

Partial stabilization of stochastic systems with application to rotating rigid bodies^{*}

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Abstract: This paper addresses the problem of stabilizing a part of variables for control systems described by stochastic differential equations of the Itô type. The considered problem is related to the asymptotic stability property of invariant sets and has important applications in mechanics and engineering. Sufficient conditions for the asymptotic stability of an invariant set are proposed by using a stochastic version of LaSalle's invariance principle. These conditions are applied for constructing the state feedback controllers in the problem of single-axis stabilization of a rigid body. The cases of control torques generated by jet engines and rotors are considered as illustrations of the proposed control design methodology.

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1. INTRODUCTION

The concept of stability with respect to a part of variables was introduced by Lyapunov and systematically investigated in the monographs by Rumyantsev and Oziraner (1987), Vorotnikov (1998), Zuyev (2015), and other authors.

An effective method for the study of stability problems for systems with random actions is the stochastic Lyapunov's direct method. In the paper by Sharov (1978), stability conditions with respect to a part of variables and conditions for partial asymptotic stability of Itô's stochastic differential equations have been obtained. The work by Socha and Zhu (2018) addresses stability issues for hybrid systems described by stochastic differential equations with three types of Markovian switching processes. For these processes, criteria of exponential y -stability have been obtained in the linear and nonlinear framework.

The paper by Ignatyev (2013) is devoted to the study of partial asymptotic stability for stochastic differential equations with time-invariant vector fields. A modification of the Barbashin–Krasovskii theorem for the case of stochastic asymptotic stability of the trivial equilibrium with respect to a part of variables has been proposed in the above-mentioned paper. The concepts of stability of solutions with respect to a part of variables and stability of an invariant set, being very similar, are not identical. For systems with random actions, namely, for a system of the Itô-type differential equations, Stanzhitskii (2001) obtained conditions for the invariance of sets and their stochastic stability. Note that these results do not imply

the asymptotic stability of an invariant set, and the study of limit sets for solutions to the stochastic differential equations remains a challenging problem (see, e.g., Øksendal (2003) and references therein). For dynamical systems with multivalued flows on a metric space, asymptotic stability conditions of invariant sets have been derived in the paper by Zuyev (2003).

Our goal is to propose asymptotic stability conditions for invariant sets of nonlinear stochastic differential equations and to apply such conditions to the partial stabilization problem of stochastic control systems. This paper is organized as follows. The notions of partial stability and partial asymptotic stability are introduced in Section 2. Sufficient conditions for partial asymptotic stability are presented in Section 3. These results are applied for the single-axis stabilization of a rotating rigid body with jet controls and rotors (flywheels) in Sections 4 and 5, respectively.

2. PARTIAL STABILITY OF STOCHASTIC SYSTEMS

In this section, we present some basic definitions related to partial stability of stochastic systems.

Consider a system of stochastic differential equations of the Itô type:

$$dx(t) = f(x)dt + \sigma(x)dW(t), \quad x \in \mathbb{R}^n. \quad (1)$$

Here $x = (x_1, \dots, x_n)^T$ is the state vector of the system, the functions $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times k}$ are assumed to be Lipschitz continuous on each compact set $K \subset \mathbb{R}^n$. We treat $X = (x_1, \dots, x_n)^T$ and $f(x) = (f_1(x), \dots, f_n(x))^T$ as column vectors, and $\sigma(x)$ as $n \times k$ matrix. System (1) is subject to the k -dimensional Wiener process $W(t)$ whose components $w_j(t)$ ($j = 1, 2, \dots, k$) are independent one-dimensional Wiener processes. Under these conditions,

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there exists a unique strictly Markov process $x(t; x^0, s)$ which is a solution of system (1) under the initial condition $x(s; x^0, s) = x^0$.

Definition 1. A nonempty set $M \subset \mathbb{R}^n$ is called:

- *forward invariant* for system (1) if $x^0 \in M$ and $s \in \mathbb{R}$ imply $x(t; x^0, s)$ for all $t \geq s$ almost surely;
- *backward invariant* for system (1) if $x^0 \in M$ and $s \in \mathbb{R}$ imply $x(t; x^0, s)$ for all $t \leq s$ almost surely;
- *invariant* for system (1) if it is forward and backward invariant.

We assume in the sequel that the state vector of system (1) can be written as $x = (y^T, z^T)^T$ with $y = (y_1, \dots, y_m)^T \in \mathbb{R}^m$ and $z = (z_1, \dots, z_p)^T \in \mathbb{R}^p$, $m + p = n$.

Definition 2. The set $M = \{x \in \mathbb{R}^n \mid y = 0\}$ is called stable in probability if, for any $s \geq 0$, $\epsilon_1 > 0$ and $\epsilon_2 > 0$, there exists a $\delta > 0$ such that the following inequality holds for all initial data $x^0 \in \mathbb{R}^n$ with $\|y^0\| < \delta$:

$$P\{\sup_{t \geq s} \|y(t; x^0, s)\| > \epsilon_1\} < \epsilon_2. \quad (2)$$

Definition 3. The set $M = \{x \in \mathbb{R}^n \mid y = 0\}$ is called asymptotically stable in probability if it is stable in probability and, for any $\epsilon > 0$, there is a $\Delta = \Delta(\epsilon) > 0$ such that

$$P\{\lim_{t \rightarrow +\infty} \|y(t; y^0, s)\| = 0\} > 1 - \epsilon, \quad \forall x^0 \in \mathbb{R}^n : \|y^0\| < \Delta. \quad (3)$$

In order to study the stability of invariant sets, we consider the class of functions $V : \mathbb{R}^n \rightarrow \mathbb{R}^+ = [0; +\infty)$, $V(0) = 0$, which are twice continuously differentiable. We relate with system (1) the differential operator

$$\mathcal{L} = \sum_{i=1}^n f_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \quad (4)$$

where $[a_{ij}] = \sigma \sigma^T$. Let us also introduce the class \mathcal{K} of comparison functions, consisting of all continuous strictly increasing functions $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\alpha(0) = 0$.

3. MAIN RESULT

In this section, sufficient conditions for the asymptotic stability of invariant sets of system (1) will be derived.

Theorem 1. Let the set $M = \{x \in \mathbb{R}^n \mid y = 0\}$ be invariant for system (1), and let $V \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R}^+)$ satisfy the following properties:

- 1) $\alpha_1(\|y\|) \leq V(x) \leq \alpha_2(\|y\|)$ for all $x \in \mathbb{R}^n$ with some $\alpha_1, \alpha_2 \in \mathcal{K}$;
- 2) $\mathcal{L}V(x) \leq 0$ for all $x \in \mathbb{R}^n$;
- 3) there exists a $\Delta > 0$ such that $x(t; x^0, s)$ is bounded for $t \geq s$ almost surely, provided that $\|y^0\| < \Delta$;
- 4) The set $\{x \in \mathbb{R}^n \mid \mathcal{L}V(x) = 0\} \setminus M$ is not forward invariant for (1).

Then the set $M = \{x \in \mathbb{R}^n \mid y = 0\}$ is asymptotically stable in probability for system (1).

Proof. Let us take an arbitrary $\epsilon > 0$. Let $\|y^0\| < \epsilon$, and let

$$\tau_\epsilon = \inf_{t \geq s} \{t \mid \|y(t; x^0, s)\| > \epsilon\}, \quad \tau_\epsilon(t) = \min(\tau_\epsilon, t).$$

Using a special case of Dynkin's formula (see, e.g., Øksendal (2003)), we obtain:

$$\mathbb{E}V(x(\tau_\epsilon(t); x^0, s)) = V(x^0) + \mathbb{E} \int_s^{\tau_\epsilon(t)} \mathcal{L}V(x(u; x^0, s)) du.$$

Therefore,

$$\mathbb{E}V(x(\tau_\epsilon(t); x^0, s)) \leq V(x^0) \quad \text{for all } t > s, \quad (5)$$

where \mathbb{E} is the expectation with respect to the probability measure $P_{x^0, s}\{\cdot\}$

Let us rewrite inequality (5) in the following form:

$$\int_{\tau_\epsilon < t} \alpha_1(\|y(\tau_\epsilon; x^0, s)\|) P_{x^0, s}(d\omega) + \int_{\tau_\epsilon \geq t} \alpha_1(\|y(\tau_\epsilon; x^0, s)\|) P_{x^0, s}(d\omega) \leq V(x^0).$$

So, $\alpha_1(\epsilon) P_{x^0, s}\{\tau_\epsilon < t\} \leq V(x^0)$. This implies, because of the continuity of $V(x)$ and the equality $V(0) = 0$, that

$$\lim_{y^0 \rightarrow 0} P_{x^0, s}\{\tau_\epsilon < t\} = 0,$$

so, (2) holds.

It remains to prove that

$$\lim_{y^0 \rightarrow 0} P\{\lim_{t \rightarrow \infty} \|y(t; x^0, s)\| = 0\} = 1.$$

For this purpose we select the subset \mathcal{B} of the set of sample trajectories of the process $x(t; x^0, s)$ such that $\tau_\epsilon(t) = t$ holds for each component $y_i(t; x^0, s)$ of $x(t; x^0, s)$ for all $t \geq s$.

Then

$$\lim_{y^0 \rightarrow 0} P(\mathcal{B}) = 1. \quad (6)$$

By the assumption 3) of Theorem 1, there exists a $\Delta > 0$ such that all solutions $x(t; x^0, s)$ of system (1) with $\|y^0\| < \Delta$ are bounded for $t > s$. Let $\|y^0\| < \Delta$, then let us show that $\lim_{t \rightarrow \infty} \|y(t; x^0, s)\| = 0$.

Due to the boundedness of $x(t; x^0, s)$, there exists a sequence $t_k \rightarrow +\infty$:

$$x(t_k; x^0, s) \rightarrow x^* = (y^*, z^*) \in G^+,$$

where G^+ is the set of ω -limit points of the solution $x(t; x^0, s)$.

Since the conditions of the stochastic version of LaSalle's invariance principle are satisfied (see Mao (1999, 2000)), $G^+ \subset \{x \mid \mathcal{L}V(x) = 0\}$.

Now we show that $y^* = 0$. By the invariance of the set G^+ , we obtain $x(t; x^*, s) \in G^+$ for all $t \geq s$. Suppose that $y^* \neq 0$. Since the set M is invariant and $x^* \in G^+ \subset \{x \mid \mathcal{L}V(x) = 0\}$, then $x(t; x^*, s) \in \{x \mid \mathcal{L}V(x) = 0\} \setminus M$, $t \geq s$.

The above property contradicts the assumption 4) of Theorem 1. Thus, $y^* = 0$ and

$$\lim_{t \rightarrow +\infty} \|y(t; x^0, s)\| = 0. \quad (7)$$

Then (7) implies the assertion of Theorem 1. \square

4. SINGLE-AXIS STABILIZATION OF A SATELLITE BY JET TORQUES

Consider the problem of stabilizing the motion of a rigid body around its center of mass under the action of jet control torques without taking into account the change of mass. The equations of motion can be written in the Euler–Poisson form with random disturbances as follows:

$$\begin{aligned} d\omega_1 &= \left(\frac{A_2 - A_3}{A_1} \omega_2 \omega_3 + u_1 \right) dt + \omega_1 \sigma dW(t), \\ d\omega_2 &= \left(\frac{A_3 - A_1}{A_2} \omega_1 \omega_3 + u_2 \right) dt + \omega_2 \sigma dW(t), \\ d\omega_3 &= \left(\frac{A_1 - A_2}{A_3} \omega_1 \omega_2 \right) dt, \\ d\nu_1 &= (\omega_3 \nu_2 - \omega_2 \nu_3) dt, \\ d\nu_2 &= (\omega_1 \nu_3 - \omega_3 \nu_1) dt, \\ d\nu_3 &= (\omega_2 \nu_1 - \omega_1 \nu_2) dt. \end{aligned} \quad (8)$$

Here $(\omega_1, \omega_2, \omega_3)$ are projections of the angular velocity vector ω on the corresponding principal axes of inertia of the body, (ν_1, ν_2, ν_3) are projections of a fixed vector ν onto the corresponding principal inertia axes, (A_1, A_2, A_3) are principal central moments of inertia of the body, and (u_1, u_2) are control torques. In the further analysis, we will assume that all variables and parameters are dimensionless.

System (8) with $u_1 = u_2 = 0$ admits the following particular solution:

$$\begin{aligned} \omega_1 = \omega_2 = \omega_3 &= 0, \\ \nu_1 = \nu_2 = 0, \nu_3 &= 1. \end{aligned} \quad (9)$$

Solution (9) corresponds to the equilibrium position at which the third principal axis of inertia of the body is directed along the ν vector. We apply Theorem 1 to stabilize the set $M = \{(\omega_1, \omega_2, \omega_3, \nu_1, \nu_2, \nu_3) | \nu_1 = \nu_2 = \omega_1 = \omega_2 = 0\}$ of system (8). This goal corresponds to the problem of single-axis stabilization of the rigid body, i.e. the projections ν_1, ν_2 and their derivatives $\dot{\nu}_1, \dot{\nu}_2$ should be small and tending to zero as $t \rightarrow +\infty$, while the other components of the solutions are assumed to be bounded.

Let us take the following control Lyapunov function candidate:

$$V = \frac{1}{2}(A_1 \omega_1^2 + A_2 \omega_2^2) + \frac{1}{2}(\nu_1^2 + \nu_2^2).$$

Then we define the feedback control as follows:

$$\begin{aligned} u_1 &= \omega_2 \omega_3 - \frac{1}{A_1} \nu_2 \nu_3 - \left\{ \frac{|A_1 - A_2|}{2A_1} |\omega_3| \right. \\ &\quad \left. + A_1 h + \frac{1}{2} \sigma \sigma^T \right\} \omega_1, \\ u_2 &= -\omega_1 \omega_3 + \frac{1}{A_2} \nu_1 \nu_3 - \left\{ \frac{|A_1 - A_2|}{2A_2} |\omega_3| \right. \\ &\quad \left. + A_2 h + \frac{1}{2} \sigma \sigma^T \right\} \omega_2, \end{aligned} \quad (10)$$

where the function h will be defined below.

By computing the action of the \mathcal{L} operator on V , we get:

$$\begin{aligned} \mathcal{L}V &= -\frac{|A_1 - A_2|}{2} |\omega_3| \omega_1^2 - \frac{|A_1 - A_2|}{2} |\omega_3| \omega_2^2 \\ &\quad - A_1^2 h \omega_1^2 - A_2^2 h \omega_2^2 \leq 0. \end{aligned}$$

Thus, the second condition of Theorem 1 is satisfied. All the solutions of system (8) are bounded with respect to

the variables ν_i due to the geometric integral $\nu_1^2 + \nu_2^2 + \nu_3^2 = \text{const}$. It remains to verify the boundedness of solutions with respect to ω_i . For this purpose we exploit Theorem 39.1 from Rumyantsev and Oziraner (1987). According to that theorem, it suffices to construct a function $W(x)$, which is positive definite with respect to all ω_i , does not increase along the trajectories of the closed-loop system (8), (10), and satisfies the condition $W(x) \rightarrow \infty$ as $|\omega| \rightarrow \infty$.

Let us define

$$W(x) = \frac{1}{2}(A_1 \omega_1^2 + A_2 \omega_2^2 + A_3 \omega_3^2) + \frac{1}{2}(\nu_1^2 + \nu_2^2).$$

Then

$$\begin{aligned} \mathcal{L}W(x) &= (A_1 - A_2) \omega_1 \omega_2 \omega_3 - \frac{|A_1 - A_2|}{2} |\omega_3| (\omega_1^2 + \omega_2^2) \\ &\quad - h(A_1^2 \omega_1^2 + A_2^2 \omega_2^2). \end{aligned}$$

According to Theorem 39.1 of Rumyantsev and Oziraner (1987), for the ω_i -boundedness of the solutions it suffices to satisfy the inequality $\mathcal{L}W_0(x) \leq 0$, so the function $h(x) > 0$ is chosen from the following condition:

$$h(x)(A_1^2 \omega_1^2 + A_2^2 \omega_2^2) \geq (A_1 - A_2) \omega_1 \omega_2 \omega_3. \quad (11)$$

Using the inequality $2A_1 A_2 \omega_1 \omega_2 \leq A_1^2 \omega_1^2 + A_2^2 \omega_2^2$, we conclude that to satisfy (11) it suffices to take

$$h(x) = \left| \frac{A_1 - A_2}{2A_1 A_2} \omega_3 \right| + \varepsilon, \quad (12)$$

where ε is any positive number.

Let us now check the invariance of the set M . Thus we analyze the solutions of the system (8), (10) for an arbitrary initial data from the set M . Let us rewrite the system (8), (10), substituting $\nu_1 = \nu_2 = \omega_1 = \omega_2 = 0$:

$$d\omega_3 = 0, \quad d\nu_3 = 0. \quad (13)$$

Then the solutions of the closed-loop system satisfy

$$\omega_3(t) = \text{const} \quad \text{and} \quad \nu_3(t) = \text{const} \quad (14)$$

almost surely. It remains to show that the condition 4) of Theorem 1 is satisfied. The set $M_v = \{x | \mathcal{L}V(x) = 0\}$ has the form $M_v = \{x | \omega_1 = \omega_2 = 0\}$, i.e.

$$\begin{aligned} x(t) \in M_v : \quad & d\nu_1 = \omega_3 \nu_2 dt, \quad d\nu_2 = -\omega_3 \nu_1 dt, \\ & 0 = -\frac{1}{A_1} \nu_2 \nu_3 dt, \\ & 0 = \frac{1}{A_2} \nu_1 \nu_3, \quad d\omega_3 = 0. \end{aligned} \quad (15)$$

It is easy to verify that, for the initial conditions close enough to (9), all the trajectories of the closed-loop system (8) with (10) satisfy the property $\nu_1 = \nu_2 = 0$. Moreover, the solutions of system (15) are $\nu_1(t) = 0$, $\nu_2(t) = 0$, $\omega_3(t) = \text{const}$, $\nu_3(t) = \text{const}$. The same solutions satisfy system (13). So, $M_v \setminus M$ does not contain any positive trajectory of the considered closed-loop system.

All the conditions of Theorem 1 are satisfied, so the invariant set $M = \{(\omega_1, \omega_2, \omega_3, \nu_1, \nu_2, \nu_3) | \nu_1 = \nu_2 = \omega_1 = \omega_2 = 0\}$ of the closed-loop system (8), (10) is asymptotically stable in probability.

Below we present numerical simulations of the closed-loop system (8), (10) with the following dimensionless parameters: $A_1 = 1, A_2 = 3, A_3 = 2, \varepsilon = 0.01$. These simulations have been performed in Maple by using the ItoProcess function.

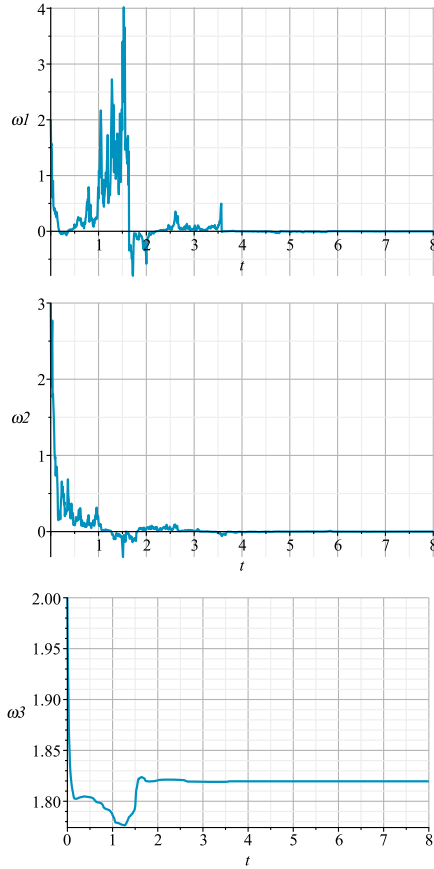


Fig. 1. Components $\omega_i(t)$ of a sample path of the closed-loop system (8), (10).

From Figs. 1 and 2 it is clear that the components $\omega_1, \omega_2, \nu_1, \nu_2$ tend to zero for large values of t , while the coordinates ω_3, ν_3 tend to some limit values. Thus, the limit motions of the body are uniform rotations around a fixed orientation vector, which is collinear to the third principal axis of inertia.

5. SINGLE-AXIS STABILIZATION OF A SATELLITE USING TWO ROTORS

The equations of motion of a rigid body containing a pair of symmetrical rotors with random effects can be written as follows:

$$\begin{aligned}
 d\omega_1 &= \left(\frac{A_2 - A_3}{A_1 - I_1} \omega_2 \omega_3 + \frac{I_2 \Omega_2}{A_1 - I_1} \omega_3 - \frac{1}{A_1 - I_1} u_1 \right) dt \\
 &\quad + \omega_1 \sigma dW(t), \\
 d\omega_2 &= \left(\frac{A_3 - A_1}{A_2 - I_2} \omega_1 \omega_3 - \frac{I_1 \Omega_1}{A_2 - I_2} \omega_3 - \frac{1}{A_2 - I_2} u_2 \right) dt \\
 &\quad + \omega_2 \sigma dW(t), \\
 d\omega_3 &= \left(\frac{A_1 - A_2}{A_3} \omega_1 \omega_2 + \frac{I_1 \Omega_1}{A_3} \omega_2 - \frac{I_2 \Omega_2}{A_3} \omega_1 \right) dt, \\
 d\Omega_1 &= \left(\frac{1}{I_2} u_1 - \frac{A_2 - A_3}{A_1 - I_1} \omega_2 \omega_3 - \frac{I_2 \Omega_2}{A_1 - I_1} \omega_3 + \frac{1}{A_1 - I_1} u_1 \right) dt \\
 &\quad - \omega_1 \sigma dW(t), \\
 d\Omega_2 &= \left(\frac{1}{I_1} u_2 - \frac{A_3 - A_1}{A_2 - I_2} \omega_1 \omega_3 + \frac{I_1 \Omega_1}{A_2 - I_2} \omega_3 + \frac{1}{A_2 - I_2} u_2 \right) dt \\
 &\quad - \omega_2 \sigma dW(t), \\
 d\nu_1 &= (\omega_3 \nu_2 - \omega_2 \nu_3) dt, \\
 d\nu_2 &= (\omega_1 \nu_3 - \omega_3 \nu_1) dt, \\
 d\nu_3 &= (\omega_2 \nu_1 - \omega_1 \nu_2) dt.
 \end{aligned} \tag{16}$$

Here ω_i are coordinates of the angular velocity vector of the carrier body in the main coordinate frame, Ω_1, Ω_2 are

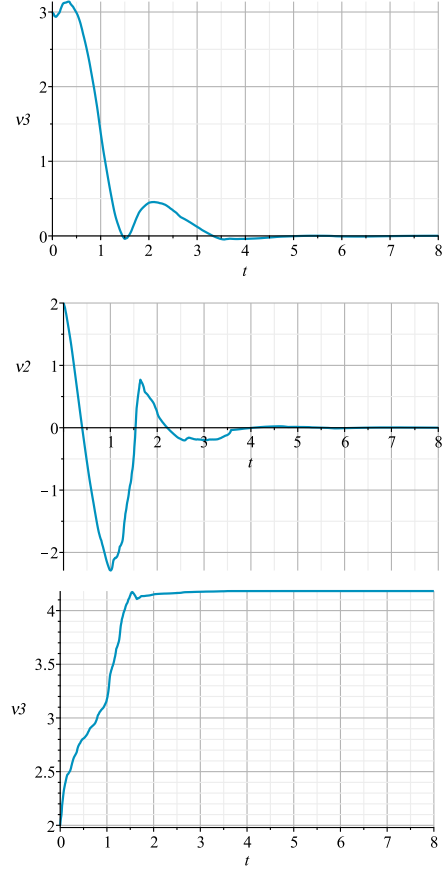


Fig. 2. Components $\nu_i(t)$ of a sample path of the closed-loop system (8), (10).

relative angular velocities of the first and second rotor, respectively, A_i are principal moments of inertia of the whole system consisting of the carrier body and rotors, I_1, I_2 are moments of inertia of the first and second rotor, respectively, and u_1, u_2 are the control torques applied to the first and the second rotor, respectively. We assume that $A_1 > I_1, A_2 > I_2$. System (16) is a stochastic version of the equations studied in Zuyev (2001). Note that the rotating rigid body with a rotor has been considered in the book on nonholonomic mechanics by Bloch (2003) as a mathematical model of a satellite.

Control system (16) with $u_1 = u_2 = 0$ admits the following equilibrium:

$$\begin{aligned}
 \omega_i &= 0, \quad \Omega_1 = const, \quad \Omega_2 = const, \\
 \nu_1 &= \nu_2 = 0, \quad \nu_3 = 1.
 \end{aligned} \tag{17}$$

The considered mechanical system has the following integrals:

$$\begin{aligned}
 W_1 &= (A_1 \omega_1 + I_1 \Omega_1)^2 + (A_2 \omega_2 + I_2 \Omega_2)^2 + (A_3 \omega_3)^2 = const, \\
 W_2 &= (A_1 \omega_1 + I_1 \Omega_1) \nu_1 + (A_2 \omega_2 + I_2 \Omega_2) \nu_2 + (A_3 \omega_3) \nu_3 = const, \\
 W_3 &= \nu_1^2 + \nu_2^2 + \nu_3^2 = const.
 \end{aligned}$$

For the existence of the integral of moments, it is necessary for the vector σ , which characterizes random actions, to satisfy the condition $\sigma \in N$, where N is the invariant subspace of the linear operator Q corresponding to zero eigenvalue. The linear operator Q is given by the Hessian matrix of W_1 :

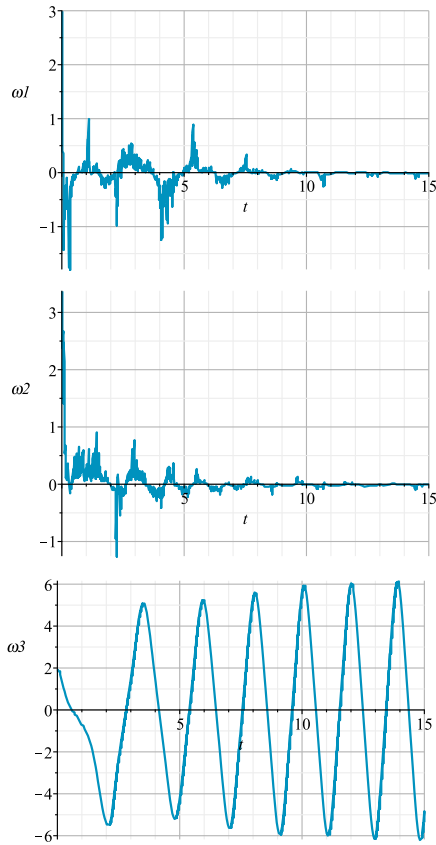


Fig. 3. Components $\omega_i(t)$ of a sample path of the closed-loop system (16), (19).

$$Q = \begin{pmatrix} 2A_1^2 & 0 & 0 & 2A_1I_1 & 0 \\ 0 & 2A_2^2 & 0 & 0 & 2A_2I_2 \\ 0 & 0 & 2A_3^2 & 0 & 0 \\ 2I_1A_1 & 0 & 0 & 2I_2^2 & 0 \\ 0 & 2I_2A_2 & 0 & 0 & 2I_2^2 \end{pmatrix}. \quad (18)$$

Let us compute the eigenvectors of Q corresponding to zero eigenvalue. These vectors are

$$x = \left(-\frac{I_1}{A_1}\Omega_1, -\frac{I_2}{A_2}\Omega_2, 0, \Omega_1, \Omega_2 \right)^T.$$

Thus the set N has the form $N = \{(\omega_1, \omega_2, \omega_3, \Omega_1, \Omega_2)^T |$

$$\omega_1 = -\frac{I_1}{A_1}\Omega_1, \omega_2 = -\frac{I_2}{A_2}\Omega_2, \omega_3 = 0\}.$$

To stabilize the set $M = \{(\omega_1, \omega_2, \omega_3, \Omega_1, \Omega_2, \nu_1, \nu_2, \nu_3) | \nu_1 = \nu_2 = \omega_1 = \omega_2 = 0\}$ of system (16), we apply Theorem 1 with the following control Lyapunov function candidate: $2V(x) = (A_1 - I_1)\omega_1^2 + (A_2 - I_2)\omega_2^2 + \nu_1^2 + \nu_2^2$.

Let us define the feedback law as follows:

$$\begin{aligned} u_1 &= \nu_2\nu_3 + (A_2\omega_2 + I_2\Omega_2)\omega_3 \\ &+ \omega_1\left(h + \frac{1}{2}\sigma\sigma^T(A_1 - I_1) + \frac{|A_1 - A_2|}{2}|\omega_3|\right), \\ u_2 &= -\nu_1\nu_3 - (A_1\omega_1 + I_1\Omega_1)\omega_3 \\ &+ \omega_2\left(h + \frac{1}{2}\sigma\sigma^T(A_2 - I_2) + \frac{|A_1 - A_2|}{2}|\omega_3|\right). \end{aligned} \quad (19)$$

As in the previous example, we check the conditions of Theorem 1. We have:

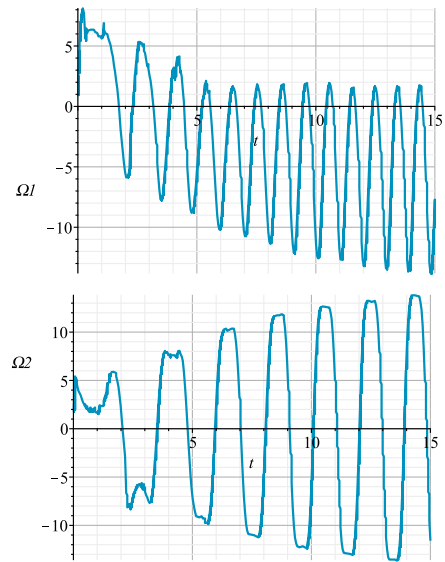


Fig. 4. Components $\Omega_i(t)$ of a sample path of the closed-loop system (16), (19).

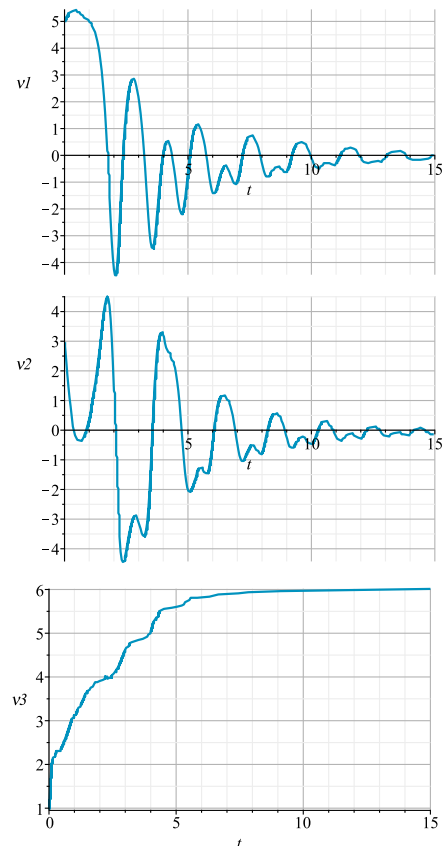


Fig. 5. Components $\nu_i(t)$ of a sample path of the closed-loop system (16), (19).

$$\mathcal{L}V(x) = -\omega_1^2\left(h + \frac{|A_1 - A_2|}{2}|\omega_3|\right) - \omega_2^2\left(h + \frac{|A_1 - A_2|}{2}|\omega_3|\right) \leq 0.$$

By exploiting the integrals W_1, W_2, W_3 and the condition $\mathcal{L}V(x) \leq 0$, we conclude about the boundedness of the solutions of system (16) with the feedback law (19).

The closed-loop system (16), (19) on the set M takes the form:

$$\begin{aligned} d\omega_3 &= 0, \\ d\Omega_1 &= \Omega_2\omega_3 dt, \\ d\Omega_2 &= \Omega_1\omega_3 dt, \\ d\nu_3 &= 0. \end{aligned} \quad (20)$$

Let us find the solutions of system (20):

$$\begin{aligned} \omega_3(t) &= c_1 \sqrt{\frac{(A-1-I_1)(A_2-I_2)}{I_1 I_2}} \neq 0, \\ \nu_3(t) &= c_2, \\ \Omega_1(t) &= c_3 \cos(c_1 t) + c_4 \sin(c_1 t), \\ \Omega_2 &= \sqrt{\frac{I_2(A_2-I_2)}{I_1(A_1-I_1)}} c_4 \cos(c_1 t) - c_3 \sin(c_1 t). \end{aligned} \quad (21)$$

So, for arbitrary initial conditions from the set M , the expressions (21) define the solutions of system (20) in M , which proves the invariance of the set M .

The set M_v has the form: $M_v = \{x | \omega_1 = \omega_2 = 0\}$, i.e.

$$\begin{aligned} x(t) \in M_v : \quad & d\nu_1 = \omega_3 \nu_2 dt, \\ & d\nu_2 = -\omega_3 \nu_1 dt, \\ & \nu_2 \nu_3 = 0, \\ & \nu_1 \nu_3 = 0, d\omega_3 = 0, d\nu_3 = 0. \end{aligned} \quad (22)$$

For the initial conditions sufficiently close to the equilibrium position (17), all the solutions of system (16) with control (19) possess the property $\nu_1 = \nu_2 = 0$. Then the corresponding solution of system (22) is given by the relations $\nu_1(t) = 0, \nu_2(t) = 0$ together with (21). The same solution satisfies system (20). This means that $M_v \setminus M$ does not contain any semi-trajectory of the considered closed-loop system, which means that the last condition of Theorem 1 is satisfied.

Thus, the set

$M = \{(\omega_1, \omega_2, \omega_3, \Omega_1, \Omega_2, \nu_1, \nu_2, \nu_3) | \nu_i = \omega_i = 0, i = 1, 2\}$ is asymptotically stable in probability for the closed-loop system (16) with controls (19) by Theorem 1. This conclusion is also illustrated by the results of numerical simulations of the closed-loop system (16), (19) with the parameters $A_1 = 10, A_2 = 31, A_3 = 22, I_1 = 8, I_2 = 27, \varepsilon = 0.001$.

Figs. 3–5 show that the stabilized variables tend to zero as $t \rightarrow +\infty$.

6. CONCLUSIONS

In this paper, the idea of the Barbashin–Krasovskii theorem and LaSalle’s invariance principle has been extended to characterize the asymptotic stability property of invariant sets of stochastic differential equations. It is shown that this result can be applied to the partial stabilization problem for nonlinear control systems with stochastic effects. It should be emphasized that the control systems, considered in Sections 4 and 5, are not stabilizable in the classical sense with respect to all variables because of the presence of the geometric integral $\nu_1^2 + \nu_2^2 + \nu_3^2 = const$. Thus, only partial stabilization is possible in the considered cases, and numerical simulations illustrate the efficiency of the proposed controllers.

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