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Axial gravity and anomalies of fermions

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Abstract

We consider a Dirac fermion in a metric-axial-tensor (MAT) background. By regulating it with Pauli-Villars fields we analyze and compute its full anomaly structure. Appropriate limits of the MAT background allows to recover the anomalies of Dirac and Weyl fermions in the usual curved spacetime, obtaining in particular the trace anomaly of a chiral fermion, which has been the object of recent analyses.

1 Introduction

A metric-axial-tensor (MAT) background for Dirac fermions has been recently constructed in [1, 2], with the main purpose of addressing anomalies, especially in a suitable chiral limit. It generalizes to curved space the approach used by Bardeen to study vector and axial couplings of Dirac fermions to gauge fields and analyze their anomalies [3]. The metric-axial-tensor is defined by

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + \gamma^5 f_{\mu\nu} \tag{1}$$

and induces similar axial extensions (i.e. with a γ^5 component) to the other geometrical quantities, like the vierbein $\hat{e}^a_\mu$ and the spin connection $\hat{\omega}_{\mu ab}$. A massless Dirac fermion coupled to the MAT background has a lagrangian of the form

$$\mathcal{L} = -\overline{\psi}\gamma^a\sqrt{\hat{g}}\hat{e}_a^\mu\hat{\nabla}_\mu\psi \tag{2}$$

with covariant derivative

$$\hat{\nabla}_\mu = \partial_\mu + \frac{1}{4}\hat{\omega}_{\mu ab}\gamma^{ab} \tag{3}$$

where $\gamma^{ab} = \frac{1}{2}[\gamma^a, \gamma^b]$. All quantities with a hat contain an axial extension with γ^5 and appear always sandwiched between the Dirac spinors $\overline{\psi}$ and ψ. For details we refer to [1, 2], especially appendix B of the latter.

For our purposes it is more convenient and transparent to split the Dirac fermion ψ into its two independent and Lorentz irreducible chiral components λ and ρ of opposite chiralities, $\psi = \lambda + \rho$. We use the conventions of [4, 5] for spinors and gamma matrices. In particular our chiral spinors satisfy $\gamma^5\lambda = \lambda$ and $\gamma^5\rho = -\rho$. Then the lagrangian takes the form

$$\mathcal{L} = -\sqrt{g_+}\,\overline{\lambda}\gamma^a e_a^{+\mu}\nabla_\mu^+\lambda - \sqrt{g_-}\,\overline{\rho}\gamma^a e_a^{-\mu}\nabla_\mu^-\rho \tag{4}$$

where $g^{\pm}_{\mu\nu} = g_{\mu\nu} \pm f_{\mu\nu}$ are two different effective metrics, with related compatible vierbeins, spin connections, and covariant derivatives (which we indicate with the $\pm$ sub/superscripts). This happens since the γ^5 matrices acting on chiral fermions are substituted by the corresponding eigenvalues. One could be more general, allowing also for the spacetime points to have an axial extension, as outlined in [2], but the present formulation is sufficient for our purposes.

The limit $f_{\mu\nu} = 0$ recovers the standard massless Dirac fermion in the metric $g_{\mu\nu}$. Setting $g_{\mu\nu} = f_{\mu\nu} \to \frac{1}{2} g_{\mu\nu}$ produces instead a left handed chiral fermion λ coupled to the final metric $g_{\mu\nu}$, while the other chirality is projected out (it remains coupled to the singular metric $g^{-}_{\mu\nu} = 0$). A less singular limit, which keeps a free propagating right-handed fermion, is to consider the "collapsing limit" [2], which consists in setting $g^{+}_{\mu\nu} = g_{\mu\nu}$ and $g^{-}_{\mu\nu} = \eta_{\mu\nu}$, with $\eta_{\mu\nu}$ the flat Minkowski metric.

The Dirac theory in the MAT background has several symmetries which may become anomalous, and limits on the background can be used to recover the anomalies of a chiral fermion, as in the Bardeen method. The classical symmetries of the model are: diffeomorphisms, local Lorentz transformation and Weyl rescalings, together with their axial extensions, all of which are background symmetries since they act on the MAT fields as well. In addition, the model admits global vector and axial U(1) symmetries, that rotate the spinors by a phase. This global $U(1)_V \times U(1)_A$ symmetry group does not act on the MAT background or any other background, as we do not couple the model to the corresponding abelian gauge fields, though that could be done as well. We will review shortly these symmetries, compute systematically all of their anomalies, and then study the chiral limit.

To compute the anomalies we use a Pauli-Villars (PV) regularization, where the mass term of the PV fields is the source of the anomalies. If the mass term can be chosen to be symmetric under a given symmetry, then there will be no anomalies in that symmetry. Otherwise the classical breaking due to the nonsymmetric mass term sources the one-loop anomaly. Here we use the scheme of [6, 7], that casts the anomalies in the form of a regulated Fujikawa jacobian [8, 9], which allows the use of well-known heat kernel formulae for the explicit final evaluation. The variation of local counterterms, that parametrize the relation to different regularization schemes, as for example those identified by different mass terms, can in general be employed to cancel or shift the anomalies to different sectors. This method has already been applied successfully to several contexts in the past, as the case of two-dimensional *b*-*c* systems [10], which bear some analogies to the four-dimensional Weyl fermion case, or the more exotic model of chiral bosons [11]. It is the same method used more recently in [4, 5, 12] to address the trace anomalies of a Weyl fermion.

Before starting our systematic treatment, let us discuss the possible form of the mass term to be used in the PV sector. This mass term is quite arbitrary, as long as it regulates correctly the theory by giving rise to an invertible matrix T in field space, to be defined shortly. Given this arbitrariness, one would like to choose it in the most symmetrical way as possible, to preserve the maximal number of symmetries at the quantum level. The choice is essentially between the Dirac and Majorana masses, suitably coupled to the MAT background. Both are legal. However choosing a Majorana mass will simplify drastically the calculations, as it allows to maintain anomaly free the diffeomorphisms and the local Lorentz symmetry, together with their axial extensions. This happens as the Majorana mass keeps a split structure for the couplings of the chiral irreducible components of the Dirac fermion to the effective metrics $g^{\pm}_{\mu\nu}$, while producing anomalies in the Weyl and $U(1)$ symmetries and their axial extensions only. Thus, we will choose a Majorana mass for computing the complete set of anomalies of a Dirac fermion in the MAT background. We will comment briefly also on the Dirac mass, which turns out to be much less symmetric since it destroys all of the axial symmetries. It could be employed as well, but calculations become much more cumbersome, producing more anomalies then necessary, that eventually must be cured by adding countertems to the effective action. However, let us recall once more that any choice of the PV mass term is valid, as local counterterms can be added to the effective action to recover the same final result, independently of the regularization scheme adopted. This arbitrariness is a general feature of the renormalization process of QFTs.

Now, let us briefly describe our method of calculation. A lagrangian for the fields φ

$$\mathcal{L} = \frac{1}{2}\varphi^T T\mathcal{O}\varphi \ , \tag{5}$$

which is invariant under a linear symmetry $\delta\varphi = K\varphi$, that may act also on the background fields contained in T and $\mathcal{O}$, is regulated by PV fields ϕ with lagrangian

$$\mathcal{L}_{PV} = \frac{1}{2}\phi^T T\mathcal{O}\phi + \frac{1}{2}M\phi^T T\phi \tag{6}$$

where M is a mass parameter which identifies the mass matrix T. The latter permits the explicit identification of the differential operator $\mathcal{O}$. In fermionic theories the operator $\mathcal{O}^2$ appears as the regulator. Indeed, one may verify that the non-invariance of the mass term under the symmetry of the massless action $\delta\phi = K\phi$ produces an anomalous variation of the regulated effective action Γ, that survives in the $M \to \infty$ limit. In our hypercondensed notation[1] it reads

$$\begin{aligned} i\delta\Gamma = i\langle\delta S\rangle &= \lim_{M\to\infty} iM\langle\phi^T(TK + \frac{1}{2}\delta T)\phi\rangle = -\lim_{M\to\infty} \mathrm{Tr}\left[\left(K + \frac{1}{2}T^{-1}\delta T\right)\left(1 + \frac{\mathcal{O}}{M}\right)^{-1}\right] \\ &= -\lim_{M\to\infty} \mathrm{Tr}\left[\left(K + \frac{1}{2}T^{-1}\delta T + \frac{1}{2}\frac{\delta\mathcal{O}}{M}\right)\left(1 - \frac{\mathcal{O}^2}{M^2}\right)^{-1}\right] . \end{aligned} \tag{7}$$

The function of the regulator $\mathcal{O}^2$ inside the trace can now be substituted with an exponential function, that gives an equivalent regularization and produces the same anomaly, so that one finds the Fujikawa-like formula

$$i\delta\Gamma = i\langle\delta S\rangle = -\lim_{M\to\infty} \mathrm{Tr}[J e^{i\frac{\mathcal{O}^2}{M^2}}] \tag{8}$$

where $J = K + \frac{1}{2}T^{-1}\delta T + \frac{1}{2}\frac{\delta\mathcal{O}}{M}$ may be identified as the infinitesimal part of a Fujikawa jacobian. As described, it is entirely due to the non invariance of the PV mass term. We use a factor of i in the exponential to stress that we employ a minkowskian time in the heat kernel. Now, heat kernel formulae may be directly applied. In particular, we need the heat kernel coefficient $a_2(\mathcal{O}^2)$, that we indicate in the notation of appendix B of [5]. Details on this PV scheme may be found in [6, 7], and recapitulated in [4, 5] as well.

2 Majorana mass

In this section we regulate the Dirac theory in the MAT background with PV fields with a lagrangian of the same form as (4), but augmented by a Majorana mass, that we choose to be

[1]The sum over (suppressed) indices includes spacetime integration, so that the lagrangian identifies directly the action.

coupled as

$$\Delta_M \mathcal{L} = \frac{M}{2}\sqrt{g_+}\,(\lambda^T C\lambda + h.c.) + \frac{M}{2}\sqrt{g_-}\,(\rho^T C\rho + h.c.) \tag{9}$$

where $h.c.$ denotes hermitian conjugation and C is the charge conjugation matrix that satisfies $C\gamma^a C^{-1} = -\gamma^{aT}$. For notational simplicity we use the same symbols for the PV fields and the original variables, since no confusion can arise in the following. The advantage of this specific mass term is that it is invariant under diffeomorphisms, and thus guarantees the absence of gravitational anomalies [13] (the stress tensor remains covariantly conserved). In fact, inspection of the action shows that this symmetry can be extended to the axial diffeomorphisms as well, guaranteeing the covariant conservation of a corresponding axial stress tensor. Let us elaborate more extensively on this point. The usual change of coordinates $x^\mu \to x^\mu - \xi^\mu(x)$ induce the standard transformation law on the fields as generated by the Lie derivative $\mathcal{L}_\xi$

$$\begin{aligned} \delta e_\mu^{\pm a} &= \xi^\nu \partial_\nu e_\mu^{\pm a} + (\partial_\mu \xi^\nu) e_\nu^{\pm a} \equiv \mathcal{L}_\xi e_\mu^{\pm a} \\ \delta\psi &= \xi^\mu \partial_\mu \psi \equiv \mathcal{L}_\xi \psi \ . \end{aligned} \tag{10}$$

However, one can define chiral transformation rules that leave the entire massive action invariant. One may define left infinitesimal diffeomorphisms generated by a vector field $\xi_+^\mu(x)$

$$\begin{aligned} \delta e_\mu^{+a} &= \mathcal{L}_{\xi_+} e_\mu^{+a} \\ \delta\lambda &= \mathcal{L}_{\xi_+}\lambda \\ \delta e_\mu^{-a} &= 0 \\ \delta\rho &= 0 \end{aligned} \tag{11}$$

and right infinitesimal diffeomorphisms generated by a vector field $\xi_-^\mu(x)$

$$\begin{aligned} \delta e_\mu^{+a} &= 0 \\ \delta\lambda &= 0 \\ \delta e_\mu^{-a} &= \mathcal{L}_{\xi_-} e_\mu^{-a} \\ \delta\rho &= \mathcal{L}_{\xi_-}\rho \ . \end{aligned} \tag{12}$$

It is only the sum with local parameters identified, i.e. with $\xi^\mu = \xi_+^\mu = \xi_-^\mu$, that plays the role of the geometrical transformation induced by the translation of the spacetime point x^μ described above. Nevertheless, they are independent symmetries of the massless and massive actions. They acquire a clear geometrical meaning once the spacetime point x^μ is extended to have an axial partner [2], but we do not need to do that for the scope of the present investigation. These symmetries imply that the stress tensor and its axial partner satisfy suitable covariant conservation laws. Invariance of the mass term, and thus invariance of the full PV action, implies that these symmetries are not anomalous at the quantum level.

Similarly, the action and the mass term are invariant under the local Lorentz symmetries that act independently on the + and − sector. On the + sector (the left-handed sector) the left-handed local Lorentz symmetry acts nontrivially by

$$\begin{aligned} \delta e_\mu^{+a} &= \omega^{+a}{}_b e_\mu^{+b} \\ \delta\lambda &= \frac{1}{4}\omega_{ab}^+ \gamma^{ab}\lambda \\ \delta e_\mu^{-a} &= 0 \\ \delta\rho &= 0 \end{aligned} \tag{13}$$

where $\omega^+_{ab} = -\omega^+_{ba}$ are local parameters. Similarly, on the right sector one has

$$\begin{aligned} \delta e^{+a}_\mu &= 0 \\ \delta\lambda &= 0 \\ \delta e^{-a}_\mu &= \omega^{-a}{}_b e^{-b}_\mu \\ \delta\rho &= \frac{1}{4}\omega^-_{ab}\gamma^{ab}\rho\ . \end{aligned} \tag{14}$$

Evidently, these are full symmetries of the total PV lagrangian, including the mass term. The invariance of the regulating fields guarantees that the stress tensor and its axial companion remain symmetric at the quantum level.

The only possible anomalies appear in the Weyl and axial Weyl symmetries, and in the vector and axial $U(1)$ symmetries. It is again more convenient to consider their $\pm$ linear combinations, that act separately on the chiral sectors of the theory. The infinitesimal Weyl symmetries are defined by

$$\begin{aligned} \delta e^{\pm a}_\mu &= \sigma^\pm e^{\pm a}_\mu \\ \delta\lambda &= -\frac{3}{2}\sigma^+\lambda \\ \delta\rho &= -\frac{3}{2}\sigma^-\rho \end{aligned} \tag{15}$$

where $\sigma^\pm$ are the two independent Weyl local parameters. The mass terms breaks them explicitly

$$\delta\Delta_M\mathcal{L} = \sigma^+\frac{M}{2}\sqrt{g_+}\,(\lambda^T C\lambda + h.c.) + \sigma^-\frac{M}{2}\sqrt{g_-}\,(\rho^T C\rho + h.c.) \tag{16}$$

causing anomalies to appear. For the global $U(1)_L \times U(1)_R$ symmetries, with independent infinitesimal parameters $\alpha^\pm$, we have the transformation rules

$$\begin{aligned} \delta\lambda &= i\alpha^+\lambda \\ \delta\rho &= i\alpha^-\rho \end{aligned} \tag{17}$$

and the PV mass term is again responsible for their breaking

$$\delta\Delta_M\mathcal{L} = i\alpha^+ M\sqrt{g_+}\,(\lambda^T C\lambda - h.c.) + i\alpha^- M\sqrt{g_-}\,(\rho^T C\rho - h.c.)\ . \tag{18}$$

Before computing the anomalies, let us cast the lagrangian with the Dirac mass term using the Dirac basis of spinors ψ and ψ_c, so to recognize the operators in (6) and identify our regulator $\mathcal{O}^2$. We prefer to use $\psi_c = C^{-1}\overline{\psi}^T$ rather than $\overline{\psi}$, as the former has the same index structure of ψ, and thus lives in the same spinor space. The massless lagrangian (2) with the addition of the Dirac mass term (9) fixes the PV lagrangian

$$\mathcal{L}_{PV} = \frac{1}{2}\psi_c^T C\sqrt{\bar{\hat{g}}}\hat{\nabla\!\!\!/}\psi + \frac{1}{2}\psi^T C\sqrt{\hat{g}}\bar{\hat{\nabla\!\!\!/}}\psi_c + \frac{M}{2}(\psi^T\sqrt{\hat{g}}C\psi + \psi_c^T\sqrt{\bar{\hat{g}}}C\psi_c) \tag{19}$$

where a bar indicates a sign change in the axial extension (e.g. $\bar{\hat{g}}_{\mu\nu} = g_{\mu\nu} - \gamma^5 f_{\mu\nu}$) and $\hat{\nabla\!\!\!/} = \gamma^a\hat{e}^\mu_a\hat{\nabla}_\mu$, so that on the field basis $\phi = \begin{pmatrix}\psi \\ \psi_c\end{pmatrix}$ one finds

$$T\mathcal{O} = \begin{pmatrix} 0 & C\sqrt{\hat{g}}\bar{\hat{\nabla\!\!\!/}} \\ C\sqrt{\bar{\hat{g}}}\hat{\nabla\!\!\!/} & 0 \end{pmatrix}\ ,\quad T = \begin{pmatrix} \sqrt{\hat{g}}C & 0 \\ 0 & \sqrt{\bar{\hat{g}}}C \end{pmatrix}\ ,\quad \mathcal{O} = \begin{pmatrix} 0 & \bar{\hat{\nabla\!\!\!/}} \\ \hat{\nabla\!\!\!/} & 0 \end{pmatrix} \tag{20}$$

and the regulator

$$\mathcal{O}^2 = \begin{pmatrix} \hat{\bar{\nabla\!\!\!/}}\hat{\nabla\!\!\!/} & 0 \\ 0 & \hat{\nabla\!\!\!/}\hat{\bar{\nabla\!\!\!/}} \end{pmatrix} . \tag{21}$$

Its structure is perhaps more transparent when the Dirac fermions are split in their chiral parts

$$\phi = \begin{pmatrix} \lambda \\ \rho \\ \rho_c \\ \lambda_c \end{pmatrix} \tag{22}$$

and one recognizes a block diagonal regulator

$$\mathcal{O}^2 = \begin{pmatrix} \mathcal{O}^2_\lambda & 0 & 0 & 0 \\ 0 & \mathcal{O}^2_\rho & 0 & 0 \\ 0 & 0 & \mathcal{O}^2_{\rho_c} & 0 \\ 0 & 0 & 0 & \mathcal{O}^2_{\lambda_c} \end{pmatrix} \tag{23}$$

with entries

$$\begin{aligned} \mathcal{O}^2_\lambda &= \nabla\!\!\!/^2_+ P_+ \,, \qquad & \mathcal{O}^2_{\lambda_c} &= \nabla\!\!\!/^2_+ P_- \\ \mathcal{O}^2_\rho &= \nabla\!\!\!/^2_- P_- \,, \qquad & \mathcal{O}^2_{\rho_c} &= \nabla\!\!\!/^2_- P_+ \,. \end{aligned} \tag{24}$$

where we have used the left/right chiral projectors $P_+ = P_L = \frac{\mathbb{1}+\gamma^5}{2}$ and $P_- = P_R = \frac{\mathbb{1}-\gamma^5}{2}$, and denoted by $\nabla\!\!\!/_\pm = \gamma^a e_a^{\pm\mu} \nabla^\pm_\mu$ the Dirac operators coupled to the $\pm$ effective vierbeins.

Let us now compute the anomalies. For the Weyl symmetries we get anomalies in the traces of the stress tensors, defined by varying the action under the two effective vierbeins $e^\pm_{\mu a}$

$$T^{\mu a}_\pm(x) = \frac{1}{\sqrt{g_\pm}} \frac{\delta S}{\delta e^\pm_{\mu a}(x)} . \tag{25}$$

In each chiral sector we use the corresponding chiral metric, and related vierbein, to perform covariant operations and take traces, and the calculation is just a double copy of the one presented in [4]. Recalling (8) one identifies the structure of the breaking term J, entirely due to the PV mass. Once inserted into the "Fujikawa trace", it is computed by using the heat kernel coefficients $a_2(\mathcal{O}^2)$ for the regulators $\mathcal{O}^2$ due to the PV fields. All the steps have been discussed in details in [4, 5], where in particular it was noticed that the term $\delta\mathcal{O}$ in J does not contribute to the functional trace. In the present situation we find for the traces of the stress tensors on the MAT background

$$\begin{aligned} \langle T^\mu_{+\mu} \rangle &= -\frac{1}{2\,(4\pi)^2} \Big[\mathrm{tr}[P_+ a_2(\mathcal{O}^2_\lambda)] + \mathrm{tr}[P_- a_2(\mathcal{O}^2_{\lambda_c})] \Big] \\ \langle T^\mu_{-\mu} \rangle &= -\frac{1}{2\,(4\pi)^2} \Big[\mathrm{tr}[P_- a_2(\mathcal{O}^2_\rho)] + \mathrm{tr}[P_+ a_2(\mathcal{O}^2_{\rho_c})] \Big] \end{aligned} \tag{26}$$

where the remaining final dimensional traces are traces on the gamma matrices. The projectors on the regulators can be dropped, as they get absorbed by the explicit projectors already present in (26). Thus, one may use $\mathcal{O}^2_\lambda = \mathcal{O}^2_{\lambda_c} = \nabla\!\!\!/^2_+$ and $\mathcal{O}^2_\rho = \mathcal{O}^2_{\rho_c} = \nabla\!\!\!/^2_-$ to simplify the anomaly expressions to

$$\langle T^\mu_{\pm\mu} \rangle = -\frac{1}{2\,(4\pi)^2} \mathrm{tr}[a_2(\nabla\!\!\!/^2_\pm)] \tag{27}$$

and one finds the following trace anomalies on the MAT background

$$\langle T^\mu_{\pm\mu}\rangle = \frac{1}{720\,(4\pi)^2}\Big(7R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} + 8R_{\mu\nu}R^{\mu\nu} - 5R^2 + 12\Box R\Big)(g_\pm) \tag{28}$$

where the functional dependence on $g_\pm$ reminds that all the geometrical quantities and covariant operations are computed using the effective metric $g^\pm_{\mu\nu}$.

We now compute the $U(1)_L \times U(1)_R$ anomalies. Evidently, we are going to find again a split form. By the Noether theorem one finds the covariantly conserved Noether currents

$$\delta S = \int d^4x\sqrt{g_+}\,\alpha_+ \nabla^+_\mu J^\mu_+ + \int d^4x\sqrt{g_-}\,\alpha_- \nabla^-_\mu J^\mu_- \tag{29}$$

where the constants $\alpha_\pm$ in (17) are extended to arbitrary functions, with the currents taking the explicit form $J^\mu_+ = i\overline{\lambda}\gamma^a e_a^{+\mu}\lambda$ and $J^\mu_- = i\overline{\rho}\gamma^a e_a^{-\mu}\rho$. We compute their anomalies with the PV regularization and find

$$\begin{aligned} \nabla^+_\mu\langle J^\mu_+\rangle &= \frac{i}{(4\pi)^2}\Big[\mathrm{tr}[P_+ a_2(\mathcal{O}^2_\lambda)] - \mathrm{tr}[P_- a_2(\mathcal{O}^2_{\lambda_c})]\Big] \\ \nabla^-_\mu\langle J^\mu_-\rangle &= \frac{i}{(4\pi)^2}\Big[\mathrm{tr}[P_- a_2(\mathcal{O}^2_\rho)] - \mathrm{tr}[P_+ a_2(\mathcal{O}^2_{\rho_c})]\Big] \end{aligned} \tag{30}$$

that once more can be simplified to

$$\nabla^\pm_\mu\langle J^\mu_\pm\rangle = \pm\frac{i}{(4\pi)^2}\mathrm{tr}[\gamma^5\, a_2(\nabla\!\!\!/^2_\pm)]\ . \tag{31}$$

Their evaluation in terms of the heat kernel coefficients produces anomalies proportional to the Pontryagin density of the effective metrics

$$\nabla^\pm_\mu\langle J^\mu_\pm\rangle = \mp\frac{1}{48\,(4\pi)^2}\sqrt{g_\pm}\epsilon_{\alpha\beta\gamma\delta}R_{\mu\nu}{}^{\alpha\beta}R^{\mu\nu\gamma\delta}(g_\pm)\ . \tag{32}$$

Eqs. (28) and (32) are our final results for the anomalies of a Dirac fermion on a MAT background. All other symmetries are anomaly free.

We have evaluated these anomalies using traces with chiral projectors of the heat kernel coefficient $a_2(\nabla\!\!\!/^2)$, associated to the covariant square of the Dirac operator in a background metric $g_{\mu\nu}$. For completeness, we list this coefficient and related traces

$$a_2(\nabla\!\!\!/^2) = \frac{1}{180}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu}) + \frac{1}{288}R^2 - \frac{1}{120}\Box R + \frac{1}{192}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} \tag{33}$$

$$\begin{aligned} \mathrm{tr}[P_\pm a_2(\nabla\!\!\!/^2)] =& -\frac{1}{720}(7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 8R_{\mu\nu}R^{\mu\nu} - 5R^2 + 12\Box R) \\ &\pm\frac{i}{96}\sqrt{g}\epsilon_{\alpha\beta\gamma\delta}R_{\mu\nu}{}^{\alpha\beta}R^{\mu\nu\gamma\delta} \end{aligned} \tag{34}$$

where $\mathcal{R}_{\mu\nu} = R_{\mu\nu ab}\gamma^{ab}$. One may deduce them from [14, 15], for example. They are useful in studying intermediate results leading to the evaluation of (26) and (30), and have appeared in the anomaly context already in [16, 17].

3 Limits of the MAT background

We now discuss the limits on the MAT background to recover the usual theories of Dirac and Weyl fermions in a curved spacetime and their anomalies.

Setting $f_{\mu\nu} = 0$ reproduces the standard coupling of a massless Dirac fermion to a curved background and corresponds to identify the two effective metrics $g^{+}_{\mu\nu} = g^{-}_{\mu\nu}$. The final stress tensor becomes the sum of the two chiral stress tensors, and acquires the sum of the two trace anomalies in (28). Thus, one recovers the usual trace anomaly of a Dirac field

$$\langle T^{\mu}{}_{\mu} \rangle = \frac{1}{360\,(4\pi)^2}\Big(7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 8R_{\mu\nu}R^{\mu\nu} - 5R^2 + 12\Box R\Big)\ . \tag{35}$$

Similarly, for the two U(1) symmetries, one obtains

$$\nabla_\mu \langle J^\mu_{\pm} \rangle = \mp \frac{1}{48\,(4\pi)^2}\sqrt{g}\epsilon_{\alpha\beta\gamma\delta}R_{\mu\nu}{}^{\alpha\beta}R^{\mu\nu\gamma\delta} \tag{36}$$

which gets translated into the covariant conservation of the vector current $J^\mu_{_V} = J^\mu_+ + J^\mu_-$, together with the anomalous conservation of the axial current $J^\mu_{_A} = J^\mu_+ - J^\mu_-$, with the well-known Pontryagin contribution [18, 19]

$$\nabla_\mu \langle J^\mu_{_V} \rangle = 0\ , \qquad \nabla_\mu \langle J^\mu_{_A} \rangle = -\frac{1}{24\,(4\pi)^2}\sqrt{g}\epsilon_{\alpha\beta\gamma\delta}R_{\mu\nu}{}^{\alpha\beta}R^{\mu\nu\gamma\delta}\ . \tag{37}$$

Let us now study the case of the Weyl fermion λ. This is obtained by taking the collapsing limit in which the effective metric $g^{-}_{\mu\nu}$ becomes flat ($g^{-}_{\mu\nu} = \eta_{\mu\nu}$ and $g^{+}_{\mu\nu} = g_{\mu\nu}$), so that the independent right-handed fermion ρ decouples completely from the background. Therefore, only the chiral left-handed part part contributes to the stress tensor, producing for the trace anomaly half of the result above. Similarly, one finds the anomalous conservation of the $U(1)$ current J^μ_+, the only one that remains coupled to the curved background, with the expected Pontryagin contribution. To summarize, we find for a left-handed Weyl fermion the following anomalies

$$\begin{aligned} \langle T^{\mu}{}_{\mu} \rangle &= \frac{1}{720\,(4\pi)^2}\Big(7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 8R_{\mu\nu}R^{\mu\nu} - 5R^2 + 12\Box R\Big) \\ \nabla_\mu \langle J^\mu_{+} \rangle &= -\frac{1}{48\,(4\pi)^2}\sqrt{g}\epsilon_{\alpha\beta\gamma\delta}R_{\mu\nu}{}^{\alpha\beta}R^{\mu\nu\gamma\delta}\ . \end{aligned} \tag{38}$$

These results confirm the absence of a Pontryagin term in the trace anomaly of a Weyl fermion, as calculated in [4] and confirmed in [20]. The Pontryagin term sits only in the chiral anomaly.

4 Dirac mass

In this section we wish to give a brief description of a different regularization, namely the one obtained by using a Dirac mass for the PV fields. In a flat background a Dirac mass is given by

$$-M\bar{\psi}\psi = \frac{1}{2}M(\psi_c^T C\psi + \psi^T C\psi_c)\ . \tag{39}$$

As well-known this term breaks the $U(1)_A$ axial symmetry while maintaining the $U(1)_V$ vector symmetry. This continues to be the case also when one tries to MAT-covariantize it. There are various options to couple the Dirac mass to the MAT geometry. One may choose to use in the mass term only the metric $g_{\mu\nu}$, without any axial extension, so that

$$\Delta_D \mathcal{L} = -\sqrt{g} M \bar{\psi}\psi = \frac{\sqrt{g}}{2} M (\psi_c^T C \psi + \psi^T C \psi_c) \tag{40}$$

has the virtue of preserving the vector-like diffeomorphisms and vector-like local Lorentz transformations on top of the $U(1)_V$ symmetry, while breaking all of their axial extensions. It also breaks both vector and axial Weyl symmetries, which are then expected to be anomalous as well. Counterterms should eventually be introduced to achieve the equivalence with our previous results. Other choices are also possible, as for example $\sqrt{g} \to \frac{1}{2}(\sqrt{g_+} + \sqrt{g_-})$, which shares the same property of preserving the vector-like diffeomorphisms and local Lorentz transformations.

In the following we just wish to derive the regulators to be used for computing the anomalies in this new scheme. We add to the lagrangian (2) written in a symmetric form

$$\mathcal{L} = \frac{1}{2}\psi_c^T C \sqrt{\bar{\hat{g}}}\hat{\nabla}\psi + \frac{1}{2}\psi^T C \sqrt{\hat{g}}\bar{\hat{\nabla}}\psi_c \tag{41}$$

the mass term (40), and comparing it with (6) we find on the field basis $\phi = \begin{pmatrix} \psi \\ \psi_c \end{pmatrix}$

$$T\mathcal{O} = \begin{pmatrix} 0 & C\sqrt{\hat{g}}\bar{\hat{\nabla}} \\ C\sqrt{\bar{\hat{g}}}\hat{\nabla} & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & \sqrt{g}C \\ \sqrt{g}C & 0 \end{pmatrix}, \quad \mathcal{O} = \begin{pmatrix} \sqrt{\frac{\bar{\hat{g}}}{g}}\hat{\nabla} & 0 \\ 0 & \sqrt{\frac{\hat{g}}{g}}\bar{\hat{\nabla}} \end{pmatrix}. \tag{42}$$

Thus, we get the regulator

$$\mathcal{O}^2 = \begin{pmatrix} \sqrt{\frac{\bar{\hat{g}}}{g}}\hat{\nabla}\sqrt{\frac{\bar{\hat{g}}}{g}}\hat{\nabla} & 0 \\ 0 & \sqrt{\frac{\hat{g}}{g}}\bar{\hat{\nabla}}\sqrt{\frac{\hat{g}}{g}}\bar{\hat{\nabla}} \end{pmatrix} \tag{43}$$

with the differential operators acting on everything placed on their right hand side. This regulator $\mathcal{O}^2$ is difficult to work with, but it has the virtue of being covariant under vector diffeomorphisms, making it somewhat manageable after all. Its structure is again more transparent when splitting the Dirac fermion into its chiral parts, so that on the basis

$$\phi = \begin{pmatrix} \lambda \\ \rho \\ \rho_c \\ \lambda_c \end{pmatrix} \tag{44}$$

one finds a block diagonal regulator

$$\mathcal{O}^2 = \begin{pmatrix} \mathcal{O}^2_\lambda & 0 & 0 & 0 \\ 0 & \mathcal{O}^2_\rho & 0 & 0 \\ 0 & 0 & \mathcal{O}^2_{\rho_c} & 0 \\ 0 & 0 & 0 & \mathcal{O}^2_{\lambda_c} \end{pmatrix} \tag{45}$$

with entries

$$\mathcal{O}^2_\lambda = \sqrt{\frac{g_-}{g}}\nabla\!\!\!\!/_-\sqrt{\frac{g_+}{g}}\nabla\!\!\!\!/_+ P_+ \, , \qquad \mathcal{O}^2_\rho = \sqrt{\frac{g_+}{g}}\nabla\!\!\!\!/_+\sqrt{\frac{g_-}{g}}\nabla\!\!\!\!/_- P_-$$
$$\mathcal{O}^2_{\lambda_c} = \sqrt{\frac{g_-}{g}}\nabla\!\!\!\!/_-\sqrt{\frac{g_+}{g}}\nabla\!\!\!\!/_+ P_- \, , \qquad \mathcal{O}^2_{\rho_c} = \sqrt{\frac{g_+}{g}}\nabla\!\!\!\!/_+\sqrt{\frac{g_-}{g}}\nabla\!\!\!\!/_- P_+ \, . \tag{46}$$

The functions $\sqrt{\frac{g_\mp}{g}}$ are scalar functions under the vector-like diffeomorphism, so that these regulators are covariant under that symmetry. The projectors $P_\pm$ take just the unit value on the corresponding chiral spinor space, but we have kept them to remember on which space the different regulators act. A systematic analysis of all the anomalies, including the axial gravitational anomaly, may be feasible in this scheme, at least when treating $f_{\mu\nu}$ as a perturbation.

5 Conclusions

We have studied the full set of anomalies of a Dirac fermion coupled to the MAT background formulated recently in [1, 2]. This result has allowed a rederivation of the anomalies of a Weyl fermion coupled to a curved spacetime, including the trace anomaly. Our result for the trace anomaly agrees with the one calculated in [4] and reproduced with different methods in [20].

These findings however are at odds with the original claim of ref. [21], reconfirmed also in [1, 2] where the notion of the MAT background was developed precisely for the purpose of studying the anomalies. Let us comment a bit more on this point. The presence of a Pontryagin term, which satisfies the consistency conditions for trace anomalies [22] and would constitute a type-B anomaly in the classification of [23], was conjectured to be a realistic possibility in [24], see also [25, 26, 27]. On the other hand, it is known that CFTs do not support nonlocal parity-odd terms in the correlation function of three stress tensors [28, 29], thus hinting at the absence of such a contribution. Our explicit calculation within the MAT background shows indeed that such terms are absent, thereby confirming the findings of refs. [4] and [20]. The analogous case of a Weyl fermion in a gauge background has also been studied more recently in [5, 12], where it was found that parity-odd terms do not contribute to the trace anomaly in that context as well.

Retracing our calculation of the trace anomaly, and observing the formulae in (26), one may notice that an imaginary term proportional to the Pontryagin density would indeed arise in the trace anomalies if the contribution from the regulators of the charge conjugated fields were neglected. However, there is no justification for dropping those terms. A close analogy is given by the two-dimensional *b*-*c* system, whose gravitational and trace anomalies have been computed in [10] with the same methods employed here. In that paper, the contributions from the regulator of the c field and that of the b field must be added together, and they sum up to produce the correct final anomaly. It would not be correct to drop, say the b contribution to find the anomaly of the c field. Said differently, the propagator of the *b*-*c* system contains information on both fields, and they cannot be split artificially. Similarly, in the Weyl fermion case both helicities $h = \pm\frac{1}{2}$ (described by the Weyl fermion and its hermitian conjugate) circulate in the loop that produces the anomalies, and their contributions cannot be split in any legal way. This is consistent with four dimensional CPT, that requires both helicities to be present in a massless relativistic QFT.

A preprint has recently appeared [30], suggesting that the methods we use for the anomaly calculations are not fit to detect parity-odd terms in the trace anomaly. We reject those criticisms, which we find unfounded. We find it important to reiterate that in principle any regularization scheme can be used to define a QFT, with different schemes producing perhaps different results for the anomalies, that however must be related by adding local counterterms to the effective action. The use of massive PV fields with a Majorana mass is certainly legal for regulating Weyl fields. We have used it here to compute systematically all the anomalies within the same regularization scheme, finding in particular the correct and well-known consistent $U(1)$ chiral anomaly. The Majorana mass couples the two helicities of a Weyl fermion, thus breaking its U(1) symmetry that indeed becomes anomalous. It also breaks the Weyl local scaling symmetry, thus causing a trace anomaly to appear as well. Other regularizations may of course be used, though some of them might be too cumbersome to carry out effectively the calculations (we have illustrated briefly the case of PV fields with Dirac mass in the last section). As a final comment on the views expressed in [30], we wish to stress that the theory of a chiral fermion in 4 dimensions may be described equivalently in two ways. One description makes use of a Weyl spinor and its hermitian conjugate (this is how we have proceeded in the present paper). Alternatively, one may use a Majorana spinor. The latter has the same field content of the former, as it casts together the two irreps of the Lorentz group ((1/2, 0) for the Weyl spinor plus its complex conjugate (0, 1/2) for the hermitian conjugate Weyl spinor) into a single spinor, making the resulting Majorana spinor reducible under the Lorentz group. Lorentz invariance fixes uniquely their actions, which are totally equivalent. A mass term can be added in both schemes, it is the so-called Majorana mass, which has the property of breaking the chiral $U(1)$ symmetry associated to the conservation of a fermion number (correlated to the helicity and identified with the lepton number in neutrino applications). References where these matters are clearly explained are the textbooks [31, 32, 33], for example. Thus, we find unfounded the suggestion that our methods are suitable for Majorana fermions but not for Weyl fermions.

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