_{1/2})$. Since the Laplace operator is conformally covariant in dimension two, v_ε is also harmonic with respect to the Euclidean metric. Therefore, it follows from separation of variables, that we can expand v_ε in Fourier modes, where we suppress the index ε.

$$v = a + b\log(r) + \sum_{n\in\mathbb{Z}^*}(c_n r^n + d_n r^{-n})e^{in\theta}.$$

Using the L^2-normalization of u_ε and orthogonality, we can show that

$$\sum_{n>0}\frac{c_n^2}{2n+2} + \sum_{n<0}\frac{d_n^2}{2n+2} \leq C. \tag{4.4.13}$$

Indeed, we have that

$$\sum_{n>0}\int_\varepsilon^1 (c_n r^n + d_n r^{-n})^2 r dr \leq C,$$

and for $\varepsilon \leq 1/2$ we can use Young's inequality to find

$$\begin{aligned}
&\int_\varepsilon^1 (c_n r^n + d_n r^{-n})^2 r dr \\
&\quad = c_n^2 \int_\varepsilon^1 r^{2n+1} dr + 2c_n d_n \int_\varepsilon^1 r dr + d_n^2 \int_\varepsilon^1 r^{-2n+1} dr \\
&\quad = \frac{c_n^2}{2n+2}(1-\varepsilon^{2n+2}) + c_n d_n(1-\varepsilon^2) + \frac{d_n^2}{2n-2}(\varepsilon^{-2n+2}-1) \\
&\quad \geq \frac{c_n^2}{2n+2}(1-\varepsilon^{2n+2}-(n+1)\delta_n) + \frac{d_n^2}{2n-2}\left(\varepsilon^{-2n+2}-1-\frac{n-1}{\delta_n}\right) \\
&\quad \geq \frac{c_n^2}{8(n+1)} + \frac{d_n^2}{2n-2}(\varepsilon^{-2n+2}-2) \\
&\quad \geq \frac{c_n^2}{8(n+1)},
\end{aligned}$$

with $\delta_n = 1/(2(n+1))$. Of course, the same computation applies to negative n, so that we obtain the same kind of bound for the d_n's. From (4.4.13), we find that

$$h_1 = \sum_{n>0} c_n r^n e^{in\theta} + \sum_{n<0} d_n r^{-n} e^{in\theta}$$

extends to a harmonic function on all of B_1, which is bounded in L^2, whence in $C^\infty(B_{1/2})$. Therefore, we are left with bounding the tangential derivative of the harmonic function

$$h_2 = v - h_1 - a = b\log(r) + \sum_{n<0} c_n r^n e^{in\theta} + \sum_{n>0} d_n r^{-n} e^{in\theta}.$$

In a first step, we use that the quantity

$$\rho\int_{\partial B_\rho}\left((\partial_T h_2)^2 - (\partial_r h_2)^2\right) d\mathcal{H}^1$$

is independent of ρ, what can be verified by a straightforward computation. For $\rho \to \infty$ the term $\rho\int_{\partial B_\rho}(\partial_T h_2)^2 d\mathcal{H}^1$ vanishes, since the integrand decays at least like ρ^{-3}. For the other term, note that $\partial_r \log(r)$ and $\partial_r(h_2 - b\log(r))$ are orthogonal in $L^2(\partial B_\rho)$. Therefore, we have

$$\begin{aligned}\int_{\partial B_\rho} (\partial_r h_2)^2 d\mathcal{H}^1 &= b^2 \int_{\partial B_\rho} (\partial_r \log(r))^2 d\mathcal{H}^1 + \int_{\partial B_\rho} (\partial_r(h_2 - b\log(r)))^2 d\mathcal{H}^1 \\ &= \frac{2\pi b^2}{\rho} + O(\rho^{-3})\end{aligned}$$

since the integrand of the second summand decays at least like ρ^{-4} as $\rho \to \infty$. In conclusion,

$$\rho \int_{\partial B_\rho} ((\partial_T h_2)^2 - (\partial_r h_2)^2) d\mathcal{H}^1 = 2\pi b^2 \tag{4.4.14}$$

for any $\rho \geq \varepsilon$. In order to obtain a bound on b, we estimate the L^2-norm of the radial derivative of h_2 on ∂B_ε from below. Using orthogonality as above, we find that

$$\frac{2\pi b^2}{\varepsilon} = \int_{\partial B_\varepsilon} (b\,\partial_r \log(r))^2 d\mathcal{H}^1 \leq \int_{\partial B_\varepsilon} (\partial_r h_2)^2 d\mathcal{H}^1 \leq C\varepsilon, \tag{4.4.15}$$

where the last inequality makes use of the bound (4.4.12). Combining (4.4.14) and (4.4.15) yields

$$\int_{\partial B_\varepsilon} (\partial_T h_2)^2 d\mathcal{H}^1 = \int_{\partial B_\varepsilon} (\partial_r h_2)^2 d\mathcal{H}^1 + \frac{2\pi b^2}{\varepsilon} \leq C\varepsilon.$$ □

We will not invoke Lemma 4.4.10 directly, but rather use the following consequence.

Corollary 4.4.16. *We have*

$$\int_{B_\varepsilon} |\nabla \tilde{u}_\varepsilon|^2 \leq C\varepsilon^2. \tag{4.4.17}$$

Proof. Without loss of generality, we may assume that

$$\int_{\partial B_\varepsilon} u_\varepsilon d\mathcal{H}^1 = 0, \tag{4.4.18}$$

since subtracting a constant only results in subtracting a constant from $\tilde{u}_\varepsilon$ in B_ε. In particular, it does not change the energy of $\tilde{u}_\varepsilon$ in B_ε. We use $\hat{u}_\varepsilon(r,\theta) = \frac{r}{\varepsilon} u(\varepsilon, \theta)$ as a competitor. In order to estimate its energy, we use that (4.4.18), the Poincaré inequality, and Lemma 4.4.10 imply

$$\int_{\partial B_\varepsilon} |u_\varepsilon|^2 d\mathcal{H}^1 \leq C\varepsilon^2 \int_{\partial B_\varepsilon} (\partial_T u_\varepsilon)^2 d\mathcal{H}^1 \leq C\varepsilon^3. \tag{4.4.19}$$

Therefore, we get

$$\begin{aligned}\int_{B_\varepsilon} |\nabla \hat{u}_\varepsilon|^2 &\leq \frac{C}{\varepsilon^2} \int_0^{2\pi} \int_0^\varepsilon \left(|u_\varepsilon|^2(\varepsilon,\theta) + (\partial_\theta u_\varepsilon)^2(\varepsilon,\theta)\right) r dr d\theta \\ &\leq \frac{C}{\varepsilon} \int_{\partial B_\varepsilon} |u_\varepsilon|^2 d\mathcal{H}^1 + C\varepsilon \int_{\partial B_\varepsilon} (\partial_T u_\varepsilon)^2 d\mathcal{H}^1 \\ &\leq C\varepsilon^2,\end{aligned}$$

where we have used (4.4.19) and Lemma 4.4.10. □

Proof of Theorem 4.4.6. For convenience we normalize area$(\Sigma) = 1$. We use the function $\tilde{u}_\varepsilon$ from above as a test function for $\lambda_1(\Sigma)$. From the maximum principle and the bound (4.2.13), we find that

$$\left| \int_\Sigma \tilde{u}_\varepsilon \right| \le \int_{B_\varepsilon} |\tilde{u}_\varepsilon| \le C|\log(\varepsilon)|\varepsilon^2$$

From Corollary 4.4.16, we find

$$\int_\Sigma |\nabla \tilde{u}_\varepsilon|^2 = \mu_1(\Sigma \setminus B_\varepsilon) \int_{\Sigma \setminus B_\varepsilon} |\tilde{u}_\varepsilon|^2 + \int_{B_\varepsilon} |\nabla \tilde{u}_\varepsilon|^2 \le \mu_1(\Sigma \setminus B_\varepsilon) + C\varepsilon^2,$$

using the normalization $\int_{\Sigma \setminus B_\varepsilon} |\tilde{u}_\varepsilon|^2 = 1$. Therefore, we can estimate $\lambda_1(\Sigma)$ from above by

$$\lambda_1(\Sigma) \le \frac{\int_\Sigma |\nabla \tilde{u}_\varepsilon|^2}{\int_\Sigma |\tilde{u}_\varepsilon|^2 - \left(\int_\Sigma \tilde{u}_\varepsilon\right)^2} \le \frac{\mu_1(\Sigma \setminus B_\varepsilon) + C\varepsilon^2}{1 - C|\log(\varepsilon)|\varepsilon^2},$$

which implies

$$\mu_1(\Sigma \setminus B_\varepsilon) \ge \lambda_1(\Sigma) - C|\log(\varepsilon)|\varepsilon^2,$$

proving the assertion. □

CHAPTER 5

Extremal metrics for Laplace eigenvalues in perturbed conformal classes on products

5.1. Introduction

For a closed manifold M we are interested in the eigenvalues of the Laplace operator considered as functionals of the metric.

We denote by

$$\mathcal{R} := \{g \ : \ g \text{ is a Riemannian metric on } M \text{ with } \operatorname{vol}(M,g) = 1\}$$

the space of all unit volume Riemannian metrics on M endowed with the C^∞-topology, i.e. the smallest topology containing any C^k-topology. The group $C^\infty_+(M)$ of positive smooth functions acts via (normalized) pointwise multiplication on $\mathcal{R}$,

$$\phi.g := \operatorname{vol}(M,\phi g)^{-2/n}\phi g, \tag{5.1.1}$$

so that $\operatorname{vol}(M,\phi.g) = 1$. The quotient space

$$\mathcal{C} = C^\infty_+(M)\backslash\mathcal{R}$$

is the space of all conformal structures on M.

Since M is compact, the spectrum of Δ_g consists of eigenvalues of finite multiplicity only for any $g \in \mathcal{R}$. We list these as

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots, \tag{5.1.2}$$

where we repeat an eigenvalue as often as its multiplicity requires.

In recent years there has been much interest in finding extremal metrics for eigenvalues λ_k considered either as functionals

$$\lambda_k \colon \mathcal{R} \to \mathbb{R} \tag{5.1.3}$$

or

$$\lambda_k \colon [g] \to \mathbb{R}, \tag{5.1.4}$$

where

$$[g] = \{\phi g \ : \ \phi \in C^\infty_+(M)\}$$

denotes the conformal class of a metric g, see for instance [**ESGJ06, FS16, Kok14, Nad96, Pet14**], and references therein. These functionals will not be smooth but only Lipschitz, therefore extremality has to be defined in an appropriate way, see below.

One reason to study these extremal metrics is their intimate connection to other classical objects from differential geometry. For (5.1.3), these are minimal surfaces in spheres, and for (5.1.4) these are sphere-valued harmonic maps with constant density, so called *eigenmaps*. There has been a lot of effort in the past to understand, which manifolds admit eigenmaps or even minimal isometric immersions into spheres, see for instance [**Ura93**, Chap. 6] for a general overview over classical results for eigenmaps including the generalized Do Carmo–Wallach theorem, and [**Bry82, Law70**] to mention only the two most classical results.

Before we state our results, we have to introduce some notation. Let M be a smooth, closed manifold.

A smooth map $u\colon M \to S^\ell$ is called an *eigenmap*, if it is harmonic, i.e.

$$\Delta u = |\nabla u|^2 u, \tag{5.1.5}$$

and has constant density $|\nabla u|^2 = const.$ In other words, the components of u are all eigenfunctions corresponding to the same eigenvalue. Note that most Riemannian manifolds do not admit eigenmaps, since the spectrum is generically simple by [**Uhl76**, Theorem 8]. Even more, the spectrum of a generic metric in a conformal class is simple [**BW80, GLSD14, Uhl76**]. Moreover, we would like to point out that it is not clear at all whether eigenmaps exist in the presence of large multiplicty.

Theorem 5.1.6. *Let (M,g) be a closed Riemannian manifold of dimension $\dim(M) \geq 3$, and assume*

(i) There is a a non-constant eigenmap $u\colon (M,g) \to S^1$,
or
(ii) $(M,g) = (N \times S^\ell, g_N + g_{st.})$, where $g_{st.}$ denotes the round metric of curvature 1 on S^ℓ.

Then there is a neighbourhood U of $[g]$ in $\mathcal{C}$, such that for any $c \in U$, there is a representative $h \in c$, such that (M,h) admits a non-constant eigenmap to S^1 respectively S^ℓ.

An obvious question is then, whether the set of conformal structures admitting non-constant eigenmaps is always non-empty. We answer this at least in the following case.

Corollary 5.1.7. *Assume $\phi\colon M \to S^1$ is a submersion. Then the set $\mathcal{E} \subset \mathcal{C}$ of conformal structures admitting non-trivial eigenmaps to S^1 is open and non-empty.*

Remark 5.1.8. It is not clear, whether $\mathcal{E}$ is also closed. This question is related to possible degenerations of n-harmonic maps, as it will become clear from the proof.

Not every manifold admits a submersion to S^1. In fact, there are topological obstructions to the existence of such a map.

More precisely, since S^1 is a $K(\mathbb{Z},1)$, a submersion gives rise to a non-trivial element in $H^1(M,\mathbb{Z})$. Moreover, the differentials of local lifts of the submersion to $\mathbb{R}$, give rise to a globally defined nowhere vanishing 1-form. In particular, M needs to have $\chi(M) = 0$.

As mentioned above, the existence of an eigenmap $u\colon (M,h) \to S^\ell$ for a metric $h \in [g]$ implies that h is extremal for some of the functionals λ_k on $[g]$. Therefore, Theorem 5.1.6 and Corollary 5.1.7 have the following consequences for the existence of extremal metrics.

Corollary 5.1.9. *Under the assumptions of Theorem 5.1.6, there is a neighbourhood U of $[g]$ in $\mathcal{C}$, such that for any $c \in U$, there is a representative $h \in c$, such that (M,h) is extremal for some eigenvalue functional on c.*

Corollary 5.1.10. *Under the assumptions of Corollary 5.1.7, the set $\mathcal{E} \subset \mathcal{C}$ of conformal structures admitting extremal metrics for some eigenvalue functional on conformal classes is open and non-empty.*

The proof of Theorem 5.1.6 is rather simple once the correct conformally invariant formulation of the assertion is found.

This is as follows. Let n be the dimension of M. Then a smooth map into a sphere is called *n-harmonic*, if it is a critical point of the *n-energy*

$$E_n[u] = \int_M |du|^n dV_g,$$

which is a conformally invariant functional. These are precisely the solutions of the equation

$$-\operatorname{div}(|\nabla u|^{n-2}\nabla u) = |\nabla u|^n u. \tag{5.1.11}$$

From (5.1.5) and (5.1.11) it is evident, that an eigenmap defines an n-harmonic map, which has $\nabla u \neq 0$ everywhere. The crucial observation is that also the converse holds up to changing the metric conformally, see Lemma 5.3.21.

Therefore, we will be concerned with n-harmonic maps with nowhere vanishing derivative.

In order to deduce Corollary 5.1.7 from Theorem 5.1.6, it suffices to find a single non-trivial eigenmap $u\colon (M,g) \to S^1$ for some metric g. This turns out to be very easy using that M is a mapping torus.

In Section 5.2 we discuss the necessary preliminaries on n-harmonic maps and Laplace eigenvalues. Section 5.3 contains the proofs.

5.2. Preliminaries

First, we explain the notion of extremal metrics and its connection to eigenmaps.

5.2.1. Extremal metrics for eigenvalue functionals. In presence of multiplicity, the functionals λ_k are not differentiable, but only Lipschitz. However, it turns out that for any analytic deformation, left and right derivatives exist. Using this El Soufi–Ilias introduced a notion of extremal metrics for these functionals.

Definition 5.2.1 ([**ESI08**, Definition 4.1])**.** A metric g is called *extremal* for the functional λ_k restricted to the conformal class $[g]$ of g, if for any analytic family of metrics $(g_t) \subset [g]$, with $g_0 = g$, and $\mathrm{vol}(M,g_0) = \mathrm{vol}(M,g_t)$, we have

$$\left.\frac{d}{dt}\right|_{t=0^-} \lambda_k(g_t) \cdot \left.\frac{d}{dt}\right|_{t=0+} \lambda_k(g_t) \leq 0.$$

We have

Theorem 5.2.2 ([**ESI08**, Theorem 4.1])**.** *The metric g is extremal for some eigenvalue λ_k on $[g]$ if and only if there is a eigenmap $u\colon (M,g) \to S^\ell$ given by $\lambda_k(g)$-eigenfunctions and either $\lambda_{k-1}(g) < \lambda_k(g)$, or $\lambda_k(g) < \lambda_{k+1}(g)$.*

5.2.2. Background on n-harmonic maps. First of all we need some background on the existence of n-harmonic maps. We call a map $u \in W^{1,n}(M, S^\ell)$ *weakly n-harmonic,* if it is a weak solution of

$$-\operatorname{div}(|\nabla u|^{n-2}\nabla u) = |\nabla u|^n u. \tag{5.2.3}$$

We assume that we have fixed a CW-structure on M, and denote by $M^{(l)}$ its l-skeleton. Let $v\colon M \to S^\ell$ be a Lipschitz map, where $l < n = \dim M$. Denote by $v^{(l)}$ the restriction of v to the l-skeleton of M. The *l-homotopy type of v* is the homotopy type of $v^{(l)}$.

Theorem 5.2.4 ([**Whi88**, Theorem 3.4])**.** *Let v be as above. Then there exists a weakly n-harmonic map $u\colon M \to S^\ell$, with well-defined l-homotopy type, which agrees with the l-homotopy type of v. Moreover, u minimizes the n-energy among all such maps.*

We do not elaborate here on how the l-homotopy type is defined for maps in $W^{1,n}(M, S^\ell)$. For our purposes this is not necessary, since the map u is actually continuous.

Theorem 5.2.5. *Let $u \in W^{1,n}(M, S^\ell)$ be a weakly n-harmonic map, which is a minimizer for its own l-homotopy type. There is a constant C depending on an upper bound on the n-energy of u, and on the bounds of the sectional curvature and injectivity radius of M, such that $\|u\|_{C^{1,\alpha}} \leq C$.*

Proof. Let $x \in M$ and $r > 0$ be small enough. If $v \in W^{1,n}(B(x,r), S^\ell)$ with $u = v$ on $\partial B(x,r)$, we can consider the map $w \in W^{1,n}(M, S^\ell)$ given by u in $M \setminus B(x,r)$ and by v in $B(x,r)$. It is shown in [**PV15**, Theorem 2.8] that the l-homotopy type of w agrees with the l-homotopy type of u. In particular, we need to have

$$\int_{B(x,r)} |du|^n dV_g \leq \int_{B(x,r)} |dv|^n dV_g,$$

which means that u is a *minimizing* n-harmonic map. Therefore, the assertion is follows e.g. from [**NVV14**, Theorem 2.19]. □

In particular, these estimates are uniform as g varies over a compact set of $\mathcal{R}$, as long as the energy stays bounded.

At points, in which we do not have a lack of ellipticity, we actually get higher regularity.

Theorem 5.2.6. *A weakly n-harmonic map $u \in C^{1,\alpha}$ is smooth near points with $\nabla u \neq 0$.*

This follows from standard techniques for quasilinear elliptic equations. For completeness, we give a proof in Section 5.3.1.

The main reason for the restrictive assumptions in item (ii) of Theorem 5.1.6 is that the above results do not imply that for a sequence $g_k \to g$ we can find a sequence of n-harmonic maps u_k (w.r.t. g_k), such that $u_k \to u$, for a given n-harmonic map u.

In the case of maps to the circle, this problem does not appear, thanks to

Theorem 5.2.7 ([**Ver12**, Theorem A])**.** *Up to rotations of S^1, we have that n-harmonic maps $u\colon M \to S^1$ are unique in their homotopy class.*

5.3. Proofs

5.3.1. Higher regularity of n-harmonic maps. In this section we give a proof of Theorem 5.2.5. We start with $W^{2,2}$-regularity. The proof follows using standard techniques, since under our assumptions the equation is of the form

$$-(Lu)(x) - b(x)u(x) = 0, \tag{5.3.1}$$

with L a quasilinear operator, which is elliptic at u (as demonstrated in Lemma 5.3.14 below) and $b \in L^\infty$.

Lemma 5.3.2. *Let $U \subset M$ be open and $u\colon (U,g) \to S^\ell$ be weakly n-harmonic. Assume that $u \in C^{1,\alpha}(U, S^\ell)$ with $\nabla u \neq 0$ everywhere in u. Then we have $u \in W^{2,2}_{loc}(U, S^\ell)$.*

Proof. For simplicity, we focus on the case $g_{ij} = \delta_{ij}$ and denote the usual differential of u in Euclidean Space by Du. The general case follows along the same lines but with some more notation.

Take open subsets $W \subset\subset V \subset\subset U$, and a cut-off function η which is 1 in W, and has $\operatorname{supp}\eta \subset V$. We show that $u \in W^{2,2}(W, S^\ell)$. We use the test functions given by $\phi^k = -D_s^{-h}(\eta^2 D_s^h u^k)$, where D_s^h denotes the difference quotient operator in coordinate direction s,

$$D_s^h\phi(x) = \frac{1}{h}(\phi(x + he_s) - \phi(x)). \tag{5.3.3}$$

To handle notation, let us write

$$F_k^\alpha(Du) = |Du|^{n-2}\partial_\alpha u^k, \tag{5.3.4}$$

and

$$G_k(u, Du) = |Du|^n u^k. \tag{5.3.5}$$

Then we have

$$-\int_U F_k^\alpha(Du)\partial_\alpha D_s^{-h}(\eta^2 D_s^h u^k) = -\int_U G_k(u, Du) D_s^{-h}(\eta^2 D_s^h u^k), \tag{5.3.6}$$

Note that this is well-defined thanks to Hölder's inequality. For the left hand side of (5.3.6), we have

$$-\int_U F_k^\alpha(Du)\partial_\alpha D_s^{-h}(\eta^2 D_s^h u^k) = \int_U D_s^h(F_k^\alpha(Du))\partial_\alpha(\eta^2 D_s^h u^k). \tag{5.3.7}$$

We can write

$$\begin{aligned} D_s^h F_k^\alpha(Du) &= \frac{1}{h}\int_0^1 \frac{d}{dt} F_k^\alpha(Du + thD_s^h Du)dt \\ &= \frac{1}{h}\int_0^1 \frac{\partial F_k^\alpha}{\partial q_\beta^l}(Du + thD_s^h Du) h D_s^h \partial_\beta u^l dt \\ &= \int_0^1 \frac{\partial F_k^\alpha}{\partial q_\beta^l}(Du + thD_s^h Du)dt\, D_s^h \partial_\beta u^l dt \\ &=: \theta_{kl}^{\alpha\beta}(Du) D_s^h \partial_\beta u^l. \end{aligned} \tag{5.3.8}$$

Note that this is well defined pointwise, since $u \in C^{1,\alpha}$. The condition $|Du| \geq c > 0$, implies that $\theta_{kl}^{\alpha\beta}$ are uniformly super strongly elliptic for $h \ll 1$, as demonstrated below. Since the coefficients θ are uniformly super strongly elliptic, we have

$$\int_U \eta^2 |D_s^h Du|^2 \leq C \int_U \eta^2 \theta_{kl}^{\alpha\beta}(Du)(D_s^h \partial_\alpha u^k)(D_s^h \partial_\beta u^l). \tag{5.3.9}$$

Moreover, since θ and $|D\eta|$ are bounded, we can estimate

$$\begin{aligned} \left| \int_U \theta_{kl}^{\alpha\beta}(D_s^h \partial_\beta u^l)(D_s^h u^k)\eta \partial_\alpha \eta \right| &\leq C \int_U |D_s^h Du||D_s^h u|\eta \\ &\leq C\varepsilon \int_U \eta^2 |D_s^h Du|^2 + \frac{C}{\varepsilon} \int_V |D_s^h u|^2 \\ &\leq C\varepsilon \int_U \eta^2 |D_s^h Du|^2 + \frac{C}{\varepsilon} \int_U |Du|^2, \end{aligned} \tag{5.3.10}$$

where we have used Young's inequality and $u \in W^{1,2}$. Combining the last two estimates with (5.3.6) and (5.3.7), we find that we can choose ε sufficiently small so that

$$\int_U \eta^2 |D_s^h Du|^2 \leq C \int_U |Du|^2 + C \left| \int_U G_k(u, Du) D_s^{-h}(\eta^2 D_s^h u^k) \right|. \tag{5.3.11}$$

To estimate the last summand above, we note that $u, |Du| \in L^\infty$, implies $G_k(u, Du) \in L^\infty$, hence

$$\begin{aligned} \left| \int_U G_k(u, Du) D_s^{-h}(\eta^2 D_s^h u^k) \right| &\leq C \int_U |D_s^{-h}(\eta^2 D_s^h u^k)| \\ &\leq \frac{C}{\varepsilon} \operatorname{vol}(U) + C\varepsilon \int_U |D_s^{-h}(\eta^2 D_s^h u^k)|^2 \\ &\leq \frac{C}{\varepsilon} + C\varepsilon \int_U |D(\eta^2 D_s^h u^k)|^2 \\ &\leq \frac{C}{\varepsilon}(1 + \int_U |Du|^2) + C\varepsilon \int_U \eta^2 |D_s^h Du^k|^2. \end{aligned} \tag{5.3.12}$$

For ε sufficiently small, we can absorb the last term, and find

$$\int_V |D_s^h Du|^2 \leq \int_U \eta^2 |D_s^h Du|^2 \leq \frac{C}{\varepsilon}(1 + \int_U |Du|^2). \tag{5.3.13}$$

Thus $u \in W_{loc}^{2,2}(U, S^\ell)$. $\square$

We still need to justify that the coefficients $\theta_{kl}^{\alpha\beta}$ are uniformly super strongly elliptic.

Lemma 5.3.14. *There is $h_0 > 0$ depending on $\|Du\|_{C^{0,\alpha}}$ such that we have $\theta_{kl}^{\alpha\beta} A_\alpha^k A_\beta^l \geq \nu |A|^2$, for any h with $|h| \leq h_0$ and $\nu = \nu(c)$, where $|Du|^2 \geq c$.*

Proof. We have

$$\frac{\partial F_k^\alpha}{\partial q_\beta^l}(q) = |q|^{n-4}(|q|^2 \delta^{\alpha\beta} \delta_{kl} + (n-2) q_\alpha^k q_\beta^l). \tag{5.3.15}$$

Thus, it is not very hard to see that

$$
\begin{aligned}
\frac{\partial F_k^\alpha}{\partial q_\beta^l}(q) A_\alpha^k A_\beta^l &= |q|^{n-4}(|q|^2\delta^{\alpha\beta}\delta_{kl}A_\alpha^k A_\beta^l + (n-2)q_\alpha^k q_\beta^l A_\alpha^k A_\beta^l) \\
&\geq |q|^{n-2}|A|^2 \\
&\geq 2\nu|A|^2,
\end{aligned} \tag{5.3.16}
$$

as long as $|q|^2 \geq (2\nu)^{2/(n-2)}$. Since $Du \in C^{0,\alpha}$, and $|Du|^2 \geq c$, we can choose $h_0 \ll 1$, such that $|(1-t)Du(x+he_s) + tDu(x)|^2 \geq c/2$, for all x, and $|h| \leq h_0$. Clearly, this implies

$$
\begin{aligned}
\theta_{kl}^{\alpha\beta}(Du)(x)A_\alpha^k A_\beta^l &= \int_0^1 \frac{\partial F_k^\alpha}{\partial q_\beta^l}((1-t)Du(x+he_s) + tDu(x))A_\alpha^k A_\beta^l dt \\
&\geq \int_0^1 \nu|A|^2 dt \\
&\geq \nu|A|^2,
\end{aligned}
$$

for $\nu = c^{(n-2)/2}/2$. □

In the next step we derive the equation for $\partial_\alpha u^k$ and apply Schauder estimates to gain higher regularity. In particular, this completes the proof of Theorem 5.2.6

Lemma 5.3.17. *Under the above assumptions, the function u is smooth.*

PROOF. Write

$$
\vartheta_{kl}^{\alpha\beta} = \frac{\partial F_k^\alpha}{\partial q_\beta^l}. \tag{5.3.18}
$$

By the calculation above, these coefficients are uniformly super strongly elliptic at u. We test the equation for u^k with $\partial_\alpha \phi^k$ for some test function ϕ and integrate by parts in order to find

$$
\int_U \vartheta_{kl}^{\alpha\beta}(\nabla u)\partial_{\beta\gamma}u^l \partial_\alpha \phi^k = \int_U \partial_\gamma G_k(u,\nabla u)\phi^k \tag{5.3.19}
$$

In other words, $v = \partial_\gamma u$ is a weak solution to

$$
-\operatorname{div}(\vartheta(Du)v) = \partial_\gamma G(u, Du). \tag{5.3.20}
$$

Since $|Du|^2 \geq c > 0$, the right hand side of this equation is in $C^{k,\alpha}$, once we have $u \in C^{k+1,\alpha}$. In this case the left hand side has coefficients in $C^{k,\alpha}$, thus it follows that $v \in C^{k+1,\alpha}$ and thus $u \in C^{k+2,\alpha}$. Since we know $u \in C^{1,\alpha}$, we can start this bootstrap argument at $k = 0$, and get $u \in C^\infty$. □

5.3.2. Proofs of main results.

We start with the following simple but crucial observation.

Lemma 5.3.21. *Let $u\colon (M,g) \to S^\ell$ be a smooth n-harmonic map with $du \neq 0$ everyhwere. Then there is metric g' conformal to g, such that $u\colon (M,g') \to S^\ell$ is an eigenmap.*

Proof. Define $g' = |du|_g^2 g$. Since we assumed $du \neq 0$ everywhere, this defines a smooth metric, which is conformal to g. Then $|du|_{g'}^2 = |du_g|^{-2}|du|_g^2 = 1$. Finally, u solves

$$-\operatorname{div}_g(|du|_g^{n-2}\nabla u) = |du|_g^n u,$$

which can also be written as

$$\Delta_{g'} u = -\frac{1}{|du|_g^n}\operatorname{div}_g(|du|_g^{n-2}\nabla u) = u,$$

hence $u\colon (M, g') \to S^\ell$ is an eigenmap. □

In order to prove Theorem 5.1.6 it now suffices to show that metrics close to the initial metric g on M also admit smooth n-harmonic maps with nowhere vanishing derivative.

Proof of Theorem 5.1.6 (i). Let $u\colon (M, g) \to S^1$ be an eigenmap and assume that the assertion of the theorem was not correct. This means that any neighbourhood $U \subset \mathcal{C}$ of $[g]$ contains a conformal class which does not contain any representative which admits an eigenmap to S^1. Let $U_k \subset \mathcal{R}$ be a sequence of open neighbourhoods of g with $\cap_{k\in\mathbb{N}} U_k = \{g\}$. (Such a sequence exists since the C^∞-topology on $\mathcal{R}$ is first countable and Hausdorff.) Denote by $\pi\colon \mathcal{R} \to \mathcal{C}$ the quotient map and observe that this is an open map. In particular, the sets $\pi(U_k) \subset \mathcal{C}$ are open and we can find $g_k \in U_k$ such that no metric in $[g_k]$ admits an eigenmap to S^1. By Lemma 5.3.21 this implies that g_k itself cannot admit a nowhere vanishing n-harmonic map to S^1.

We now plan to use Theorem 5.2.4 to obtain weakly n-harmonic maps $u_k\colon (M, g_k) \to S^1$ which are close to u for k sufficiently large. By assumption the u_k have some point x_k with $du_k(x_k) = 0$. This forces u to have a critical point as well, which gives the desired contradiction.

More precisely, we apply Theorem 5.2.4 to $u\colon (M, g_k) \to S^1$ and obtain n-harmonic representatives $u_k\colon (M, g_k) \to S^1$ of $[u]$. If $du_k \neq 0$ everywhere, Theorem 5.2.6 implies that u_k is a smooth n-harmonic map from (M, g_k) to S^1 with nowhere vanishing derivative contradicting the construction of g_k in the preceding paragraph. Therefore, we can find $x_k \in M$ such that $du_k(x_k) = 0$. Since $\dim(M) \geq 3$ and $S^1 \simeq K(\mathbb{Z}, 1)$, we have that $w \simeq u$ if and only if their l-homotopy type agrees for some $l \geq 2$. In particular, we have that

$$\int_M |du_k|^n dV_{g_k} \leq \int_M |du|^n dV_{g_k} \leq C \int_M |du|^n dV_g,$$

so that we are in the position to apply Theorem 5.2.5.

By taking a subsequence if necessary, we may assume that $x_k \to x$. Thanks to Theorem 5.2.5 and the compact embedding $C^{1,\alpha}(M) \hookrightarrow C^{1,\beta}(M)$ for $\beta < \alpha$, we can extract a further subsequence, such that $u_k \to v$ in $C^{1,\beta}(M,g)$. We have

$$\begin{aligned}
\int_M |dv|_g dV_g &= \lim_{k\to\infty} \int_M |dv|_{g_k} dV_{g_k} \\
&\leq \lim_{k\to\infty} \left(\int_M |du_k|_{g_k} dV_{g_k} + \int_M \left||dv|_{g_k} - |du_k|_{g_k}\right| dV_{g_k} \right) \\
&\leq \lim_{k\to\infty} \left(\int_M |dw|_{g_k} dV_{g_k} + C d_{C^{1,\beta}(M,g_k)}(v,u_k) \right) \\
&\leq \lim_{k\to\infty} \int_M |dw|_{g_k} dV_{g_k} + \lim_{k\to\infty} C d_{C^{1,\beta}(M,g)}(v,u_k) \\
&= \int_M |dw|_g dV_g,
\end{aligned}$$

for any $w \simeq u$. It follows, that v is n-harmonic and homotopic to u. Thus it follows from Theorem 5.2.7 that there is $A \in SO(2)$, such that $A \circ v = u$. Then $A \circ u_k \to u$ in $C^{1,\beta}(M)$. It follows, that

$$|du(x)| \leq \limsup_{k\to\infty} C d(x,x_k)^\beta = 0,$$

contradicting the assumption on u. $\square$

In order to adapt the strategy from above for more general situations, we need to understand whether there exist eigenmaps $u\colon (M,g) \to S^\ell$, which can be approximated through n-harmonic maps for any sequence of metrics $g_k \to g$.

This is precisely what we do now for product metrics $g_{st.} + g_N$ on $S^\ell \times N$. The natural candidate here is the projection map onto S^ℓ. In what follows n will denote the dimension of N, so that the dimension of $N \times S^\ell$ is $n + l$.

Proposition 5.3.22. *Let $g = g_N + g_{st.}$ be a product metric on $N \times S^\ell$, with $g_{st.}$ the round metric of curvature 1 on S^ℓ. The projection $u\colon N \times S^\ell \to S^\ell$ onto the second factor is the unique minimizer for the $(n+l)$-energy in its l-homotopy class up to rotations of S^ℓ.*

Proof. Let $v\colon N \times S^\ell \to S^\ell$ be a Lipschitz map whose restriction to the l-skeleton of $N \times S^\ell$ is homotopic to the restriction of the projection $N \times S^\ell \to S^\ell$ to the ℓ-skeleton. We want to estimate

$$\int_{N\times S^\ell} |dv|_g^{n+\ell} dV_g \tag{5.3.23}$$

from below.

We have

$$\int_{N\times S^\ell}|dv|_g^{n+\ell}dV_g$$

$$\begin{aligned}
&= \int_N\int_{S^\ell}(|\nabla^N v|^2+|\nabla^{S^\ell}v|^2)^{(n+\ell)/2}(x,\theta)d\theta dx \\
&\geq \int_N\int_{S^\ell}|\nabla^{S^\ell}v|^{n+\ell}(x,\theta)d\theta dx \\
&\geq (\ell+1)\omega_{\ell+1})^{-n/\ell}\int_N\left(\int_{S^\ell}|\nabla^{S^\ell}v|^\ell(x,\theta)d\theta\right)^{(n+\ell)/\ell}dx,
\end{aligned} \tag{5.3.24}$$

where we have used Hölder's inequality in the last step. Equality holds in the above inequalities if and only if $|\nabla^M v|^2=0$ and $|\nabla^{S^\ell}v|^2=const.$

In order to estimate the remaining integral in the last line of (5.3.24) we use that the maps $v(x,\cdot)\colon S^\ell\to S^\ell$ have degree 1. This can be seen by inspecting the l-homotopy type of v: If we endow S^ℓ with the CW-structure consisting of a single 0- and a single ℓ-cell, we have $(N\times S^\ell)^{(\ell)}=N^{(\ell)}\times\{\theta_0\}\cup\{x_0\}\times S^\ell=N^{(\ell)}\vee S^\ell$ with $\theta_0\in S^\ell$ and $x_0\in N$ corresponding to the 0-cells. The projection onto S^ℓ restricts to the map $N^{(\ell)}\vee S^\ell\to S^\ell$ that collapses the first summand and is the identity on S^ℓ. In particular, we find that for any v, such that $v^{(\ell)}$ is homotopic to the map described above, the degree of $v(x_0,\cdot)\colon S^\ell\to S^\ell$ equals 1. Since N is connected, $v(x,\cdot)\simeq v(x_0,\cdot)$ for any x, thus $\deg v(x,\cdot)=1$ for any $x\in N$.

This implies, that

$$\int_{S^\ell}|\nabla^{S^\ell}v|^l(x,\theta)d\theta\geq(\ell+1)\omega_{\ell+1}\ell^{\ell/2}|\deg v(x,\cdot)|=(\ell+1)\omega_{\ell+1}l^{\ell/2}. \tag{5.3.25}$$

Here, equality holds if and only if $v(x,\cdot)$ is conformal. Combining (5.3.24) and (5.3.25), we find

$$\int_{N\times S^\ell}|dv|_g^{n+\ell}dV_g\geq \operatorname{vol}(N)(\ell+1)\omega_{\ell+1}l^{(n+\ell)/2}, \tag{5.3.26}$$

with equality if and only if $|\nabla^M v|^2=0$, and $|\nabla^{S^\ell}v|^2=const.$, and $v(x,\cdot)$ is conformal. It follows in this case that $v(x,\theta)=\tilde v(\theta)$ with $\tilde v\colon S^\ell\to S^\ell$ of degree 1. Observe, that $u\colon M\times S^\ell\to S^\ell$ realizes the equality in (5.3.26). Therefore,

$$\inf_v\int_{N\times S^\ell}|dv|^{n+\ell}dV_g=\operatorname{vol}(N)(\ell+1)\omega_{\ell+1}l^{(n+\ell)/2}, \tag{5.3.27}$$

where the infimum is taken over all Lipschitz maps v having the l-homotopy of u. In particular, by the equality discussion above, minimizers need to be $(n+l)$-harmonic maps $v(x,\theta)=\tilde v(\theta)$, with $|\nabla v|^2=const.$ Therefore, $\tilde v$ defines a harmonic selfmap of S^ℓ with constant density. Since $\tilde v$ is non-trivial, it follows that $|\nabla\tilde v|^2\geq\lambda_1(S^\ell)=l$. Consequently, equality in (5.3.26) is only achieved by maps of the form $A\circ u$, with $A\in O(l+1)$. □

Using Proposition 5.3.22 instead of Theorem 5.2.7, assertion (*ii*) of Theorem 5.1.6 follows along the same lines as assertion (*i*.)

Proof of Corollary 5.1.7. Let $f\colon M\to S^1$ be a submersion. Since M is compact this is a proper submersion. Moreover, f has to be surjective, since otherwise M

would be contractible. It follows by Ehresmann's lemma that $f\colon M \to S^1$ is a fibre bundle, $F \to M \to S^1$, with F a smooth $(n-1)$-dimensional manifold. As a consequence there is a diffeomorphism $\phi\colon F \to F$, such that M is obtained as the mapping torus corresponding to ϕ, i.e.

$$M \cong (F \times [0,1])\,/(x,0) \sim (\phi(x),1).$$

Choose a metric g_0 on F, which is invariant under ϕ. We claim that the metric $g_1 = g_0 + dt^2$ defined on $F \times [0,1]$ descends to a smooth metric g on M. Clearly, g_1 descends to a metric g on M, we only need to check that it is smooth. This is clear near all points (x,t) with $t \neq 0,1$. We have coordinates with values in $F \times (-\varepsilon,\varepsilon)$ near the $t=0$-slice as follows.

$$(x,t) \mapsto \begin{cases} (x,t-1) & \text{if } \ t \leq 1 \\ (\phi(x),t) & \text{if } \ t > 0. \end{cases} \tag{5.3.28}$$

In these coordinates g is given by $g_0 + dt^2$, since g_0 is ϕ-invariant.

It remains to show that (M,g) admits an eigenmap. Define $u\colon F \times [0,1]/(x,0) \sim (f(x),1) \to S^1$ by $(x,t) \mapsto t$. With respect to g this is a Riemannian submersion. Moreover, it follows from (5.3.28) that u has totally geodesic fibres. Thus u is an eigenmap. □

CHAPTER 6

Regularity of extremal metrics for Laplace eigenvalues in a conformal class

6.1. Introduction

For a closed Riemannian manifold (M, g), the Laplace operator $\Delta = \Delta_g$ acting on functions has discrete spectrum, which we denote by

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots. \tag{6.1.1}$$

In recent years there has been much interest in finding extremal metrics for eigenvalues λ_k considered either as functionals

$$\lambda_k \colon \mathcal{R} \to \mathbb{R} \tag{6.1.2}$$

or

$$\lambda_k \colon [g] \to \mathbb{R}, \tag{6.1.3}$$

see for instance [**ESGJ06, FS16, Kok14, Nad96, Pet14**], and references therein. Here we denote by $\mathcal{R}$ the space of all unit volume metrics, and $\mathcal{C} = C^\infty_+ \setminus \mathcal{R}$ the space of all conformal structures. These functionals will not be smooth but only Lipschitz, therefore extremality has to be defined in appropriate way, see below.

One reason to study these extremal metrics is their intimate connection to other classical objects from differential geometry. For (6.1.2) these are minimal surfaces in spheres, and for (6.1.3) these are sphere-valued harmonic maps with constant density. There has been a lot of effort in the past to understand, which manifolds admit eigenmaps or even minimal isometric immersions into spheres, see for instance [**Ura93**, Chap. 6] for a general overview over classical results for eigenmaps including the generalized Do Carmo–Wallach theorem, and [**Bry82, Law70**] to mention only the two most classical results.

We prove two regularity results for metrics that are extremal for eigenvalue functionals in a conformal class.

We need to make the following technical assumption, which among others guarentees the existence of eigenavalues and eigenfunctions. Given a non-negative function $\phi \in L^{n/2}(M)$, we will always assume that the embedding

$$W^{1,2}(M, \phi g) \hookrightarrow L^2(M, \phi g) \tag{6.1.4}$$

is compact.

Note that $0 < c \leq \phi \leq C$ almost everywhere implies that (6.1.4) is compact.

The first result is concerned with higher regularity.

Theorem 6.1.5. *Let ϕg be extremal for one of the functionals λ_k on $[g]$. Assume that the embedding* (6.1.4) *is compact and that $\lambda_k(\phi g) > 0$. If $\phi \in C^{0,\alpha}$, then ϕ is smooth in $\{\phi > 0\}$.*

Given this it is natural to ask how large the set at which ϕ is not Hölder continuous can be.

Theorem 6.1.6. *Let ϕg be extremal for λ_k on $[g]$. Assume that the embedding* (6.1.4) *is compact and that $\lambda_k(\phi g) > 0$. Then ϕ is Hölder continuous in the interior of its support away from at most $k-1$ points. If there is a constant $C > 0$, such that $1/C \leq \phi \leq C$ almost everywhere, ϕ is Hölder continuous everywhere.*

Remarks 6.1.7. 1.) In contrast to the two dimensional situation, the assumption $\lambda_k > 0$ can not be removed as the following example shows. Let A, B be two closed sets that admit disjoint open neighborhoods U_A, U_B, respectively. For any two L^∞-functions ϕ_A, ϕ_B that satisfy $\int_A \phi_A^{n/2} + \int_B \phi_B^{n/2} = 1$, and $\phi = \phi_A, \phi_B \geq c > 0$ almost everywhere, consider the metric ϕg. Then the embedding (6.1.4) is compact and the first eigenvalue vanishes. In particular the functional λ_1 attains a global minimum at ϕg.
2.) If the support of ϕ is not too disconnected, we get that $\lambda_k > 0$, see Lemma 6.2.6

These results have an analogue in two dimensions, which can be found in the most general form in [**Kok14**]. Compared to the two dimensional case we have a possibly larger singular set. This is related to the fact that Theorem 6.1.5 is connected to the regularity of n-harmonic maps rather than to harmonic maps as in two dimensions.

In Section 6.2 we discuss the necessary preliminaries on n-harmonic maps and Laplace eigenvalues. Section 6.3 contains the proofs. We start with the calculation of the variational formula for the eigenvalue under mild regularity assumptions. We then explain the connection to (minimizing) n-harmonic maps and finally prove the regularity of these.

6.2. Preliminaries and Setting

In this section we give explain the setting we will work in. We also give some background on extremal metrics and n-harmonic maps, that was mostly already covered in f Section 5.2 from Chapter 5. Some of it comes in a different formulation here, that is more convenient for the purpose of this chapter.

6.2.1. Setting. We will work on closed manifold M of dimenson n. We fix a smooth unit volume metric g on M and consider its unit volume conformal class

$$[g] := \left\{ \phi g \mid \phi \in L^{n/2}(M),\ \phi \geq 0 \text{ a.e.}, \int_M \phi^{n/2} dV_g = 1 \right\}. \tag{6.2.1}$$

Eigenvalues of the Laplace operator associated to a singular metric ϕg can be defined to be those λ for which a non-trivial weak solution to

$$\Delta_{\phi g} u = \lambda u \tag{6.2.2}$$

exists. Note that the weak formulation does not require to take any derivatives of ϕ. If we assume that the embedding

$$W^{1,2}(M, \phi g) \hookrightarrow L^2(M, \phi g) \tag{6.2.3}$$

is compact, eigenvalues can also be characterized as min-max values as usual. For instance, the k-th eigenvalue can be charecterized by

$$\lambda_k = \inf_{\{u \in W^{1,2}(M,\phi g)\ :\ u \perp E_{k-1} 0\}} \frac{\int |du|_g^2 \phi^{n/2-1} dV_g}{\int |u|^2 \phi^{n/2} dV_g}, \tag{6.2.4}$$

where $E_{k-1} \subset L^2(M, \phi g)$ is the subspace spanned by the first $k-1$ eigenfunctions. Using this charcterization the standard arguments relying on the direct method give the existence of a sequence of eigenvalues

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty. \tag{6.2.5}$$

Observe that the eigenvalue 0 can have multiplicity larger than one, e.g. if the support of ϕ is not connected. Moreover, any eigenvalue has finite multiplicity.

Note the following elementary observation, related to the assumption of the positivity of λ_k in Theorem 6.1.5 and Theorem 6.1.6.

Lemma 6.2.6. *Assume that ϕ is continuous. If the support of ϕ has at most k components, then $\lambda_k(M, \phi g) > 0$.*

PROOF. If $\lambda_k = 0$, the eigenvalue equations implies

$$\int_M |\nabla u|_g^2 \phi^{n/2-1} dV_g = 0, \tag{6.2.7}$$

for any λ_l-eigenfunction with $l \leq k$. Thus, $|\nabla u|_g^2 \phi^{n/2-1} = 0$ almost everywhere, which implies that u is constant on each component of $\operatorname{supp} \phi$. But the dimension of the space of locally constant functions is exactly the number of components of $\operatorname{supp} \phi$. Therefore, $\operatorname{supp} \phi$ has at least $k+1$ components. □

6.2.2. Extremal metrics for eigenvalue functionals. We will use the following definition of extremal metrics which is equivalent to that in Section 5.2 from Chapter 5

Definition 6.2.8 ([**ESI08**])**.** A metric $g_0 = \phi g$ is called *extremal* for the functional λ_k restricted to the conformal class $[g]$, if for any smooth family of metrics $g_t = \phi_t g_0$, with $\phi_0 = g$, and $\operatorname{vol}(M, g_0) = \operatorname{vol}(M, g)$, we have

$$\lambda_k(g_t) \leq \lambda_k(g_0) + o(t) \tag{6.2.9}$$

or

$$\lambda_k(g_t) \geq \lambda_k(g_0) - o(t), \tag{6.2.10}$$

as $t \to 0$.

Recall that a (smooth) map $u\colon (M, g) \to S^l$ is called *harmonic*, if

$$\Delta u = |\nabla u|^2 u.$$

If the energy density $|\nabla u|^2$ is constant, we call u an *eigenmap*.

Theorem 6.2.11 ([**ESI08**, Theorem 4.1])**.** *A smooth metric g is extremal for some eigenvalue λ_k on $[g]$ if and only if there is an eigenmap $u\colon (M, g) \to S^l$ given by $\lambda_k(g)$-eigenfunctions and either $\lambda_{k-1}(g) < \lambda_k(g)$, or $\lambda_k(g) < \lambda_{k+1}(g)$.*

Below, we will show that singular extremal metrics still give rise to eigenmaps.

6.2.3. Background on n-harmonic maps. We will need two regularity results for n-harmonic maps. Let us first of all recall the definition again.

Definition 6.2.12. We call a map $u \in W^{1,n}(M, S^l)$ *weakly n-harmonic,* if it is a weak solution of

$$-\operatorname{div}(|\nabla u|^{n-2}\nabla u) = |\nabla u|^n u. \tag{6.2.13}$$

Note, that (6.2.13) is the Euler Lagrange equation arising from the n-energy

$$E_n[u] = \int_M |du|^n dV_g,$$

if only variations in the codomain are take into account.

The first regularity result we need is the following.

Theorem 6.2.14. *Let $u \in W^{1,n}(\Omega, S^l)$ be a weakly n-harmonic map. Then $u \in C^{1,\alpha}(\Omega, S^l)$ for some $\alpha > 0$.*

As we proved in Chapter 5, at points, in which we do not have a lack of ellipticity, we actually get higher regularity.

Theorem 6.2.15 (Theorem 5.2.6)**.** *A weakly n-harmonic map $u \in C^{1,\alpha}(\Omega, S^l)$ is smooth near points with $\nabla u \neq 0$.*

6.3. Proofs

The starting point of the proof is to show that an extremal metrics still come along with an associated eigenmap, even if they are not a priori assumed to be smooth.

6.3.1. Variation formula for the eigenvalues. We follow the approach from [**FS16**] which avoids the use of analytic perturbation theory. The arguments are mainly along the lines of [**FS16**], but extended to higher dimensions and extremal metrics. Some more care is required in our situation. There are mainly two reasons for this. This first one is the possible lack of higher order elliptic estimates for the Laplacian of the singular metric ϕg. The second one is the additional gradient term appearing in the quadratic form q below.

Let $g_0 = \phi g$, where g is smooth and $\phi \in L^{n/2}(M, g)$ with $\phi \geq 0$ almost everywhere, and $\int_M \phi^{n/2} dV_g = 1$. Recall, that we will assume throughout this section, that the embedding

$$W^{1,2}(M, \phi g) \hookrightarrow L^2(M, \phi g) \tag{6.3.1}$$

is compact. Let $t \mapsto g_t = \phi(t) g_0$ be a smooth family of conformal metrics with $\phi_0 = 1$, and $\operatorname{vol}(M, g_t) = 1$, that is $\int \dot{\phi}_t dV_t = 0$. Here and below we indicate by sub- or superscripts t that a metric quantity refers to the metric g_t. If not explicitly indicated differently, the reference metric is the smooth metric g. Note that smoothness of $t \mapsto \phi(t)$ in particular implies that the embeddings

$$W^{1,2}(M, g_t) \hookrightarrow L^2(M, g_t) \tag{6.3.2}$$

are compact for sufficiently small t.

Lemma 6.3.3. *The map $t \mapsto \lambda_k(g_t)$ is Lipschitz. In points t_0 where its derivative exists, we have*

$$\dot{\lambda_k}(g_{t_0}) = \Big(\frac{n}{2} - 1\Big) \int_M |du_0|^2_{t_0} \dot{\phi_{t_0}} dV_{t_0} - \frac{n}{2}\lambda_k(g_{t_0}) \int_M |u_0|^2 \dot{\phi_{t_0}} dV_{t_0}, \tag{6.3.4}$$

where u_0 is a normalized $\lambda_k(g_{t_0})$-eigenfunction.

Proof. Let $t_1 \leq t_2$. Without loss of generality, we may assume that $\lambda_k(g_{t_1}) \leq \lambda_k(g_{t_2})$, and $\phi_{t_1} = 1$. Denote by P_t the orthogonal projection in $L^2(M, g_t)$ onto the space of spanned by the first k eigenfunctions of (M, g_t). Take u an eigenfunction for $\lambda_k(g_{t_1})$ with $\|u\|_{L^2(M,g_{t_1})} = 1$. We want to use $\bar{u} = u - P_{t_2}(u)$ as a test function for $\lambda_k(g_{t_2})$. We have

$$\begin{aligned} \left| \int_M |du|^2_{t_1} dV_{t_1} - \int_M |du|^2 dV_{t_2} \right| &= \left| \int_M (1 - \phi_{t_2}^{n/2}) |du|^2_{t_1} dV_{t_1} \right| \\ &\leq C \|1 - \phi_{t_2}^{n/2}\|_{L^\infty} \\ &\leq C|t_2 - t_1|. \end{aligned} \tag{6.3.5}$$

Since g_t is smooth in t, we also have

$$|P_{t_2}(u)| \leq C|t_2 - t_1|. \tag{6.3.6}$$

This easily implies

$$\left| \int_M |\bar{u}|^2 dV_{t_1} - \int_M |\bar{u}|^2 dV_{t_2} \right| \leq C|t_2 - t_1|. \tag{6.3.7}$$

Elementary calculations then lead to

$$|R(\bar{u}, g_{t_1}) - R(\bar{u}, g_{t_2})| \leq C|t_2 - t_1|. \tag{6.3.8}$$

Therefore,

$$|\lambda_k(g_{t_2}) - \lambda_k(g_{t_1})| \leq |R(\bar{u}, g_{t_2}) - R(\bar{u}, g_{t_1})| \leq C|t_2 - t_1|. \tag{6.3.9}$$

This shows that $t \mapsto \lambda_k(g_t)$ is Lipschitz and thus differentiable almost everywhere. For simplicity, we assume that the derivative at $t = 0$ exists. In order to compute the derivative, consider the function

$$f(t) = \int_M |du_t|^2_t dV_t - \lambda_k(g_t) \int_M |u_t|^2 dV_t, \tag{6.3.10}$$

where u_0 is a $\lambda_1(g_0)$-eigenfunction and $u_t = u_0 - P_t(u_0)$, so that u_t is an admissible test function for $\lambda_k(g_t)$. Then, $f(t) \geq 0$ and $f(0) = 0$. Thus, if $\lambda_k(g_t)$ is differentiable at $t = 0$, also f is differentiable at $t = 0$, and we have

$$\begin{aligned} 0 &= \frac{d}{dt} f(t)\Big|_{t=0} \\ &= 2\int_M du_0 \cdot d\dot{u_0} dV_0 + \Big(\frac{n}{2} - 1\Big) \int_M |\dot{u_0}|^2 \dot{\phi_0} dV_0 - \dot{\lambda_k}(g_0) \int_M |u_0|^2 dV_0 \\ &\quad - 2\lambda_k(g_0) \int_M u_0 \dot{u_0} dV_0 - \frac{n}{2}\lambda_k(g_0) \int_M |u_0|^2 \dot{\phi_0} dV_0. \end{aligned} \tag{6.3.11}$$

Since u_0 is a $\lambda_k(g_0)$-eigenfunction, rearranging and integration by parts gives

$$\dot{\lambda}_k(g_0) = \left(\frac{n}{2}-1\right)\int_M |du_0|_0^2 \dot{\phi}_0 dV_0 - \frac{n}{2}\lambda_k(g_0)\int_M |u_0|^2\dot{\phi}_0 dV_0. \tag{6.3.12}$$ □

For a metric ϕg and a $\lambda_k(\phi g)$ eigenfunction u define

$$q(u) = \frac{n}{2}\lambda_k(\phi g)|u|^2 - \left(\frac{n}{2}-1\right)|du|^2_{\phi g}, \tag{6.3.13}$$

and note that we have $\int_M q(u)dV_{\phi g} = \lambda_k(g)\int_M |u|^2 dV_{\phi g}$, and

$$q(\mu u) = |\mu|^2 q(u), \tag{6.3.14}$$

for any $\mu \in \mathbb{R}$.

Lemma 6.3.15. *Let ϕg be an extremal metric for λ_k on $[g]$. Assume that the embedding (6.1.4) is compact. Then there is a family of $\lambda_k(\phi g)$-eigenfunctions $u_1, \dots u_l$, such that*

$$\sum_i q(u_i) = 1 \tag{6.3.16}$$

on the support of ϕ.

Proof. By (6.3.14) it suffices to show that the constant function 1 lies in the convex hull of $\{q(u) \ : \ u \in E(\lambda_k(\phi g))\}$. We denote this convex hull by C. We argue by contradiction and assume $1 \notin C$. By the Hahn–Banach separation theorem and since $C \cup \{1\}$ lies in a finite dimensional subspace, and since $\{1\}$ is compact, we can find an L^2 function η strictly separating C and 1, that is $\int_M \eta dV_{\phi g} < \alpha$, and $\int_M \eta q(u) dV_{\phi g} > \alpha$ for any $u \in C$, and some $\alpha \in \mathbb{R}$. Since $C \cup \{1\}$ lies in a finite dimensional subspace, we may assume that η is smooth by approximation. Using the scaling properties of $q(u)$, it is not very hard to see that, that we may assume $\alpha = 0$: Write $\psi = \eta - \alpha$, then we have $\int_M \psi dV_g < 0$. Moreover, by (6.3.14), $\int_M \psi q(u) > 0$ if and only if $\int_M \psi q(\mu u) > 0$ for some $\mu \neq 0$. We have

$$\begin{aligned} \int \psi q(\mu u) &= \int_M \eta q(\mu u) - \alpha \int_M q(\mu u) \\ &> \alpha(1 - |\mu|^2 \int_M |u|^2) \\ &> 0 \end{aligned} \tag{6.3.17}$$

provided μ is sufficiently small or large (depending on the sign of α).

Since C lies in a finite dimensional space, we can find $\delta > 0$, such that

$$\int_M \psi q_0(u) dV_{\phi g} > 2\delta$$

for any $u \in C$, with $\int_M |u|^2 = 1$.

Consider a smooth volume preserving variation $g_t = \phi_t \phi g$ of g with $\dot{\phi}_0 = \psi - \int_M \psi dV_{\phi g}$. For $|t|$ small enough we will then still have $\int_M \psi dV_t < 0$, and $\int_M \psi q_t(u) dV_t > \delta$ for any normalized $u \in E(\lambda_k(g_t))$. The first of these assertions is obvious. For the second observe that otherwise there would be a sequence $t_l \to 0$, such that $\int_M \psi q(u) dV_{g_t} \leq \delta$ for some normalized $u_l \in E(\lambda_1(g_{t_k}))$.

Since $1/2 \leq \phi_t \leq 2$ for t small enough (u_l) is bounded in $W^{1,2}(M, \phi g)$ with $\|u\|_{L^2(M,\phi g)} \geq 1/2$. By the compactness of the embedding (6.1.4), we may assume that $u_l \to u$ weakly in $W^{1,2}(M, \phi g)$ and strongly in $L^2(M, \phi g)$. We claim that u is a normalized $\lambda_k(\phi g)$-eigenfunction with $\int_M \psi q(u) dV_{g_t} \leq \delta$, which is a contradiction. Write $\phi_l = \phi_{t_l}$, such that $\phi_l \to 1$ in L^∞. Then $\phi_l^{n/2-1} \nabla^0 u_l \rightharpoonup \nabla^0 u$ in $L^2(M, \phi g)$ and also $\phi_l^{n/2} u_l \to u$ in $L^2(M, \phi g)$, so that we can pass to the limit in the equations

$$\int \nabla^0 v \cdot \nabla^0 u_l \phi_l^{n/2-1} dV_0 = \lambda_l \int \phi v u \phi_l^{n/2} dV_0, \tag{6.3.18}$$

where $\lambda_l \to \lambda_k(\phi g)$. This implies that u is a normalized $\lambda_k(\phi g)$-eigenfunction. We now show that

$$\int \psi q_l(u_l) \phi_l^{n/2} dV_0 \to \int \psi q_0(u) dV_0. \tag{6.3.19}$$

Invoking the arguments from above again, we only have to take care of the gradient term in q. This can be done via integration by parts.

We have

$$\begin{aligned} \int_M \psi |du_l|^2 \phi_l^{n/2-1} dV_0 &= \int_M d(\psi u_l) \cdot du_l \phi_l^{n/2-1} dV_0 - \int_M u_l d\psi \cdot du_l \phi_l^{n/2-1} dV_0 \\ &= \lambda_k(\phi_l g) \int_M \psi u_l^2 \phi_l^{n/2} dV_0 - \int_M u_l d\psi \cdot du_l \phi_l^{n/2-1} dV_0. \end{aligned} \tag{6.3.20}$$

Now, $u_l du_l \rightharpoonup u du$ in $L^2(M, \phi g)$, hence, by the same arguments as above,

$$\int_M \psi |du_l|^2 \phi_l^{n/2-1} dV_0 \to \int_M \psi |du_l|^2 dV_0. \tag{6.3.21}$$

By the fundamental theorem of calculus for Lipschitz functions and (6.2.9), we have

$$\int_0^t \dot{\lambda}_k(g_s) ds = \lambda_k(g_t) - \lambda_k(g_0) \leq o(t). \tag{6.3.22}$$

Therefore, there is a sequence $t_l \searrow 0$, such that $\dot{\lambda}_k(g_t)$ exists for any $t = t_l$ and has $\dot{\lambda}_k(g_{t_l}) \leq o(1)$, as $l \to \infty$.

We assume now that (6.2.9) holds for the family g_t. The case (6.2.10) follows with a few obivious modifications. From the formula for the derivative of λ_k, and using (6.3.19), we find

$$\int_M q(u) \dot{\phi}_0 dV_0 = \lim_{l \to \infty} \int_M q(u_l) \dot{\phi}_{t_l} dV_{g_{t_l}} = \lim_{k \to \infty} \dot{\lambda}_k(g_{t_l}) \leq 0. \tag{6.3.23}$$

But this is a contradiction to

$$\int_M q(u) \dot{\phi}_0 dV_{g_0} = \int_M q(u) \psi dV_{g_0} - \int_M \psi dV_{g_0} \int_M q(u) dV_{g_0} > 0, \tag{6.3.24}$$

for any non-trivial u. $\square$

6.3.2. Extremal metrics and n-harmonic maps. As an application of the variation formula we now derive the connection of extremal metrics and n-harmonic maps.

Lemma 6.3.25. *Let ϕg be extremal for λ_k on $[g]$ with $\lambda_k(\phi g) > 0$ and assume that the embedding (6.1.4) is compact. Denote by Ω the interior of $\operatorname{supp}\phi$. Up to scaling, the family $u_1, \dots u_l$ from above defines a weakly harmonic map*

$$u\colon (\Omega, \phi g) \to S^{k-1} \tag{6.3.26}$$

with constant density.

This is exactly as in [**ESI03**] but in integrated form. For completness, we give the proof.

Proof. We have for any test function η, that

$$\begin{aligned}
\sum_i \int_M d(u_i^2)\cdot d\eta\phi^{n/2-1}dV_g &= 2\sum_i \int_M u_i du_i \cdot d\eta\phi^{n/2-1}dV_g \\
&= 2\sum_i \int_M du_i \cdot d(u_i\eta)\phi^{n/2-1}dV_g - 2\sum_i\int_M \eta|du_i|^2\phi^{n/2-1}dV_g \\
&= 2\sum_i \int_M \lambda_k(\phi g)\eta u_i^2\phi^{n/2}dV_g - 2\sum_i\int_M \eta|du_i|^2\phi^{n/2-1}dV_g \\
&= \frac{4}{n}\sum_i \int_M \eta q_{\phi g}(u_i)\phi^{n/2}dV_g - \frac{4}{n}\sum_i\int_M \eta|du_i|^2\phi^{n/2-1}dV_g.
\end{aligned} \tag{6.3.27}$$

Similarly, we have

$$\begin{aligned}
&\sum_i \int_M \eta|du_i|^2\phi^{n/2-1}dV_g \\
&\quad = \sum_i \int_M \lambda_k(\phi g)\eta u_i^2\phi^{n/2}dV_g - \frac{1}{2}\sum_i \int_M d(u_i^2)\cdot d\eta\phi^{n/2-1}dV_g.
\end{aligned} \tag{6.3.28}$$

Combining these two calculatons we find

$$\frac{n-2}{4}\int_M d\left(\sum_i u_i^2\right)\cdot d\eta\phi^{n/2-1}dV_g = \int_M \eta\left(1-\lambda_k\sum_i u_i^2\right)\phi^{n/2}dV_g \tag{6.3.29}$$

Thus the function $f = \sum_i u_i^2 - \frac{1}{\lambda_k}$ satisfies the weak form of

$$\frac{n-2}{4}\Delta_{\phi g} f = -\lambda_k(\phi g) f. \tag{6.3.30}$$

By assumption, $\lambda_k(\phi g)$ is positive and we conclude that $f = 0$ in $L^2(M, \phi g)$, which means that $f = 0$ almost everywhere on $\operatorname{supp}\phi$. $\square$

Proposition 6.3.31. *Let ϕg be extremal for λ_k on $[g]$ such that the embedding (6.1.4) is compact. Let $u\colon (M, \phi g) \to S^l$ be the associated eigenmap. Then $u\colon (M, g) \to S^l$ is n-harmonic and on the support of ϕ, we have*

$$\phi = \frac{1}{\lambda_k(\phi g)}\sum_i |du_i|_g^2. \tag{6.3.32}$$

Proof. Since $u\colon (\Omega, \phi g) \to S^k$ has constant density it defines an n-harmonic map from $(\Omega, \phi g)$ to S^l, thus, by conformal invariance, also from (Ω, g) to S^l.

The components of u solve

$$\Delta_{\phi g} u_i = \lambda_k(\phi g) u_i. \tag{6.3.33}$$

Let Ω be the interior of $\operatorname{supp}\phi$. Take $\eta \in C^\infty$, and test the equation above with ηu_i, then

$$\int_M du_i \cdot d(\eta u_i)\phi^{n/2-1} dV_g = \lambda_k(\phi g) \int_M \eta u_i^2 \phi^{n/2} dV_g.$$

Since $\sum_k u_i^2 = 1$, summing over i yields

$$\int_M \eta \sum_i |du_i|_g^2 \phi^{n/2-1} dV_g = \lambda_k(\phi g) \int_M \eta \phi^{n/2} dV_g.$$

Since η was arbitrary, this means that

$$\sum_i |du_i|_g^2 \phi^{n/2-1} = \lambda_k(\phi g)\phi^{n/2}$$

almost everywhere in $\operatorname{supp}\phi$. Therefore,

$$\phi = \frac{1}{\lambda_1(\phi g)} \sum_i |du_i|_g^2 \tag{6.3.34}$$

holds almost everywhere on $\operatorname{supp}\phi$. □

In order tob obtain regularity for ϕ, we can use the n-harmonic map $u\colon (M, g) \to S^l$, where g is smooth.

Proof of Theorem 6.1.5. By Proposition 6.3.31, it suffices to show that u is smooth in the interior of $\operatorname{supp}\phi$, which we denote by Ω again. It follows from [**MY96**] that $u \in C^{1,\alpha}_{loc}(\Omega)$. Therefore we can apply Theorem 5.2.6 and conclude that u is smooth in Ω. since $\phi \in C^0$, there is a neighbourhood U of x such that $\phi \geq c > 0$ in U. □

CHAPTER 7

Regularity of conformal metrics with large first eigenvalue

7.1. Introduction

Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. For a fixed metric g we consider conformally related metrics of the form $u^{4/(n-2)}g$, where u is a smooth positive function. The volumes of a measurable set $\Omega \subseteq M$ with respect to g and $u^{4/(n-2)}g$ are related by

$$\operatorname{vol}(\Omega, u^{4/(n-2)}g) = \int_\Omega u^{2^\star} dV_g,$$

where $2^\star = 2n/(n-2)$ is the critical Sobolev exponent for the embedding $W^{1,2} \hookrightarrow L^p$ and V_g denotes the volume measure of g.

The scalar curvature transforms according to the semilinear elliptic equation

$$4\frac{n-1}{n-2}\Delta_g u + R_g u = R_{u^{4/(n-2)}g} u^{2^\star - 1} \tag{7.1.1}$$

which is of critical nonlinearity. Here R_g and $R_{u^{4/(n-2)}g}$ denote the scalar curvature of the metric g respectively $u^{4/(n-2)}g$, and $\Delta_g = \Delta$ the (positive) Laplace operator of (M, g).

Thus one may view the scalar curvature as a 'Laplacian of the metric' when one considers only a fixed conformal class. In view of this analogy one may ask whether L^p-bounds on the scalar curvature imply $W^{2,p}$-bounds on the conformal factors u. In general, this is not true, see e.g. [**CGW94**]. For $p > n/2$, several results in this direction are known under certain additional assumptions. Classical examples of such assumptions are that the first eigenvalue of the Laplace operator is bounded away from 0 and that additionally (M, g) has dimension three [**BPY89, CY90**] or $(M, g) = (S^n, g_{st.})$ [**CY89**]. The assumptions on the geometry are made in order to rule out a blow-up at a single point. In the case of the sphere, the result only holds true up to pulling back the conformal factors by conformal transformations. The large group of conformal diffeomorphisms of S^n makes it possible to avoid blow-ups. In dimension three a possible blow-up can be analyzed carefully and eventually ruled out.

We prove a result in the spirit of the results in [**BPY89, CY89, CY90, Gur93**], but instead of geometric assumptions we assume that the first eigenvalue is sufficiently large in order to rule out a possible blow-up.

Denote by ω_n the n-dimensional Euclidean volume of the unit ball. Our main result is

Theorem 7.1.2. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$ and u a smooth positive function. Consider the conformal metric $\tilde{g} = u^{4/(n-2)}g$ and denote by $\tilde{R}$ its scalar curvature. Assume that* *(i)* $\mathrm{vol}(M, \tilde{g}) = 1$,

(ii) $\int_M |\tilde{R}|^p u^{2^\star} dV_g \leq A$ *for some* $n/2 < p < \infty$,

(iii) $\lambda_1(M, \tilde{g}) \geq B > n\left((n+1)\omega_{n+1}\right)^{2/n}$.

Then there exist constants $C_1, C_2, C_3 > 0$ *depending on* (M, g) *and* A, B *such that* $C_1 \leq u \leq C_2$ *and* $\|u\|_{W^{2,p}(M,g)} \leq C_3$.

Observe that once the L^∞-bounds on u and u^{-1} are established, the bound $\|u\|_{W^{2,p}(M,g)} \leq C_3$ is a consequence of the standard elliptic estimates in L^p-spaces applied to equation (7.1.1), see e.g. [**Mor66**, Theorem 6.4.8]

The geometric significance of the constant B in assumption (*iii*) is that we have $\lambda_1(S^n, g_{st.}) \,\mathrm{vol}(S^n, g_{st.})^{2/n} = n\left((n+1)\omega_{n+1}\right)^{2/n}$, where $g_{st.}$ denotes the round metric on S^n of curvature 1.

Remark 7.1.3. Due to a result of Petrides [**Pet15**], any conformal class except for the standard conformal class on S^n admits a smooth metric $\tilde{g}$ with unit volume and $\lambda_1(M, \tilde{g}) > n\left((n+1)\omega_{n+1}\right)^{2/n}$. See also [**CES03**], where a related but weaker result is proved.

Theorem 7.1.2 has some immediate and interesting consequences. As mentioned above, for A sufficiently large we can find a constant $B > n\left((n+1)\omega_{n+1}\right)^{2/n}$ such that there is at least one metric in the conformal class of g satisfying the assumptions of Theorem 7.1.2 with these constants A, B. For such A, B it is possible to find a positive and Hölder continuous function u, such that $u^{4/(n-2)}g$ maximizes λ_1 among all unit volume metrics in the conformal class of g satisfying the same L^p-bound on the scalar curvature, see Theorem 7.3.1

Another consequence is a compactness result for sets of isospectral metrics within a conformal class, which satisfy in addition the assumptions of Theorem 7.1.2, see Theorem 7.3.4

In Section 7.2 we explain the proof of Theorem 7.1.2. Afterwards, in Section 7.3, we briefly discuss the above mentioned applications.

7.2. Proof of the theorem

The main argument for the proof of Theorem 7.1.2 is that assumption (*iii*) rules out a possible blow-up. Once this is established, the result follows from arguments which are seen as fairly standard by now. These arguments are based on the Moser iteration scheme. For convenience, we explain how to lift the integrability in order to a find bound on $\|u\|_{2^\star+\varepsilon}$ for some $\varepsilon > 0$. This is proved in different ways in various places, but we could not locate a reference stating precisely what we need. Once this is done, the L^∞-bounds on u and u^{-1} follow from a Harnack inequality established by Trudinger in [**Tru68**]. Given the L^∞-bounds the result follows from standard elliptic theory.

In general, all constants called C may differ from line to line and will depend on (M, g) and the data A, B.

7.2.1. A Volume non-concentration result. We start with a few preparations. From now on all metric quantities refer to the fixed background metric g if not explicitly stated differently. Recall the definition of p-capacities,

Definition 7.2.1. For a pair (E,F) of subsets $E \subset\subset \mathring{F} \subseteq M$ we define the *p-capacity* by

$$\mathrm{Cap}_p(E,F) := \inf \int_M |df|^p dV_g,$$

where the infimum is taken over all Lipschitz functions $f\colon M \to \mathbb{R}$ which are 1 on E and 0 outside F.

Note that since

$$\int_M |df|^n_{\tilde g} dV_{\tilde g} = \int_M |df|^n_g dV_g, \tag{7.2.2}$$

whenever g and $\tilde g$ are conformally related, the n-capacity is conformally invariant. We will use the following frequently.

Lemma 7.2.3. *Let $R > 0$, then for $p \le n$ and any point $x \in M$, the p-capacity satisfies* $\lim_{r\to 0} \mathrm{Cap}_p(B(x,r), B(x,R)) = 0$.

Proof. Observe that Hölder's inequality implies that it suffices to consider the case $p = n$. Moreover, it clearly suffices to prove the statement only for some small $R < inj(M)$. Let $0 < r < R$ and define $\psi_{r,R}$ by

$$\psi_{r,R}(z) = \begin{cases} \frac{\log(d_g(x,z)R^{-1})}{\log(rR^{-1})} & \text{if } r \le d_g(x,z) \le R \\ 1 & \text{if } d_g(x,z) \le r \\ 0 & \text{if } d_g(x,z) \ge R. \end{cases}$$

If g is flat in $B(x,R)$, we have

$$\int_{B(x,R)} |\nabla \psi_{r,R}|^n dV_g = \omega_n \left(\log\left(\frac{R}{r}\right)\right)^{1-n}.$$

In general, for $R > 0$ such that g is comparable on $B(x,R)$ to the Euclidean metric on $B(0,R)$, we have

$$\int_{B(x,R)} |\nabla \psi_{r,R}|^n dV_g \le C \left(\log\left(\frac{R}{r}\right)\right)^{1-n}.$$

We conclude

$$\lim_{r\to 0} \int_{B(x,R)} |\nabla \psi_{r,R}|^n = 0,$$

thus $\lim_{r\to 0} \mathrm{Cap}_n(B(x,r), B(x,R)) = 0$. $\square$

Before we can prove the volume non-concentration result, we need the following observation about conformal immersions which also appears in [**ESI86**].

Lemma 7.2.4. *Let $\Phi\colon (M,g) \to (S^n, g_{st.})$ be a conformal immersion. Denote by $z^1, \dots, z^{n+1}$ the standard coordinate funtions of $\mathbb{R}^{n+1}$ restricted to S^n. Then*

$$\Phi^* g_{st.} = \frac{1}{n} \sum_{i=1}^{n+1} |\nabla(z^i \circ \Phi)|^2 g. \tag{7.2.5}$$

PROOF. First, observe that for $\gamma \in SO(n+1)$ we have

$$(\gamma \circ \Phi)^* g_{st.} = \Phi^* g_{st.}$$

and

$$\frac{1}{n}\sum_{i=1}^{n+1} |\nabla(z^i \circ \Phi)|^2 g = \frac{1}{n}\sum_{i=1}^{n+1} |\nabla(z^i \circ \gamma \circ \Phi)|^2 g.$$

Thus it suffices to compute both sides of (7.2.5) at a point $x \in M$ with $\Phi(x) = N$, where $N = (0, \dots, 0, 1) \in S^n$ denotes the northpole. In this case we may use the functions $z^1, \dots, z^n$ as coordinates about N. We thus have coordinates $z^i \circ \Phi$, $i = 1, \dots, n$ about x, such that $D\Phi(x) = \mathrm{id}$ in these coordinates. Moreover, since Φ is conformal, there is positive constant a such that $g_{jk}(x) = a\delta_{jk}$. Thus we find

$$\frac{1}{n}\sum_{i=1}^{n+1} |\nabla(z^i \circ \Phi)|^2(x) g_{jk}(x) = \frac{a}{n}\sum_{i=1}^{n+1} \frac{1}{a}|\nabla z^i|^2(N)\delta_{jk} = \delta_{jk} = \left(\Phi^* g_{st.}\right)_{jk}(x). \qquad \square$$

The next proposition states that we can control the volume of small balls uniformly in terms of the lower bound B on the first eigenvalue.

Proposition 7.2.6. *Let u be a function satisfying assumptions (i) and (iii) in Theorem 7.1.2. For any $\delta > 0$, there is a radius $r = r(M, g, \delta, B) > 0$, such that*

$$\int_{B(x,r)} u^{2^\star} dV_g < \delta$$

for any $x \in M$.

PROOF. The idea of the proof is based on arguments due to Kokarev, who proved the same result in dimension two in [**Kok14**]. Kokarev used ideas developed by Nadirashvili in [**Nad96**].

Assume the statement is not correct. Then we can find $\delta > 0$ together with a sequence $x_k \in M$ of points and a sequence u_k of smooth positive functions such that $\mathrm{vol}(M, u_k^{4/(n-2)} g) = 1$, $\lambda_1(M, u_k^{4/(n-2)} g) \geq B$, and $\int_{B(x_k, 1/k)} u_k^{2^\star} dV_g \geq \delta$. We denote by g_k the metric $u_k^{4/(n-2)} g$.

Up to extracting a subsequence we can assume that the probability measures V_{g_k} converge to a Radon probability measure μ in the weak*-topology. Moreover, we may also assume that $x_k \to x$. We claim that $\mu(\{x\}) > 0$. In fact, take a sequence $\eta_l \in C_c^\infty(B(x, 2/l))$ with $0 \leq \eta_l \leq 1$ and $\eta_l(x) = 1$. Then, by dominated convergence,

$$\mu(\{x\}) = \lim_{l \to \infty} \int_M \eta_l d\mu.$$

So fix η_l as above and assume in addition that $\eta_l = 1$ on $B(x, 1/l)$. We thus have

$$\begin{aligned}\mu(\{x\}) &= \lim_{l \to \infty} \lim_{k \to \infty} \int_{B(x,2/l)} \eta_l u_k^{2^\star} dV_g \geq \lim_{l \to \infty} \lim_{k \to \infty} \int_{B(x,1/l)} u_k^{2^\star} dV_g \\ &\geq \lim_{l \to \infty} \lim_{k \to \infty} \int_{B(x_k,1/k)} u_k^{2^\star} dV_g \geq \lim_{k \to \infty} \int_{B(x_k,1/k)} u_k^{2^\star} dV_g \geq \delta.\end{aligned}$$

We consider two cases: The first case is $\mu = \delta_x$ for some $x \in M$. In the remaining case we can find for a point $x \in M$ with $\mu(\{x\}) > 0$ a radius $R > 0$ such that $\mu(M \setminus B(x, 2R)) > 0$. Let us start with the second case which is the easier one.

Take a ball $B(x, 2R)$ as described above. By Lemma 7.2.3 we find $r > 0$ such that $\mathrm{Cap}_n(B(x, r), B(x, R)) < \varepsilon$. Thus we can choose a Lipschitz function ψ supported in $B(x, R)$, which satisfies $0 \leq \psi \leq 1$, $\psi = 1$ on $B(x, r)$ and $\int_M |\nabla \psi|^n dV_g < \varepsilon$.

Denote by α_k the mean (with respect to g_k) of ψ, i.e.

$$\alpha_k = \int_M \psi dV_{g_k}.$$

By the min-max principle, Hölder's inequality and the conformal invariance (7.2.2), we find

$$\begin{aligned}
\lambda_1(g_k) \int_M (\psi - \alpha_k)^2 dV_{g_k} \leq & \int_M |d\psi|^2_{g_k} dV_{g_k} \\
\leq & \left(\int_M |d\psi|^n_{g_k} dV_{g_k} \right)^{2/n} (\mathrm{vol}_{g_k}(\operatorname{supp} \psi))^{(n-2)/n} \\
= & \left(\int_M |\nabla \psi|^n dV_g \right)^{2/n} (\mathrm{vol}_{g_k}(\operatorname{supp} \psi))^{(n-2)/n} \\
\leq & \varepsilon^{2/n}.
\end{aligned}$$

We can estimate the left-hand-side from below by $\lambda_1(g_k)$ times

$$\alpha_k^2 \int_{M \setminus B(x,R)} u_k^{2^\star} dV_g + (1 - \alpha_k)^2 \int_{B(x,r)} u_k^{2^\star} dV_g.$$

Let us investigate both terms as $k \to \infty$. We have

$$\liminf_{k \to \infty} \int_{M \setminus B(x,R)} u_k^{2^\star} dV_g \geq \mu(M \setminus B(x, 2R)),$$

and

$$\liminf_{k \to \infty} \int_{B(x,r)} u_k^{2^\star} dV_g \geq \mu(\{x\}).$$

Up to extracting a subsequence we may assume $\alpha_k \to \alpha$ as $k \to \infty$. By construction $\alpha \in [0, 1]$, thus we find

$$\begin{aligned}
\limsup_{k \to \infty} \lambda_1(g_k) \leq & \frac{\varepsilon^{2/n}}{\alpha^2 \mu(M \setminus B(x, 2R)) + (1 - \alpha)^2 \mu(\{x\})} \\
\leq & \frac{4\varepsilon^{2/n}}{\min\{\mu(M \setminus B(x, 2R)), \mu(\{x\})\}}.
\end{aligned}$$

And thus $\limsup_{k \to \infty} \lambda_1(g_k) = 0$, a contradiction.

The case $\mu = \delta_x$ is slightly more involved. In a first step we observe that we may assume without loss of generality that g is flat near x. This observation is motivated by the arguments in [**CES03**].

Given any $\varepsilon > 0$ we can replace g by a another metric g' which is flat near x and $(1+\varepsilon)$-quasiisometric to g, i.e. $(1+\varepsilon)^{-2} g(v, v) \leq g'(v, v) \leq (1+\varepsilon)^2 g(v, v)$ for all non-zero tangent vectors v, see e.g. [**CES03**, Lemma 2.3]. Then for each k the metric $u_k^{4/(n-2)} g'$

is $(1+\varepsilon)$-quasiisometric to the metric $u_k^{4/(n-2)}g$. Rescale the functions u_k to obtain new functions u_k' such that we have $\mathrm{vol}(M,(u_k')^{4/(n-2)}\,g')=1$. Since the volumes of $(M,u_k^{4/(n-2)}g)$ and $(M,u_k^{4/(n-2)}g')$ are controlled by

$$(1+\varepsilon)^{-n}\leq\frac{\mathrm{vol}(M,u_k^{4/(n-2)}g')}{\mathrm{vol}(M,u_k^{4/(n-2)}g)}\leq(1+\varepsilon)^n, \tag{7.2.7}$$

we find that the ratios $u_k^{4/(n-2)}/(u_k')^{4/(n-2)}$ are uniformly bounded from above and below by $(1+\varepsilon)^2$ respectively $(1+\varepsilon)^{-2}$. This implies that $u_k^{4/(n-2)}g$ is $(1+\varepsilon)^2$-quasiisometric to $(u')_k^{4/(n-2)}g'$, thus

$$\lambda_1\left(\left(u_k'\right)^{4/(n-2)}g'\right)\geq(1+\varepsilon)^{-4(n+1)}\lambda_1\left(u_k^{4/(n-2)}g\right).$$

In particular, for ε sufficiently small we can find a constant $B'>n\left((n+1)\omega_{n+1}\right)^{2/n}$, such that $\lambda_1(M,(u_k')^{4/(n-2)}\,g')\geq B'$.

Similarly as in (7.2.7), we have for any measurable subset $\Omega\subseteq M$ that

$$\mathrm{vol}(\Omega,(u_k')^{4/(n-2)}g')\leq(1+\varepsilon)^{2n}\,\mathrm{vol}(\Omega,u_k^{4/(n-2)}g),$$

since $u_k^{4/(n-2)}g$ is $(1+\varepsilon)^2$-quasiisometric to $(u_k')^{4/(n-2)}g'$. Applying this to subsets of $M\setminus\{x\}$ easily implies that $(u_k')^{2^\star}V_{g'}\rightharpoonup^*\delta_x$. In more detail, if ν is the weak*-limit of a subsequence of $(u_k')^{2^\star}V_{g'}$, we have for any open $\Omega\subset\subset M\setminus\{x\}$ that

$$\begin{aligned}\nu(\Omega)&\leq\liminf_{k\to\infty}\mathrm{vol}(\Omega,(u_k')^{4/(n-2)}g')\\&\leq\liminf_{k\to\infty}(1+\varepsilon)^{2n}\,\mathrm{vol}(\Omega,(u_k)^{4/(n-2)}g)\\&\leq\limsup_{k\to\infty}(1+\varepsilon)^{2n}\,\mathrm{vol}(\overline{\Omega},(u_k)^{4/(n-2)}g)\\&\leq(1+\varepsilon)^{2n}\mu(\overline{\Omega})\\&=0,\end{aligned}$$

since $u_k^{2^\star}V_g\rightharpoonup^*\mu=\delta_x$. This implies $\nu(M\setminus\{x\})=0$ and thus $\nu=\delta_x$. We have shown that the limit of any weakly*-convergent subsequence of $(u_k')^{2^\star}V_{g'}$ has to be δ_x. Since every subsequence of $(u_k')^{2^\star}V_{g'}$ has a weakly*-convergent subsequence, it follows that $(u_k')^{2^\star}V_{g'}\rightharpoonup^*\delta_x$. Now it suffices to show that such a sequence (u_k') can not exist on (M,g').

Let Ω be a conformally flat neighborhood of x. Choose a conformal immersion $\Phi\colon(\Omega,g)\to(S^n,g_{st.})$. By diminishing Ω if necessary we may assume that Φ is an embedding. Fix $\varepsilon>0$ and choose a function $\psi\in W_0^{1,\infty}(\Omega)$ with $\int_\Omega|\nabla\psi|^n dV<\varepsilon$, $0\leq\psi\leq1$, and $\psi(x)=1$, which is possible thanks to Lemma 7.2.3. By Lemma Lemma 7.2.13 below, we find $s_k\in\mathrm{Conf}(S^n)$ such that

$$\int_\Omega\psi\cdot(z^i\circ s_k\circ\Phi)dV_{g_k}=0$$

for all $i = 1, \dots, n$. Using the functions $\psi \cdot (z^i \circ s_k \circ \Phi)$ as test functions and summing over all i yields

$$
\begin{aligned}
\lambda_1(g_k) \int_\Omega \psi^2 dV_{g_k} =& \lambda_1(g_k) \sum_{i=1}^{n+1} \int_\Omega (\psi z^i)^2 dV_{g_k} \\
\leq& \sum_{i=1}^{n+1} \int_\Omega |d(\psi \cdot (z^i \circ s_k \circ \Phi))|^2_{g_k} dV_{g_k} \\
=& \sum_{i=1}^{n+1} \int_\Omega |d\psi|^2_{g_k} (z^i \circ s_k \circ \Phi)^2 dV_{g_k} \\
&+ 2 \sum_{i=1}^{n+1} \int_\Omega (z^i \circ s_k \circ \Phi) \psi \langle d\psi, d(z^i \circ s_k \circ \Phi)\rangle_{g_k} dV_{g_k} \\
&+ \sum_{i=1}^{n+1} \int_\Omega \psi^2 |d(z^i \circ s_k \circ \Phi)|^2_{g_k} dV_{g_k}.
\end{aligned} \tag{7.2.8}
$$

By Hölder's inequality and conformal invariance, the first summand in (7.2.8) can be controlled as follows

$$
\begin{aligned}
\sum_{i=1}^{n+1} \int_\Omega |d\psi|^2_{g_k} (z^i \circ s_k \circ \Phi)^2 dV_{g_k} &= \int_\Omega |d\psi|^2_{g_k} dV_{g_k} \\
&\leq \left(\int_\Omega |d\psi|^n_{g_k} dV_{g_k} \right)^{2/n} \mathrm{vol}_{g_k}(\Omega)^{(n-2)/n} \\
&\leq \left(\int_\Omega |\nabla \psi| dV_g \right)^{2/n} \leq \varepsilon^{2/n}.
\end{aligned} \tag{7.2.9}
$$

For the second summand in (7.2.8) notice that

$$
\begin{aligned}
\int_\Omega |d(z^i \circ s_k \circ \Phi)|^n_{g_k} dV_{g_k} &= \int_{s_k \circ \Phi(\Omega)} |\nabla z^i|^n dV_{g_{st.}} \\
&\leq C \,\mathrm{vol}(s_k \circ \Phi(\Omega)) \leq C,
\end{aligned} \tag{7.2.10}
$$

for a constant $C = C(n)$, thanks to conformal invariance. This implies

$$
\begin{aligned}
&\sum_{i=1}^{n+1} \int_\Omega (z^i \circ s_k \circ \Phi) \psi \langle d\psi, d(z^i \circ s_k \circ \Phi)\rangle_{g_k} dV_{g_k} \\
\leq& \sum_{i=1}^{n+1} \sup_{x \in \Omega} |(z^i \circ s_k \circ \Phi)| \int_\Omega |d\psi|_{g_k} |d(z^i \circ s_k \circ \Phi)|_{g_k} dV_{g_k} \\
\leq& \sum_{i=1}^{n+1} \left(\int_\Omega |d\psi|^n_{g_k} dV_{g_k} \right)^{1/n} \left(\int_\Omega |d(z^i \circ s_k \circ \Phi)|^n_{g_k} dV_{g_k} \right)^{1/n} \mathrm{vol}_{g_k}(\Omega)^{(n-2)/n} \\
\leq& C\varepsilon^{1/n}.
\end{aligned} \tag{7.2.11}
$$

The last summand in (7.2.8) is estimated using Lemma 7.2.4,

$$\begin{aligned}(7.2.12)\qquad \sum_{i=1}^{n+1}\int_\Omega \psi^2|d(z^i\circ s_k\circ\Phi)|^2_{g_k}dV_{g_k} \leq &\left(\int_\Omega\left(\sum_{i=1}^{n+1}|d(z^i\circ s_k\circ\Phi)|^2_{g_k}\right)^{n/2}dV_{g_k}\right)^{2/n}\\ =&n\operatorname{vol}((s_k\circ\Phi)(\Omega))^{2/n}\leq n((n+1)\omega_{n+1})^{2/n}.\end{aligned}$$

Combining (7.2.8), (7.2.9), (7.2.11) and (7.2.12), we conclude

$$\begin{aligned}\limsup_{k\to\infty}\lambda_1(u_k^{4/(n-2)}g) =&\limsup_{k\to\infty}\lambda_1(u_k^{4/(n-2)}g)\int_\Omega\psi dV_{g_k}\\ \leq&\varepsilon^{2/n}+C\varepsilon^{1/n}+n\left((n+1)\omega_{n+1}\right)^{2/n},\end{aligned}$$

which proves our claim. □

Next, we give the version of the Hersch lemma, which we have used in the proof of Proposition 7.2.6.

Lemma 7.2.13 (Hersch lemma)**.** *Let μ be a continuous Radon measure on M, $\psi\in W_0^{1,\infty}(\Omega)$, where $\Omega\subseteq M$. Moreover, assume $0\leq\psi\leq 1$. Then for any fixed conformal map $\Phi\colon(\Omega,g)\to(S^n,g_{st.})$ there exists a conformal diffeomorphsim $s\in\operatorname{Conf}(S^n)$ such that*

$$\int_\Omega\psi\cdot(z^i\circ s\circ\Phi)d\mu=0,$$

for all $i=1,\dots,n+1$.

This can be proved in complete analogy to the original Hersch lemma, see [**Her70**]. For convenience of the reader we recall the main idea of the elegant argument.

Denote by s_x the stereographic projection onto T_xS^n. For $x\in S^n$ and $\lambda\in\mathbb{R}_{>0}$ we have conformal diffeomorphisms $g_{x,\lambda}$ of S^n given by $g_{x,\lambda}(y)=s_x^{-1}(\lambda s_x(y))$ for $y\neq -x$ and $g_{x,\lambda}(-x)=-x$. Consider the map $C\colon D^{n+1}\to D^{n+1}$ given by

$$C(\lambda x)=\left(\int_\Omega\psi d\mu\right)^{-1}\int_\Omega\psi\cdot(z\circ g_{x,1-\lambda}\circ\Phi)d\mu,$$

where $\lambda\in[0,1)$, and $x\in S^n$. It is easily checked that this extends to a continous map $\overline{C}\colon\overline{D}^{n+1}\to\overline{D}^{n+1}$, which restricts to the identity on S^n. It follows, that there have to be x,λ such that $C(\lambda x)=0$.

7.2.2. Higher integrability of the conformal factors. The next step is to improve the integrability of the conformal factors. Once this is done a Harnack inequality due to Trudinger [**Tru68**] gives L^∞-bounds and Theorem 7.1.2 follows from the standard elliptic estimates.

Lemma 7.2.14. *For u as in Theorem 7.1.2, there are $D,\varepsilon>0$ depending on (M,g) and A,B such that*

$$\int_M u^{2^\star+\varepsilon}dV_g\leq D.$$

Proof. We follow the proof of Lemma 2.3 in [**Gur93**].

Of course, it suffices to prove that there are constants $\varepsilon = \varepsilon(n) > 0$, $r = r(M, g, A, B) > 0$, and $C = C(M, g, A, B)$ such that $\int_{B(x,r)} u^{2^\star+\varepsilon} dV_g \le C$ for any $x \in M$.

For $r > 0$ to be chosen later we take a smooth function $\eta\colon M \to [0,1]$ with $\eta = 1$ on $B(x, r/2)$, $\eta = 0$ outside $B(x, r)$, and $|\nabla\eta| \le C/r$. If we choose $\varepsilon > 0$ such that $2 + 2\varepsilon \le 2^\star$, we get from the Sobolev inequality that

$$\begin{aligned}\left(\int_M \left(\eta u^{1+\varepsilon}\right)^{2^\star} dV_g\right)^{2/2^\star} &\le C\left(\int_M |\nabla(\eta u^{1+\varepsilon})|^2 dV_g + \int_M \eta^2 u^{2+2\varepsilon} dV_g\right)\\ &\le C\int_M |\nabla(\eta u^{1+\varepsilon})|^2 dV_g + C\left(\int_M u^{2^\star} dV_g\right)^{(2+2\varepsilon)/2^\star}\\ &\le C\int_M |\nabla(\eta u^{1+\varepsilon})|^2 dV_g + C,\end{aligned} \tag{7.2.15}$$

using Hölder's inequality and the volume bound.

In order to estimate the first summand above we multiply equation (7.1.1) by $\eta^2 u^{1+2\varepsilon}$ and integrate by parts in order to find

$$\begin{aligned}(1+\varepsilon)\int_M \eta^2 u^{2\varepsilon}|\nabla u|^2 dV_g = \frac{n-2}{4(n-1)}&\left(\int_M \tilde{R}\eta^2 u^{2^\star+2\varepsilon} dV_g - \int_M R\eta^2 u^{2+2\varepsilon} dV_g\right)\\ &- 2\int_M \eta u^{1+2\varepsilon}\nabla\eta\nabla u dV_g - \varepsilon\int_M \eta^2 u^{2\varepsilon}|\nabla u|^2 dV_g.\end{aligned}$$

Inserting this into

$$\begin{aligned}\int_M |\nabla(\eta u^{1+\varepsilon})|^2 dV_g = &\int_M u^{2+2\varepsilon}|\nabla\eta|^2 dV_g + 2(1+\varepsilon)\int_M \eta u^{1+2\varepsilon}\nabla\eta\nabla u dV_g\\ &+ (1+\varepsilon)^2\int_M \eta^2 u^{2\varepsilon}|\nabla u|^2 dV_g\end{aligned}$$

gives

$$\begin{aligned}\int_M |\nabla(\eta u^{1+\varepsilon})|^2 dV_g &= \int_M u^{2+2\varepsilon}|\nabla\eta|^2 dV_g - (1+\varepsilon)\varepsilon\int_M \eta^2 u^{2\varepsilon}|\nabla u|^2 dV_g\\ &\quad + (1+\varepsilon)\frac{n-2}{4(n-1)}\left(\int_M \tilde{R}\eta^2 u^{2^\star+2\varepsilon} dV_g - \int_M R\eta^2 u^{2+2\varepsilon} dV_g\right)\\ &\le \int_M u^{2+2\varepsilon}|\nabla\eta|^2 dV_g + C\left(\int_M \tilde{R}\eta^2 u^{2^\star+2\varepsilon} dV_g - \int_M R\eta^2 u^{2+2\varepsilon} dV_g\right).\end{aligned} \tag{7.2.16}$$

Let us discuss all summands above separately. By the choice of η and the volume bound, we have, using Hölder's inequality,

$$\int_M u^{2+2\varepsilon}|\nabla\eta|^2 dV_g \le \frac{C}{r^2}. \tag{7.2.17}$$

By Hölder's inequality and the volume bound again, we find

$$\int_M \tilde{R}\eta^2 u^{2^\star+2\varepsilon} dV_g \le \left(\int_M |\tilde{R}|^p u^{2^\star} dV_g\right)^{1/p}\left(\int_M (\eta u^\varepsilon)^{2p/(p-1)} u^{2^\star} dV_g\right)^{(p-1)/p}.$$

Observe that $p > n/2$ implies $q = n(p-1)/(p(n-2)) > 1$. Thus by Hölder's inequality

$$\int_M (\eta u^\varepsilon)^{2p/(p-1)} u^{2^\star} dV_g \le \left(\int_M \left(\eta u^{1+\varepsilon}\right)^{2^\star} dV_g\right)^{\frac{1}{q}} \left(\int_{B(x,r)} u^{2^\star} dV_g\right)^{(q-1)/q},$$

which implies

$$\int_M \tilde{R}\eta^2 u^{2^\star+2\varepsilon} dV_g \le A^{1/p} \left(\int_M \left(\eta u^{1+\varepsilon}\right)^{2^\star} dV_g\right)^{2/2^\star} \left(\int_{B(x,r)} u^{2^\star} dV_g\right)^{(2p-n)/np}.$$

Since $p > n/2$ and using Proposition 7.2.6, we find $r = r(M, g, A, B)$, such that

$$\left(\int_{B(x,r)} u^{2^\star} dV_g\right)^{(2p-n)/np} \le \frac{1}{2A^{1/p}C}.$$

For such r we conclude that

$$C \int_M \tilde{R}\eta^2 u^{2^\star+2\varepsilon} dV_g \le \frac{1}{2} \left(\int_M \left(\eta u^{1+\varepsilon}\right)^{2^\star} dV_g\right)^{2/2^\star}. \tag{7.2.18}$$

The last summand is controlled by the volume bound

$$\int_M R\eta^2 u^{2+2\varepsilon} dV_g \le C \int_M u^{2+2\varepsilon} dV_g \le C. \tag{7.2.19}$$

Combining (7.2.15)-(7.2.19) we conclude

$$\left(\int_M \left(\eta u^{1+\varepsilon} dV_g\right)^{2^\star}\right)^{2/2^\star} \le C + \frac{C}{r^2} + \frac{1}{2} \left(\int_M \left(\eta u^{1+\varepsilon}\right)^{2^\star} dV_g\right)^{2/2^\star},$$

and thus

$$\int_{B(x,r/2)} u^{2^\star(1+\varepsilon)} dV_g \le C,$$

with $\varepsilon = \varepsilon(n)$, $r = r(M, g, A, B)$, and $C = C(M, g, A, B, r)$. □

In order to prove Theorem 7.1.2 we need the following Harnack inequality, which can be found in [**Tru68**].

Lemma 7.2.20. *Let $u \in W^{1,2}(M, g)$ be a non-negative solution of the elliptic equation $\Delta u = fu$, with $f \in L^q(M, g)$ for some $q > n/2$. Then there is a constant $C = C(M, g, \|u\|_{L^2(M,g)}, \|f\|_{L^q(M,g)})$ such that*

$$C^{-1} \le u \le C.$$

We now turn to the proof of Theorem 7.1.2.

Proof of Theorem 7.1.2. For $n/2 < q < p$, we have

$$\begin{aligned}\int_M |\tilde{R}|^q u^{(2^\star-2)q} dV_g &= \int_M |\tilde{R}|^q u^{(2^\star-2)q-2^\star} u^{2^\star} dV_g \\ &\le \left(\int_M |\tilde{R}|^p u^{2^\star} dV_g\right)^{q/p} \left(\int_M u^{((2^\star-2)q-2^\star)p/(p-q)+2^\star} dV_g\right)^{(p-q)/p}.\end{aligned}$$

Observe that $((2^\star - 2)q - 2^\star)p/(p-q) + 2^\star \to 2^\star$, as $q \to n/2$. Thus we can find $q > n/2$, such that $((2^\star - 2)q - 2^\star)p/(p-q) + 2^\star \le 2^\star + \varepsilon$, for ε given by Lemma 7.2.14. For such q we have

$$\int_M |\tilde{R}u^{(2^\star-2)}|^q dV_g \le D^{(p-q)/p} A^{q/p}.$$

In particular, we can apply Lemma 7.2.20 to

$$4\frac{n-1}{n-2}\Delta u = \left(\tilde{R}u^{2^\star-2} - R\right) u \tag{7.2.21}$$

and conclude that we have $C_1 \le |u| \le C_2$ with C_1, C_2 depending on M, g, A, B, D. This in turn implies that the right hand side of (7.2.21) is bounded in $L^p(M,g)$. Thus the standard elliptic estimates [**Mor66**, Theorem 6.4.8] in L^p-spaces imply that $\|u\|_{2,p} \le C_3$. □

7.3. Applications

We discuss two applications of Theorem 7.1.2, one of which was in fact a motivation for studying this problem.

7.3.1. Conformal spectrum. Thanks to the work of Li–Yau [**LY82**], El Soufi–Ilias [**ESI86**] and Korevaar [**Kor93**] the scale invariant quantities $\lambda_k(M,g)\operatorname{vol}(M,g)^{2/n}$ are bounded within a fixed conformal class. Thus it is a natural question to ask whether there are metrics realizing $\sup_\phi \lambda_1(M,\phi g)\operatorname{vol}(M,\phi g)^{2/n}$, where the supremum is taken over all smooth positive functions ϕ. In dimension two the conformal covariance of the Laplace operator simplifies the situation tremendously, but it remains a very difficult problem which was resolved only recently (see [**Kok14, Pet14**].) Also, it follows from the appendix in [**CY90**] that there are Hölder continuous maximizers in dimension two, if one additionally imposes L^p curvature bounds for $p > 1$. Theorem 7.1.2 generalizes this partially to higher dimensions.

For $p > n/2$, $A > 0$ and $B > n((n+1)\omega_{n+1})^{2/n}$, denote by $[g]_{A,B}$ the subset of the conformal class $[g]$ consisting of all metrics of the form $u^{4/n-2}g$, such that $u \in W^{2,p}$ with $\operatorname{vol}(M, u^{4/n-2}g) = 1$, $\int_M |R_{u^{4/(n-2)}g}|^p u^{2^\star} dV_g \le A$ and $\lambda_1(M, u^{4/n-2}g) \ge B$. Thanks to [**Pet15**], there are A, B as above such that $[g]_{A,B}$ is non-empty.

We have

Theorem 7.3.1. *Let $p > n/2$, $A > 0$ and $B > n((n+1)\omega_{n+1})^{2/n}$, such that $[g]_{A,B} \neq \emptyset$. Then there is a Hölder continuous positive function u, such that $u^{4/(n-2)}g \in [g]_{A,B}$ and $\lambda_1(M, u^{4/(n-2)}g) = \sup_{h\in[g]_{A,B}} \lambda_1(h)$.*

Proof. Let (u_k) be a sequence of functions such that $u_k^{4/(n-2)}g \in [g]_{A,B}$ and

$$\lim_{k\to\infty} \lambda_1(M, u_k^{4/(n-2)}g) = \sup_{h\in[g]_{A,B}} \lambda_1(M,h).$$

Due to Theorem 7.1.2, (u_k) is bounded in $W^{2,p}$. Thus we have a subsequence (not relabeled) $u_k \rightharpoonup u_*$ in $W^{2,p}$. By the standard embedding results for Sobolev spaces we have that $W^{2,p} \hookrightarrow C^{0,\alpha}$ for some $\alpha > 0$, since $p > n/2$. Thanks to the Theorem of Arzela–Ascoli, the embedding $C^{0,\alpha} \hookrightarrow C^{0,\beta}$ is compact for $\beta < \alpha$. Thus we can extract a further subsequence (again not relabeled) such that $u_k \to u_*$ in $C^{0,\beta}$. Since the functional

λ_1 is continuous with respect to convergence in C^0, we see that $\lambda_1(M, u_*^{4/(n-2)}g) = \sup_{h\in[g]_{A,B}} \lambda_1(h)$.

It remains to prove that $u_*^{4/(n-2)}g \in [g]_{A,B}$. Since we have uniform upper and lower bounds for u_k and thus also for u_*, we can write thanks to (7.1.1) that

$$R_k = \frac{4(n-1)/(n-2)\Delta u_k + Ru_k}{u_k^{2^\star-1}} \tag{7.3.2}$$

and similarly for R_*. Since $\Delta u_k \rightharpoonup \Delta u_*$ in $L^p(M,g)$ and $u_k \to u_*$ in C^0, this implies that $R_k \rightharpoonup R_*$ in $L^p(M,g)$. Moreover, by the uniform upper and lower bounds on u_*, this implies that $R_k \rightharpoonup R_*$ in $L^p(M, u_*^{4/(n-2)}g)$. This implies

$$\int_M |R_*|^p u^{2^\star} dV_g \leq \liminf_{k\to\infty} \int_M |R_k|^p u^{2^\star} dV_g = \liminf_{k\to\infty} \int_M |R_k|^p u_k^{2^\star} dV_g \tag{7.3.3}$$

and thus $u_*^{4/(n-2)}g \in [g]_{A,B}$. $\square$

7.3.2. Isospectral metrics. For a fixed metric g denote by $\mathcal{I}(g)$ the set of all metrics on M isospectral to g. With the same arguments as in in the proof of Theorem 7.3.1 we find

Theorem 7.3.4. *Let (M,g) and $A > 0$ be such that $\mathrm{vol}(M,g) = 1$, $\lambda_1(M,g) > n((n+1)\omega_{n+1})^{2/n}$ and $g \in [g]_{A,\lambda_1(M,g)}$. Then the set $\mathcal{I}(g) \cap [g]_{A,\lambda_1(M,g)}$ is precompact in $C^{0,\alpha}$ for some $\alpha > 0$.*

CHAPTER 8

The systole of large genus minimal surfaces in positive Ricci curvature

8.1. Introduction

In 1985 Choi and Schoen [**CS85**] proved that the space of compact embedded minimal surfaces with bounded genus in closed ambient three manifolds of positive Ricci curvature with bounded genus is compact in the C^k topology for any $k \geq 2$. Conversely, in the present paper we want to study properties of minimal surfaces in such ambient manifolds if the genus becomes unbounded.

Our main result shows that the systole tends to zero as the genus goes to infinity. Recall that the *systole* of a closed surface Σ is defined to be

$$\mathrm{sys}(\Sigma) := \inf\{\mathrm{length}(c) \ : \ c\colon S^1 \to M \text{ non-contracitble}\}.$$

Similarly, the *homology systole* is given by

$$\mathrm{hsys}(\Sigma) := \inf\{\mathrm{length}(c) \ : \ 0 \neq [c] \in H_1(\Sigma;\mathbb{Z})\}.$$

Clearly, we have

$$\mathrm{sys}(\Sigma) \leq \mathrm{hsys}(\Sigma).$$

We can now state our main result.

Theorem 8.1.1. *Assume that (S^3, g) is a three sphere with positive Ricci curvature. Let $\Sigma_j \subset M$ be a sequence of closed, embedded minimal surfaces with* $\mathrm{genus}(\Sigma_j) \to \infty$ *as $j \to \infty$. Then we have*

$$\mathrm{hsys}(\Sigma_j) \to 0,$$

as $j \to \infty$.

As an immediate corollary we get the same type of result for all ambient three-manifolds of positive Ricci curvature for the systole.

Corollary 8.1.2. *Assume (M, g) is a closed three manifold with positive Ricci curvature. Let $\Sigma_j \subset M$ be a sequence of closed, embedded minimal surfaces with $\chi(\Sigma_j) \to -\infty$ as $j \to \infty$. Then we have*

$$\mathrm{sys}(\Sigma_j) \to 0,$$

as $j \to \infty$.

For generic metrics, the compactness theorem by Choi–Schoen implies that there are at most finitely many closed, embedded, minimal surfaces of a given genus in (M, g) as above. Moreover, by recent work of Marques–Neves [**MN17**], any closed three-manifold of positive Ricci curvature admits infinitely many distinct closed, minimal hypersurfaces. Therefore, at least for generic metrics, a sequence of minimal surfaces as in Theorem 8.1.1 exists.

To put our result into some more context, we want to mention recent work by Irie–Marques–Neves[**IMN18**] and Marques–Neves–Song [**MNS17**] on the equidistribution of min-max minimal surfaces for generic metrics. Their results are based on the Weyl law for the min-max widths recently obtained by Liokumovich–Marques–Neves in [**LMN18**]. These results present a remarkable step towards understanding the asymptotic behaviour of the minimal surfaces corresponding to the min-max widths, a question raised in [**Nev14**]. As mentioned above, for generic metrics, our result gives some information about the intrinsic geometry of these surfaces. To our knowledge, it is the first result of this type.

We want to briefly discuss why our result is more subtle than one might expect at first glance. In general, one could expect that $\mathrm{sys}(S_i) \to 0$ for any (i.e. not necessarily minimal) surface S_i in M with $\mathrm{genus}(S_i) \to \infty$ at least as long as S_i is unknotted. However, if e.g. $M = S^3$, one can easily produce counterexamples to this using the Nash–Kuiper theorem: Take a surface S_γ of genus γ with systole $\mathrm{sys}(S_\gamma) \geq c_0 > 0$. By the Nash–Kuiper theorem, there is a $C^{1,\alpha}$-isometric embedding of S_γ in an arbitrarily small ball $B_\delta \subset \mathbb{R}^3$. After smoothing this and applying stereographic projection, we get a sequence of closed, unknotted surfaces of unbounded genus in S^3, which have systole uniformly bounded from below.

Moreover, the result does not hold without any assumptions on the ambient geometry.

Example 8.1.3. Denote by Σ_γ a closed surface of genus γ for $\gamma \geq 2$. It is shown in [**Tol69**] (see also [**Neu76**] for a generalization) that the three-manifold $M = S^1 \times \Sigma_\gamma$ admits fibre bundles

$$\Sigma_\delta \to M \to S^1 \tag{8.1.4}$$

for $\delta = \gamma + n(\gamma - 1)$ and $n \in \mathbb{N}$. Since $\pi_2(S^1) = 0$, the long exact sequence for homotopy groups associated to these fibrations implies that $\Sigma_\delta \to M$ is incompressible, i.e. the induced map $\pi_1(\Sigma_\delta) \to \pi_1(M)$ is injective. It follows from [**SY79**, Theorem 3.1] that there are immersed minimal surfaces S_δ in M which are diffeomorphic to Σ_δ and the induced map on π_1 is given by the inclusion of the fibres from (8.1.4). Moreover, [**FHS83**, Theorem 5.1] implies that these are not only immersions but even embeddings. Since $\pi_1(S_\delta) \to \pi_1(M)$ is injective, we have in particular that

$$\mathrm{sys}(S_\delta) \geq \mathrm{sys}(M) > 0.$$

On the other hand, it follows from [**SY79**, Theorem 5.2] that M does not admit any metric of positive scalar curvature.

Main problems and strategy. Let us for simplicity focus on the systole instead of the homology systole. We want to argue by contradiction and consider a sequence of minimal surfaces $\Sigma_j \subset \tilde{M}$ with $\mathrm{sys}(\Sigma_j) \geq c_0 > 0$ and $\mathrm{genus}(\Sigma_j) \to \infty$. In general, we would like to pass to a limit $\Sigma_j \to \mathcal{L}$ in the class of minimal laminations and argue that $\mathcal{L}$ has a stable leaf, which would easily lead to a contradiction. The problem about this is that we can only do this outside the closed set at which $|A^{\Sigma_j}|^2$ blows-up. A priori, the blow-up set could even be all of $\tilde{M}$. Work of Colding and Minicozzi gives strong structural information about the blow-up set if the surfaces in question have bounded genus. The main step of our proof is to show that the sequence Σ_j as above can be dealt with in this framework. The reason why this is not obvious is that we do not have $-\Delta_{\Sigma_j} d^2(x,\cdot) \leq 0$ globally (as it is the case for minimal surfaces in $\mathbb{R}^3$). Therefore, the assumption on $\mathrm{sys}(\Sigma_j)$ does not directly imply that there is $R = R(c_0)$ such that the instrinsic balls $B^{\Sigma_j}(x, R_0)$ are contained in disks in the extrinsic balls $B(x, R_0)$. Instead, $B^{\Sigma_j}(x, R_0)$ is contained in some disk $D_x^j \subset \Sigma_j$ but D_x^j could leave any mean convex ball $B(x, r)$. The main step is to show that this is impossible after going to a (potentially much) smaller scale.

Organization. In Section 8.2 we provide necessary background from [**CM15**] on Colding–Minicozzi lamination theory of minimal surfaces with some control on the topology. Section 8.3 contains two weak chord-arc properties for minimal surfaces contained in small extrinsic balls of an ambient three manifold. Our main result, Theorem 8.1.1, is proved in Section 8.4.

8.2. Background on Colding–Minicozzi lamination theory

Colding and Minicozzi developed a theory that describes how minimal surfaces of uniformly bounded genus in an ambient three-manifold can degenerate in the absence of curvature bounds. Our arguments are based on their results and we use this section to provide a very brief introduction to those parts of their theory that will be relevant in the present paper. We will focus here on the case of planar domains, since this is sufficient for our purposes.

We start by recalling the definition of a lamination.

Definition 8.2.1 (see Appendix B in [**CM04e**])**.** (1) A codimension one lamination on a 3-manifold M is a collection $\mathcal{L}$ of smooth disjoint surfaces Γ, the so-called leaves, such that $\cup_{\Gamma\in\mathcal{L}}\Gamma$ is closed. Furthermore, for each point $x \in M$, there exists an open neighborhood U of x and a coordinate chart, (U, Φ), with $\Phi(U) \subset \mathbb{R}^3$ so that in these coordinates the leaves in $\mathcal{L}$ pass through $\Phi(U)$ in slices of the form $(\mathbb{R} \times \{t\}) \cap \Phi(U)$.

(2) A foliation is a lamination for which the union of the leaves is all of M.
(3) A minimal lamination is a lamination whose leaves are minimal.
(4) A Lipschitz lamination is a lamination for which the chart maps Φ are Lipschitz.

Given any sequence of minimal surfaces $\Sigma_j \subset M$, we consider the *singular* or *blow-up set*

$$\mathcal{S} = \{z \in M : \inf_{\delta>0} \sup_j \sup_{B(z,\delta)} |A^{\Sigma_j}| = \infty\},$$

i.e. the points z where the curvature blows up. Up to taking a subsequence one can always pass to a limit

$$\Sigma_j \to \mathcal{L} \text{ in } M \setminus \mathcal{S},$$

where the convergence is in $C^{0,\alpha}$ and the limit lamination is a minimal Lipschitz lamination.

In the case of minimal surfaces $\Sigma_j \subset B(0, R_j) \subset \mathbb{R}^3$ with bounded genus and $\partial\Sigma_j \subset \partial B(0, R_j)$ the limit lamination has much more structure than in general, see e.g. the example in [**CM04a**].

We first consider the case of Σ_j being disks. Colding and Minicozzi proved [**CM04b, CM04c, CM04d, CM04e**] that every embedded minimal disk is either a graph of a function or is a double spiral staircase where each staircase is a multivalued graph. More precisely, they show that if the curvature is large at some point (and thus the surface is not a graph), then the surface is a double spiral staircase like the helicoid. In this case a subsequence of Σ_j converges to a lamination by parallel planes away from a singular curve $\mathcal{S}$ - see Theorem 0.1 in [**CM04e**].

A more general case than disks concerns uniformly locally simply connected (in short: ULSC) planar domains.

A sequence of minimal surfaces $\Sigma_j \subset M$ is called *uniformly locally simply connected* if given any compact $K \subset M$ there is some $r > 0$ such that

$$\Sigma_j \cap B(x, r) \text{ consists of disks for any } x \in K.$$

In the case when the sequence Σ_j consists of ULSC but not simply connected planar domain $\partial\Sigma_j \subset \partial B(0, R_j)$ and $R_j \to \infty$, we may assume that there exists some $R > 0$ such that such that

$$\text{some component of } B(0, R) \cap \Sigma_j \text{ is not a disk} \tag{8.2.2}$$

for each j. In this case a subsequence of Σ_j converges to a foliation by parallel planes away from two curves $\mathcal{S}_1$ and $\mathcal{S}_2$. These curves are disjoint, orthogonal to the leaves of the foliation and we have $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ - see Theorem 0.9 in [**CM15**].

The main local structural result we need for ULSC surfaces concerns so-called collapsed leaves, whose existence is described in the next lemma. We assume that $\Sigma_j \to \mathcal{L}'$ in $M \setminus \mathcal{S}$, where Σ_j is a ULSC sequence.

Lemma 8.2.3 (Lemma II.2.3. in [**CM15**]). *Given a point $x \in \mathcal{S} = \mathcal{S}_{\text{ulsc}}$, there exists $r_0 > 0$ so that $B_{r_0}(x) \cap \mathcal{L}'$ has a component Γ_x whose closure $\overline{\Gamma_x}$ is a smooth minimal graph containing x and with boundary in $\partial B_{r_0}(x)$ (so x is a removable singularity for Γ_x).*

The leaves of the limit foliation $\mathcal{L}'$ may not be complete. A special type of incomplete leaves are collapsed leaves. A leaf Γ of $\mathcal{L}'$ is *collapsed* if there exists some $x \in \mathcal{S}_{\text{ulsc}}$ so that Γ contains the local leaf Γ_x given by Lemma Lemma 8.2.3; see Definition II.2.9 in [**CM15**].

By [**CM15**] every leaf of $\mathcal{L}'$ whose closure contains a point in $\mathcal{S}_{\text{ulsc}}$ is collapsed.

Proposition 8.2.4 (see Section II.3. in [**CM15**]). *Each collapsed leaf Γ of $\mathcal{L}'$ has the following properties:*

(1) Given any $y \in \Gamma_{clos} \cap \mathcal{S}_{\text{ulsc}}$, there exists $r_0 > 0$ so that the closure (in $\mathbb{R}^3$) of each component of $B_{r_0}(y) \cap \Gamma$ is a compact embedded disk with boundary in $\partial B_{r_0}(y)$. Furthermore, $B_{r_0}(y) \cap \Gamma$ must contain the component Γ_y given by Lemma 8.2.3 and Γ_y is the only component of $B_{r_0}(y) \cap \Gamma$ with y in its closure.
(2) Γ is a limit leaf.
(3) Γ extends to a complete minimal surface in M.

The sequences Σ_j appearing in this manuscript will all be ULSC. This is equivalent to the fact that the singular set $\mathcal{S}$ is given by $\mathcal{S}_{\text{ulsc}}$, i.e. $\mathcal{S} = \mathcal{S}_{\text{ulsc}}$. Although we will not directly apply the results for non-ULSC surfaces here, some of our arguments (in particular the proof of Lemma 8.4.15) are inspired by those in [**CM15**] for this case.

8.3. Chord arc properties

We need two weak chord-arc properties for minimal surfaces contained in small extrinsic balls of an ambient three manifold. Given $x \in M$ and $r > 0$, we write $B(x, r)$ for the metric ball in (M, g). If $z \in \Sigma$ and $r > 0$, we denote by $B^{\Sigma}(z, r)$ the metric ball of radius in r in Σ with respect to the induced Riemannian metric.

Let (M, g) be a closed Riemannian three manifold. For $R_0 > 0$ sufficiently small, we consider minimal embedded disks Σ in $B = B(x_0, R_0)$ for some $x_0 \in M$. By $\Sigma_{x_0,r}$ we denote the connected component of $\Sigma \cap B(x_0, r)$ that contains x_0.

Theorem 8.3.1. *Let $\Sigma \subset B$ be an embedded minimal disk with $x_0 \in \Sigma$. There is $\alpha > 0$ such that if $B^{\Sigma}(x_0, R) \subset \Sigma \setminus \partial\Sigma$, then $\Sigma_{x_0,\alpha R} \subset B^{\Sigma}(x_0, R/2)$.*

This is proved in [**CM08**] for minimal disks in $\mathbb{R}^3$ and in [**MR06**] under a technical assumption on Σ which might not be satisfied if Σ has points of positive curvature.

The proof of Theorem 8.3.1 is exactly as the proof of [**CM08**, Proposition 1.1]. This does not use that intrinsic subballs $B^{\Sigma}(x, R) \subset \Sigma$ of a minimal disk Σ are disks again, but only that $\Sigma_{x,r}$ is disk provided that $\partial\Sigma \cap B(x, r) = \emptyset$.

We also need a related chord-arc property for uniformly locally simply connected surfaces.

Theorem 8.3.2. *Let $\Sigma \subset B(x, R)$ be a minimal surface with $x \in \Sigma$. Assume that there is $r > 0$, such that $\Sigma \cap B(y, r)$ consists only of disks for any $y \in B(x, R - r)$. Then, given $k \in \mathbb{N}$ such that $kr \leq R$ there is $\beta_k > 0$ such that if $B^{\Sigma}(x, \beta_k r) \cap \partial\Sigma = \emptyset$, then $\Sigma \not\subseteq B(x, kr)$.*

This is stated in [**CM15**, Appendix B.1] with intrinsic instead of extrinsic balls. In our setting, intrinsic balls that are contained in a disk may not be disks themselves. The version stated above is proved as in [**CM15**] with some easy changes using Theorem 8.3.1.

8.4. Proof of the main result

Throughout this section let (M, g) be a closed three manifold with positive Ricci curvature. In order to prove Theorem 8.1.1, we want to argue by contradiction. Therefore, we study properties of a sequence $\Sigma_j \subset (M, g)$ of closed minimal surfaces with $\text{sys}(\Sigma_j) \geq c_0 > 0$ (or $\text{hsys}(\Sigma_j) \geq c_0$). More precisely, we will be concerned with a limit lamination

$$\Sigma_j \to \mathcal{L} \text{ in } M \setminus \mathcal{S}$$

of such a sequence.

We start with a simple observation concerning the maximum of the curvature of a sequence of minimal surfaces in M with unbounded genus. It says, that for a sequence of minimal surfaces of unbounded genus $\Sigma_j \subset M$, we necessarily have $\mathcal{S} \neq \emptyset$.

For simplicity, we will focus on the case of simply connected M, i.e. M is diffeomorphic to S^3, from here on.

Lemma 8.4.1. *Let $\Sigma_j \subset (M, g)$ be a sequence of closed, embedded minimal surfaces with $|\chi(\Sigma_j)| \to \infty$. Then there is a sequence of points $z_j \in \Sigma_j$ such that $|A^{\Sigma_j}|^2(z_j) \to \infty$.*

Proof. Assume that there is a constant $C > 0$, such that

$$\sup_{\Sigma_j} |A^{\Sigma_j}|^2 \le C. \tag{8.4.2}$$

By scaling we may for simplicity assume that $|\sec(M)| \le 1$. Thus, by minimality and the theorem of Gauß–Bonnet, the total curvature satisfies

$$\begin{aligned} \int_{\Sigma_j} |A^{\Sigma_j}|^2 d\mu_{\Sigma_j} &= -2 \int_{\Sigma_j} (K^{\Sigma_j} - \sec(T_x\Sigma_j)) d\mu_{\Sigma_j}(x) \\ &\ge 4\pi|\chi(\Sigma_j)| - 2\,\text{area}(\Sigma_j). \end{aligned} \tag{8.4.3}$$

On the other hand we have

$$\int_{\Sigma_j} |A^{\Sigma_j}|^2 d\mu_{\Sigma_j} \le C\,\text{area}(\Sigma_j) \tag{8.4.4}$$

by assumption. Combining (8.4.3) and (8.4.4), we obtain

$$4\pi|\chi(\Sigma_j)| \le (C + 2)\,\text{area}(\Sigma_j).$$

By assumption the left hand side tends to infinity, therefore we find that

$$\text{area}(\Sigma_j) \to \infty$$

as $j \to \infty$.

The pointwise curvature bound (8.4.2) allows us to pass to a subsequence (not relabeled) such that

$$\Sigma_j \to \mathcal{L} \text{ in } C^{0,\alpha},$$

where $\mathcal{L}$ is a Lipschitz lamination, whose leaves are smooth, complete minimal surfaces. Moreover, since $\text{area}(\Sigma_j) \to \infty$, a standard argument shows that there needs to be at least one leaf Γ with stable universal cover, see e.g. the proof of Theorem 1.3 in [**CKM17**] and the references therein. It follows from [**FCS80**] and [**SY83**], that $\tilde{\Gamma}$ is diffeomorphic to S^2. Since S^3 does not contain any embedded real projective plane, we need to have $\tilde{\Gamma} = \Gamma$. In particular, Γ is a closed, two sided, stable minimal surface in M, which gives the desired contradiction. ☐

By Lemma 8.4.1, in order to prove Theorem 8.1.1 we are forced to study the structure of a limit lamination of Σ_j in the presence of a non-empty singular set. Before we turn to this, we need to recall the following elementary topological lemma.

Lemma 8.4.5. *Let Σ be a surface, $x \in \Sigma$, and $R > 0$, then $\pi_1(B^\Sigma(x, R), x)$ is generated by curves of length at most $3R$.*

Since the Hurewicz homomorphism $\pi_1(B^\Sigma(x, R), x) \to H_1(B^\Sigma(x, R); \mathbb{Z})$ is surjective, we immediately get the following corollary.

Corollary 8.4.6. *Let Σ be a surface, $x \in \Sigma$, and $R > 0$, then $H_1(B^\Sigma(x, R); \mathbb{Z})$ is generated by curves of length at most $3R$.*

For convenience of the reader we give a brief sketch of the argument.

Proof of Lemma 8.4.5. Let $c\colon S^1 \to B^\Sigma(x,R)$ be a loop based at x. Choose a subdivision

$$0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1,$$

such that

$$\mathrm{length}(c_{|[t_i,t_{i+1}]}) \le R.$$

Fix curves $d_i\colon I \to B^\Sigma(x,R)$ with $d_i(0) = x$ and $d_i(1) = c(t_i)$ and such that

$$\mathrm{length}(d_i) \le R.$$

We can then write

$$\begin{aligned} c = \left(c_{|[t_{k-1},t_k]} * d_{k-1}\right) * \left(\bar{d}_{k-1} * c_{|[t_{k-2},t_{k-1}]} * d_{k-2}\right) * \\ \cdots * \left(\bar{d}_2 * c_{|[t_1,t_2]} * d_1\right) * \left(\bar{d}_1 * c_{|[t_0,t_1]}\right), \end{aligned}$$

which implies the assertion. □

We now fix $r_0 > 0$ such that the results from Section 8.3 apply in any ball $B(x,r_0)$. In particular, any ball $B(x,r) \subset M$ with $r \le r_0$ is assumed to have strictly mean convex boundary.

Lemma 8.4.7. *Let $\Sigma \subset M$ be a closed minimal surface such that all non-separating curves have length at least l_0. There is $l_1 \le \min(r_0, l_0/2)$ depending on M and l_0 with the following property. Let c be a curve in Σ which is contained in some ball $B(x, r_0/2)$ but non-contractible in $\Sigma \cap B(x,r_0)$ and assume that any other curve d with these two properties satisfies*

$$\mathrm{length}(c) \le 2\,\mathrm{length}(d).$$

If we have

$$\mathrm{length}(c) \le l_1$$

the two connected components Σ_1 and Σ_2 of $\Sigma \setminus c$ satisfy

$$\Sigma_i \cap \partial B(x,r_0) \ne \emptyset \tag{8.4.8}$$

for $i = 1,2$.

Proof. Write $R_0 = \mathrm{length}(c)/8$ and assume that $R_0 \le r_0/2$ and $\mathrm{length}(c) \le l_0/2$. Let $y \in \Sigma \cap B(x, r_0/2)$. We claim that there is a unique disk $D_y \subset \Sigma \cap B(x,r_0)$ with

$$B^\Sigma(y,R_0) \subset D_y \tag{8.4.9}$$

and

$$\partial D_y \subset \partial B^\Sigma(y,R_0). \tag{8.4.10}$$

By Lemma 8.4.5, if there is a non-contractible curve σ in $B^\Sigma(y,R_0)$, we can find a simple closed, non-contractible curve σ' with

$$\mathrm{length}(\sigma') \le 3R_0 < \mathrm{length}(c)/2. \tag{8.4.11}$$

By assumption, σ' has to be contractible in $\Sigma \cap B(x,r_0)$. In particular, there is a disk $D_{\sigma'} \subset B(x,r_0) \cap \Sigma$ with boundary σ'. We can iterate this argument until we obtain the desired disk $D_y \supset B^\Sigma(y,R_0)$. If Σ is not a sphere it follows immediately, that such a disk is unique. In the case of Σ being a sphere there are two such disks in Σ. However, by the choice of r_0 not both of these disks can be entirely contained in $B(x,r_0)$.

It follows from Theorem 8.3.1 and the convex hull property, that we can find some small $\alpha > 0$ such that

$$\Sigma \cap B(y, \alpha R_0) \text{ consists of disks for any } y \in B(x, r_0/2). \tag{8.4.12}$$

Choose $z \in c$ and take $k \in \mathbb{N}$ such that $k\alpha \geq 9$. In particular, by Theorem 8.3.2, there is $\beta_k > 1$ such that the connected component of $B^\Sigma(z, \beta_k \alpha R_0) \cap \Sigma_i \cap B(x, r_0)$ containing z is either all of Σ_i or intersects $\partial B(z, 9R_0)$ non-trivially. If we can rule out the former case it follows from the convex hull property that

$$\Sigma_i \cap \partial B(x, r_0) \neq \emptyset, \tag{8.4.13}$$

since

$$c \subset B(z, 8R_0)$$

If Σ_i is a disk we can not have $\Sigma_i \subset B^\Sigma(z, \beta_k \alpha R_0) \cap \Sigma_i \cap B(x, r_0)$ since this would imply that c is contractible in $\Sigma \cap B(x, r_0)$ contradicting the assumption. If Σ_i is not a disk it contains at least one non-separating curve d, since $\partial \Sigma_i$ is connected. For l_1 and hence R_0 sufficiently small, we can not have $d \subset B^\Sigma(x, \beta_k \alpha R_0)$, since, by Corollary 8.4.6, this would imply that we could find a non-separating curve d' having

$$\text{length}(d') \leq 3\beta_k \alpha R_0 < l_0 \tag{8.4.14}$$

contradicting the assumptions. □

Below, we will solve a Plateau problem in $M \setminus \Sigma$ with boundary given by a curve c as above. In this situation, Lemma 8.4.7 implies that Σ is a useful barrier.

Lemma 8.4.15. *Given l_0 there is l_2 depending on l_0 and M such that if $\Sigma \subset M$ is a closed minimal surface all of whose non-separating curves have length at least l_0, then all separating curves in Σ which are non-contractible in balls $B(x, r_0)$ have length at least l_2.*

We will apply this to two types of curves. On the one hand, applied to homologically trivial non-contractible curves, this implies that the homology systole of a sequence Σ_j tends to 0 if we can show that the systole does so. On the other hand, we will apply it to short curves bounding (large) disks in Σ_j in order to understand the convergence of Σ_j to a limit lamination.

Proof. We argue by contradiction and assume that we can find a sequence of minimal surfaces Σ_j such that

(1) All non-separating curves in Σ_j have length at least l_0
(2) There is a mean convex ball $B(x, r_0)$ and curves $c_j \subset \Sigma_j \cap B(x, r_0)$ which are separating in Σ_j, non-contractible in $\Sigma_j \cap B(x, r_0)$, and

$$\text{length}(c_j) \to 0.$$

By choosing a different c_j if necessary we may in addition assume that any separating curve $d_j \subset \Sigma_j$ which is contained in some mean convex ball $B(y, r_0)$ and non-contractible in $\Sigma_j \cap B(y, r_0)$ satisfies

$$\text{length}(c_j) \leq 2\,\text{length}(d_j)$$

Since M is simply connected, Σ_j separates M into two mean-convex connected components,

$$M \setminus \Sigma_j = M_j^1 \cup M_j^2.$$

Clearly, c is null homologous in both of them. In addition, we claim that least one of M_j^1 and M_j^2 has the following property: If length$(c_j) \leq l_1$ from Lemma 8.4.7, then any surface $S \subset M_j^i$ with $\partial S = c$ has

$$S \cap \partial B(x, r_0) \neq \emptyset. \tag{8.4.16}$$

If this was not the case, we would find $S_j^1 \subset M_j^1 \cap B(x, r_0)$ and $S_j^2 \subset M_j^2 \cap B(x, r_0)$ such that $\partial S_j^i = c$. The surface $S_j = S_j^1 \cup S_j^2 \subset B(x, r_0)$ is a closed surface and separates $B(x, r_0)$ into two connected components. Moreover (8.4.16) does not hold for S, so that one of these components is contained in $B(x, r_0 - \delta)$ for some small $\delta > 0$. By construction, this component contains a component of $\Sigma_j \setminus c_j$ contradicting Lemma 8.4.7.

Let M_j^1 be the component having property (8.4.16). By [**HS79**] we can find a stable minimal surface $\Gamma_j \subset M_j^1$ with $\partial \Gamma_j = c_j$ which minimizes area among all surfaces in M_j^1 which have boundary c_j. Up to taking a subsequence, we may assume that $c_j \subset B(x, r_j)$ for radii $r_j \to 0$. It follows from (8.4.16) that

$$\Gamma_j \cap \partial B(x, r_0) \neq \emptyset \tag{8.4.17}$$

for any j. Moreover, by the curvature estimates [**Sch83**], there is a constant C such that

$$\sup_{\Gamma_j \cap (M \setminus B(x,r))} (r - r_j)^2 |A^{\Gamma_j}|^2 \leq C \tag{8.4.18}$$

for any $r > r_j$. In particular, we can pass to a subsequence such that

$$\Gamma_j \to \mathcal{L} \tag{8.4.19}$$

in $C^{0,\alpha}_{loc}(M \setminus \{x\})$, where $\mathcal{L}$ is a minimal Lipschitz lamination. Since Γ_j is stable, the same argument as in [**CKM17**, Lemma 4.1] implies that the lamination $\mathcal{L}$ extends to a lamination $\tilde{\mathcal{L}}$ across $\{x\}$ with stable leaves. From (8.4.17), we find that there is a leaf $\bar{\Gamma} \subset \tilde{\mathcal{L}}$ with

$$\bar{\Gamma} \cap \partial B(x, r_0) \neq \emptyset. \tag{8.4.20}$$

In particular, $\bar{\Gamma}$ is non-empty. Moreover, since M is simply connected, $\bar{\Gamma}$ is two-sided. Since M has positive Ricci curvature, this is a contradiction since $\bar{\Gamma}$ is a non-empty, two-sided, closed, stable minimal surface in M. $\square$

Remark 8.4.21. For curves that are non-contracitble in $\Sigma \cap B(x, r)$ but contractible in Σ, it should be possible to extend Lemma 8.4.7 to bumpy metrics of positive scalar curvature. In this situation one component of $\Sigma_j \setminus c_j$ is a planar domain and one can write large parts of this component as graph over Γ_j. This can then be used to construct a non-trivial Jacobi field on Γ

Proof of Theorem 8.1.1. We argue by contradiction and assume that we have sequence of minimal surfaces $\Sigma_j \subset M$ with

$$\operatorname{hsys}(\Sigma_j) \geq c_0 > 0$$

for some positive constant c_0. We claim that this implies, that there is $r_1 > 0$ such that

$$\Sigma_j \cap B(x, r_1) \text{ consists of disks for any } x \in M. \tag{8.4.22}$$

In fact, if we apply Lemma 8.4.15 to Σ_j we get some $l_2 > 0$ such that all curves in Σ_j of length at most l_2 are contractible in some mean convex ball $B(x, r_0)$. In particular, it follows from Lemma 8.4.5 that any intrinsic ball $B^\Sigma(z, l_2/3)$ is contained in some disk D_z with

$$B^\Sigma(z, l_2/3) \subset D_z \subset \Sigma_j \cap B(z, r_0).$$

The claim now easily follows with $r_1 = \alpha l_2/3$ from Theorem 8.3.1, where also $\alpha > 0$ is from Theorem 8.3.1.

Thanks to (8.4.22) and [**Whi15**], we can pass to a subsequence such that

$$\Sigma_j \to \mathcal{L} \text{ in } M \setminus \mathcal{S}$$

outside the singular set $\mathcal{S}$ which is contained in a union of C^1-curves. It follows from Lemma 8.4.1, that $\mathcal{S} \neq \emptyset$. In particular, we can pick $x \in \mathcal{S}$ and the associated collapsed leaf Γ_x. Moreover, since Γ_x is a limit leaf of $\mathcal{L}$ it is stable by [**MPR10**]. It follows from Proposition 8.2.4 that Γ_x extends to a complete minimal surface $\bar{\Gamma}$ in M and that $\mathcal{S} \cap \bar{\Gamma}$ is discrete. In particular, by [**FCS80**] and [**SY83**] once again, also $\bar{\Gamma}$ is stable and its universal cover is diffeomorphic to S^2. Since M is simply connected, it does not contain any one-sided surfaces and we conclude that $\bar{\Gamma}$ is a two-sided, closed, stable minimal surface in M. This is clearly a contradiction, since M has positive Ricci curvature. □

Proof of Corollary 8.1.2. Let $\Sigma_j \subset M$ be a sequence of minimal surfaces with $\chi(\Sigma_j) \to -\infty$. We denote by $\pi \colon \tilde{M} \to M$ the universal covering and consider $\bar{\Sigma}_j := \pi^{-1}(\Sigma_j) \subset \tilde{M} \cong S^3$. Clearly, $\bar{\Sigma}_j$ is a sequence of minimal surfaces. Moroever, since $\tilde{M}$ has positive Ricci curvature, the surfaces $\bar{\Sigma}_j$ are connected and have

$$\chi(\bar{\Sigma}_j) \le \chi(\Sigma_j) \to -\infty,$$

so that

$$\operatorname{genus}(\bar{\Sigma}_j) \to \infty,$$

since the covering map π restricts to covering maps $\bar{\Sigma}_j \to \Sigma_j$. We can then apply Theorem 8.1.1 to the sequence $\bar{\Sigma}_j$ and in particular find that

$$\operatorname{sys}(\bar{\Sigma}_j) \to 0.$$

Since $\pi \colon \bar{\Sigma}_j \to \Sigma_j$ induces an injective map on fundamental groups, this implies

$$\operatorname{sys}(\Sigma_j) \le \operatorname{sys}(\bar{\Sigma}_j) \to 0.$$ □

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