Unmeasured Bohmian properties and their measurement through local-in-position weak values for assessing non-contextual quantum dynamics

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Abstract. We present a thorough discussion about the significance of unmeasured properties, defined in terms of Bohmian trajectories, to provide information on the dynamics of general quantum systems. We prove the existence of a one-to-one correspondence between any unmeasured Bohmian property and a local-in-position weak value of the corresponding observable. Consequently, as long as weak values are experimentally accessible, unmeasured Bohmian properties are measurable and always yield a positive semi-definite output value probability distribution. Importantly, while generalized (positive-operator valued) measurements of multi-time expectation values happen to be contextual, local-in-position weak values are, by construction, free from quantum backaction effects. Therefore, we conclude that unmeasured Bohmian properties, measurable through weak values, can be understood as a genuine representation of the intrinsic dynamics of quantum systems. The dwell time, the work distribution or the high-frequency electrical current of quantum systems are shown to be paradigmatic examples of the physical soundness of unmeasured Bohmian properties.

1. Introduction

Properties of classical systems are well defined regardless of whether these properties are being measured or not. Therefore, evaluating a property of a system at time t_1 and correlating the outcome with the value of the same (or another) property at a later time t_2 provides an unequivocal way of representing the dynamics of classical systems. In quantum mechanics, however, Bell [1] as well as Kochen and Specker [2], showed that measurements cannot be thought of as simply revealing the underlying properties of the system in a way that is independent of the context in which the observable is measured. The result of correlating the outcome of measuring an observable at time t_1 with that at t_2 of the same (or another) observable depends, in general, upon the specification of the experimental setup. This property of quantum mechanics is known as contextuality and the unavoidable contamination that measurements induce on the subsequent evolution of quantum systems is commonly referred to as the quantum backaction [3, 4].

Take for example the problem that arises, due to quantum backaction, when trying to define a quantum work distribution. One can think of evaluating quantum work by means of a two-time projective measurement protocol of the energy [5, 6]. However, the first measurement projects the initial state into an energy eigenstate and hence it prevents the possibility of capturing any coherent evolution of the energy beyond that of a Hamiltonian eigenstate. Whilst a number of alternative protocols have been proposed to alleviate this problem (involving weak and collective measurements [7, 8, 9, 10]), the backaction of the measuring apparatus on the measured system is, in all existing protocols, an undesired side-effect that yields a list of incompatible definitions of quantum work. This has culminated in a "no-go" theorem that states that, in fact, there cannot exist a (super)operator for work that simultaneously satisfies all the physical properties required [11].

It is then essential to look for dynamic properties of quantum systems that are not contaminated by the measuring apparatus. One could naively think that avoiding the quantum backaction is possible by simply making the coupling between the system and measuring apparatus very weak, e.g., using indirect or weak measurements [12, 13]. However, as it will be proven here for general observables, aside from the trivial case where the state of the system is an eigenstate of one of the measured observables, this is not possible.

The difficulties in getting apparatus independent information about the dynamics of quantum systems is rooted in the foundations of orthodox quantum mechanics: quantum states cannot be directly associated to a defined value of a given property unless such property is explicitly measured. The only exception is when the state itself is an eigensate of the measured property. There are certainly many physical scenarios whose conceptual understanding will be greatly simplified by the possibility of talking about the physical reality of the properties of quantum objects without the necessity of directly measuring them. Besides the already mentioned quantum work distribution, one could think, e.g., in the time needed by an electron to cross a tunneling region without the need to locate scintillating screens in the middle of the path of the electron, at the beginning and end of the tunneling region.

It is well known among the scientific community that the association of the reality of a property of a quantum system to the fact of being measured (the so-called "eigenvalue-eigenstate link") is a "deliberate theoretical choice" that is "not forced on us by experimental facts" [14]. In this respect, there are other quantum theories, in empirical agreement with the orthodox theory, where the reality of the properties of quantum objects is independent of the measurement [14]. These other theories allow to evaluate properties of quantum systems even when they are not measured (i.e., just as in classical mechanics). There is thus a renewed interest in defining unmeasured properties within such "quantum theories without observers" [15, 16, 17]. We use here the so called Bohmian theory, which introduces the reality of the position of quantum objects at an ontological level independently of its measurement. From the unmeasured position of quantum objects, it is then straightforward to define any other unmeasured (Bohmian) property [18, 19, 20].

For static properties (involving one-time measurements), it is well-known that ensemble values of orthodox properties become independent of the measuring protocol and, hence, that they are identical

to the unmeasured Bohmian properties. Due to quantum backaction, the situation becomes more complicated when dealing with dynamic properties that involve sequential (in time) measurements. In this paper, we show that unmeasured Bohmian properties are, by construction, always linked to a positive probability distribution. It is also easy to show that unmeasured Bohmian two-time correlation functions agree with the measured orthodox predictions when the involved property of the quantum system is well-defined (i.e., the quantum system is defined by a property eigenstate before one of the two measurements). Importantly, we show that unmeasured Bohmian properties can be identified with local-in-position weak values as defined by Aharonov-Albert-Vaidman [21] and hence that the unmeasured Bohmian properties are indeed measurable properties. However, contrarily to generalized (positive-operator valued) measurements of multi-time expectation values which are contextually dependent on the measuring protocol, local-in-position weak values are, by construction, free from quantum backaction effects. Therefore, unmeasured Bohmian properties, measurable through weak values, can be understood as a genuine representation of the unperturbed (intrinsic) dynamics of quantum systems.

This paper is structured as follows. In section 2 we will discuss one-time and two-time expectation values defined in terms of positive-operator valued measures. We will show how the contextuality is present in the two time expectation values, whilst one time expectation values are independent of the measuring apparatus. In section 3 we will discuss unmeasured properties of quantum systems derived from the ontology of Bohmian mechanics. We will identify circumstances where these ensemble values of unmeasured properties are equal to the orthodox ones. In section 4 we will provide a connection between the unmeasured properties and the local-in-position weak values. This shows that these unmeasured properties can actually be measured. In section 5 we will discuss some paradigmatic examples that demonstrate the soundness of the unmeasured Bohmian properties: the quantum work distribution, the high frequency quantum noise and the tunneling time. A summary and conclusions of the work are presented in section 6.

2. Measured properties

To describe the indirect or weak measurement of a property of a quantum system we consider, as usual, three different Hilbert spaces [12]. First, the quantum system of interest from which we want to get information which is described by the collective degree of freedom x. Second, the ancilla, which interacts directly with the system and is represented by the collective degree of freedom y. And third, the pointer, which interacts directly with the ancilla and is represented by the collective degree of freedom z. Then, the weak or indirect measurement of the properties of the system is, in fact, a strong measurement of the properties of the ancilla. That is, while the ancilla-pointer interaction must be strong, and hence a given position of the pointer z after a measurement is linked to a single position of the ancilla y (position eigenstate of y), the system-ancilla interaction can be more general and thus a given position of the ancilla y after the measurement cannot always be linked to a single eigenstate of the system. For the sake of clarity we will omit the reference to the pointer unless necessary.

2.1. One-time (measured) expectation values

Let us first consider the expectation value of a property of a quantum system at time t_1, i.e.:‡.

$$\langle y(t_1)\rangle = \int dy_k y_k P(y_k), \tag{1}$$

where $P(y_k)$ is the probability of reading-out a particular value $y_k = z_k$. In order to relate the probability distribution of outcomes $P(y_k)$ with the degrees of freedom of the system x, we here follow a generalized quantum Von Neumann measurement protocol for weak (generalized) measurements [12, 22, 23]. We assume the full state of the system-ancilla-pointer to be initially

‡ All integrals are definite integrals over all the possible values of the variables x, y and z $(-\infty, +\infty)$

described by a separable state vector:

$$|\Psi(0)\rangle = \sum_{i=1} c_i(0)|s_i\rangle \otimes \int a(y,0)|y\rangle dy \otimes \int f(z,0)|z\rangle dz, \tag{2}$$

where the system state vector $|\psi(0)\rangle = \sum_{i=1} c_i(0)|s_i\rangle$ has been defined using the eigenstates $|s_i\rangle$ of the operator $\hat{S}$ of interest, with $\hat{S}|s_i\rangle = s_i|s_i\rangle$. Without the loss of generality, we chose here a discrete and nondegenerate spectrum $\{s_1, s_2, s_3....\}$ of the operator $\hat{S}$. The (ancilla) state vector $|\phi_W(0)\rangle = \int a(y,0)|y\rangle dy$ interacts with the system and also with the (pointer) state vector $|\phi_P(0)\rangle = \int f(z,0)|z\rangle dz$.

First, a pre-measurement (unitary) evolution from $t = 0$ to t_1 entangles the ancilla with the system and the pointer with the ancilla as follows (a more detailed derivation can be found in Appendix A):

$$|\Psi(t_1)\rangle = \sum_{i=1} c_i|s_i\rangle \otimes \int a(y-\lambda s_i)|y\rangle dy \otimes \int f(z-y)|z\rangle dz. \tag{3}$$

The original ancilla wave function $a(y,0)$ splits into several wave functions $a(y-\lambda s_i)$ with $i = 1, 2, ...$ We have defined λ as a macroscopic parameter with dimensions of $[y]/[S]$ that relates y to s_i. The shape of $a(y_k - \lambda s_i)$ is arbitrary and includes, in particular, strong (projective measurement) interactions when $\int\ dy\ a(y-\lambda s_i)a(y-\lambda s_j) = \delta_{i,j}$ and weaker (non-projective measurement) interactions when $\int\ dy\ a(y-\lambda s_i)a(y-\lambda s_j) \neq \delta_{i,j}$. We have defined $\delta_{i,j}$ as a Kronecker delta function. The only two conditions imposed on the ancilla wave functions $a(y-\lambda s_i)$ to be representative of an indirect or weak measurement are: (i) $\int y|a(y-\lambda s_i)|^2 dy = \lambda s_i\ \ \forall i$, which implies that the center of mass of $|a(y-\lambda s_i)|^2$ is λs_i, and (ii) $\int |a(y-\lambda s_i)|^2 dy = 1\ \ \forall i$, which simply states that the ancilla wave function is well normalized.

Secondly, the read-out process is described by the non-unitary operator $\hat{I}_S \otimes \hat{I}_W \otimes \hat{\mathbb{P}}_{z_k}$, where $\hat{I}_S$ is the identity operator defined in the system Hilbert space, $\hat{I}_W$ is the identity operator defined in the ancilla Hilbert space, and $\hat{\mathbb{P}}_{z_k} = |z_k\rangle\langle z_k|$ is a projector acting on the Hilbert space of the pointer. As mentioned previously, it becomes now evident that a weak or indirect measurement of a system is just a strong measurement of an ancilla that is entangled with the system. The non-unitary operator $\hat{I}_S \otimes \hat{I}_W \otimes \hat{\mathbb{P}}_{z_k}$ causes the collapse of the pointer wave function providing the read-out value $z_k = y_k$ and the measured state becomes $|\Psi_k(t_1)\rangle = \sum_{i=1} c_i a(y_k - \lambda s_i)|s_i\rangle \otimes |y_k\rangle \otimes |z_k\rangle$. Therefore, the state of the system can be effectively represented by:

$$|\psi_k(t_1)\rangle = \sum_{i=1} a(y_k - \lambda s_i)c_i|s_i\rangle, \tag{4}$$

where the subscript k indicates (the perturbation that the system has suffered due to) the measurement of the pointer value $z_k = y_k$.

The probability of measuring a particular pointer position y_k can be then easily evaluated from Born's rule $P(y_k) = \langle\Psi_k(t_1)|\Psi_k(t_1)\rangle = \langle\psi_k(t_1)|\psi_k(t_1)\rangle$ applied to the non-normalized state in (4). While the probability distribution $P(y_k) = \sum_{i=1} |c_i|^2|a(y_k - \lambda s_i)|^2$ clearly depends on the type of ancilla that we are considering, the expectation value in (1),

$$\langle y(t_1)\rangle = \int y_k \sum_{i=1} |c_i|^2|a(y_k - \lambda s_i)|^2 dy_k = \sum_{i=1} |c_i|^2 \lambda s_i = \lambda\ \langle\hat{S}\rangle, \tag{5}$$

only depends on the system state $|\psi(t_1)\rangle = \sum_{i=1} c_i|s_i\rangle$. As we already anticipated, expectation values of static (one-time) properties provide information of the system without any contamination from the measuring apparatus.

2.2. Two-time (measured) expectation values

One can now generalize the above measurement scheme to account for a second measurement of another observable $\hat{G}$ at time $t_2 > t_1$. By repeatedly reading-out the positions y_k (at t_1) and y_ω

(at t_2) for a large number of identically prepared experiments, we can compute the corresponding two-time correlation function $\langle y(t_2)y(t_1)\rangle$ as:

$$\langle y(t_2)y(t_1)\rangle = \int dy_k \int dy_\omega \; y_k y_\omega P(y_\omega, y_k), \tag{6}$$

where $P(y_\omega, y_k)$ is the joint probability of subsequently reading-out the values y_k and y_ω at times t_1 and t_2, respectively.

To evaluate $P(y_\omega, y_k)$ in (6) we simply need to apply the above protocol to the final state in (4). We first let the state in (4) to evolve freely from t_1 till t_2 according to the time-evolution operator $\hat{U} = \exp(i\hat{H}(t_2 - t_1)/\hbar)$, where $\hat{H}$ is the Hamiltonian that dictates the evolution of the system degrees of freedom x in the absence of any interaction with the ancilla and pointer degrees of freedom. For convenience, we write the state of the system in terms of the eigenstates of the operator $\hat{G}$, i.e., $|g_j\rangle$, using the transformation $|s_k\rangle = \sum_j \beta_{k,j}|g_j\rangle$. We then rewrite the state of the system $\hat{U}|s_i\rangle = \sum_k \gamma_{i,k}|s_k\rangle$ in terms of the new basis as $\hat{U}|s_i\rangle = \sum_{k,j} \gamma_{i,k}\beta_{k,j}|g_j\rangle$. More compactly, $\hat{U}|s_i\rangle = \sum_j c_{j,i}|g_j\rangle$, where $c_{j,i} = \sum_k \gamma_{i,k}\beta_{k,j}$. Therefore, the state of the system right before the second pre-measurement can be written as the (non-normalized) state:

$$|\psi_k(t_2)\rangle = \sum_{i,j} a(y_k - \lambda s_i)c_i c_{j,i}|g_j\rangle. \tag{7}$$

Subsequently, under the assumption that there is no correlation between the ancilla degrees of freedom at times t_1 and t_2, the system state vector in (7) undergoes a second pre-measurement evolution and the system becomes entangled again with the ancilla and the pointer wave functions (see also Appendix A):

$$|\Psi_k(t_2)\rangle = \sum_{i,j} a(y_k - \lambda s_i)c_i c_{j,i}|g_j\rangle \otimes \int a(y - \lambda g_j)|y\rangle dy \otimes \int f(z - y)|z\rangle dz, \tag{8}$$

where now $a(y - \lambda g_j)$ is the pointer wave function displaced by λg_j.

The read-out of the pointer position (for an output value y_ω) at time t_2 is described again by a non-unitary operator $\hat{I}_S \otimes \hat{I}_W \otimes \hat{\mathbb{P}}_{z_w}$ with $\hat{\mathbb{P}}_{z_w} = |z_w\rangle\langle z_w|$. This non-unitary operator causes the collapse of the state in (8) into $|\Psi_{k,\omega}(t_2)\rangle = \sum_{i,j} c_i c_{j,i} a(y_k - \lambda s_i)a(y_\omega - \lambda g_j)|g_j\rangle \otimes |y_\omega\rangle \otimes |z_w\rangle$, and so the state of the system can be effectively written as:

$$|\psi_{k,\omega}(t_2)\rangle = \sum_{i,j} c_i c_{j,i} a(y_k - \lambda s_i)a(y_\omega - \lambda g_j)|g_j\rangle. \tag{9}$$

Born's rule can be used again to write the probability $P(y_\omega, y_k) = \langle\Psi_{k,\omega}(t_2)|\Psi_{k,\omega}(t_2)\rangle = \langle\psi_{k,\omega}(t_2)|\psi_{k,\omega}(t_2)\rangle$ of subsequently measuring y_k and y_ω as:

$$P(y_\omega, y_k) = \sum_j \sum_{i,i'} c^*_{i'} c_i c^*_{j,i'} c_{j,i} a^*(y_k - \lambda s_{i'}) \; a(y_k - \lambda s_i)|a(y_\omega - \lambda g_j)|^2. \tag{10}$$

By introducing the probability $P(y_\omega, y_k)$ in (10) into (6) we finally get:

$$\langle y(t_2)y(t_1)\rangle = \lambda \sum_{i,i'} \int dy_k y_k a(y_k - \lambda s_i)a^*(y_k - \lambda s_{i'})\langle\psi(t_1)|s_{i'}\rangle\langle s_{i'}|\hat{U}^\dagger\hat{G}\hat{U}|s_i\rangle\langle s_i|\psi(t_1)\rangle, \tag{11}$$

where we have used $\int dy_\omega \; y_\omega \; |a(y_\omega - \lambda g_j)|^2 = \lambda g_j$ and $\hat{G} = \sum_j g_j|g_j\rangle\langle g_j|$ together with $c_i = \langle s_i|\psi(t_1)\rangle$ and $c_{j,i} = \langle g_j|U|s_i\rangle$.

Expression (11) is completely general and describes the expectation value of the two-time correlation function of $\hat{S}$ and $\hat{G}$ at times t_1 and t_2. At this point what is significant is that in (11)

we have not been able to eliminate the dependence of the ancilla degrees of freedom $a(y_k - \lambda s_i)$ and $a^*(y_k - \lambda s_{i'})$ on $\langle y(t_2)y(t_1)\rangle$. Therefore, contrarily to what happens to one-time expectation values in (5), different types of measurements (ancillas) will provide different time-correlation functions. Therefore, multiple-time correlation functions such as $\langle y(t_2)y(t_1)\rangle$ are not universal properties of quantum systems, but are contextual properties that depend on the measuring apparatus itself.

The only scenario where the outcome of the second measurement does not depend on the first measurement is when the initial state of the system is an eigenstate of the operator $\hat{S}$, i.e., $|\psi(t_1)\rangle = |s_k\rangle$. Then the first measurement always yields the same output result $y_k = \lambda s_k$ without having perturbed the state of the system, and hence the second measurement happens to be independent of the first measurement. Mathematically this can be stated as:

$$\langle y(t_2)y(t_1)\rangle = \lambda^2 s_k \langle s_k|\hat{U}^T\hat{G}\hat{U}|s_k\rangle = \lambda^2\langle\hat{G}(t_2)\rangle\langle\hat{S}(t_1)\rangle, \tag{12}$$

where we have used that $\langle\psi(t_1)|s_{i'}\rangle\langle s_i|\psi(t_1)\rangle = \langle s_k|s_{i'}\rangle\langle s_i|s_k\rangle = \delta_{i,k'}\delta_{i',k'}$ and that $\int dy y|a(y-\lambda s)|^2 = \lambda s$. Equivalently, if $\langle s_{i'}|\hat{U}^\dagger\hat{G}\hat{U}|s_i\rangle = g_w\delta_{i',i}$ which means that the evolved state $\hat{U}|s_i\rangle$ is an eigenstate of $\hat{G}$, then (11) can be also written as $\langle y(t_2)y(t_1)\rangle = \lambda^2 g_w\langle\psi(t_1)|\hat{S}|\psi(t_1)\rangle = \lambda^2\langle\hat{G}(t_2)\rangle\langle\hat{S}(t_1)\rangle$. In these two scenarios, since the results $\langle\hat{G}(t_2)\rangle$ and $\langle\hat{S}(t_1)\rangle$ are non-contextual, then the two-time correlation function in (12) represents also a non-conextual correlation function. Unfortunately, this result is not general enough and is invalid in many practical situations, where initial states are coherent superpositions of observable eigenstates §.

2.2.1. Ideally-weak measurements: At this point, one could naively think that avoiding the quantum backaction is possible by simply making the coupling between the system and measuring apparatus very weak. To show that this line of thought is wrong, we define an ideally-weak measurement as the one where the system-ancilla coupling is minimized. This is mathematically equivalent to making the support of the ancilla wave function (in y) much larger than the support of system wave function (in λs) i.e. $y \gg \lambda s_k$. In this limit we can assume a first order Taylor approximation so that the ancilla wave packet can be written as $a(y_k - \lambda s_i) \approx a(y_k) - \lambda s_i\frac{\partial a(y_k)}{\partial y_k}$. As discussed in Appendix B, the general result in (11) reduces to $\langle y(t_2)y(t_1)\rangle = \lambda^2\mathrm{Re}[\langle\psi(t_1)|\hat{U}^T\hat{G}\hat{U}\hat{S}|\psi(t_1)\rangle]$. Defining the Heisenberg operators $\hat{G}(t_2) = \hat{U}^T\hat{G}\hat{U}$ and $\hat{S}(t_1) = \hat{S}$, then in the ideally-weak measurement regime we can write the two-time correlation function as:

$$\langle y(t_2)y(t_1)\rangle = \lambda^2\mathrm{Re}[\langle\psi(t_1)|\hat{G}(t_2)\hat{S}(t_1)|\psi(t_1)\rangle]. \tag{13}$$

Now, we want to demonstrate that the expectation value in (13) can be only understood as the result of a non-negligible perturbation of the measuring apparatus on the state of the system. To see that, we first rewrite the general state in (9) using the above mentioned Taylor series expansion (more details can be found in Appendix C):

$$|\psi_{k,\omega}(t_2)\rangle = \left(a(y_\omega)a(y_k)\hat{U} + \lambda^2\frac{\partial a}{\partial y_\omega}\frac{\partial a}{\partial y_k}\hat{G}\hat{U}\hat{S} - \lambda\frac{\partial a}{\partial y_\omega}a(y_k)\hat{U}\hat{S} - \lambda\frac{\partial a}{\partial y_k}a(y_\omega)\hat{G}\hat{U}\right)|\psi(t_1)\rangle. \tag{14}$$

For simplicity, we defined $\partial a/\partial y \equiv \partial a(y)/\partial y$. Erroneously assuming $\partial a(y)/\partial y = 0$, one could then think that the state of the system after the two measurements can be approximated only by the first term in (14) as $|\tilde{\psi}_{k,\omega}(t_2)\rangle \approx a(y_\omega)a(y_k)\hat{U}|\psi(t_1)\rangle$. This approximation would indeed imply that the state of the system has not been perturbed during the two-time measurement. However, $|\tilde{\psi}_{k,\omega}(t_2)\rangle$ does not yield the result in (13) but a separable probability $P(y_\omega, y_k) = \langle\tilde{\psi}_{k,\omega}(t_2)|\tilde{\psi}_{k,\omega}(t_2)\rangle \approx |a(y_\omega)|^2|a(y_k)|^2 = P(y_\omega)P(y_k)$ that leads to $\langle y(t_2)y(t_1)\rangle \approx \lambda^2\langle\hat{G}\rangle\langle\hat{S}\rangle$.

To understand why the approximation $|\psi_{k,\omega}(t_2)\rangle \approx |\tilde{\psi}_{k,\omega}(t_2)\rangle$ yields a wrong result, let us assume a system where either $\int y_\omega P(y_\omega)dy_\omega = 0$ or $\int y_k P(y_k)dy_k = 0$ so that $\langle y(t_2)y(t_1)\rangle \approx \lambda^2\langle\hat{G}\rangle\langle\hat{S}\rangle =$

§ In quantum systems prepared by collapsing the system state into one of an eigenstate of the operator $\hat{S}$ at $t = 0$ and measuring the system by such operator $\hat{S}$ without time evolution, the trivial result $\langle y(t_2)y(0)\rangle = \lambda^2\langle\hat{G}(t_2)\rangle\langle\hat{S}(0)\rangle$ is obtained.

0. Then, we consider the very rare output results $y_k \to \infty$ and $y_\omega \to \infty$, corresponding to $a(y_\omega)a(y_k)\hat{U}|\psi(t_1)\rangle \to 0$ (because $a(y_\omega), a(y_k) \to 0$). Then, the other terms in (14) can no longer be neglected as they provide a non-zero contribution to $\langle y(t_2)y(t_1)\rangle$. The rare events associated to $y \to \infty$, and hence to a large perturbation of the system, provide physical non-zero correlations and are responsible for providing non-zero correlations in (13) ‖.

In conclusion, two-time measurements do entail, in general, a non-negligible perturbation on the state of the system, and therefore cannot provide non-contextual information of the dynamics of quantum systems. This is an important result that is in contrast with the naive thought that ideally-weak measurements can be used to avoid the quantum backaction of the measurement apparatus.

3. Unmeasured (Bohmian) properties

A natural definition of the dynamics of the system that is, in no circumstances, contaminated by the backaction of the measuring apparatus can be established by relying on "quantum theories without observers", that is, theories where quantum objects have real properties associated independently of whether a measurement is being carried out or not. Here we will focus on the unmeasured properties defined in Bohmian mechanics theory, which introduces the reality of the position of quantum objects at an ontological level (independently of its measurement).

3.1. One-time (unmeasured) expectation values

In Bohmian mechanics the expectation value of a property S is defined as an ensemble average of single-experiment realizations as:

$$\langle S\rangle_B = \lim_{M\to\infty} \frac{1}{M} S_B(x^i(t_1)) = \lim_{M\to\infty} \frac{1}{M} \int \; S_B(x) \left(\sum_{i=1}^{M} \delta[x - x^i(t_1)] \right) dx = \int dx \; |\psi(x,t)|^2 \; S_B(x), \tag{15}$$

where $x^i(t)$ are the Bohmian trajectories (corresponding to each $i-th$ experiment) defined through the the so-called quantum equilibrium condition [19],

$$|\psi(x,t)|^2 = \lim_{M\to\infty} \frac{1}{M} \sum_{i=1}^{M} \delta[x - x^i(t)]. \tag{16}$$

In (16), there is an implicit spatial integral involved in the computation of the Dirac delta function $\delta[x - x^i(t)]$. In (15) we have also defined the (local-in-position) unmeasured Bohmian property:

$$S_B(x) = \mathrm{Re}\left[\frac{\psi^*(x,t) S(x) \psi(x,t)}{|\psi(x,t)|^2} \right], \tag{17}$$

where $S(x)\delta[x - x'] = \langle x|\hat{S}|x'\rangle$ is the position representation of the operator $\hat{S}$. According to (17), the results in (15) and (5) are identical up to an irrelevant constant factor, i.e., $\langle S\rangle_B = \langle S\rangle$.

Since we are interested in discussing unmeasured properties, we have avoided to introduce the interaction of the system with any measurement apparatus. In other words, although the Bohmian theory obviously allows us to includ the degrees of freedom y and z and their interaction with x discussed in section 2, we have intentionally neglected such interaction in the present discussion ¶

‖ Obviously, we can always redesign our experiment to have $\int y_\omega P(y_\omega) dy_\omega \neq 0$, $\int y_k P(y_k) dy_k \neq 0$, and both much more large (in absolute value) than the contributions from very large values of y mentioned above. Then, it will be true that the two-time correlations can be approximated by the uninteresting classical-like result $\langle y(t_2)y(t_1)\rangle \approx \langle y(t_2)\rangle\langle y(t_1)\rangle$. In this last case, since we have shown that $\langle y(t)\rangle$ is non-contextual, then $\langle y(t_2)y(t_1)\rangle$ is also non-contextual.

¶ We insist that getting properties of a quantum system without measuring is possible in Bohmian mechanics but not in the orthodox theory, where the properties of systems are created during the measurement process itself.

Therefore, the probability distribution of the different outcomes is also independent of the measuring apparatus and is given by

$$P_B(s) = \lim_{M\to\infty} \frac{1}{M} \sum_{i=1}^{M} \delta[s - S_B(x^i(t))], \tag{18}$$

where the possibility that different trajectories, $x^i(t) \neq x^j(t)$, provide the same Bohmian property value, $S_B(x^i(t)) = S_B(x^j(t))$, is taken into account.

Equation (17) can be either read as the required condition to ensure that the unmeasured Bohmian results exactly reproduce the measured orthodox ones or, on the contrary, a required condition to ensure that the measured orthodox results reproduce the unmeasured Bohmian ones. For one-time measurements, ensemble results are unaffected by the backaction of the measuring apparatus and the distinction between unmeasured Bohmian properties and orthodox measured ones is nonexistent. However, as it will be shown below, for the multiple-time measurements, a fundamental difference emerges between measured and unmeasured expectation values.

3.2. Two-time (unmeasured) expectation values

Since the position of a quantum object is mathematically well-defined at any time in Bohmian mechanics, one can then easily write the unmeasured expectation value of two-time correlation functions as:

$$\langle G(t_2)S(t_1)\rangle_B = \lim_{M\to\infty} \frac{1}{M} \sum_{i=1}^{M} G_B(x^i(t_2))S_B(x^i(t_1)). \tag{19}$$

where $S_B(x^i(t_1))$ and $G_B(x^i(t_2))$ defined in (17) are respectively *unmeasured* values of the properties S and G for a particular experiment i at times t_1 and t_2.

The probability distribution of the corresponding ensemble of experiments can be written as:

$$P_B(s,g) = \lim_{M\to\infty} \frac{1}{M} \sum_{i=1}^{M} \delta[s - S_B(x^i(t_1))]\delta[g - G_B(x^i(t_2))]. \tag{20}$$

Note that by construction the probability $P_B(s,g)$ is positive semi-definite and normalized. From a conceptual point of view, no negative probabilities appear because both properties $S_B(x^i(t_1))$ and $G_B(x^i(t_2))$ are perfectly well-defined (ontologically real) in the Bohmian theory [+].

When a quantum object is defined by an eigenstate of an operator, then, the orthodox theory states that the property associated to such operator is well defined and equal to the eigenvalue. In those cases, the measurement of such properties does not imply any perturbation on the quantum state and the same result is obtained in all repetitions of the experiment. This result is mathematically expressed in Eq. (12). As there is no perturbation on the state of the system due to the measurement backaction, one should expect to reproduce the same precise result using unmeasured (Bohmian) properties. To see that, we consider that the quantum object is prepared in an eigenstate $|s_0\rangle$ of the operator $\hat{S}$. Then, we always obtain $s = s_0 = S_B$ where we have used $S(x)\psi(x,t) = s_0\psi(x,t)$ in (17). Then, Eq. (19) can be simply written as:

$$\langle G(t_2)S(t_1)\rangle_B = s_0 \left(\lim_{M\to\infty} \frac{1}{M} \sum_{i=1}^{M} G_B(x^i(t_2)) \right) = \langle G(t_2)\rangle\langle S(t_1)\rangle. \tag{21}$$

Identically, if we know that the final state before the second measurement is an eigenstate g_0 of the second operator $\hat{G}$, we get $\langle G(t_2)S(t_1)\rangle_B = \langle G(t_2)\rangle\langle S(t_1)\rangle$. As expected, the unmeasured result (21) is thus identical to the measured one in (12) up to an irrelevant constant.

[+] On the contrary, in the orthodox definition of simultaneous properties, e.g., the Wigner distribution function, one is looking for the probability distribution of two properties that are not simultaneously well-defined (real) in a quantum object, e.g. the position and the momentum, and hence it can easily yield negative probabilities.

4. Measuring unmeasured Bohmian properties using local-in-position weak values

There is currently a great interest in the ability of weak values to provide simultaneous information of non-commuting operators, viz., in the present context $[\hat{S},\hat{G}] \neq 0$. The measurement of weak values requires, however, a very specific measurement protocol: i) the state of the quantum system is prepared (pre-selected) in an arbitrary initial state $|\psi\rangle$, ii) this state is weakly measured to provide imprecise information on the property S, iii) the resulting state is strongly measured to collapse it into an eigenstate of the operator $\hat{G}$ providing precise information on the property G, iv) only those final results associated to the state $|g_\omega\rangle$ are post-selected and contribute to the ensemble average.

Mathematically, the definition of the weak value ${}_{g_\omega}\langle\hat{S}\rangle_{\psi(t)}$ post-selected by $|g_\omega\rangle$ and pre-selected by $|\psi\rangle$ was first given by Aharonov-Albert-Vaidman [21], and reads:

$$ {}_{g_\omega}\langle\hat{S}\rangle_{\psi(t)} = \frac{\langle g_\omega|\hat{S}|\psi(t)\rangle}{\langle g_\omega|\psi(t)\rangle}. \tag{22}$$

The physical meaning of (22) can be easily understood by rewriting the one-time expectation value of $\hat{S}$ in terms of the weak values in (22) as:

$$\langle\hat{S}\rangle = \langle\psi(t)|\hat{S}|\psi(t)\rangle = \sum_\omega \langle\psi(t)|g_\omega\rangle\langle g_\omega|\hat{S}|\psi(t)\rangle = \sum_\omega |\langle\psi(t)|g_\omega\rangle|^2 \ {}_{g_\omega}\langle\hat{S}\rangle_{\psi(t)}, \tag{23}$$

where we have used the identity $\sum_\omega |g_\omega\rangle\langle g_\omega| = 1$ to introduce the role of the second measurement. Since we can define $|\psi\rangle = \sum_\omega c_\omega|g_\omega\rangle$ with $c_\omega = \langle g_\omega|\psi\rangle$, the interpretation of (23) is that the mean value $\langle\hat{S}\rangle$ is a sum of weak values ${}_{g_\omega}\langle\hat{S}\rangle_{\psi(t)}$ weighted by c_ω. The experimental accessibility of (22) is discussed in Appendix D. At this point, it is very instructive to rewrite (23) when the quantum state is post-selected using position eigenstates $|g_w\rangle = |x\rangle$. Then, we can rewrite the real part of (23) as:

$$\langle\hat{S}\rangle = \langle\psi|\hat{S}|\psi\rangle = \int dx\ |\psi(x,t)|^2\ \mathrm{Re}\left({}_x\langle\hat{S}\rangle_{\psi(t)}\right). \tag{24}$$

A comparison of expressions (24) and (15) reveals the identity:

$$S_B(x) \equiv \mathrm{Re}\left[\frac{\psi^*(x,t)S(x)\psi(x,t)}{|\psi(x,t)|^2}\right] = \mathrm{Re}\left[{}_x\langle\hat{S}\rangle_{\psi(t)}\right]. \tag{25}$$

That is, local-in-position weak values are identical to unmeasured Bohmian properties. Therefore, the equivalence in (25) shows that local-in-position weak values are, by construction, free from the measurement backaction and hence provide intrinsic information of the system (without contamination from the measuring apparatus). This idea was first conceived by Wiseman when he identified the local-in-position weak value of the momentum of a particle with the (unmeasured) Bohmian velocity of such particle[24].

In practice, one can measure the Bohmian velocity field [24, 25] and then reconstruct the unmeasured Bohmian trajectories [26]. From the knowledge of such trajectories the whole dynamics picture of the unmeasured quantum system, including mean values and fluctuations of any observable of interest, can be then accessed. In particular, looking at (19) and (25), unmeasured two-time correlation functions can be written in terms of local-in-position weak values as:

$$\begin{aligned}\langle G(t_2)S(t_1)\rangle_B &= \lim_{M\to\infty}\frac{1}{M}\sum_{i=1}^{M} G_B(x^i(t_2))S_B(x^i(t_1)) \\ &= \lim_{M\to\infty}\frac{1}{M}\sum_{i=1}^{M}\mathrm{Re}\left[{}_{x^i(t_2)}\langle\hat{G}\rangle_{\psi(t_2)}\right]\mathrm{Re}\left[\ {}_{x^i(t_1)}\langle\hat{S}\rangle_{\psi(t_1)}\right].\end{aligned} \tag{26}$$

We notice that the wavefunction $\psi(t_2)$ in the definition of the second weak value corresponds to an experimens with a unitary evolution of the initial state $\psi(t_0)$ from t_0 till t_2 (without measurement at t_1). The local-in-position weak values have to be evaluated at the positions $x^i(t_1)$ and $x^i(t_2)$, respectively, which are linked by the Bohmian trajectory $x^i(t)$.

In our opinion, the equivalence in (25) tells us that the great interest in measuring weak values to provide a deeper understanding of quantum systems can be also understood as an interest to measure unmeasured properties.

5. Examples of the soundness of unmeasured Bohmian properties

We now want to show that the identification of local-in-position weak values with unmeasured Bohmian properties leads to an extraordinarily useful tool to unveil the intrinsic dynamics of quantum systems (not accessible from the orthodox way of thinking). This is exemplified with what we believe constitute three paradigmatic *unmeasured* properties.

5.1. The quantum work distribution

The quantum work is the basic ingredient in the development of quantum thermodynamics which is one of the most important topics in the field of open quantum systems. Quantum thermodynamics is essential in developing new quantum technologies such as quantum heat engines. It also plays a fundamental role in the consistency of the second law of thermodynamics in the quantum regime. However, there are many issues that are still being investigated, most notably related to the definition of work and heat. The problem is that these thermodynamic variables are not observables related to Hermitian super operators, but are trajectory (history) dependent [27, 28, 29]. This has culminated in the so-called "no-go" theorem that states that in fact there cannot exist a (super)operator for work that simultaneously satisfies all the physical properties required from it [11].

As indicated in the introduction, evaluating quantum work by means of a two-time projective measurement protocol of the energy will imply that the first measurement projects the initial state into an energy eigenstate, hence preventing the possibility of capturing any coherent evolution of the energy. The alternative orthodox protocols for the evaluation of the quantum work like Gaussian measurements [7, 8], weak measurements [9, 10], collective measurements [11], etc. do all suffer from quantum contextuality, which provides as many different work definitions as measurement schemes exist.

The problem appears due to the requirement of the orthdox theory to include a measuring apparatus that in practice does not exist. In other words, we are not interested in the direct measurement of work, but on using this dynamics information of the quantum (sub)system in conjunction with quantum thermodynamic equations to compute, e.g., the temperature variation of a larger system involving a macroscopic thermodynamic environment. We are thus seeking for an *unmeasured* value of work. This idea has been already introduced in Refs. [30, 31].

For a detailed discussion of the Bohmian quantum work, the reader can follow the previous works of some of the co-authors in [31] which presents an alternative microscopic definition of quantum work based on Bohmian mechanics. The aforementioned approach circumvents the problem of the unavoidable contextuality of quantum work in the orthodox theory. For the purpose of this paper, we follow here the original Kobe's development [30]. We define a single particle* wave function solution of the following Schrödinger equation:

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}=\left(\frac{(-i\hbar\vec{\nabla}-q\vec{A}(\vec{r},t))^2}{2m}+qV(\vec{r},t)\right)\psi(\vec{r},t). \tag{27}$$

where $\vec{r}$ is defined as a vector in the ordinary three dimensional space, $\vec{\nabla}$ is the gradient operator and $-i\hbar\vec{\nabla}-q\vec{A}(\vec{r},t)$ is the canonical momentum with $A(\vec{r},t)$ the electromagnetic vector potential. When the wave function is written in polar form as $\psi(\vec{r},t)=R(\vec{r},t)\exp\left(\frac{i\mathcal{S}(\vec{r},t)}{\hbar}\right)$ where $R(\vec{r},t)$ and $\mathcal{S}(\vec{r},t)$ are the modulus and phase, respectively, the real part of (27) evaluated along the Bohmian trajectory

* A many body treatment will not add any new insight on the discussion.

$\vec{r} = \vec{r}^{\,i}(t)$ for the i-th experiment, gives us the following equation for the unmeasured power:

$$\frac{d\mathcal{E}(\vec{r}^{\,i}(t))}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\vec{v}(\vec{r}^{\,i}(t),t)^2 + Q(\vec{r}^{\,i}(t),t)\right) = q\vec{v}(\vec{r}^{\,i}(t),t)\vec{E}(\vec{r}^{\,i}(t),t) + \frac{\partial Q(\vec{r}^{\,i}(t),t)}{\partial t}. \tag{28}$$

Here $\mathcal{E}(\vec{r}^{\,i}(t),t)$ is the unmeasured energy of the system, $v(\vec{r}^{\,i}(t),t)$ is the Bohmian velocity, $Q(\vec{r}^{\,i}(t),t)$ is the quantum potential and $\vec{E}(\vec{r}^{\,i}(t),t)$ is the electric field. While we have considered an external electromagnetic field interacting with the quantum system, no measuring apparatus is accounted for in (28). Thus, from (28), we can describe the unmeasured work represented by the wave function $\psi(\vec{r},t)$ and the trajectory $\vec{r}^{\,i}(t)$, during the time-interval $t_2 - t_1$ by just subtracting the initial energy $\mathcal{E}(\vec{r}^{\,i}(t_1),t_1)$ to the final one $\mathcal{E}(\vec{r}^{\,i}(t_2),t_2)$. As we have already mentioned, this result corresponds to the single experiment labelled by the superscript i. Getting ensemble values of the work just requires repeating the previous procedure for different initial positions of the particles, according to the quantum equilibrium hypothesis [19].

As stressed along the paper, the very crucial aspect of this unmeasured work is its measurability in the laboratory using the local-in-position weak values. This can be expressed as follows,

$$\mathrm{Re}\left({}_{\vec{r}^{\,i}(t)}\langle\hat{\mathcal{E}}\rangle_{\psi(t)}\right) = \mathrm{Re}\left(\frac{\langle\vec{r}^{\,i}(t)|\hat{H}|\psi(t)\rangle}{\langle\vec{r}^{\,i}(t)|\psi(t)\rangle}\right). \tag{29}$$

Now using the Hamiltonian from (27) in (29) and writing the wave function in the polar form to evaluate $\langle\vec{r}^{\,i}(t)|\hat{H}|\psi(t)\rangle$, we can rewrite (29) as:

$${}_{\vec{r}^{\,i}(t)}\langle\hat{\mathcal{E}}\rangle_{\psi(t)} = \left[\frac{-\hbar^2}{2m}\frac{\nabla^2 R(\vec{r},t)}{R(\vec{r},t)} + \frac{(\vec{\nabla}S(\vec{r},t))^2}{2m}\right]_{r=r^i(t)} = Q(\vec{r}^{\,i}(t),t) + \frac{1}{2}m\vec{v}(\vec{r}^{\,i}(t),t)^2, \tag{30}$$

where ∇^2 is the Laplacian operator. By construction, ${}_{\vec{r}^{\,i}(t)}\langle\hat{\mathcal{E}}\rangle_{\psi(t)}$ is real-valued. Thus, from (28), it is straightforward to see that,

$${}_{\vec{r}^{\,i}(t)}\langle\hat{\mathcal{E}}\rangle_{\psi(t)} = \mathcal{E}(\vec{r}^{\,i}(t),t). \tag{31}$$

Therefore we conclude that the unmeasured Bohmian energy is equal to the local-in-position weak value of the energy and hence that it can be, in principle, measured experimentally. Given a collection of weak values of the energy at times t_1 and t_2, one can then easily evaluate the quantum work distribution,

$$W^i(t_2,t_1) = {}_{\vec{r}^{\,i}(t_2)}\langle\hat{\mathcal{E}}\rangle_{\psi(t_2)} - {}_{\vec{r}^{\,i}(t_1)}\langle\hat{\mathcal{E}}\rangle_{\psi(t_1)}, \tag{32}$$

as well as the corresponding expectation value:

$$\langle W(t_2,t_1)\rangle = \lim_{M\to\infty}\frac{1}{M}\sum_{i=1}^{M}\left[{}_{\vec{r}^{\,i}(t_2)}\langle\hat{\mathcal{E}}\rangle_{\psi(t_2)} - {}_{\vec{r}^{\,i}(t_1)}\langle\hat{\mathcal{E}}\rangle_{\psi(t_1)}\right]. \tag{33}$$

where M is the total number of considered experiments $i = 1, 2, .., M$.

5.2. The quantum noise at high frequencies

Electron devices are meant to manipulate information in terms of scalar potentials and output electrical currents. State-of-the-art electron devices are nowadays entering into the nanometer scale and operating at frequencies of hundreds of GHz. At such time and length scales, a full quantum treatment of the electrical current is mandatory. The total current at such frequencies is the sum of the conduction (flux of particles) plus the displacement (time-derivative of the electric field) components [32, 33, 34, 35]. The displacement current on a surface of an electron device is different from zero whenever electrons are able to modify the electric field on it (independently on how far the electrons are from that surface). Therefore, while in DC this quantity is zero after time averaging, at high

frequencies a proper solution of Maxwell's equations is needed to know the interplay between scalar potentials and electrical currents.

Typically we are interested in getting information of the total electrical current in a surface S of the active region The pertinent point in our discussion here is that in a realistic experimental scenario, there is no measuring apparatus on the surface S of the active device region where the dynamics of electrons is simulated. Instead, the ammeter is usually located at a macroscopic distance far from the nanoscale device and connected to it through macroscopic cables with an amount of electrons given by the Avogadro number. Still, the current on surface S coincides with the current in the ammeter because of the conservation of the total (particle plus displacement) current. In this respect, one author of this work has recently demonstrated that the electrical current is very weakly perturbed by the electrons of the cables and surroundings [36]. Again, to simplify the discussion, we consider the *unmeasured* value of the total electrical current when only one electron is present in the active region. Once we get the Bohmian trajectory $\vec{r}^{\,i}(t)$ of an electron moving through the device by solving a transport equation like the one in (27), the total current generated by this electron in a surface S is given by:

$$I^i(t) = \int_S \vec{J}^{\,i}_c(\vec{r},t)\cdot \mathrm{d}\vec{s} + \int_S \epsilon(\vec{r})\frac{\mathrm{d}\vec{E}^{\,i}(\vec{r},t)}{\mathrm{d}t}\cdot \mathrm{d}\vec{s}, \tag{34}$$

where $\epsilon(\bar{r},t)$ is the (inhomogeneous) electric permittivity. The current (particle) density of the i-th experiment is given by $\vec{J}^{\,i}_c(\vec{r},t) = q\vec{v}(\vec{r}^{\,i}(t),t)\delta[\vec{r}-\vec{r}^{\,i}(t)]$ with q the electron charge. The electric field $\vec{E}^{\,i}(\vec{r},t)$ is just the solution of the Gauss equation♯ with the proper boundary conditions for a charge density given by $Q^i(\vec{r},t) = q\delta[\vec{r}-\vec{r}^{\,i}(t)]$. In principle, the integration of the density current in (34) on the surface S has a dependence on its position. However, such dependence disappears in practical two-terminal scenarios, due to the total current and its conservation law. It can be proven that in a two terminal device of distance L with metallic contacts of surface $S = W\cdot H$, if $L \ll W, H$ (width and height) the total current can be written as [32, 33, 37]:

$$I^i(t) = \frac{q}{L}v^i_x(t), \tag{35}$$

where $v^i_x(t)$ is the velocity in the transport direction of the considered electron in the active region in the i-th experiment †† Outside the active region, the electron is screened and its contribution to the total current can be neglected.

Now using (35) we will try to arrive at the weak value of the current and see if it is equivalent to the unmeasured current $I^i(t)$. The Bohmian velocity $v_x(t)$ is evaluated by the Bohmian guidance equation [38] as follows,

$$v_x(t) = \frac{\hbar}{m}\mathrm{Im}\left[\frac{\frac{\partial}{\partial x}\psi(\vec{r},t)}{\psi(\vec{r},t)}\right] = \frac{1}{m}\mathrm{Re}\left[\frac{\langle\vec{r}|\hat{P}_x|\psi(t)}{\langle\vec{r}|\psi(t)\rangle}\right], \tag{36}$$

where we have used $\langle\vec{r}|\hat{P}_x|\psi(t) = -i\hbar\frac{\partial}{\partial x}\psi(\vec{r},t)$. Now evaluating the velocity in (36) for a particular trajectory $\vec{r} = \vec{r}^{\,i}(t)$ we get the current for an i-th bohmain experiment as in (35) which can be rewritten using (36) as,

$$I^i(t) = \frac{q}{L}v^i_x(t) = \frac{q}{mL}\mathrm{Re}\left[\frac{\langle\vec{r}^{\,i}(t)|\hat{P}_x|\psi(t)}{\langle\vec{r}^{\,i}(t)|\psi(t)\rangle}\right] = \frac{q}{mL}\mathrm{Re}\left[{}_{\vec{r}^{\,i}(t)}\langle\hat{P}_x\rangle_{\psi(t)}\right]. \tag{37}$$

♯ Notice that this electric field $\vec{E}^{\,i}(\vec{r},t)$ in (34) is different from the electric field $\vec{E}(\vec{r},t)$ used in (28) in the previous section. Here, the electron located at $\vec{r} = \vec{r}^{\,i}(t)$ generates an electric field $\vec{E}^{\,i}(\vec{r},t)$ which leads to a density current everywhere inside the device. In (28), on the other hand, $\vec{E}(\vec{r},t)$ was an external electric field without self-consistence with the electric charge of the considered electron.

†† Typically, hundreds of electrons have to be considered in a realistic device and the total current in the i-th experiment can be computed by just adding the contribution of each electrons $I^i(t) = \sum_{n=1}^{N_e} I^i_n(t)$ with N_e the number of electrons inside the volume $L\cdot W\cdot H$ at time t.

Thus the unmeasured current is equivalent to the local-in-position weak measurement of the momentum operator $\hat{P}$. From the information of $I^i(t)$ in (37), when $i = 1, ..., M$ experiments are considered, we can compute the ensemble value of the current as,

$$\langle I(t)\rangle = \lim_{M\to\infty} \frac{1}{M}\sum_{i=1}^{M} \frac{q}{mL}\mathrm{Re}\left[_{\vec{r}^i(t)}\langle \hat{P}_x\rangle_{\psi(t)}\right], \tag{38}$$

and the autocorrelations of the total current as discussed in section 4. The Fourier transform of such correlations provides the power spectral density of the fluctuations of the current at high frequencies.

$$\mathrm{PSD}(\omega) = \lim_{M\to\infty} \frac{1}{M}\sum_{i=1}^{M} \left(\frac{q}{mL}\right)^2 \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau}\mathrm{Re}\left[_{\vec{r}^i(t_2)}\langle \hat{P}_x\rangle_{\psi(t_2)}\right]\mathrm{Re}\left[_{\vec{r}^i(t_1)}\langle \hat{P}_x\rangle_{\psi(t_1)}\right]. \tag{39}$$

where we have assume that we are dealing with a wide-sense stationary process where the correlation depends only on the time difference $\tau = t_2 - t_1$. Some of the authors of this paper have elaborated these ideas into the simulator named BITLLES awhich is an acronym for Bohmian interacting transport in non-equilibrium electronic structures [39, 40, 41, 42, 43, 44, 45].

5.3. The quantum dwell time

Measuring the time spent by a particle within a particular region $\vec{a} < \vec{r} < \vec{b}$ requires measuring the time t_1 at which the particle enters that region and, later, the time t_2 at which the particle leaves it. As we have already seen, the measurement of the position of the particle implies the perturbation of the state of the system in most general circumstances. Thus, any subsequent measurement of the position is generally influenced by the first measurement. In spite of its controversial definition in orthodox quantum theory, the concept of dwell time is necessary, for example, to evaluate the maximum working frequency of state-of-the-art transistors and hence the performance of modern computers [46]. In this respect, it is important to notice that, when using the information of the dwell time in the evaluation of the performance of computers, there are no position detectors at the two ends ($\vec{a}$ and $\vec{b}$) of the active region of the transistors. A valid question is then which of the two dwell times, the measured or the unmeasured one, provide a better estimation of the maximum working frequency of transistors. Anyhow, what is definitely true is that to estimate the maximum working frequency of transistors nobody evaluates dwell times by means of projective measurements [47, 48].

The *unmeasured* value of the dwell time can be easily computed from *unmeasured* Bohmian trajectories $\vec{r}^i(t)$. Again, for simplicity, we only consider one electron inside the active region in each experiment. The expectation value of the unmeasured dwell time can be defined as:

$$\tau_D = \lim_{M\to\infty} \frac{1}{M}\sum_{i=1}^{M} \tau^i, \tag{40}$$

where τ^i is defined as the time spent by the (unmeasured) i-th Bohmian trajectory inside the region $\vec{a} < \vec{r} < \vec{b}$, i.e.:

$$\tau^i = \int_0^{\infty} dt\Theta[\vec{r}^i(t) - \vec{a}]\Theta[\vec{b} - \vec{r}^i(t)]. \tag{41}$$

The above expression can be rewritten as:

$$\tau_D = \lim_{M\to\infty} \frac{1}{M}\sum_{i=1}^{M} \int_0^{\infty} dt \int_{\vec{a}}^{\vec{b}} \delta[\vec{r} - \vec{r}^i(t)]d\vec{r}, \tag{42}$$

and, making use of the quantum equilibrium condition [19] in (42), we already get the well-know expression:

$$\tau_D = \int_0^{\infty} dt \int_{\vec{a}}^{\vec{b}} |\psi(\vec{r},t)|^2 d\vec{r}, \tag{43}$$

which is certainly an unmeasured property of the quantum system as there is no contamination from the measuring apparatus. Notice that the experimental validation of the above arguments requires the unmeasured Bohmian trajectories that can be reconstructed from the unmeasured Bohmian velocity field understood as a local-in-position weak value of the momentum [24], as we have done in the previous section.

Certainly, there exist many orthodox protocols to compute either the dwell time or the tunneling time [49, 50, 51, 52, 53, 54]. For example, one can make use of a physical clock to measure the time elapsed during the tunneling [51, 52, 53, 54]. Larmor precession was precisely introduced to measure the time associated with scattering events [52, 54]. Anyhow, what is essential here is that the scientific community has been persistent in looking for observables of dynamical properties whose expectation value is free from any contamination from the (physically nonexistent) measuring apparatus. This is exactly what *unmeasured* properties defined in this paper are meant for.

6. Conclusions:

In this paper we have introduced the concept of unmeasured (Bohmian) properties as directly related to the concept of *beable* originally introduced by J. S. Bell to talk about properties of quantum systems whose existence is independent of any experimental setup [14]. More specifically, we have defined unmeasured (Bohmian) properties as the beables of quantum systems that are not interacting with any measuring apparatus.

We have shown that these unmeasured properties are indeed measurable through local-in-position weak values [21]. Importantly, and contrarily to generalized or indirect measurements of time-correlation functions, we have shown that local-in-position weak values are free from quantum backaction effects and thus allow to define non-contextual time-correlation functions, i.e., correlation functions that are independent of the experimental setup. As a result, the intrinsic dynamics of quantum systems can be genuinely formulated in terms of unmeasured properties and experimentally assessed through weak values. This is a relevant statement, as the scientific community is persistently looking for dynamical properties whose expectation value is free from the contamination of measuring apparatuses [48, 55].

A paradigmatic example of the ability of unmeasured Bohmian properties to describe the dynamics of quantum systems without the contamination of the measuring apparatus is the Bohmian velocity of a quantum system. Its experimental assessment through weak-values was first proposed by Wiseman [24]. From the Bohmian velocities it is then easy to reconstruct trajectories in space [25], which, as shown here, can be used to reconstruct the entire dynamics of a quantum system. In particular, this includes the evaluation of unmeasured dwell times, quantum work statistics, or the power spectral density of current-current correlation functions.

Finally, let us distill the precise merit of the correspondence between unmeasured Bohmian properties and local-in-position weak values that we have shown. Strictly speaking (i.e., at the ontological level), the unmeasured Bohmian properties $S_B(x)$ are defined for a single experimental realization (associated to the trajectory $x^i(t)$) that can be associated to the position x at time t. On the contrary, local-in-position weak values are obtained in the laboratory, not from a single experiment, but from a (post-selected) ensemble of weak measurements of the property associated to $\hat{S}$ (i.e, for a large set of identically-prepared quantum systems). Therefore, the above identification does not provide a demonstration of the (ontological) existence of unmeasured Bohmian properties, but it provides solid arguments in favour of considering them in the discussion of the intrinsic dynamics of quantum systems. An analogy with the wave function can help to clarify this point. Strictly speaking, the wave function can be reconstructed from weak values (or quantum tomography) [56], but not directly measured in a single experiment [57]. Despite the ontological difficulties in justifying the reality or not of the wave function, nobody doubts about the practical utility of the wave function when simulating quantum systems. Similarly, in this work we claim that the identification between unmeasured Bohmian properties and local-in-position weak values motivates the use of the former to

assess the intrinsic (not contaminated by the measuring apparatus) dynamics of quantum systems.

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Appendix A. Interaction evolution of the combined state vector in (2)

A measurement requires an entanglement between the system and the ancilla. Such entanglement can be obtained through a unitary interaction of the ancilla with the system given by the Von Neumann time-evolution operator $\exp(\frac{-i}{\hbar}\hat{S}\otimes\hat{P}_a)$ where $\hat{S}$ is the system operator (the observable we want to measure) and $\hat{P}_a$ is the momentum operator for the ancilla. As described in the text, a weak measurement requires an additional unitary entanglement between the ancilla and the pointer given by the Von Neumann time-evolution operator on the ancilla and pointer degrees of freedom given by $\exp(\frac{-i}{\hbar}\hat{Y}\otimes\hat{P}_p)$ where $\hat{Y}$ is the ancilla operator (observable that we effectively measure) and $\hat{P}_p$ is the momentum operator associated to the pointer. Noticing that the ancilla and the pointer wave functions are represented in the position representation, the process of the pre-measurement described in the text can be mathematically described as follows,

$$|\Psi(t_1)\rangle = \exp\left(\frac{-i}{\hbar}\hat{Y}\otimes\hat{P}_p\right)\cdot\exp\left(\frac{-i}{\hbar}\hat{S}\otimes\hat{P}_a\right)|\Psi(0)\rangle. \tag{A.1}$$

when applying the first operator $\exp(\frac{-i}{\hbar}\hat{S}\otimes\hat{P}_a)$ on the initial state $|\Psi(0)\rangle$ given by (2), we get:

$$|\Psi(t_1)\rangle = \exp\left(\frac{-i}{\hbar}\hat{Y}\otimes\hat{P}_p\right)\sum_i c_i|s_i\rangle\otimes\int a(y-\lambda s_i)|y\rangle dy\otimes\int f(z,0)|z\rangle dz, \tag{A.2}$$

where we have used,

$$\begin{aligned}
&\exp\left(\frac{-i}{\hbar}\hat{S}\otimes\hat{P}_a\right)\sum_i c_i|s_i\rangle\otimes\int a(y,0)|y\rangle dy\\
&=\sum_i c_i|s_i\rangle\otimes\int\int|y'\rangle\langle y'|\exp(-\lambda s_i\hat{P}_a)|y\rangle a(y,0)dy\,dy'\\
&=\sum_i c_i|s_i\rangle\otimes\int\exp\left(-\lambda s_i\frac{\partial}{\partial y}\right)a(y,0)|y\rangle dy\\
&=\sum_i c_i|s_i\rangle\otimes\int\left(1-\lambda s_i\frac{\partial}{\partial y}+\lambda^2 s_i^2\frac{\partial^2}{\partial y^2}-...\right)a(y,0)|y\rangle dy\\
&=\sum_i c_i|s_i\rangle\otimes\int a(y-\lambda s_i)|y\rangle dy.
\end{aligned} \tag{A.3}$$

The second operator $\exp(\frac{-i}{\hbar}\hat{Y}\otimes\hat{P}_p)$ on the state given by (A.2) can be finally written as:

$$|\Psi(t_1)\rangle = \exp\left(\frac{-i}{\hbar}\hat{Y}\otimes\hat{P}_p\right)\sum_i c_i|s_i\rangle\otimes\int a(y-\lambda s_i)|y\rangle dy\otimes\int f(z,0)|z\rangle dz$$
$$= \sum_i c_i|s_i\rangle\otimes\int a(y-\lambda s_i)|y\rangle dy\otimes\int f(z-y)|z\rangle dz, \tag{A.4}$$

where, as done in the previous step, we have used

$$\exp\left(\frac{-i}{\hbar}\hat{Y}\otimes\hat{P}_p\right)\int a(y-\lambda s_i)|y\rangle dy\otimes\int f(z,0)|z\rangle dz$$
$$= \int a(y-\lambda s_i)|y\rangle dy\otimes\int \exp\left(-y\frac{\partial}{\partial z}\right) f(z,0)|z\rangle dz$$
$$= \int a(y-\lambda s_i)|y\rangle dy\otimes\int \left(1-y\frac{\partial}{\partial z}+y^2\frac{\partial^2}{\partial z^2}-...\right) f(z,0)|z\rangle dz$$
$$= \int a(y-\lambda s_i)|y\rangle dy\otimes\int f(z-y)|z\rangle dz. \tag{A.5}$$

Appendix B. Two-time correlation functions for an ideally-weak measurement.

For scenarios where the ancilla wave packet has a support much more larger than support of the system wave packet, $y \gg \lambda s$, one can consider a first order Taylor's expansion of the ancilla wave function $a(y_k-\lambda s)$ around y giving,

$$a(y_k-\lambda s) \approx a(y_k) - \lambda\frac{\partial a(y_k)}{\partial y_k}s. \tag{B.1}$$

The measuring protocol satisfying the above Taylor expansion is what we named *ideal weak measurements* in the text. The above condition for its definition written above can be equivalently written as $\left|\frac{\partial a(y_k)}{\partial y_k}\right| \gg \left|\frac{\lambda}{2}\frac{\partial^2 a(y_k)}{\partial y_k^2}s\right|$.

The evaluation of the main result in (11) for the ideal weak measurements used here, requires the evaluation of the integral $\int dy_k\ y_k a(y_k-\lambda s_i)a(y_k-\lambda s_{i'})^*$, which can be evaluated when (B.1) is considered as:

$$\int dy_k\ y_k a(y_k-\lambda s_i)a^*(y_k-\lambda s_{i'}) = \int dy_k\ y_k\left(a(y_k)-\lambda\frac{\partial a(y_k)}{\partial y_k}s_i\right)\left(a^*(y_k)-\lambda\frac{\partial a^*(y_k)}{\partial y_k}s_{i'}\right)$$
$$= \int dy_k\ y_k a(y_k)a^*(y_k) - \lambda s_i\int_{-\infty}^{\infty} dy_k y_k a^*(y_k)\frac{\partial a(y_k)}{\partial y_k}$$
$$- \lambda s_{i'}\int dy_k y_k a(y_k)\frac{\partial a^*(y_k)}{\partial y_k} + \lambda^2 s_i\ s_{i'}\int dy_k y_k\frac{\partial a^*(y_k)}{\partial y_k}\frac{\partial a(y_k)}{\partial y_k}$$
$$= \lambda\frac{1}{2}(s_i+s_{i'}). \tag{B.2}$$

In the evaluation of (B.2) we have considered that ancilla wave function is real (not complex), $a(y_k-\lambda s_i) = a^*(y_k-\lambda s_i)$. We have also used the identity:

$$\int dy\ y\ a(y)\frac{\partial a(y)}{\partial y} = -1/2, \tag{B.3}$$

because when integrating by parts, we get:$\int dy\ y\ a(y)\frac{\partial a(y)}{\partial y} = -\int dy a(y)a(y) - \int dy\ y\ \frac{\partial a(y)}{\partial y}a(y)$. Identically, in the evaluation of (B.2), we have used:

$$\int dy\ y\ \frac{\partial a(y)}{\partial y}\frac{\partial a(y)}{\partial y} = 0, \tag{B.4}$$

and:

$$\int dy\ y\ a(y)a(y) = 0. \tag{B.5}$$

Finally, putting (B.2) into the integral (11), we get:

$$\begin{aligned}\langle y(t_2)y(t_1)\rangle &= \frac{\lambda^2}{2}\langle\Psi(t_1)|\hat{U}^T\hat{G}\hat{U}\hat{S}|\psi(t_1)\rangle + \frac{\lambda^2}{2}\langle\Psi(t_1)|\hat{S}\hat{U}^T\hat{G}\hat{U}|\psi(t_1)\rangle \\ &= \lambda^2\mathrm{Re}[\langle\Psi(t_1)|\hat{U}^T\hat{G}\hat{U}\hat{S}|\psi(t_1)\rangle],\end{aligned} \tag{B.6}$$

where we have used the identities $\hat{S} = \sum_i s_i|s_i\rangle\langle s_i|$ and $\hat{G} = \sum_j g_j|g_j\rangle\langle g_j|$, and $1 = \sum_i |s_i\rangle\langle s_i|$ and $\hat{G} = \sum_j |g_j\rangle\langle g_j|$. This last result reproduces (13) in the main text.

Appendix C. The perturbed state for an ideally weak measurement

To understand if the result (B.6) is contaminated or not by the measurement, it is relevant to rewrite the general state in the system space in (9), according to the Taylor series used above for the ancilla wave function:

$$\begin{aligned}|\psi_{k,\omega}(t_2)\rangle &= \sum_{j,i}|g_j\rangle a(y_\omega - \lambda g_j)\langle g_j|\hat{U}|s_i\rangle a(y_k - \lambda s_i)\langle s_i|\psi(t_1)\rangle \\ &= \sum_{j,i}|g_j\rangle\left(a(y_\omega) - \lambda\frac{\partial a(y_\omega)}{\partial y_\omega}g_j\right)\langle g_j|\hat{U}|s_i\rangle\left(a(y_k) - \lambda\frac{\partial a(y_k)}{\partial y_k}s_i\right)\langle s_i|\psi(t_1)\rangle,\end{aligned} \tag{C.1}$$

which can be easily expanded to read:

$$\begin{aligned}|\psi_{k,\omega}(t_2)\rangle &= a(y_\omega)a(y_k)\sum_{j,i}|g_j\rangle\langle g_j|\hat{U}|s_i\rangle\langle s_i|\psi(t_1)\rangle - \lambda\frac{\partial a(y_\omega)}{\partial y_\omega}a(y_k)\sum_{j,i}|g_j\rangle\langle g_j|\hat{U}|s_i\rangle s_i\langle s_i|\psi(t_1)\rangle \\ &-\lambda\frac{\partial a(y_k)}{\partial y_k}a(y_\omega)\sum_{j,i}|g_j\rangle g_j\langle g_j|\hat{U}|s_i\rangle\langle s_i|\psi(t_1)\rangle + \lambda^2\frac{\partial a(y_\omega)}{\partial y_\omega}\frac{\partial a(y_k)}{\partial y_k}\sum_{j,i}|g_j\rangle g_j\langle g_j|\hat{U}|s_i\rangle s_i\langle s_i|\psi(t_1)\rangle.\end{aligned} \tag{C.2}$$

Using now $\hat{S} = \sum_i s_i|s_i\rangle\langle s_i|$ and $\hat{G} = \sum_i g_i|s_i\rangle\langle s_i|$ (C.2) reduces to:

$$\begin{aligned}|\psi_{k,\omega}(t_2)\rangle &= a(y_\omega)a(y_k)\hat{U}|\psi(t_1)\rangle - \lambda\frac{\partial a(y_\omega)}{\partial y_\omega}a(y_k)\hat{U}\hat{S}|\psi(t_1)\rangle - \lambda\frac{\partial a(y_k)}{\partial y_k}a(y_\omega)\hat{G}\hat{U}|\psi(t_1)\rangle \\ &+ \lambda^2\frac{\partial a(y_\omega)}{\partial y_\omega}\frac{\partial a(y_k)}{\partial y_k}\hat{G}\hat{U}\hat{S}|\psi(t_1)\rangle.\end{aligned} \tag{C.3}$$

This is the result found in (14) in the main text. To help in the discussion of the text, it is interesting now to discuss when the coefficient in the first term in (C.3) can be comparable to the coefficients of the other terms for the particular case of a gaussian ancilla wave packet $a(y_k) = \left(\frac{1}{\pi\sigma^2}\right)^{1/4}\exp\left(\frac{-y_k^2}{2\sigma^2}\right)$. In particular we consider when, $a(y_\omega)a(y_k) = \lambda^2\frac{\partial a(y_\omega)}{\partial y_\omega}\frac{\partial a(y_k)}{\partial y_k}$ under the simplifying condition $y = y_k = y_\omega$. We get $y \approx \sigma^2/\lambda$ which certifies that for small values of the measurement outputs y we can reasonable neglect the second, third and forth terms of (14), but this is not possible for very large values of y. The perturbation of the state in the ideally-weak measurement is hidden in this very large values of y (very rare experimental results).

Appendix D. Experimental evaluation of the weak values

Starting from the two time probability $P(y_\omega, y_k)$ in (10) we want to compute the Aharanov-Albert-Vaidman expression of the weak values [21] as follows,

$$\mathrm{Re}\left({}_{g_\omega}\langle\hat{S}\rangle_{\psi(t_2)}\right) = \mathrm{Re}\left(\frac{\langle g_\omega|\hat{S}|\psi(t_2)\rangle}{\langle g_\omega|\psi(t_2)\rangle}\right) \approx \frac{1}{\lambda}\frac{\int dy_k\ y_k P(y_\omega, y_k)}{\int dy_k P(y_\omega, y_k)}, \tag{D.1}$$

As is well-known in the literature, the right hand side of (D.1) is a good approximation to compute the weak values, but not an exact result. For the interest of this paper about unmeasured properties we can rephrase the above sentence saying that the original Aharanov-Albert-Vaidman expression of the weak values is certainly not contaminated by the measuring apparatus by construction, while the approximation in the right hand side of (D.1) can still contain such contamination.

We can use the first order Taylor expansion for the first ancilla wave function $a(y_k - \lambda s_i) \approx a(y_k) - \lambda s_i \frac{\partial a(y_k)}{\partial y_k}$ and a Kronecker delta function for the second ancilla wave function $|a(y_\omega - \lambda g_j)|^2 = |\delta_{y_\omega, \lambda g_j}|^2$. Using this in (10) and defining $g_w = \frac{y_w}{\lambda}$ we get,

$$P(y_\omega, y_k) = \sum_{i,i'} c^*_{i'} c_i c^*_{w,i'} c_{w,i} \left(a^*(y_k) - \lambda s_{i'} \frac{\partial a^*(y_k)}{\partial y_k}\right)\left(a(y_k) - \lambda s_i \frac{\partial a(y_k)}{\partial y_k}\right). \tag{D.2}$$

Using (B.2) in (D.2) we can evaluate the numerator in (D.1) as,

$$\begin{aligned}\int dy_k\ y_k P(y_\omega, y_k) &= \sum_{i,i'} c^*_{i'} c_i c^*_{w,i'} c_{w,i} \lambda \frac{1}{2}(s_i + s'_i) \\ &= \sum_{i,i'} \langle\psi(t_1)|s_{i'}\rangle\langle s_i|\psi(t_1)\rangle\langle s_{i'}|\hat{U}^\dagger|g_\omega\rangle\langle g_\omega|\hat{U}|s_i\rangle \lambda\frac{1}{2}(s_i + s_{i'}) \\ &= \frac{\lambda}{2}\langle\psi(t_1)|U^\dagger|g_\omega\rangle\langle g_\omega|\hat{U}\hat{S}|\psi(t_1)\rangle + \frac{\lambda}{2}\langle g_\omega|\hat{U}|\psi(t_1)\rangle\langle\psi(t_1)|\hat{S}\hat{U}^\dagger|g_\omega\rangle,\end{aligned} \tag{D.3}$$

where the dependence on the second measurement is indicated by the term $|g_w\rangle$. We have used the identity devloped in (B.2) and also $\hat{S} = \sum_i |s_i\rangle s_i\langle s_i| = \sum_i |s_{i'}\rangle s_{i'}\langle s_{i'}|$ and $\sum_i |s_i\rangle\langle s_i| = \sum_{i'} |s'_i\rangle\langle s_{i'}| = 1$ and we allow a unitary evolution described by $\hat{U}$ between the measurements. The commutation of the operators S and U in (D.3) requires some further discussion.

Our goal of achieving the approximation in (D.1) necessitates that the operator $\hat{S}$ should commute with the unitary operator $\hat{U}$. For the case of work, the operator $\hat{S}$ is the hamiltonian $\hat{H}$ which commutes with the unitary operator. In the case of the momentum operator $\hat{P}$ using the assumption of a flat potential leads to the commutation of the momentum operator $\hat{P}$ with the unitary operator (as far as the time between the weak and strong measurements are done within a short time interval $t_2 - t_1$). Under this assumption, we will get:

$$\int dy_k\ y_k P(y_\omega, y_k) = \mathrm{Re}\left[\lambda\langle\psi(t_2)|g_\omega\rangle\langle g_\omega|\hat{S}|\psi(t_2)\rangle\right] \tag{D.4}$$

We evaluate now the denominator of (D.1)as follows,

$$\int dy_k P(y_\omega, y_k) = \int dy_k \sum_{i,i'} c^*_{i'} c_i c^*_{w,i'} c_{w,i} \left(a^*(y_k) - \lambda s_{i'} \frac{\partial a^*(y_k)}{\partial y_k}\right)\left(a(y_k) - \lambda s_i \frac{\partial a(y_k)}{\partial y_k}\right). \tag{D.5}$$

Noting that $\int dy_k |a(y_k)|^2 = 1$ and $\int dy_k a(y_k)\frac{\partial a(y_k)}{\partial y_k} = 0$ we get,

$$\int dy_k P(y_\omega, y_k) = \int dy_k \sum_{i,i'} c^*_{i'} c_i c^*_{w,i'} c_{w,i} \left(|a(y_k)|^2 + \lambda^2 s_i s_{i'} \left| \frac{\partial a(y_k)}{\partial y_k} \right|^2 \right). \tag{D.6}$$

Again, one is tempted to argue that in general $c^*_{i'} c_i |a(y_k)|^2 \gg \lambda^2 c^*_{i'} c_i s_i s_{i'} \left| \frac{\partial a(y_k)}{\partial y_k} \right|^2$ so that the last term can be neglected. However, as a test, by using $a(y_k) = \left(\frac{1}{\pi\sigma^2}\right)^{1/4} \exp\left(\frac{-y_k^2}{2\sigma^2}\right)$ we can check for which values of $y = y_k$ this is true. We get that the second coefficient becomes comparable to the first one when $y \approx \sigma^2/\lambda$.

The solution to the above-mentioned source of contamination in the denominator is quite simple from an experimental point of view. In a real experiment, the rare events corresponding to $y > \sigma^2/\lambda$ will not provide a significant contribution. Notice that we are approximating here the marginal probability in (D.6) with $P(y_\omega, y_k) \approx 0$ for $y_k \to \infty$. The same approximation cannot be done when dealing with the correlation functions computed in the text and in Appendix C. Then, we can evaluate the denominator as

$$\begin{aligned} \int dy_k P(y_\omega, y_k) &= \sum_{i,i'} c^*_{i'} c_i c^*_{w,i'} c_{w,i} = \sum_{i,i'} \langle \psi(t_1)|s_{i'}\rangle\langle s_i|\psi(t_1)\rangle\langle s_{i'}|\hat{U}^\dagger|g_\omega\rangle\langle g_\omega|\hat{U}|s_i\rangle \\ &= \sum_{i,i'} \langle \psi(t_1)|s_{i'}\rangle\langle s_{i'}|\hat{U}^\dagger|g_\omega\rangle\langle g_\omega|\hat{U}|s_i\rangle\langle s_i|\psi(t_1)\rangle = \langle \psi(t_1)|\hat{U}^\dagger|g_\omega\rangle\langle g_\omega|\hat{U}|\psi(t_1)\rangle. \end{aligned} \tag{D.7}$$

Finally, using (D.7) and (D.4), the weak value of a general operator $\hat{S}$ is given as,

$$\mathrm{Re}\left({}_{g_\omega}\langle \hat{S} \rangle_{\psi(t_2)} \right) = \mathrm{Re}\left(\frac{\langle g_\omega|\hat{S}|\psi(t_2)\rangle}{\langle g_\omega|\psi(t_2)\rangle} \right) \approx \frac{1}{\lambda} \frac{\int dy_k \; y_k P(y_\omega, y_k)}{\int dy_k P(y_\omega, y_k)}. \tag{D.8}$$

which is the result anticipated from D.1. The role of λ is just providing the correct dimensional consistency.