

## Partial Stability Concept in Extremum Seeking Problems<sup>★</sup>

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**Abstract:** The paper deals with the extremum seeking problem for a class of cost functions depending only on a part of state variables of a control system. This problem is related to the concept of partial asymptotic stability and analyzed by Lyapunov's direct method and averaging schemes. Sufficient conditions for the practical partial stability of a system with oscillating inputs are derived with the use of Lie bracket approximation techniques. These conditions are exploited to describe a broad class of extremum-seeking controllers ensuring the partial stability of the set of minima of a cost function. The obtained theoretical results are illustrated by the Brockett integrator and rotating rigid body.

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*Keywords:* extremum seeking, partial stability, output stabilization, Lie bracket approximation, control-affine systems, oscillating control

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### 1. INTRODUCTION

Extremum seeking has become an important branch of modern control theory because of challenging theoretical features and various practical applications. The goal of extremum seeking control is to optimize the steady-state performance of a control system using the output measurements. The main motivation behind this problem statement is to reduce the amount of information needed for the control design. In particular, an optimal operating point as well as analytical expression of the output (cost) function are assumed to be unknown. During the past couple of decades, several important approaches for the extremum seeking control design have been developed (see, e.g., Krstić and Wang (2000); Krstić and Ariyur (2003); Guay and Zhang (2003); Tan et al. (2010); Nešić et al. (2010); Dürr et al. (2013); Guay and Dochain (2015); Haring and Johansen (2017); Dürr et al. (2017); Suttner and Dashkovskiy (2017); Scheinker and Krstić (2017); Grushkovskaya et al. (2018a,b)). The above approaches assume that the cost function depends essentially on all state variables, and/or that the system admits an asymptotically stable steady-state. However, these assumptions can be redundant for various applied problems, for which it is important (or even only possible) to optimize the system with respect to a prescribed part of state variables, and consequently to stabilize the system only with respect to these variables. In particular, such problems arise if the cost function depends on a part of system variables, if only partial output measurements are available for control design, or if the partial stabilization is sufficient for correct system operation. As a simple example, one can imagine

the problem of tracking a planar target by a multi-DOF robot (see, e.g., Cochran et al. (2009); Matveev et al. (2011); Khong et al. (2014); Mandić and Mišković (2015)). The goal of this paper is to introduce the problems of partial extremum seeking, in which the goal is to optimize the system performance with respect to a part of state variables only. Such problem statement allows to consider a broader class of systems and applications. The contribution of this paper is twofold. First, we generalize the Lie bracket approximation approach (see, e.g., Moreau and Aeyels (2000); Dürr et al. (2013); Dürr et al. (2017)) and techniques introduced in Grushkovskaya et al. (2018b) to input-affine systems whose Lie bracket system has a partially asymptotically stable manifold. To solve the problem under consideration, we attract methods of partial stability theory, which dates back to Lyapunov and has been developed in the works of Malkin (1952); Rumyantsev and Oziraner (1987); Vorotnikov (2012); Zuyev (2000, 2003); Kovalev et al. (2009); Grushkovskaya and Zuyev (2015) and others (see Vorotnikov (2005) for a review). Second, we consider a class of extremum seeking problems, in which the system has to be optimized with respect to a prescribed part of variables. Up to our best knowledge, such problem statement has not been considered before.

The rest of the paper is organized as follows. Section 1.1 contains some notations and definitions which will be used throughout the paper. In Section 2.1, we extend the Lie bracket approximation approach assuming that the corresponding Lie bracket system is partially asymptotically stable, and derive conditions for practical partial asymptotic stability. These results are applied to extremum seeking problems in Section 2.2. In Section 3, we consider several examples illustrating the proposed approach and some possible extensions.

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1.1 Notations and definitions

Consider the system

$$\dot{x} = f^\varepsilon(t, x), \quad x \in \mathbb{R}^n, \quad t \geq 0, \quad (1)$$

where  $f : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ , and  $\varepsilon > 0$  is a parameter. We will split the components of the state vector  $x$  as  $x = (y^\top, z^\top)^\top \in \mathbb{R}^n$  with  $y \in \mathbb{R}^{n_1}$ ,  $z \in \mathbb{R}^{n_2}$ ,  $n_1 + n_2 = n$ . With a slight abuse of notations, the column  $x$  will be also denoted as  $x = (y, z)$ . Throughout the text,  $B_\delta(x^*)$  and  $\overline{B_\delta(x^*)} = B_\delta(x^*) \cup \partial B_\delta(x^*)$  denote the  $\delta$ -neighborhood of an  $x^* \in \mathbb{R}^n$  and its closure, respectively. Notation  $\varphi \in \mathcal{K}$  means that a function  $\varphi$  belongs to the class  $\mathcal{K}$ , i.e.  $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a continuous strictly increasing function,  $\varphi(0) = 0$ . For  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $x \in \mathbb{R}^n$ , we denote the directional derivative as  $L_g f(x) = \lim_{s \rightarrow 0} \frac{f(x+sg(x)) - f(x)}{s}$ , and  $[f, g](x) = L_f g(x) - L_g f(x)$  is the Lie bracket. We will use the following definition, which extends the notion of *partial asymptotic stability* (Rumyantsev and Oziraner (1987); Vorotnikov (2012); Zuyev (2003)) to systems with parameters of the form (1).

*Definition 1.* For  $y^* \in \mathbb{R}^{n_1}$ , the set  $D^* = \{x = (y, z) \in \mathbb{R}^n : y = y^*\}$  is *practically uniformly y-asymptotically stable for system (1)*, if it is:

- *practically uniformly y-stable for system (1)*, i.e., for every  $\rho > 0$ , there exist  $\delta > 0, \bar{\varepsilon} > 0$  such that the following property holds for all  $t_0 \geq 0, z(t_0) \in \mathbb{R}^{n_2}, \varepsilon \in (0, \bar{\varepsilon})$ :

- if  $y(t_0) \in B_\delta(y^*)$  then  $y(t) \in B_\rho(y^*)$  for all  $t \in [t_0, \infty)$ ;

- *practically uniformly y-attractive for system (1)*, i.e., for some  $\delta > 0$ , for every  $\rho > 0$ , there are  $t_1 \geq 0, \bar{\varepsilon} > 0$  such that the following property holds for all  $t_0 \geq 0, z(t_0) \in \mathbb{R}^{n_2}, \varepsilon \in (0, \bar{\varepsilon})$ : if  $y(t_0) \in B_\delta(y^*)$  then  $y(t) \in B_\rho(y^*)$  for all  $t \in [t_0 + t_1, \infty)$ .

If the attractivity property holds for any  $\hat{\delta} > 0$ , then  $y^*$  is called to be *semi-globally practically uniformly y-asymptotically stable for system (1)*. For systems independent of  $\varepsilon$ , we omit the terms “practically” and “semi”. In case  $n_1 = n, n_2 = 0$ , the above definition coincides with a well-known definition of practical asymptotic stability (Moreau and Aeyels (2000); Dürr et al. (2013)). Up to our best knowledge, the proposed definition of *practical partial stability* is introduced here for the first time.

2. MAIN RESULTS

2.1 Lie bracket approximation & partial stability

In this section, we extend the Lie bracket approximation approach to partially asymptotically stable systems. Namely, we consider the system

$$\dot{x} = f_0(x) + \sum_{i=1}^m f_i(x)u_i, \quad x \in \mathbb{R}^n, \quad (2)$$

where  $u_i = \frac{1}{\sqrt{\varepsilon}} w_i\left(\frac{t}{\varepsilon}\right)$ ,  $w_i\left(\frac{t}{\varepsilon}\right)$  are  $\varepsilon$ -periodic continuous functions with some  $\varepsilon > 0$ , and  $\int_0^\varepsilon w_i\left(\frac{t}{\varepsilon}\right) dt = 0$ . We assume that there exists a  $W > 0$  such that  $\max_{1 \leq i \leq m, 0 \leq t \leq \varepsilon} w_i\left(\frac{t}{\varepsilon}\right) \leq W$  for each  $\varepsilon > 0$ .

Consider also the so-called Lie bracket system

$$\dot{\bar{x}} = f_0(\bar{x}) + \sum_{i < j, i, j = 1}^m [f_i, f_j](\bar{x})\nu_{ij}, \quad \bar{x} \in \mathbb{R}^n, \quad (3)$$

where  $\nu_{ij} = \frac{1}{\varepsilon^2} \int_0^\varepsilon \int_0^\tau w_j\left(\frac{\tau}{\varepsilon}\right) w_i\left(\frac{s}{\varepsilon}\right) ds d\tau$ . Denote  $x = (y, z)$ ,  $\bar{x} = (\bar{y}, \bar{z})$ ,  $y, \bar{y} \in \mathbb{R}^{n_1}$ ,  $z, \bar{z} \in \mathbb{R}^{n_2}$ ,  $n_1 + n_2 = n$ .

*Assumption 1.* Let  $D_1 \subseteq \mathbb{R}^{n_1}$  and  $D_2 \subseteq \mathbb{R}^{n_2}$  be domains, and let  $y^* \in D_1, D = \{(y, z) \in \mathbb{R}^n : y \in D_1, z \in D_2\}, D^* = \{x = (y, z) \in D : y = y^*\}$ . We suppose that:

A1.1)  $f_0, f_1, \dots, f_m \in C^2(D \setminus D^*; \mathbb{R}^n)$ ;

A1.2) for any compact  $\tilde{D}_1 \subset D_1$ , the functions  $f_i, L_{f_j} f_i, L_{f_i} L_{f_j} f_i \in C(D; \mathbb{R}^n)$  are bounded for all  $y \in \tilde{D}_1, z \in D_2, i, j, l \in \{0, \dots, m\}$ ;

A1.3) if  $x(t) \in D, t \in I = [t_0, t_1]$  is a solution of (2) such that  $\inf_{t \in I} \text{dist}(y(t), \partial D_1) > 0$  then  $\inf_{t \in I} \text{dist}(z(t), \partial D_2) > 0$ .

Here  $\text{dist}(\xi, X)$  denotes the Euclidian distance between a point  $\xi \in \mathbb{R}^{n_k}$  and a set  $X \subset \mathbb{R}^{n_k}$ . If both  $D_2 \subset \mathbb{R}^{n_2}$  and  $z(t) \in D_2, t \in I$ , are unbounded, we will follow the convention that  $\inf_{t \in I} \text{dist}(z(t), \partial D_2) = 0$ . Note that A1.3) is a reformulation of the standard  $z$ -extendability assumption in partial stability theory (see, e.g., Rumyantsev and Oziraner (1987)). For the case  $D_2 = \mathbb{R}^{n_2}$ , this assumption means that  $z(t)$  cannot escape to infinity in finite time whenever  $y(t)$  remains bounded. The above assumption is usually satisfied in well-posed practical problems without blow-up of solutions.

The first main result of the paper is as follows.

*Theorem 1.* Let  $D_1 \subseteq \mathbb{R}^{n_1}, D_2 \subseteq \mathbb{R}^{n_2}$  be such that Assumption 1 is satisfied,  $y^* \in D_1$ , and let there exist a function  $V(x) \in C^2(D)$  such that the following conditions hold for all  $x = (y, z) \in D$ :

$$1.1) \alpha_1(\|y - y^*\|) \leq V(x) \leq \alpha_2(\|y - y^*\|),$$

$$1.2) L_{\bar{f}} V(x) \leq -\alpha_3(\|y - y^*\|).$$

Here  $\bar{f}(x) = f_0(x) + \sum_{i < j} [f_i, f_j](x)\nu_{ij}$  is the right-hand side of system (3), and  $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}$ .

Then  $D^* = \{x = (y, z) \in D : y = y^*\}$  is *practically y-asymptotically stable for (2) with the initial conditions from the set  $D_0 = \{(y, z) \in \mathbb{R}^n : \|y - y^*\| \leq \delta, z \in D_2\}$ , where  $\delta \in (0, \alpha_2^{-1}(\alpha_1(\text{dist}(y^*, \partial D_1)))$ .*

The proof of Theorem 1 is in Appendix A. Note that the assumptions of Theorem 1 are more general than those used in Grushkovskaya et al. (2018b), so that the proof of this result extends the approaches of Grushkovskaya et al. (2018b) to a broader class of systems.

The next results follow from the proof of Theorem 1.

*Corollary 1.* If the conditions of Theorem 1 hold with 1.1) replaced by  $\alpha_1(\|y - y^*\|) \leq V(x) \leq \tilde{\alpha}_2(\|x - x^*\|), x^* = (y^*, z^*), z^* \in D_2$ , where  $\tilde{\alpha}_2 \in \mathcal{K}$ , then the set  $D^*$  is *practically y-attractive in  $D_0$  for system (2) provided that there exist  $\delta > 0, c_\delta \in (0, \alpha_1(\text{dist}(y^*, \partial D_1)))$  such that  $\alpha_2(\|x - x^*\|) \leq c_\delta$  for all  $x \in D_0$ .*

*Corollary 2.* If the conditions of Theorem 1 hold with the function  $V$  depending on the  $y$ -variable only, then the assertion of Theorem 1 holds even if the  $z$ -components of the functions from A1.2) are unbounded.

*Remark 1.* Under some additional assumptions on the function  $V$  and the vector fields of system (2), it is possible to state *classical* (instead of practical) asymptotical stability conditions and to describe the decay rate of solutions of system (2), as it was done in (Grushkovskaya et al.

(2018b)) by extending the techniques of (Grushkovskaya and Zuyev (2014); Zuyev and Grushkovskaya (2017)). We leave these studies for future work.

2.2 Partial stabilization of control-affine extremum seeking systems

In this section, we apply the proposed results to extremum seeking problems in which the goal is to optimize the system performance with respect to certain part of variables. Namely, we assume that the set of minima of a cost function  $J : \mathbb{R}^n \rightarrow \mathbb{R}$  is a hyperplane of the form  $\text{argmin } J = \{x = (y, z) : y = y^*\}$ , where the value of  $y^* \in \mathbb{R}^{n_1}$  is a priori unknown for the control design. Thus we arrive to the following problem statement.

*Problem 1.* Given a cost function  $J: \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $\text{argmin } J = \{x = (y, z) \in \mathbb{R}^n : y = y^*\}$  with some  $y^* \in \mathbb{R}^{n_1}$ . The goal is to construct a control  $u = \tilde{u}(t, J(x))$  such that the set  $\text{argmin } J$  is practically  $y$ -asymptotically stable for (2).

Such kind of problems appears, for example, if the cost  $J$  depends on the  $y$ -variables only, or if  $J$  can be represented as  $J(x) = J^*(y - y^*)\phi(z)$ , where  $J^*(\eta)$  is a positive definite function, and  $\phi(z) > 0$  for all  $z \in \mathbb{R}^{n_2}$ . The above task is relevant to the output stabilization problem, if the stabilization with respect to all variables is not possible (or not required for control purposes), and to synchronization problems, where the goal  $y = y^*$  describes synchronous motion of a multi-agent system (e.g., system of pendulums) while the  $z$ -variables stand for redundant degrees of freedom. Let us define the controls  $u_i$  as

$$u_i = \frac{1}{\sqrt{\varepsilon}} \left( g_i(J(x))w_i \left( \frac{t}{\varepsilon} \right) + g_{i+m}(J(x))w_{i+m} \left( \frac{t}{\varepsilon} \right) \right), \quad (4)$$

where  $\varepsilon > 0$ ,  $w_i, w_{i+n}$  satisfy the assumptions of section 2.1 and are such that  $\nu_{ij} = 0$  whenever  $j \neq i + m$ ,  $\nu_{ii+m} = 1$ , and the functions  $g_i, g_{i+m}$  satisfy the relation

$$g_{i+m}(z) = -\gamma_i g_i(z) \int \frac{dz}{g_i(z)^2}, \gamma_i > 0, i = \overline{1, m}. \quad (5)$$

*Theorem 2.* Let  $D_1 \subseteq \mathbb{R}^{n_1}, D_2 \subseteq \mathbb{R}^{n_2}$  be convex domains such that Assumption 1 is satisfied,  $y^* \in D_1$ , and let the function  $V(x) = J(x) - J(y^*, z)$  satisfies the conditions of Theorem 1 with  $f(x) = f_0(x) - \sum_{i=1}^m \gamma_i f_i(x) f_i^T(x) \nabla J(x)$ . Then the set  $D^* = \{x = (y, z) \in D : y = y^*\}$  is practically  $y$ -asymptotically stable in  $D_0$  for system (2) with the controls  $u_i$  given by (4)–(5).

**Proof.** Straightforward calculations show that the Lie bracket system for (2) with the controls  $u_i$  given by (4)–(5) has the form  $\dot{\tilde{x}} = f_0(\tilde{x}) - \sum_{i=1}^m \gamma_i f_i(\tilde{x}) f_i^T(\tilde{x}) \nabla J(\tilde{x})$ . Then the conditions of Theorem 1 are satisfied.

The assumptions on the cost function  $J$  required in Theorem 2 are common in extremum seeking studies for ensuring the stability with respect to all variables (cf. Tan et al. (2006); Guay and Atta (2018)). They can be relaxed for certain classes of systems, as in the next result.

*Theorem 3.* Let a control system be of the form

$$\dot{y} = \sum_{i=1}^{n_1} \tilde{f}_i(x) u_i, \quad \dot{z} = h(x, u), \quad (6)$$

where the vector fields  $\tilde{f}_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $h : \mathbb{R}^n \times \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2}$  satisfy A1.1)–A1.2). Assume that the vector

fields  $\tilde{f}_i(x) = (\tilde{f}_{i1}(x) \dots \tilde{f}_{in_1}(x))^T, i = 1, 2, \dots, n_1$ , are linearly independent at each  $x \in D$ , and the cost function  $J = J(y) : D_1 \subset \mathbb{R}^{n_1} \rightarrow \mathbb{R}$  satisfies the inequalities

$$\alpha_1(\|y - y^*\|) \leq J(y) - J(y^*) \leq \alpha_2(\|y - y^*\|), \\ \|\nabla J(y)\| \leq -\alpha_3(\|y - y^*\|)$$

with some  $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}$ .

Then the set  $D^*$  is practically  $y$ -asymptotically stable for system (6) with the controls  $u_i$  given by (4)–(5).

**Sketch of the proof.** Computing the time-derivative of  $J$  along the trajectories of the corresponding Lie bracket system for (6), we get  $\dot{J}(\tilde{y}) = -\sum_{i,j=1}^{n_1} \gamma_i \left( \frac{\partial J(\tilde{y})}{\partial y_j} \tilde{f}_{ij}(\tilde{x}) \right)^2$ .

In general,  $\dot{J}(\tilde{y})$  does not satisfy condition 1.2). However, it is easy to see that  $\dot{J}(\tilde{y}) = 0$  if and only if  $\nabla J(\tilde{y}) F(\tilde{x}) = 0$ ,

$$\text{where } F(\tilde{x}) = \begin{pmatrix} \tilde{f}_{11}(\tilde{x}) & \dots & \tilde{f}_{n_1 1}(\tilde{x}) \\ \vdots & \ddots & \vdots \\ \tilde{f}_{1 n_1}(\tilde{x}) & \dots & \tilde{f}_{n_1 n_1}(\tilde{x}) \end{pmatrix}.$$

Under the conditions of Theorem 3, the matrix  $F(\tilde{x})$  is nonsingular for all  $\tilde{x}$ , which means  $\dot{J}(\tilde{y}) = 0$  if and only if  $\tilde{y} = y^*$ . Then the practical asymptotic stability can be proved similar to Theorem 1.

3. EXAMPLES

In this section, we consider several examples illustrating the obtained results and some possible extensions. In all the examples, we use extremum seeking controls  $u = (u_1, u_2)^T$  with

$$u_1 = \gamma_1 \sqrt{\frac{\pi}{\varepsilon}} \left( g_1(J(x)) \cos \frac{2\pi t}{\varepsilon} + g_3(J(x)) \sin \frac{2\pi t}{\varepsilon} \right), \\ u_2 = \gamma_2 \sqrt{\frac{\pi}{\varepsilon}} \left( g_2(J(x)) \sin \frac{2\pi t}{\varepsilon} - g_4(J(x)) \cos \frac{2\pi t}{\varepsilon} \right), \quad (7)$$

where  $\gamma_1, \gamma_2 > 0$ , and the functions  $g_i, g_{i+2}$  satisfy (5),  $i = 1, 2$ . We exploit two types of such functions:

$$g_i(z) = \sin z, g_{i+2} = \cos z; \quad (8) \\ g_i(z) = \sqrt{\frac{1-e^{-z/4}}{1+e^{z/4}}} \sin(e^{z/4} + 2 \ln(e^{z/4} - 1)), \quad (9) \\ g_{i+2}(z) = \sqrt{\frac{1-e^{-z/4}}{1+e^{z/4}}} \cos(e^{z/4} + 2 \ln(e^{z/4} - 1)), z > 0,$$

which were introduced in (Scheinker and Krstić (2014)) and (Grushkovskaya et al. (2018b)), respectively. Note that our reason for this is not to compare the performance of these control strategies, but just to illustrate different possibilities for control design.

3.1 Partial stabilization of the Brockett integrator

As the first example, we consider Problem 1 with the extremum seeking system described by the equations

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_2 u_1 - x_1 u_2, \quad (10)$$

and the two cost functions:

$$J_1(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 1)^2, \quad (11)$$

$$J_2(x_1, x_3) = (x_1 - 4)^2 + x_3^2. \quad (12)$$

For the cost function  $J_1(x_1, x_2)$ , one can easily see that the assumptions of Theorem 3 are satisfied since the vector fields  $\tilde{f}_1 = (1, 0)^T$  and  $\tilde{f}_2 = (0, 1)^T$  are linearly independent in  $\mathbb{R}^2$ . For  $J_2(x_1, x_3)$ ,  $\tilde{f}_1 = (1, x_2)^T$  and

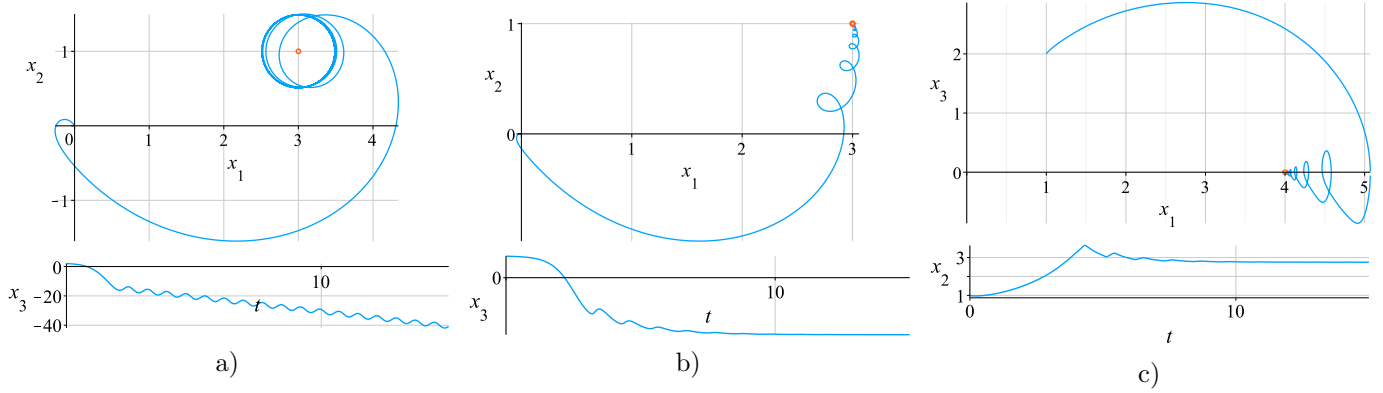


Fig. 1. Projections of the trajectories of system (10) on the  $(x_1, x_2)$ -plane (top) and the graph of  $x_3(t)$  (bottom) with controls (7),(8) (plot a)) and (7),(9) (plots b),c)). In the plots a),b), the cost function is given by (11);  $x(0) = (0, 0, 2)^\top$ ,  $y^* = (3, 1)^\top$ . In the plot c), the cost function is given by (12);  $x(0) = (1, 1, 2)^\top$ ,  $y^* = (4, 0)^\top$ .

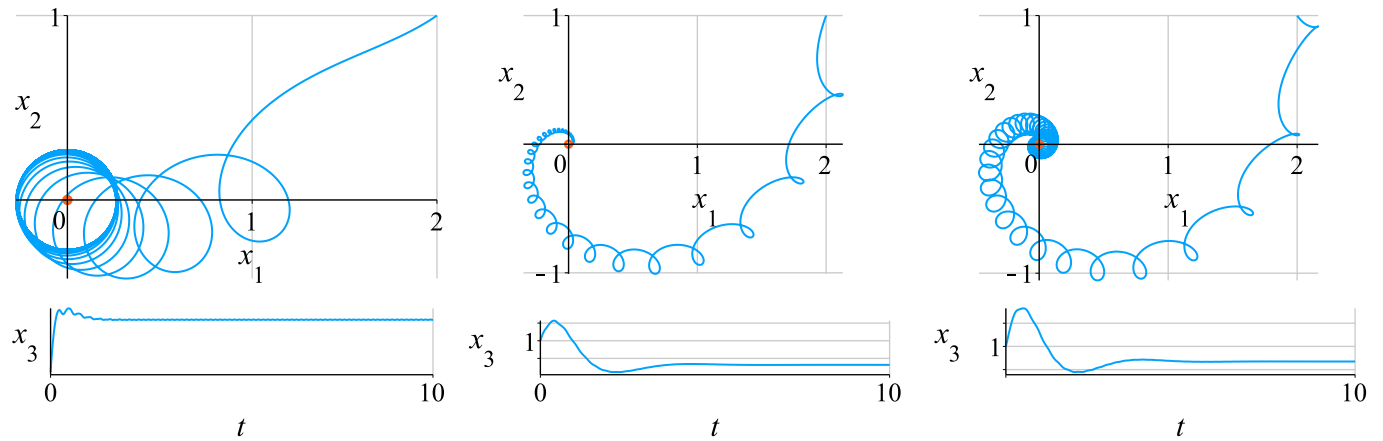


Fig. 2. Projections of the trajectories of system (13) on the  $(x_1, x_2)$ -plane (top) and graph of  $x_3(t)$  (bottom) with controls (7),(8) (plot a)) and (7),(9) (plots b),c)),  $J(x) = x_1^2 + x_2^2$  (a),b)) and  $J(x) = x_1^2 + x_2^2 + x_3^2$  (c)). Here  $A_1 = 1, A_2 = 2, A_3 = 3, \varepsilon = 0.25, x(0) = (2, 1, 1)^\top, y^* = (0, 0)^\top$ .

$\tilde{f}_2 = (0, -x_1)^\top$  are linearly independent if  $x_1 \neq 0$  which can be achieved if  $x_1(0)x_1^* > 0$  and if  $\varepsilon$  is small enough. Note that the boundedness of the vector fields of (10) holds only for controls (7),(9), since in this case it can be proved that  $x_1(t), x_2(t)$  belongs to a compact set for all  $t \geq 0$ .

Fig. 1,a) illustrates the behavior of trajectories of system (10) with the cost function (11) and controls (7),(8),  $\varepsilon = 0.75, \gamma_1 = \gamma_2 = 2$ . In this case, we observe the practical asymptotic stability property. We expect that the use of controls (7),(9) yields the classical asymptotic stability result, similarly to the one obtained in Grushkovskaya et al. (2018b). This property is illustrated in Fig. 1,b). For the cost function (12), the behavior of trajectories of system (10) with controls (7),(9) is shown in Fig. 1,c).

### 3.2 Partial stabilization of a rotating rigid body

As another example, consider the Euler equations describing the rotational motion of a rigid body:

$$\begin{aligned} \dot{x}_1 &= \frac{A_3 - A_2}{A_1} x_2 x_3 + u_1, \quad \dot{x}_2 = \frac{A_1 - A_3}{A_2} x_1 x_3 + u_2, \\ \dot{x}_3 &= \frac{A_2 - A_1}{A_3} x_1 x_2. \end{aligned} \quad (13)$$

Here  $x_1, x_2, x_3$  represent the principal components of the angular velocity vector,  $A_1, A_2, A_3 > 0$  are the main

central moments of inertia, and  $u_1, u_2$  are the control torques. Our goal is to stabilize system (13) along the  $x_3$ -axis, i.e. to  $x_1^* = x_2^* = 0$ , assuming that the cost function is  $J(x) = x_1^2 + x_2^2$ . As in the previous example, we use controls (7), (8), and (9). Then the Lie bracket system for (13) takes the form

$$\begin{aligned} \dot{\bar{x}}_1 &= \frac{A_3 - A_2}{A_1} \bar{x}_2 \bar{x}_3 - 2\bar{x}_1, \quad \dot{\bar{x}}_2 = \frac{A_1 - A_3}{A_2} \bar{x}_1 \bar{x}_3 - 2\bar{x}_2, \\ \dot{\bar{x}}_3 &= \frac{A_2 - A_1}{A_3} \bar{x}_1 \bar{x}_2. \end{aligned} \quad (14)$$

Using the Lyapunov function  $V(\bar{x}) = A_1 \bar{x}_1^2 + A_2 \bar{x}_2^2 + A_3 \bar{x}_3^2$ , one can show that  $\dot{V}(\bar{x}) = -4(A_1 \bar{x}_1^2 + A_2 \bar{x}_2^2)$ . Note that in this case condition 1.1) of Theorem 1 is not satisfied; however, using Corollary 1 we can prove the practical asymptotic attractivity. Furthermore, if  $\max\{A_1, A_2\} < A_3$  (or  $\min\{A_1, A_2\} > A_3$ ), then the conditions of Theorem 1 can be ensured with  $V(x) = \frac{A_1 - A_3}{A_3 - A_2} x_1^2 + \frac{A_2 - A_3}{A_3 - A_1} x_2^2$  (or  $V(x) = \frac{A_1}{A_2 - A_2} x_1^2 + \frac{A_2}{A_1 - A_3} x_2^2$ ) (see Fig. 2,a) and b)).

The proposed techniques for generating partially stabilizing gradient-free controllers can also be used in related problems, e.g., for partial output stabilization of control systems. In particular, assume that in the considered example only the measurements of  $J(x) = x_1^2 + x_2^2 + x_3^2$  are available. Then Corollary 1 implies that the controls (7),

(9) still can be used for steering system (13) to a neighborhood of the set  $\{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$  (see Fig. 2,c).

#### 4. CONCLUSIONS

In this paper, we have addressed the problem of extremum seeking with respect to a part of variables. To obtain practical partial asymptotic stability conditions, we have extended the Lie bracket approximation approach and the methods proposed in Grushkovskaya et al. (2018b) to control-affine systems, whose averaged system has only a partially asymptotically stable equilibrium. The obtained results have been exploited for the design of extremum seeking controllers. Besides, we have illustrated applications of the proposed techniques to partial output stabilization on the rotating rigid body example. In future work, we expect to derive classical (instead of practical) partial asymptotic stability conditions and relax assumptions on the Lyapunov function and the cost. Furthermore, we expect that the proposed approach will be of particular use for synchronization tasks.

#### REFERENCES

- Cochran, J., Siranosian, A., Ghods, N., and Krstic, M. (2009). 3-d source seeking for underactuated vehicles without position measurement. *IEEE Trans. on Robotics*, 25(1), 117–129.
- Dürr, H.B., Krstić, M., Scheinker, A., and Ebenbauer, C. (2017). Extremum seeking for dynamic maps using Lie brackets and singular perturbations. *Automatica*, 83, 91–99.
- Dürr, H.B., Stanković, M.S., Ebenbauer, C., and Johansson, K. (2013). Lie bracket approximation of extremum seeking systems. *Automatica*, 49, 1538–1552.
- Grushkovskaya, V., Michalowsky, S., Zuyev, A., May, M., and Ebenbauer, C. (2018a). A family of extremum seeking laws for a unicycle model with a moving target: theoretical and experimental studies. In *Proc. 2018 European Control Conf.*, 912–917.
- Grushkovskaya, V. and Zuyev, A. (2014). Optimal stabilization problem with minimax cost in a critical case. *IEEE Trans. Autom. Control*, 59(9), 2512–2517.
- Grushkovskaya, V. and Zuyev, A. (2015). Attractors of nonlinear dynamical systems with a weakly monotone measure. *J. of Mathematical Analysis and Applications*, 422(1), 559–570.
- Grushkovskaya, V., Zuyev, A., and Ebenbauer, C. (2018b). On a class of generating vector fields for the extremum seeking problem: Lie bracket approximation and stability properties. *Automatica*, 94, 151–160.
- Guay, M. and Atta, K.T. (2018). Dual mode extremum-seeking control via Lie-bracket averaging approximations. In *Proc. 2018 Annual American Control Conference*, 2972–2977.
- Guay, M. and Dochain, D. (2015). A time-varying extremum-seeking control approach. *Automatica*, 51, 356–363.
- Guay, M. and Zhang, T. (2003). Adaptive extremum seeking control of nonlinear dynamic systems with parametric uncertainties. *Automatica*, 39(7), 1283–1293.
- Haring, M.A. and Johansen, T.A. (2017). Asymptotic stability of perturbation-based extremum-seeking control for nonlinear systems. *IEEE Trans. Autom. Control*, 62(5), 2302–2317.
- Khong, S.Z., Tan, Y., Manzie, C., and Nešić, D. (2014). Multi-agent source seeking via discrete-time extremum seeking control. *Automatica*, 50(9), 2312–2320.
- Kovalev, A., et al. (2009). Novel qualitative methods of nonlinear mechanics and their application to the analysis of multifrequency oscillations, stability, and control problems. *Nonlinear Dynamics and Systems Theory*, 9(2), 117–145.
- Krstić, M. and Ariyur, K.B. (2003). *Real-Time optimization by Extremum Seeking Control*. Wiley-Interscience.
- Krstić, M. and Wang, H.H. (2000). Stability of extremum seeking feedback for general nonlinear dynamic systems. *Automatica*, 36(4), 595–601.
- Malkin, I.G. (1952). *Theory of Stability of Motion*. United States Atomic Energy Commission.
- Mandić, F. and Mišković, N. (2015). Tracking underwater target using extremum seeking. *IFAC-PapersOnLine*, 48(2), 149 – 154.
- Matveev, A.S., Teimoori, H., and Savkin, A.V. (2011). Navigation of a unicycle-like mobile robot for environmental extremum seeking. *Automatica*, 47(1), 85–91.
- Moreau, L. and Aeyels, D. (2000). Practical stability and stabilization. *IEEE Trans. Autom. Control*, 45(8), 1554–1558.
- Nešić, D., Tan, Y., Moase, W.H., and Manzie, C. (2010). A unifying approach to extremum seeking: Adaptive schemes based on estimation of derivatives. In *Proc. 49th IEEE Conf. on Decision and Control*, 4625–4630.
- Rumyantsev, V.V. and Oziraner, A.S. (1987). *Stability and Stabilization of Motion with Respect to Part of the Variables*. Nauka. (in Russian).
- Scheinker, A. and Krstić, M. (2014). Extremum seeking with bounded update rates. *Syst. Cont. Lett.*, 63, 25–31.
- Scheinker, A. and Krstić, M. (2017). *Model-free stabilization by extremum seeking*. Springer.
- Suttner, R. and Dashkovskiy, S. (2017). Exponential stability for extremum seeking control systems. *IFAC-PapersOnLine*, 50(1), 15464–15470.
- Tan, Y., Moase, W.H., Manzie, C., Nešić, D., and Mareels, I.M.Y. (2010). Extremum seeking from 1922 to 2010. In *Proc. 29th Chinese Control Conf.*, 14–26.
- Tan, Y., Nešić, D., and Mareels, I. (2006). On non-local stability properties of extremum seeking control. *Automatica*, 42(6), 889–903.
- Vorotnikov, V.I. (2005). Partial stability and control: The state-of-the-art and development prospects. *Automation and Remote Control*, 66(4), 511–561.
- Vorotnikov, V.I. (2012). *Partial stability and control*. Springer Science & Business Media.
- Zuyev, A. (2003). Partial asymptotic stability and stabilization of nonlinear abstract differential equations. In *Proc. 42nd IEEE Conf on Decision and Control*, volume 2, 1321–1326.
- Zuyev, A. and Grushkovskaya, V. (2017). Motion planning for control-affine systems satisfying low-order controllability conditions. *Int. J. of Control*, 90(11), 2517–2537.
- Zuyev, A.L. (2000). Stabilization of non-autonomous systems with respect to a part of variables by means of control Lyapunov functions. *J. of Automation and Information Sciences*, 4, 18–25.

Appendix A. PROOF OF THEOREM 1

Without loss of generality, assume  $t_0 = 0$ .

For any  $\delta \in (0, \alpha_2^{-1}(\alpha_1(\text{dist}(y^*, \partial D_1))))$ , let  $c_\delta = \alpha_2(\delta)$ ,  $D_0 = \{(y, z) \in \mathbb{R}^n : \|y - y^*\| \leq \delta, z \in D_2\}$ . Then  $D_0 \subseteq D' = \{x : z \in D_2, V(x) \leq c_\delta\} \subset D$ . From Assumption 1, we define

$$\begin{aligned} M_0 &= \sup_{x \in D'} \|f_0(x)\|, M_1 = \sup_{x \in D', 1 \leq i \leq m} \|f_i(x)\| \\ M_2 &= \sup_{x \in D', 0 \leq i, j \leq m} \|L_{f_j} f_i(x)\|, \\ M_3 &= \sup_{\substack{x \in D', 1 \leq i, j \leq m \\ 0 \leq l \leq m}} \|L_{f_l} L_{f_j} f_i(x)\|. \end{aligned} \tag{A.1}$$

For any  $\rho > 0$ , take  $\delta' \in (0, \alpha_2^{-1}(\alpha_1(\rho)))$  and put  $\rho' = \alpha_1^{-1}(\alpha_2(\delta'))$ ,

$$d = \min \{ \rho - \rho', \text{dist}(y^*, \partial D_1) - \alpha_1^{-1}(c_\delta) \} > 0.$$

By the conditions of Theorem 1, if  $z(0) = z^0 \in D_2$  then  $z(t) \in D_2$  for all  $t \geq 0$ . Thus, to ensure that the solutions  $x(t)$  with initial conditions  $x(0) = x^0 \in D'$  are well-defined in  $D$  for  $t \in [0, \varepsilon]$ , it suffices to define  $\varepsilon_0$  as the positive root of the equation  $M_0\varepsilon + M_1W\sqrt{\varepsilon} = d$ . Then, for each  $\varepsilon \in (0, \varepsilon_0)$ ,  $x^0 \in D'$ , and for all  $t \in [0, \varepsilon]$ ,

$$\|y(t) - y^*\| \leq tM_0 + \frac{tM_1W}{\sqrt{\varepsilon}} + \delta' < d + \delta' < \text{dist}(y^*, \partial D_1).$$

The above choice of  $\rho', d$  implies the following properties:

$$V(x^0) \leq \alpha_2(\delta') \Rightarrow \|y^0 - y^*\| < \rho' \Rightarrow \|y(t) - y^*\| < \rho, t \in [0, \varepsilon]. \tag{A.2}$$

To investigate the behavior of  $V(x)$  along the trajectories of system (2), consider the Volterra series expansion of the solution  $x(t)$  of system (2) with an arbitrary initial condition  $x(0) = x^0$  from  $D'$  on the interval  $t \in [0, \varepsilon]$ :

$$\begin{aligned} x(t) &= x^0 + tf_0(x^0) + \frac{1}{\sqrt{\varepsilon}} \sum_{i=1}^m f_i(x^0) \int_0^t w_i\left(\frac{\tau}{\varepsilon}\right) d\tau \\ &+ \frac{1}{\varepsilon} \sum_{i < j} [f_i, f_j](x^0) \int_0^t \int_0^\tau \left( w_j\left(\frac{\tau}{\varepsilon}\right) w_i\left(\frac{s}{\varepsilon}\right) \right. \\ &\left. - w_i\left(\frac{\tau}{\varepsilon}\right) w_j\left(\frac{s}{\varepsilon}\right) \right) dsd\tau + R(t), \end{aligned} \tag{A.3}$$

where

$$\begin{aligned} R(t) &= \int_0^t \int_0^\tau L_{f_0} f_0(x(s)) dsd\tau \\ &+ \frac{1}{\sqrt{\varepsilon}} \sum_{i=1}^m \int_0^t \int_0^\tau \left( L_{f_i} f_0(x(s)) w_i\left(\frac{s}{\varepsilon}\right) + L_{f_0} f_i(x(s)) \right. \\ &\times w_i\left(\frac{\tau}{\varepsilon}\right) \left. \right) dsd\tau + \frac{1}{\varepsilon} \sum_{i=1}^m L_{f_i} f_i(x^0) \int_0^t \int_0^\tau w_i\left(\frac{\tau}{\varepsilon}\right) \\ &\times w_i\left(\frac{s}{\varepsilon}\right) dsd\tau + \frac{1}{\varepsilon} \sum_{i, j=1}^m \int_0^t \int_0^\tau \int_0^s L_{f_0} L_{f_j} f_i(x(p)) w_i\left(\frac{\tau}{\varepsilon}\right) \\ &\times w_j\left(\frac{s}{\varepsilon}\right) dp dsd\tau + \frac{1}{\varepsilon^{3/2}} \sum_{i, j, l=1}^m \int_0^t \int_0^\tau \int_0^s L_{f_l} L_{f_j} f_i(x(p)) \\ &\times w_i\left(\frac{\tau}{\varepsilon}\right) w_j\left(\frac{s}{\varepsilon}\right) w_l\left(\frac{p}{\varepsilon}\right) dp dsd\tau. \end{aligned}$$

In particular, for  $t = \varepsilon$ , representation (A.3) takes the form

$$x(\varepsilon) = x^0 + \varepsilon \left( f_0(x^0) + \sum_{i < j} [f_i, f_j](x^0) \nu_{ij} \right) + R(\varepsilon), \tag{A.4}$$

and from (A.1) the remainder can be estimated as

$$\|R(\varepsilon)\| \leq \varepsilon^{3/2} \left( M_2 + \frac{W^2 m^2 M_3}{6} \right) (\sqrt{\varepsilon} + Wm) = \sigma \varepsilon^{3/2},$$

where  $\sigma = \left( M_2 + \frac{W^2 m^2 M_3}{6} \right) (\sqrt{\varepsilon} + Wm)$  is monotone with respect to  $\varepsilon$ . Next, we apply Taylor's formula to  $V(x(\varepsilon))$ :

$$\begin{aligned} V(x(\varepsilon)) &= V(x^0) + (\nabla V(x^0), x(\varepsilon) - x^0) \\ &+ \frac{1}{2} \sum_{i, j=1}^m \frac{\partial^2 V(x)}{\partial x_i \partial x_j} \Big|_{x=x^0 + \theta(x(\varepsilon) - x^0)} (x_i(\varepsilon) - x_i^0)(x_j(\varepsilon) - x_j^0), \end{aligned}$$

with some  $\theta \in (0, 1)$ . Let  $\mu_1 = \sup_{x \in D'} \|\nabla V(x)\|$ ,  $\mu_2 = 2 \sup_{x \in D'} \left\| \frac{\partial^2 V(x)}{\partial x^2} \right\| (M_0 + M_2 \sum_{i < j} \nu_{ij} + \sqrt{\varepsilon} \sigma)^2$ . Then, from (A.4) and (A.1), we conclude that

$$V(x(\varepsilon)) \leq V(x^0) + \varepsilon L_{\bar{f}} V(x^0) + \varepsilon^{3/2} \sigma \mu_1 + \varepsilon^2 \mu_2.$$

Recall that  $L_{\bar{f}} V(x) \leq -\alpha_3(\|y - y^*\|)$  in  $D$ . Thus, if  $\|y^0 - y^*\| \geq \rho'$  then

$$V(x(\varepsilon)) \leq V(x^0) - \varepsilon \alpha_3(\rho') + \varepsilon^{3/2} \sigma \mu_1 + \varepsilon^2 \mu_2.$$

Let  $\lambda \in (0, \alpha_3(\rho'))$  and let  $\varepsilon_1$  be the smallest positive root of the equation

$$\sqrt{\varepsilon} \sigma \mu_1 + \varepsilon^2 \mu_2 = \alpha_3(\rho') - \lambda.$$

Then

$$V(x(\varepsilon)) \leq V(x^0) - \varepsilon \lambda < V(x^0), \tag{A.5}$$

provided that  $\|y^0 - y^*\| \geq \rho'$ . The last inequality shows that  $x(\varepsilon) \in D'$ , and the solutions  $x(t)$  of system (2) with the initial conditions  $x(0) = x^0 \in D_0 \subset D'$  are well-defined in  $D$  for  $t \in [0, 2\varepsilon]$ . Furthermore, we conclude that there exists an  $N \in \mathbb{N} \cup \{0\}$  such that

$$\begin{aligned} \|y(j\varepsilon) - y^*\| &\geq \rho' \text{ for all } j = 0, \dots, N-1, \\ \text{and } \|y(N\varepsilon) - y^*\| &\leq \rho'. \end{aligned} \tag{A.6}$$

Indeed, assume  $\|y(j\varepsilon) - y^*\| \geq \rho'$  for all  $j \in \mathbb{N} \cup \{0\}$ . Then repeating inequality (A.5), we get  $V(x(N\varepsilon)) \leq V(x^0) - N\varepsilon\lambda$ . With an increase of  $N$ , the right-hand side of the above inequality becomes negative which contradicts  $V(x(N\varepsilon)) \geq 0$ . Thus, there exists an  $N \in \mathbb{N} \cup \{0\}$  such that (A.6) holds.

Estimate (A.2) implies that  $\|y((N+1)\varepsilon) - y^*\| \leq \rho$ . If  $\|y((N+1)\varepsilon) - y^*\| \geq \rho'$ , we apply (A.5) again and obtain

$$V((N+2)\varepsilon) < V((N+1)\varepsilon).$$

Otherwise we have  $\|y((N+2)\varepsilon) - y^*\| \leq \rho$  and repeat the procedure. Taking  $\bar{\varepsilon} = \min\{\varepsilon_0, \varepsilon_1\}$ , we conclude that, for any  $\varepsilon \in (0, \bar{\varepsilon})$ , the solutions of system (2) satisfy the following property:

if  $\|y(0) - y^*\| \leq \delta$  and  $z(0) \in D_2$  then there exists a  $t_1 > 0$  such that  $\|y(t) - y^*\| \leq \rho$  for all  $t \geq t_1$ .

Since  $\rho$  is assumed to be an arbitrary positive number, the practical  $y$ -attractivity has been proved. To prove the practical  $y$ -stability property, for any  $\rho > 0$  we take the  $\delta'$  defined as before. Then, for any  $y^0 \in B_{\delta'}(y^*) \subset B_\rho(y^*)$  and  $z^0 \in D_2$ ,  $V(x^0) \leq \alpha_2(\delta')$ . Summarizing (A.2), (A.5) and the previous argumentation, we conclude with the stability property.