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On exponential stabilization of nonholonomic systems with time-varying drift ⋆

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Abstract: A class of nonlinear control-affine systems with bounded time-varying drift is considered. It is assumed that the control vector fields together with their iterated Lie brackets satisfy Hörmander's condition in a neighborhood of the origin. Then the problem of exponential stabilization is treated by exploiting periodic time-varying feedback controls. An explicit parametrization of such controllers is proposed under a suitable non-resonance assumption. It is shown that these controllers ensure the exponential stability of the closed-loop system provided that the period is small enough. The proposed control design methodology is applied for the stabilization of an underwater vehicle model and a front-wheel drive car.

Keywords: exponential stabilization, time-varying feedback law, Lie algebra rank condition, nonholonomic system.

1. INTRODUCTION

The paper focuses on the stabilization problem for a class of nonholonomic systems in the control-affine form. As the number of control inputs in such systems can be significantly smaller than the dimension of the state vector, this causes certain challenges in control design. There exists a number of approaches which allow to stabilize *control-linear* nonholonomic systems (see, e.g., Coron (1992); Astolfi (1994); Morin et al. (1999); Zuyev (2016), and references therein). However, the stabilization problem becomes even more complicated for *control-affine* systems with unstable drift terms. Controllability properties and motion planning problems of control-affine systems were discussed, e.g., in De Luca and Oriolo (1995); Godhavn et al. (1999); Pomet (1999); Aguilar (2012); Jean and Prandi (2015); Zuyev and Grushkovskaya (2017). While rather general results have been obtained for motion planning problems, stabilization of nonholonomic systems with drift is mainly studied for specific classes of systems (see, e.g., M'Closkey and Morin (1998); Reyhanoglu et al. (1999); Bullo et al. (2000); Floquet et al. (2000); Wang et al. (2004); Yang and Yang (2010); Gao et al. (2011); Zhao and Wu (2013), and Kolmanovsky and McClamroch (1995); Michalska and Torres-Torriti (2003) for a survey). A more general class of control-affine systems was considered in Hermes (1980); Michalska and Torres-Torriti (2003), where stabilizing controllers have been proposed under the assumption that the system is strongly controllable and can be approximated by a system with nilpotent Lie algebra, and that the drift term vanishes at the origin. In this paper, we propose a class of control functions that stabilize the origin of an underactuated control-affine system with time-varying drift term. In general, we do not assume that the drift vanishes at the origin, which leads to the practical asymptotic stability of the corresponding closed-loop system. For a special class of drift terms vanishing at the origin, we show that the trajectories of the system exponentially tend to zero. We also do not involve the drift vector field in the controllability rank condition. In Section 2, we formulate the problem statement and present a novel stabilizability result as an the extension of the control design approach from (Zuyev (2016); Grushkovskaya and Zuyev (2018)). Section 3 contains the proofs. Several examples are presented in Section 4.

2. MAIN RESULTS

2.1 Problem statement

Consider a system

$$\dot{x} = g(t,x) + \sum_{i=1}^{m} f_i(x)u_i, \quad x \in D \subset \mathbb{R}^n, \tag{1}$$

where $x = (x_1, \ldots, x_n)^\top$ is the state, $u = (u_1, \ldots, u_m)^\top \in \mathbb{R}^m$ is the control, $f_i \in C^3(D;\mathbb{R}^n)$ describe the system dynamics, and $g : \mathbb{R} \times D \to \mathbb{R}^n$ is the drift term related to the system dynamics or to disturbances. In this paper,

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we propose a family of control laws for stabilizing the origin of system (1) under the assumption that the vector fields f_i together with their first- and second-order Lie brackets span the whole n-dimensional space, and the drift g satisfies certain boundedness assumptions.

Assumption 1. (Rank condition). Let $S_1 \subseteq \{1, 2, ..., m\}$, $S_2 \subseteq \{1, 2, ..., m\}^2$, $S_3 \subseteq \{1, 2, ..., m\}^3$ be sets of indices such that $|S_1| + |S_2| + |S_3| = n$ and, for each $x \in D$,

$$\operatorname{span}\big\{f_i(x), [f_{j_1}, f_{j_2}](x), [[f_{\ell_1}, f_{\ell_2}], f_{\ell_3}](x) \mid i \in S_1, (j_1, j_2) \in S_2, (\ell_1, \ell_2, \ell_3) \in S_3\big\} = \mathbb{R}^n. \tag{2}$$

Assumption 2. (Boundedness of the drift). For each compact set $\xi \subseteq D$, there exists a $\tau > 0$ and $\mathcal{M}_g \geq 0$ such that, for any $t_0 \geq 0$, $\|g(t, x)\| \leq \mathcal{M}_g$ for all $t \in [t_0, t_0 + \tau]$, $x \in \xi$.

To stabilize system (1) at $x^* = 0$, we adopt the control design approach previously proposed for the case $g(t, x) = 0$ in Zuyev et al. (2016); Grushkovskaya and Zuyev (2018). Note that the presence of non-zero drift may affect significantly the system behavior and complicates the stabilization problem. Therefore, the results of the above mentioned papers cannot be directly applied, and more sophisticated analysis is required.

2.2 Notations and definitions

Definition 1. We say that there is a *resonance of order* $N \in \mathbb{N}$ between the pairwise distinct numbers $k_1, \dots, k_n$, if there exist relatively prime integers $c_1, \dots, c_n$ such that $|c_1| + ... + |c_n| = N$ and $c_1 k_1 + ... + c_n k_n = 0$.

Similarly to the approaches of Clarke et al. (1997); Zuyev (2016), we will exploit the sampling concept. For a given $\varepsilon > 0$, define a partition π_ε of $[0, +\infty)$ into the intervals $[t_j, t_{j+1})$, $t_j = \varepsilon j$, $j = 0, 1, 2, \dots$.

Definition 2. Given a feedback $u = h(t, x)$, $h : [0, +\infty) \times D \to \mathbb{R}^m$, $\varepsilon > 0$, and $x^0 \in D$, a π_ε-solution of (1) *corresponding to x^0 and $h(t, x)$ is an absolutely continuous function $x(t) \in D$, defined for $t \in [0, +\infty)$, such that $x(0) = x^0$ and $\dot{x}(t) = f\big(x(t), h(t, x(t_j))\big)$, $t \in [t_j, t_{j+1})$, for each j=0,1,2,....*

For $f, g : \mathbb{R}^n \to \mathbb{R}^n$, $x \in \mathbb{R}^n$, the directional derivative is denoted as $L_g f(x) = \lim\limits_{s \to 0} \frac{f(x + sg(x)) - f(x)}{s}$, and $[f, g](x) = L_f g(x) - L_g f(x)$ stands for the Lie bracket. Throughout this paper, $\|a\|$ denotes the Euclidean norm of a vector $a \in \mathbb{R}^n$, and the norm of an $n \times n$-matrix $\mathcal{F}$ is defined as $\|\mathcal{F}\| = \sup_{\|y\|=1} \|\mathcal{F}y\|$.

2.3 Control functions

Given positive real numbers ε and γ, we define the control functions u_k, $k = 1, \dots, m$, as

$$\begin{aligned} u_k = h_k^\varepsilon(t, x) &= \sum_{i_1 \in S_1} a_{i_1}(x) \phi_{i_1}^{(k,\varepsilon)}(t) \\ &+ \varepsilon^{-\frac{1}{2}} \sum_{(j_1, j_2) \in S_2} \sqrt{|a_{j_1 j_2}(x)|} \phi_{j_1 j_2}^{(k,\varepsilon)}(t, x) \\ &+ \varepsilon^{-\frac{2}{3}} \sum_{(\ell_1, \ell_2, \ell_3) \in S_3} \sqrt[3]{a_{\ell_1 \ell_2 \ell_3}(x)} \phi_{\ell_1 \ell_2 \ell_3}^{(k,\varepsilon)}(t), \end{aligned} \tag{3}$$

where the state-dependent vector function

$$a(x) = \big(a_{i_1}(x)\big|_{i_1 \in S_1}, a_{j_1 j_2}(x)\big|_{(j_1, j_2) \in S_2}, a_{\ell_1 \ell_2 \ell_3}(x)\big|_{(\ell_1, \ell_2, \ell_3) \in S_3}\big)^\top \in \mathbb{R}^n$$

is chosen as

$$a(x) = -\gamma \mathcal{F}^{-1}(x) x \tag{4}$$

with some control gain $\gamma > 0$, and $\phi_{i_1}^{(k,\varepsilon)}(t) = \delta_{k i_1}$,

$$\begin{aligned} \phi_{j_1 j_2}^{(k,\varepsilon)}(t, x) &= 2\sqrt{\pi \kappa_{j_1 j_2}} \Big(\delta_{k j_1} \operatorname{sign}(a_{j_1, j_2}(x)) \cos \frac{2\pi \kappa_{j_1 j_2}}{\varepsilon} t \\ &\quad + \delta_{k j_2} \sin \frac{2\pi \kappa_{j_1 j_2}}{\varepsilon} t\Big), \\ \phi_{\ell_1 \ell_2 \ell_3}^{(k,\varepsilon)}(t) &= 2\sqrt[3]{2\pi^2 \kappa_{3\ell_1\ell_2\ell_3} \kappa_{4\ell_1\ell_2\ell_3}} \Big(\delta_{k l 3_1} \cos \frac{2\pi \kappa_{1\ell_1\ell_2\ell_3} t}{\varepsilon} \\ &\quad + \delta_{k \ell_2} \sin \frac{2\pi \kappa_{2\ell_1\ell_2\ell_3} t}{\varepsilon} \\ &\quad + \delta_{k \ell_3} \cos \frac{2\pi \kappa_{1\ell_1\ell_2\ell_3} t}{\varepsilon} \sin \frac{2\pi \kappa_{2\ell_1\ell_2\ell_3} t}{\varepsilon}\Big). \end{aligned} \tag{5}$$

Here δ_{ki} is the Kronecker delta, and the integer parameters $\kappa_{j_1 j_2}$, $\kappa_{1\ell_1\ell_2\ell_3}$, $\kappa_{2\ell_1\ell_2\ell_3}$ are specified according to the following assumption.

Assumption 3. (Absence of resonances). *The positive integers $\kappa_{j_1 j_2}$, $\kappa_{1\ell_1\ell_2\ell_3}$, $\kappa_{2\ell_1\ell_2\ell_3}$, $\kappa_{3\ell_1\ell_2\ell_3} = \kappa_{1\ell_1\ell_2\ell_3} + \kappa_{2\ell_1\ell_2\ell_3}$, and $\kappa_{4\ell_1\ell_2\ell_3} = \kappa_{2\ell_1\ell_2\ell_3} - \kappa_{1\ell_1\ell_2\ell_3}$ are pairwise distinct, and there are no third-order resonances between $\kappa_{s\ell_1\ell_2\ell_3}$ ($s = 1, \dots, 4$), except those imposed by the definition of $\kappa_{3\ell_1\ell_2\ell_3}$, $\kappa_{4\ell_1\ell_2\ell_3}$.*

2.4 Stabilization of system (1)

Consider the matrix

$$\mathcal{F}(x) = \Big(\big(f_i(x)\big)_{j_1 \in S_1} \; \big([f_{j_1}, f_{j_2}](x)\big)_{(j_1, j_2) \in S_2} \; \big([[f_{\ell_1}, f_{\ell_2}], f_{\ell_3}](x)\big)_{(\ell_1, \ell_2, \ell_3) \in S_3}\Big), \tag{6}$$

which is nonsingular in D provided that condition (2) holds. The main result of this paper is the following theorem.

Theorem 1. Let $D \subseteq \mathbb{R}^n$, $f_i \in C^3(D; \mathbb{R}^n)$, $i = 1, \dots, m$. Suppose that Assumptions 1–2 hold in D and there exists an $\alpha > 0$ such that $\|\mathcal{F}^{-1}(x)\| \leq \alpha$ for all $x \in D$, where the matrix $\mathcal{F}(x)$ is given by (6).
If the functions $u_k = h_k^\varepsilon(t, x)$, $k = 1, \dots, m$, are defined as in (3)–(5) *with the parameters satisfying Assumption 3, then for any $\delta, \rho > 0$ there exist $\gamma, \bar{\varepsilon} > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon}]$, the π_ε-solution of system* (1) *with the initial data $x(0) = x^0 \in B_\delta(0)$ is well-defined on $t \in [0, +\infty)$ and*

$$\|x(t)\| \leq \|x^0\| e^{-\lambda t} + \rho \text{ for all } t \in [t_0, t_1), \quad \text{and } \|x(t)\| \leq \rho \text{ for all } t \in [t_1, \infty),$$

with some $\lambda, t_1 > 0$.

The proof is given in Section 3.1. Note that the proof provides a constructive procedure for choosing γ, λ and $\bar{\varepsilon}$. Theorem 1 gives the *practical exponential stability* conditions of the point $x = 0$. Obviously, to stabilize system (1) in the practical sense at any other point $x^* \in D$, one can take $a(x) = -\gamma \mathcal{F}^{-1}(x)(x - x^*)$. Under some stronger assumptions on $g(t, x)$, even local exponential stability can be achieved, as stated in the following corollaries.

Corollary 1. Let $D \subseteq \mathbb{R}^n$, $f_i \in C^3(D; \mathbb{R}^n)$, $i = 1, \dots, m$. Assume that Assumption 1 holds in D and there exists an

$\alpha > 0$ such that $\|\mathcal{F}^{-1}(x)\| \le \alpha$ for all $x \in D$, where the matrix $\mathcal{F}(x)$ is given by (6). Assume also that there are $\mathcal{M}_g, \mathcal{L}_g \ge 0$ and $\delta_0 > 0$ such that

$$g(t,x) \le \mathcal{M}_g \|x\|^3,\ \Big\|g(t,x) - g(t,y)\Big\| \le \mathcal{L}_g \|x-y\|,$$

for all $t \ge 0, x, y \in B_{\delta_0}(0)$. If the functions $u_k = h_k^\varepsilon(t,x)$, $k = 1,\dots,m$, are defined as in (3)–(5) with the parameters satisfying Assumption 3, then for any $\delta > 0$ there exist $\gamma, \bar\varepsilon > 0$ such that, for any $\varepsilon \in (0,\bar\varepsilon]$, the π_ε-solution of system (1) with the initial data $x(0) = x^0 \in B_\delta(0)$ is well-defined on $t \in [0,+\infty)$ and

$$\|x(t)\| = O(e^{-\lambda t}) \text{ as } t \to \infty, \text{ with some } \lambda > 0.$$

The proof of Corollary 1 is in Section 3.2.

3. PROOFS OF THE MAIN RESULTS

3.1 Proof of Theorem 1

For any $x^0 \in D$, let $\rho, \delta, \delta' > 0$ be such that $\overline{B_\rho(0)} \subset \overline{B_\delta(0)} \subset \overline{B_{\delta'}(0)} \subset D$, and $\mathcal{M}_f = \sup_{x \in \overline{B_{\delta'}(0)}} \|f_i(x)\|$, $\mathcal{M}_g = \sup_{x\in\overline{B_{\delta'}(0)}, t\in[0,\tau]} \|g(t,x)\|$. Let $\varepsilon_0 = \min\left\{\tau, \frac{1}{\gamma}\right\}$ and $U^\varepsilon(x^0) = \max\limits_{0\le t\le\varepsilon} \sum_{i=1}^m |h_i^\varepsilon(t,x^0)|$. Here we assume that $\gamma > 0$ is fixed, since, as it will be shown later, γ can be defined independently on ε. From (Grushkovskaya and Zuyev (2018)), for every $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned} U^\varepsilon(x^0) &\le c_{u1}\gamma\|x^0\| + c_{u2}\sqrt{\frac{\gamma}{\varepsilon}\|x^0\|} + \sqrt[3]{\frac{\gamma}{\varepsilon^2}\|x^0\|}, \\ U^\varepsilon \varepsilon(x^0) &\le c_u \sqrt[3]{\varepsilon\gamma\|x^0\|}, \end{aligned} \tag{7}$$

where

$$c_{u1} = \alpha\sqrt{|S_1|}, c_{u2} = 4\sqrt{\pi\alpha}\Big(\sum_{(j_1,j_2)\in S_2} \kappa_{j_1j_2}{}^{2/3}\Big)^{3/4},$$

$$c_{u3} = 6\sqrt[3]{2\pi^2\alpha}\Big(\sum_{(\ell_1,\ell_2,\ell_3)\in S_3} |\kappa^2_{2\ell_1\ell_2\ell_3} - \kappa^2_{1\ell_1\ell_2\ell_3}|^{2/5}\Big)^{5/6},$$

and

$$c_u = c_{u1}\|x^0\|^{2/3} + c_{u2}\|x^0\|^{1/6} + c_{u3}.$$

The integral representation

$$x(t) - x^0 = \int_0^t \Big(\sum_{i=1}^m f_i(x(s)) h_i^\varepsilon(s,x^0) + g(s,x(s))\Big) ds$$

yields that, for any $x^0 \in B_\delta(0)$, $\varepsilon \in (0,\varepsilon_0)$,

$$\|x(t) - x^0\| \le \mathcal{M}_f c_u \sqrt[3]{\varepsilon\gamma\|x^0\|} + \varepsilon\mathcal{M}_g \text{ for all } t \in [0,\varepsilon].$$

For $d = \min\{\delta' - \delta, \frac{1}{2}\rho\} > 0$, let ε_1 be the smallest positive root of the equation

$$\mathcal{M}_f c_u \sqrt[3]{\varepsilon\gamma\delta'} + \varepsilon\mathcal{M}_g = d.$$

Then for any $\varepsilon \in (0, \min\{\varepsilon_0, \varepsilon_1\})$, the solutions of (1), (3) with $x(0) \in B_\delta$ are well defined in D ($\|x(t)\| \le \delta'$) for $t \in [0,\varepsilon]$, and

$$\text{if } \|x^0\| \le \frac{\rho}{2} \text{ then } \|x(t)\| \le \rho \text{ for all } t \in [0,\varepsilon]. \tag{8}$$

Then we use the Chen–Fliess series to represent the π_ε-solution of system (1) at time ε, taking into account the drift term $g(t,x)$ and formula (4):

$$\begin{aligned} x(\varepsilon) =& x^0 + \varepsilon \sum_{j_1\in S_1} f_{j_1}(x^0) a_{j_1}(x^0) \\ &+ \varepsilon \sum_{(j_1,j_2)\in S_2} [f_{j_1}, f_{j_2}](x^0) a_{j_1j_2}(x^0) \\ &+ \varepsilon \sum_{(\ell_1,\ell_2,\ell_3)\in S_3} [[f_{\ell_1}, f_{\ell_2}], f_{\ell_3}](x^0) a_{\ell_1\ell_2\ell_3}(x^0) \\ &+ \Omega(a,\varepsilon) + r_f(\varepsilon) + r_g(\varepsilon) + \int_0^\varepsilon g(s,x(s)) ds \\ =& x^0 - \gamma\varepsilon x^0 + \int_0^\varepsilon g(s,x(s))ds + \Omega(a,\varepsilon) \\ &+ r_g(\varepsilon) + r_a(\varepsilon), \end{aligned} \tag{9}$$

$$\begin{aligned} r_g(\varepsilon) =& \int_0^\varepsilon\int_0^{s_1} \sum_{j_1=1}^m L_g f_{j_1}(x(s)) h_{j_1}^\varepsilon(s_1,x^0) \mathrm{d}s_2 \mathrm{d}s_1 \\ &+ \int_0^\varepsilon\int_0^{s_1}\int_0^{s_2} \sum_{j_1,j_2=1}^m L_g L_{f_{j_2}} f_{j_1}(x(p)) \\ &\quad \times h_{j_2}^\varepsilon(s_2,x^0) h_{j_1}^\varepsilon(s_1,x^0) \mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1 \\ &+ \sum_{j_1,j_2,j_3=1}^m \int_0^\varepsilon\int_0^{s_1}\int_0^{s_2}\int_0^{s_3} L_g L_{f_{j_3}} L_{f_{j_2}} f_{j_1}(x(s_4)) \\ &\quad \times h_{j_3}^\varepsilon(s_3,x^0) h_{j_2}^\varepsilon(s_2,x^0) h_{j_1}^\varepsilon(s_1,x^0) \mathrm{d}s_4\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1, \end{aligned}$$

$$\begin{aligned} r_f(\varepsilon) =& \sum_{j_1,\dots,j_4=1}^m \int_0^\varepsilon\int_0^{s_1}\int_0^{s_2}\int_0^{s_3} L_{f_{j_4}} L_{f_{j_3}} L_{f_{j_2}} f_{j_1}(x(s_4)) \\ &\times h_{j_4}^\varepsilon(s_4,x^0) h_{j_3}^\varepsilon(s_3,x^0) h_{j_2}^\varepsilon(s_2,x^0) h_{j_1}^\varepsilon(s_1,x^0) \mathrm{d}s_4\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1. \end{aligned}$$

We omit the explicit expression for $\Omega(a,\varepsilon)$ due to the space limits. Similarly to (Grushkovskaya and Zuyev (2018); Grushkovskaya et al. (2018); Zuyev and Grushkovskaya (2019)), it can be shown that there exist $c_\Omega, c_g, c_f \ge 0$ such that, for any $x^0 \in \overline{B_\delta(0)}$,

$$\|\Omega(a,\varepsilon)\| \le c_\Omega(\varepsilon\|x^0\|)^{7/6},$$

$$\|r_g(\varepsilon)\| \le c_g M_g \varepsilon^{4/3}\|x^0\|^{1/3},\ \|r_f(\varepsilon)\| \le c_f(\varepsilon\|x^0\|)^{4/3}.$$

Applying these estimates to (9), we conclude that

$$\|x(\varepsilon)\| \le (1-\gamma\varepsilon)\|x^0\| + \sigma(\varepsilon)\varepsilon^{7/6}\|x^0\|^{1/3} + \mathcal{M}_g\varepsilon, \tag{10}$$

where $\sigma(\varepsilon) = c_\Omega\delta^{5/6} + \varepsilon^{1/6}\big(c_g M_g + c_f\delta\big)$. Assume $x^0 \in \overline{B_\delta(0)} \backslash B_{\rho/2}(0)$. Then the latter inequality can be rewritten as

$$\begin{aligned} \|x(\varepsilon)\| &\le (1-\gamma\varepsilon)\|x^0\| + \sigma\varepsilon^{7/6}\Big(\frac{2}{\rho}\Big)^{2/3}\|x^0\| + \frac{2\mathcal{M}_g}{\rho}\varepsilon\|x^0\| \\ &= (1-\varepsilon\lambda_1)\|x^0\|, \end{aligned}$$

where $\lambda_1 = \gamma - \frac{2\mathcal{M}_g}{\rho} - \sigma(\varepsilon)\varepsilon^{1/6}\Big(\frac{2}{\rho}\Big)^{2/3}$. Taking $\gamma > \frac{2\mathcal{M}_g}{\rho}$, we ensure that there exists a $\lambda_2 > 0$ such that $\gamma - \frac{2\mathcal{M}_g}{\rho} > \lambda_2$. For any $\lambda \in (0,\lambda_2)$, let $\varepsilon_2 = \min\left\{\frac{1}{\lambda}, \hat\varepsilon\right\}$, where $\hat\varepsilon$ is the smallest positive root of the equation

$$\sigma(\varepsilon)\varepsilon^{1/6}\Big(\frac{2}{\rho}\Big)^{2/3} = \lambda_2 - \lambda.$$

Then, for any $\varepsilon \in (0, \bar\varepsilon = \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2\})$, if $\|x^0\| > \frac{\rho}{2}$, then

$$\|x(\varepsilon)\| \le (1-\varepsilon\lambda)\|x^0\|.$$

Since $x^0 \in B_\delta(0)$ then $x(\varepsilon) \in B_\delta(0)$, and we repeat the above argumentation for the solutions $x(t)$ of system (1),

(3) with the initial conditions $x(\varepsilon) \in B_\delta(0)$. Thus, we conclude that there exists an $N \in \mathbb{N} \cup \{0\}$ such that

$$\|x(j\varepsilon)\| \ge \frac{\rho}{2} \text{ for all } j = 0,\dots,N-1,\ \|x(N\varepsilon)\| \le \frac{\rho}{2},$$

which implies that the solutions $x(t)$ of system (1), (3) with the initial conditions $x(0) = x^0 \in B_\delta(0)$ are well defined for all $t \in [0,(N+1)\varepsilon]$, and

$$\|x(j\varepsilon)\| \le \|x^0\|e^{-\lambda j\varepsilon} \text{ for all } j = 0,\dots,N.$$

Furthermore, $\|x\big((N+1)\varepsilon\big)\| \le \rho$ from (8). If $\|x\big((N+1)\varepsilon\big)\| \ge \frac{\rho}{2}$, we apply again the same reasoning and obtain $\big\|x\big((N+2)\varepsilon\big)\big\| \le \|x\big((N+1)\varepsilon\big)\|$. Otherwise, (8) implies $\|x\big((N+2)\varepsilon\big)\| \le \rho$. Thus, for any $\varepsilon \in (0,\bar\varepsilon)$, the solutions of system (1), (3) with the initial conditions $x(0) = x^0 \in B_\delta(0)$ satisfy the following properties:

$$\|x(0)\| \le \delta \Rightarrow \|x(t)\| \le \|x^0\|e^{-\lambda t} + \frac{\rho}{2} \text{ for all } t \ge 0,$$

and there exists a $t_1 > 0$ such that $\|x(t)\| \le \rho$ for $t \ge t_1$.

3.2 Proof of Corollary 1

As it follows from Theorem 1 and its proof, for any $\delta, \delta_0 > 0$, there exists an $\bar\varepsilon_1 > 0$ such that, for any $\varepsilon \in (0,\bar\varepsilon_1]$, the π_ε-solution of system (1) with the initial data $x(0) = x^0 \in B_\delta(0)$ is well-defined on $t \in [0,+\infty)$ and

$$\begin{aligned}\|x(t)\| \le \|x^0\|e^{-\lambda_1 t} + \delta_0 \text{ for all } t \in [t_0,t_1),\\ \text{and } \|x(t)\| \le \delta_0 \text{ for all } t \in [t_1,\infty),\end{aligned} \tag{11}$$

with some $\lambda_1, t_1 > 0$. The proof is similar to the proof of Theorem 1 with $D = B_{\delta_0}(0)$, so we just briefly describe the main differences. Let us analyze the behavior of solutions of system (1) in $B_{\delta_0}(0)$.

Let $\tilde x^0 \in B_{\delta_0}(0)$. Using the integral representation of $x(t)$, the Grönwall–Bellman inequality, estimate (7), and the assumptions on $g(t,x)$, we conclude that

$$\|x(t) - \tilde x^0\| \le c_x \sqrt[3]{\varepsilon\|\tilde x^0\|} \text{ for all } t \in [0,\varepsilon], \tag{12}$$

where $c_x = \big(\mathcal{M}_f c_u \sqrt[3]{\gamma} + \mathcal{M}_g \delta_0^2 (\varepsilon\delta_0)^{2/3}\big)e^{\mathcal{L}_f c_u \sqrt[3]{\varepsilon\gamma\delta_0} + \mathcal{L}_g}$, and $\mathcal{L}_f$ is such that $\Big\|f(x) - f(y)\Big\| \le \mathcal{L}_f\|x-y\|$ for all $x, y \in B_{\delta'}(0)$. Furthermore,

$$\begin{aligned}\|g(t,x(t))\| &\le \mathcal{M}_g\|x(t)\|^3 \le \mathcal{M}_g(\|\tilde x^0\| + \|x(t) - \tilde x^0\|)^3\\ &\le \mathcal{M}_g\|\tilde x^0\|(\delta_0^2 + c_x\sqrt[3]{\varepsilon})^3 \text{ for all } t \in [0,\varepsilon].\end{aligned} \tag{13}$$

Then the term $r_g(\varepsilon)$ in (9) can be estimated as $\|r_g(\varepsilon)\| \le \tilde c_g(\varepsilon\|\tilde x^0\|)^{4/3}$ with some $\tilde c_g > 0$. Consequently, the estimate (10) can be written as

$$\begin{aligned}\|x(\varepsilon)\| \le&(1-\gamma\varepsilon)\|x^0\| + \tilde\sigma(\varepsilon)\varepsilon^{7/6}\|x^0\|^{7/6}\\ &+ \varepsilon\mathcal{M}_g\|\tilde x^0\|(\delta_0^2 + c_x\sqrt[3]{\varepsilon})^3\\ =&\Big(1 - \varepsilon\big(\gamma - \mathcal{M}_g\delta_0^6 - \varepsilon^{1/6}\sigma_1(\varepsilon)\big)\Big)\|x^0\|.\end{aligned}$$

Here $\tilde\sigma(\varepsilon) = c_\Omega + (\varepsilon\delta_0)^{1/6}\big(\tilde c_g + c_f\big)$, $\sigma_1(\varepsilon) = \tilde\sigma(\varepsilon)\delta_0^{1/6}$. Taking $\gamma > \mathcal{M}_g\delta_0^6$, $\lambda_2 \in (0,\gamma - \mathcal{M}_g\delta_0)$, and $\tilde\varepsilon_1$ as the smallest positive root of the equation $\varepsilon^{1/6}\sigma_1(\varepsilon) = \tilde\lambda$, we obtain $\|x(\varepsilon)\| \le (1-\lambda_2\varepsilon)\|\tilde x^0\|$. Repeating the above argumentation for an arbitrary $\tilde x^0 \in B_{\delta_0}(0)$, we conclude that

$$\|x(j\varepsilon)\| \le \|\tilde x^0\|e^{-\lambda_2 j\varepsilon}, \text{ for } j = 0,1,2,\dots. \tag{14}$$

For any $t \ge 0$ and $\varepsilon \in (0,\bar\varepsilon = \min\{\tilde\varepsilon_0,\tilde\varepsilon_1\})$, we have

$$\begin{aligned}\|x(t)\| &\le \Big\|x(t) - x\Big(\Big[\frac{t}{\varepsilon}\Big]\varepsilon\Big)\Big\| + \Big\|x\Big(\Big[\frac{t}{\varepsilon}\Big]\varepsilon\Big)\Big\|\\ &\le \sqrt[3]{\Big\|x\Big(\Big[\frac{t}{\varepsilon}\Big]\varepsilon\Big)\Big\|}\Big(c_x\sqrt[3]{\varepsilon} + \Big\|x\Big(\Big[\frac{t}{\varepsilon}\Big]\varepsilon\Big)\Big\|^{2/3}\Big).\end{aligned}$$

Using (14), we obtain the following estimate:

$$\|x(t)\| \le \mu_1\sqrt[3]{\|\tilde x^0\|}e^{-\frac{\lambda_2}{3}t}, \tag{15}$$

with $\mu_1 = e^{\lambda_2\varepsilon}\big(c_x\sqrt[3]{\varepsilon} + \delta_0^{2/3}\big)$. Choosing $\bar\varepsilon = \min\{\bar\varepsilon_1,\bar\varepsilon_2\}$ and summarizing (11) and (15), we conclude that, for any $\varepsilon \in (0,\bar\varepsilon]$, there exists a $t_1 > 0$

$$\|x(t)\| \le \begin{cases}\|x^0\|e^{-\lambda_1 t} + \delta_0 \text{ for } t \in [0,t_1),\\ \mu_1\sqrt[3]{\|x(t_1)\|}e^{-\frac{\lambda_2}{3}t} \text{ for } t \in [t_1,\infty),\end{cases}$$

which proves the Corollary.

4. EXAMPLES

4.1 Underwater vehicle with drift

Consider the equations of motion for an autonomous 3D underwater vehicle studied, e.g., in Barraquand and Latombe (1989), and assume that the motion of the vehicle is also affected by external disturbances:

$$\dot x_1 = \sum_{k=1}^4 f_k(x)u_k + g(t), \tag{16}$$

where (x_1,x_2,x_3) are the coordinates of the center of mass, $(x_4,\ x_5,\ x_6)$ describe the vehicle orientation (Euler angles), u_1 is the translational velocity along the Ox_1 axis, (u_2,u_3,u_4) are the angular velocity components, and the vector fields of the unperturbed system are

$$\begin{aligned}&f_1(x) = (\cos x_5\cos x_6, \cos x_5\sin x_6, -\sin x_5, 0,0,0)^\top,\\ &f_2(x) = (0,0,0,1,0,0)^\top,\\ &f_3(x)=(0,0,0,\sin x_4 \mathrm{tg}\, x_5, \cos x_4, \sin x_4\sec x_5)^\top,\\ &f_4(x)=(0,0,0,\cos x_4\mathrm{tg}\, x_5, -\sin x_4, \cos x_4\sec x_5)^\top.\end{aligned}$$

The drift term in (16) accounts for the external disturbances caused by waves and ocean currents, and we choose the following form for $g(t)$:

$$g(t) = (0, d, a\sin(\omega t + b), 0,0,0)^\top,$$

where a, b, d, ω are some positive constants. The rank condition (2) is satisfied in the domain $D = \{x \in \mathbb{R}^6 \mid -\frac{\pi}{2} < x_5 < \frac{\pi}{2}\}$ with $S_1 = \{1,2,3,4\}$, $S_2 = \{(1,3),(1,4)\}$, $S_3 = \emptyset$. Then the matrix (6) takes the form

$$\mathcal{F}(x) = (f_1(x),\ f_2(x),\ f_3(x),\ f_4(x),\ [f_1,f_3](x),\ [f_1,f_4](x)),$$

and we may write controls (3) as $u_k = h_k^\varepsilon(t,x)$:

$$\begin{aligned}h_1^\varepsilon(t,x) =&a_1(x) + 2\ \mathrm{sign}(a_{13}(x))\sqrt{\tfrac{\pi|a_{13}(x)|}{\varepsilon}}\cos\tfrac{2\pi k_{13}t}{\varepsilon}\\ &+ 2\ \mathrm{sign}(a_{14}(x))\sqrt{\tfrac{\pi|a_{14}(x)|}{\varepsilon}}\cos\tfrac{2\pi k_{14}t}{\varepsilon},\\ h_2^\varepsilon(t,x) =&a_2(x),\\ h_3^\varepsilon(t,x) =&a_3(x) + 2\sqrt{\tfrac{\pi|a_{13}(x)|}{\varepsilon}}\sin\tfrac{2\pi k_{13}t}{\varepsilon},\\ h_4^\varepsilon(t,x) =&a_4(x) + 2\sqrt{\tfrac{\pi|a_{14}(x)|}{\varepsilon}}\sin\tfrac{2\pi k_{14}t}{\varepsilon},\end{aligned} \tag{17}$$

with $a(x) = (a_1(x),\ a_2(x),\ a_{13}(x),\ a_{14}(x))^\top = -\gamma\mathcal{F}^{-1}(x)x$. The behavior of system (16) with controls (17) is illustrated in Fig. 1a). For numerical simulations, we take

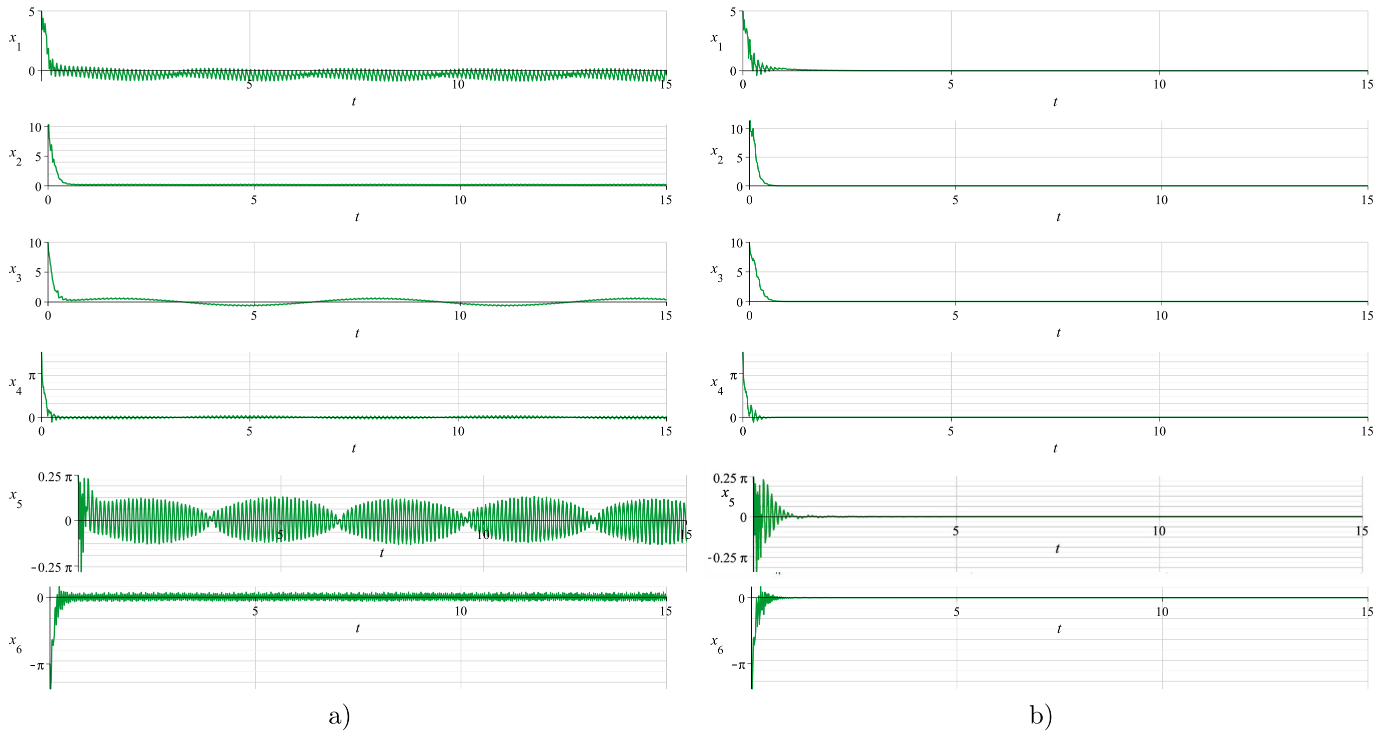

Fig. 1. Time-plots of the trajectories of system (16) with controls (17).

$g(t) = (0, 2, 5\sin t, 0, 0, 0)^\top$, $x^0 = \left(5, 10, 10, \frac{3\pi}{2}, \frac{\pi}{4}, -\pi\right)^\top$, $\varepsilon = 0.1$, $\gamma = 10$, $k_{13} = 1$, $k_{14} = 2$. To illustrate Corollary 1, assume that the drift is described by $g(t, x) = (0, x_1^3(t), x_2^3(t)\sin t, 0, 0, 0)^\top$. As it is shown in Fig.1b), the trajectories of system (16) tend asymptotically to zero in this case.

4.2 Front-wheel drive car

As an example of a nonholonomic system satisfying condition (2) with the second-order Lie brackets, consider a kinematic model of the front-wheel drive car (see, e.g., De Luca and Oriolo (1995)):

$$\dot{x}_1 = \sum_{k=1}^{2} f_k(x)u_k,$$

where (x_1, x_2) are the Cartesian coordinates of the rear axle center, the angle x_3 defines the car orientation with respect to the x_1-axis, x_4 is the steering angle, u_1, u_2 denote the driving and the steering velocity input, respectively; thus the vector fields of the system are given by

$$f_1(x) = (\cos x_3 \cos x_4, \sin x_3 \cos x_4, \sin x_4, 0)^\top,$$
$$f_2(x) = (0, 0, 0, 1)^\top.$$

It can be verified that the rank condition (2) is satisfied with $S_1 = \{1, 2\}$, $S_2 = \{(1, 2)\}$, $S_3 = \{(1, 2, 1)\}$, so that the matrix $\mathcal{F}(x) = \big(f_1(x), f_2(x), [f_1, f_2](x), \big[[f_1, f_2], f_1\big](x)\big)$ is nonsingular in $\mathbb{R}^4$. If the control input acts with an error, i.e. $u_k = h_k^\varepsilon(t, x) + n_k(t, x)$, where $n_k(t, x)$ are some disturbances, then the system equations can be interpreted as the system with drift:

$$\dot{x}_1 = \sum_{k=1}^{2} f_k(x)u_k + g(t, x), \tag{18}$$

where $g(t, x) = \sum_{k=1}^{2} f_k(x)n_k(x, t)$. According to the proposed design procedure, we take controls of the form (3):

$$\begin{aligned} h_1^\varepsilon(t, x) =& a_1(x) + 2\,\mathrm{sign}(a_{12}(x))\sqrt{\tfrac{\pi|a_{12}(x)|}{\varepsilon}}\cos\tfrac{2\pi k_{12}t}{\varepsilon} \\ &+ 2\sqrt[3]{\tfrac{2\pi^2(k_{2121}-k_{1121})a_{121}(x)}{\varepsilon^2}}\cos\tfrac{2\pi k_{1121}t}{\varepsilon} \\ &\times\left(1 + \tfrac{2\pi k_{2121}t}{\varepsilon}\sin\right), \end{aligned} \tag{19}$$

$$\begin{aligned} h_2^\varepsilon(t, x) =& a_2(x) + 2\sqrt{\tfrac{\pi|a_{12}(x)|}{\varepsilon}}\sin\tfrac{2\pi k_{12}t}{\varepsilon}, \\ &+ 2\sqrt[3]{\tfrac{2\pi^2(k_{2121}-k_{1121})a_{121}(x)}{\varepsilon^2}}\sin\tfrac{2\pi k_{2121}t}{\varepsilon} \end{aligned}$$

with

$$a(x) = (a_1(x), a_2(x), a_{12}(x), a_{121}(x))^\top = -\gamma\mathcal{F}^{-1}(x)x.$$

For the numerical simulation, we take $n_1(t, x) = 2\cos 10\pi t$, $n_2(t, x) = \sin 20\pi t$, $x^0 = \left(5, 3, -\frac{\pi}{2}, \frac{\pi}{4}\right)^\top$, $\varepsilon = 0.5$, $\gamma = 15$, $k_{12} = 7$, $k_{1121} = 3$, $k_{2121} = 1$. The corresponding plots are depicted in Fig. 2.

5. CONCLUSIONS

We have considered a class of nonholonomic systems with time-varying drift term satisfying certain boundedness assumptions. Extending the approach of Zuyev et al. (2016); Grushkovskaya and Zuyev (2018), we have obtained a family of time-periodic control functions with rather simple formulas for state-dependent coefficients. It should be emphasized that the considered systems with vanishing controls, in general, do not admit the trivial equilibrium. It is also crucial that the exponential decay estimates have been derived without assuming that the drift can be compensated by a linear combination of control vector fields.

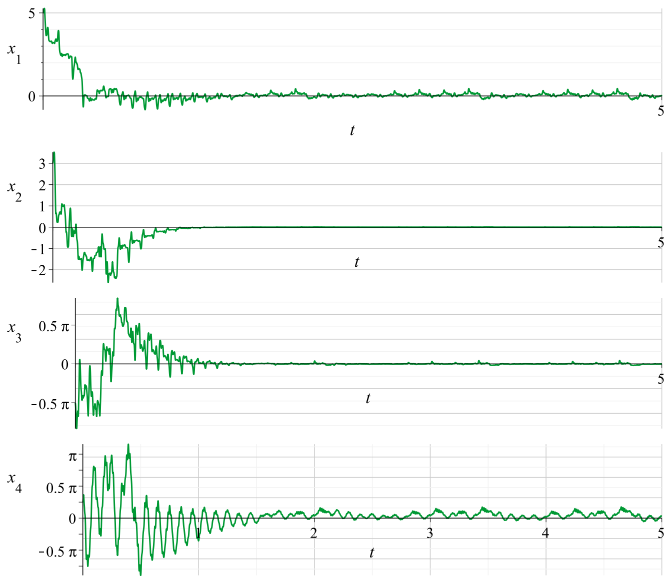

Fig. 2. Time-plots of the trajectories of system (18) with controls (19).

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