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Gauge Theories

of

Conformal Gravitation

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Zussamenfassung:

In dieser Arbeit untersuchen wir die lokal skaleninvarianten Theorien der konformen und Weyl quadratischen Gravitation und die Beziehungen zwischen ihnen. Mithilfe einer Yang-Mills inspirierten Eichtheorie leiten wir die Wirkungen ab, die diese Theorien beschreiben, und diskutieren ihre Phänomenologien. Besonderes Augenmerk wird auf die physikalischen propagierenden Freiheitsgrade in jeder Theorie gelegt, sowie auf die kosmologischen Implikationen der konformen Gravitation und die dimensionale Transmutation, die in der quadratischen Gravitation von Weyl auftritt. Das Problem der Ostrogradsky Instabilitäten, die aufgrund von Ableitungen vierter Ordnung auftreten, die in beiden Theorien vorhanden sind, wird ebenfalls diskutiert. Schließlich, mithilfe des Verfahrens der sogenannten Ricci-Eichung stellen wir fest, dass die konforme Gravitation als geeichte quadratische Weyl-Schwerkraft angesehen werden kann, und wir spekulieren über die Existenz einer alternativen Wahl der Eichung, die zu einer neuen Theorie führt, die dual zur konformen Gravitation ist.

Abstract:

In this thesis we investigate the locally scale-invariant theories of conformal and Weyl quadratic gravity, and the ensuing relationship between them. Using a Yang-Mills inspired gauge theory perspective, we derive the actions describing these theories and discuss their phenomenologies. Particular focus is put on the physically propagating degrees of freedom in each theory, as well as the cosmological implications of conformal gravity and the dimensional transmutation that occurs naturally in Weyl quadratic gravity. The issue of Ostrogradsky instabilities that arise from the fourth-order derivatives present in both theories is also discussed. Finally, using a process known as Ricci gauging, we find that conformal gravity can be viewed as gauged-fixed Weyl quadratic gravity, and we speculate on the existence of an alternative gauge fixing procedure that leads to a new theory which is dual to conformal gravity.

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Part I

Locally Weyl-Invariant Theories of Gravity

1 Introduction

Despite the countless incredible predictions of modern theoretical physics, there are still many important things that we do not understand about our universe. Perhaps chief among them is gravity's relationship with the quantum world. Over the last couple of centuries, we have steadily gained deeper and deeper insights into the workings of our world at the smallest scales. Indeed, it is not an exaggeration to say that modern particle physics written in the language of quantum field theory is the most well-tested theory ever laid out by humankind. However, the Standard Model of particle physics has absolutely nothing to say about the workings of gravity. Of course that is not to say that we lack an understanding of gravitation in general. Einstein put forth his famous theory of General Relativity over one hundred years ago and we have yet to find any significant deviations from its predictions. As recently as 2012, gravitational waves were detected in precisely the way his theory said they would appear. The fact remains however, that serious theoretical inconsistencies arise when General Relativity is viewed in conjunction with quantum mechanics. When we extrapolate our established understanding of gravity to the smallest scales, contradictions such as physical singularities and infinite probabilities abound. One of the key tenants of quantum field theory is the notion of renormalization, by which we are able to extract meaningful predictions out of a theory that would otherwise exhibit divergences. The trouble is that Einstein's theory as it stands is fundamentally non-renormalizable. It resists description in the language of particle physics, which tells us that our current description is incomplete. We have come to learn that our world is quantized and we have yet to find a description of gravity that reflects this.

As theoretical physicists have always done, we will attempt to remedy our misunderstanding by following the guiding light of symmetry. When we understand the dynamics of a system in one regime then an appropriate symmetry transformation can tell us how it behaves in another. It is our belief that the fundamental symmetries of spacetime itself are not yet properly understood and that we must enlarge the currently known group of acceptable transformations. It was through the strict enforcement of general coordinate invariance that Einstein was able to derive General Relativity, and through the notion of internal $SU(N)$ symmetries that we gained an understanding of the other fundamental forces. It is quite natural to assume that invariance under some additional symmetry will provide the answers that we seek, and since what we seek is a description of all the forces at every scale, symmetry under a change of scale is where we will base our investigations. Of course, while General Relativity has its theoretical problems, it also has many testable predictions that the theory which replaces it must reproduce in the appropriate limits. Fortunately for us, there are many theoretical and experimental bounds imposed by General Relativity and quantum field theory that will allow us to narrow our search for a full description of quantum gravity.

In the first part of this thesis, we will investigate two candidates for an improved description of gravity that are both based on the principle of invariance under local scale transformations. Before getting into the specifics of these theories, we will first present a review of the important aspects of General Relativity and Yang-Mills gauge theory in Sections 1.2 and 1.3. Next, we will go through derivations of our two theories, and discuss their phenomenologies in Chapters 2 and 3. In the second part of this thesis, we will identify a connection between these two theories in Section 4.1 and show in Chapter 5 that they have the potential to work together to further our understanding of gravitation. We will use natural units throughout this work unless specified otherwise. We will also use the metric signature $\{+,-,-,-\}$, and define the Riemann and Ricci tensors "positive" as $R^{\alpha}{}_{\beta\mu\nu} \equiv +\partial_\mu \Gamma^{\alpha}{}_{\beta\nu} + \dots$ and $R_{\mu\nu} \equiv +R^{\alpha}{}_{\mu\alpha\nu}$. Now let us begin by clearly stating our reasons to expect that the universe is insensitive to changes of scale at the fundamental level.

1.1 Motivations

It is an established fact in the world of particle physics that any theory with dimensionless couplings, such as the Standard Model, becomes insensitive to the energy scales imposed by the mass of the particles at high energies. When the momenta of any massive particle becomes very large the mass term in the propagator becomes negligible and the particle in question can be treated as massless.

$$\frac{i}{p^2 - m^2 + i\epsilon} \xrightarrow{p \gg m} \frac{i}{p^2 + i\epsilon} \tag{1.1}$$

As we move into the high energy regime in such a theory, our calculations become immune to the effects of any inherent mass scale - we pick up scale symmetry. However, this process only indicates the presence of an approximate symmetry that becomes stronger as we move into the UV, so why should we expect scale invariance to be exact? We certainly don't see scale invariance in the world around us that is filled with massive particles! If scale invariance is to be realized in nature then it must be spontaneously broken below some threshold energy. This effect is well known to exist in many areas of physics and there is no reason it cannot apply to theories involving gravity as well.

From a theoretical point of view, a local scale symmetry is much more attractive than a global one. Gauged local symmetries are the backbone of modern particle physics where they are used to describe interactions between fundamental forces and matter. On the other hand, global symmetries tend to arise as approximate symmetries of a model, and while they can certainly lead to important predictions, by themselves they do not share quite the same power to describe physical processes on the fundamental level. Their true descriptive power only comes about when we promote them to local symmetries by introducing redundant degrees of freedom in the form of gauge fields.

Since we are already discussing the difference between global and local symmetry, we would be remiss if we didn't mention the complicated relationship between gravity and global charge conservation. Though it is still a fiercely debated topic, it is becoming common knowledge among experts that global symmetries are always violated in the presence of gravity. There are plenty of formal arguments invoking the holographic principle, explicit wormhole metrics, etc., that support this seemingly strange claim, but it is perhaps easier to understand by example. Consider a classic global symmetry such as baryon number and a star with $B \approx 10^{48}$ that has collapsed into a black hole. We need only a semi-classical theory interpretation of gravity to show that the black hole will decay through thermal Hawking radiation. Since all of the baryon number charge is concentrated at the singularity and is non-local to everything outside the horizon, the thermal radiation has no way to receive information about the charge and emits with an average of $B = 0$. Eventually the the black hole will decay in this fashion until it is no longer massive enough to emit 10^{48} baryons, regardless of any theory of quantum gravity that may apply on small scales. Clearly the same logic holds for any type of globally conserved charge. Gauge symmetries on the other hand appear to be safe from this effect since they correspond to fields that exist outside of the event horizon which allow them keep track of the total charge. A formal justification of this statement rapidly gets quite technical, and since that is not the topic of this thesis, suffice to say for now that the experts agree that symmetries must be local in the presence of gravity. We refer the curious reader to works such as [Kallosh et al., 1995] for more information.

Given the powerful descriptive capabilities of local gauge symmetry and nature's apparent predilection towards scale invariance at high energies, we choose to use local scale symmetry as the guiding principle in our search for a complete description of gravity and its interactions with quantum fields. Let us now proceed by establishing what is formally meant by scale symmetry.

1.1.1 Scale, conformal, and Weyl symmetry

The terms scale-, conformal-, and Weyl-symmetric are often used interchangeably throughout the literature, but it is important to be precise about what each of these concepts means. Even if the community at large often conflates these terms without remorse, we can at least differentiate between them in this work.

A local scale transformation is simply the rescaling of the spacetime coordinates x^μ by the local factor $\lambda(x)$.

$$x^\mu \;\; \rightarrow \;\; x'^\mu = \lambda(x) x^\mu \tag{1.2}$$

This transformation is also often referred to as a dilation and its corresponding Abelian symmetry group is usually named $D(1)$. We usually consider quantum field theories that are invariant under the Poincaré group of spacetime symmetries

$ISO(1,3)$ i.e. invariant under translations, and Lorentz transformations, so it would seem natural to extend this symmetry to $G = ISO(1,3) \times D(1)$. Interestingly however, in all but a few special cases (ex. the theory of elasticity in two dimensions [Riva and Cardy, 2005]), QFTs that are invariant under G are also invariant under the full conformal group $SO(2,4)$. It turns out that the only requirements for a G-invariant theory to also be invariant under the conformal group are that the theory is unitary and that it is possible to write the trace of the energy momentum tensor $T_\mu{}^\mu = \partial_\mu \partial_\nu L^{\mu\nu}$ for some operator $L^{\mu\nu}$ [Polchinski, 1988]. The conformal group is special for a number of reasons, but most notably because it represents the largest group of spacetime symmetries that leave both the Maxwell equations and the light cone $ds^2 = 0$ invariant. In some sense, conformal symmetry is the "maximal" amount of spacetime symmetry that we can expect to see in a realistic quantum field theory. Prior to symmetry breaking and the introduction of a Higgs vacuum expectation value, even the Standard Model is conformally invariant at the classical level (up to the Higgs mass term). This means that all of the other forces in nature that we know of are at least classically conformal, so it makes sense that gravity would follow the same pattern.

Conformal transformations can be written as the infinitesimal coordinate transformation $x^\mu \to x^\mu + \varepsilon \xi^\mu$, where ε is an infinitesimal parameter and ξ^μ is a solution to the conformal Killing equations

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{1}{2} \eta_{\mu\nu} \partial^\lambda \xi_\lambda \,. \tag{1.3}$$

There are fifteen solutions to these equations which generate the conformal algebra $\mathfrak{so}(2,4)$. The first ten, which generate the Poincaré algebra, are $\xi^\mu = \alpha^\mu$ for translations and $\xi^\mu = \beta^\mu{}_\nu x^\nu$ for Lorentz transformations. The remaining five correspond to the conformal extension; one generator $\xi^\mu = \gamma x^\mu$ of dilations, and four generators $\xi^\mu = \delta_\nu (\eta^{\mu\nu} x^2 - 2x^\mu x^\nu)$ of special conformal transformations. α^μ, $\beta^{\mu\nu}$, γ, and δ^μ are arbitrary constants. Special conformal transformations are the extra operations that make the conformal group distinct from G. They can be understood as an inversion, followed by a translation, followed by another inversion of the coordinates. With a bit of hand-waving one can picture that this composition of transformations represents a unique conformally invariant operation in the limit $\delta^a \to 0$. With all of this, we can write the infinitesimal conformal transformation of coordinates as

$$x^\mu \;\; \to \;\; x'^\mu = x^\mu + \varepsilon \Big(\alpha^\mu + \beta^\mu{}_\nu x^\nu + \gamma x^\mu + \delta_\nu \big(\eta^{\mu\nu} x^2 - 2x^\mu x^\nu \big) \Big) \,. \tag{1.4}$$

When we perform this operation on a vector, its physical direction will be conserved, but not its length. This differs from the standard Poincaré transformations where both are conserved.

Next, we address the notion of Weyl symmetry. While scale and conformal symmetries are described by fundamental coordinate transformations (transformations that act directly on x^μ) Weyl symmetry acts as a point-wise rescaling of the fields themselves. For some field $\Psi(x)$, a Weyl transformation is defined as

$$\Psi(x) \quad \rightarrow \quad e^{q[\Psi]\omega(x)}\Psi(x)\,. \tag{1.5}$$

Here, $\omega(x)$ is the local scale factor and $q[\Psi]$ is the Weyl weight of the field Ψ. Weyl weight is related to mass dimension in the way that it differs for different fields, but constants (even dimensionful constants) do not transform, so they carry a weight of zero. For the metric this transformation looks like

$$g_{\mu\nu}(x) \quad \rightarrow \quad g'_{\mu\nu}(x) = e^{2\omega(x)}g_{\mu\nu}(x)\,. \tag{1.6}$$

Operator (Ψ)	Weyl Weight ($q[\Psi]$)
$\sqrt{\lvert g\rvert}$	+4
$g_{\mu\nu}$	+2
$g^{\mu\nu}$	-2
$e_{\mu a}$	+1
$e^{\mu}{}_{a}$	-1
∂_μ	0
$\langle\phi\rangle$	0
ϕ	-1
ψ	-3/2
A_μ	0

Table 1.1: Weyl weights of common operators

In theories where there is also a gauge boson associated with the symmetry then that boson will also transform in the adjoint representation of $D(1)$ as part of the Weyl transformation, but we will get to that in Section 3.2. It is important to note that Weyl-invariant theories of curved spacetime are only defined up to some conformal class of metrics since they are insensitive to the replacement $g_{\mu\nu} \rightarrow g'_{\mu\nu}$. In other words, while a theory in standard pseudo-Riemannian space may have a unique metric solution, a Weyl-invariant metric solution is only unique up to the choice of conformal factor $\omega(x)$.

There is an obvious correlation between conformal and Weyl symmetry that can be seen if we perform a conformal transformation on the metric. Since conformal transformations are nothing more than a certain type of diffeomorphism, we can use (1.4) to write them as

$$\begin{aligned} g'_{\mu\nu}(x') &= \frac{\partial x^\mu}{\partial x'^\mu}\frac{\partial x^\nu}{\partial x'^\nu}g_{\mu\nu}(x) = \gamma^2\big(1 - 2\delta_\mu x'^\mu + \delta_\mu\delta^\mu x'_\nu x'^\nu\big)^2 g_{\mu\nu}(x') \\ &= \Omega^2(x')g_{\mu\nu}(x')\,, \qquad \Omega(x') \equiv \gamma\big(1 - 2\delta_\mu x'^\mu + \delta_\mu\delta^\mu x'_\nu x'^\nu\big)\,. \end{aligned} \tag{1.7}$$

Since we now have $g'_{\mu\nu}(x') = \Omega^2(x')g_{\mu\nu}(x')$ we can just rename x' to x and make the identification $e^{2\omega(x)} = \Omega^2(x)$. With this it is easy to see that conformal invariance is just a particular case of Weyl invariance, since Weyl invariance is defined for arbitrary $\omega(x)$ and conformal invariance is only defined for the particular form of $\Omega(x)$ in (1.7). The converse of this statement is not always true, though in practice, many conformal field theories are also Weyl-invariant[1].

The difference between scale, conformal, and Weyl invariance is subtle, but can occasionally be very important. To briefly restate the distinction, scale and conformal transformations are diffeomorphisms that act directly on the coordinates with conformal transformations being the full group of scale, special conformal, and Poincaré transformations. Weyl invariance is not a coordinate transformation, but a transformation that acts directly on the fields. However, conformal transformations can be expressed as Weyl transformations for a certain form of $\Omega(x)$. Thus, Weyl symmetry always implies conformal symmetry.

Now that we have laid out some important definitions and established our motivations for describing a locally scale-invariant theory of gravity, we will take a step back and take a look at some well-established theories that will form the backbone of our work. The first of these will be the most well-tested description of gravity we have to date - General Relativity.

1.2 Overview of General Relativity

For approximately one hundred years, Einstein's theory of General Relativity has been the only widely accepted theory of gravitational interactions. Despite its theoretical drawbacks that we discussed in the previous section, it has been instrumental in our understanding of physics at large scales and has contributed many predictions from the scale of our solar system up to the extra-galactic scales of cosmology. Since we have yet to find any significant deviations from the predictions of GR, it is necessary for any more complicated theory of gravitation to reproduce these predictions in the appropriate limits. For this reason, we present the following overview of the basic principles of General Relativity. We will mostly follow the introductory texts by Amendola [2018] and Zee [2013].

General Relativity is a theory of gravity based around the concept of invariance of physical laws under general coordinate transformations. It is written in the language of differential geometry, so we begin by defining the notion of a differentiable manifold. An n-dimensional manifold $\mathcal{M}$ is nothing more than a set of infinitesimal coordinate patches $\mathcal{U}_i$ that are stuck together to form a global structure.

$$\bigcup_i \mathcal{U}_i = \mathcal{M} \tag{1.8}$$

[1] See Karananas and Monin [2016] for specific examples of conformal theories that are not Weyl-invariant.

We require that each local patch is flat $\mathcal{U}_i \cong \mathbb{R}^n$, but the whole manifold is allowed to have more a more complicated, "curved", structure where $\mathcal{M} \ncong \mathbb{R}^n$. We also define transition functions that tell us how overlapping patches relate to one another and when the whole collection of these functions is differentiable, we have a differentiable manifold. Given a set of coordinates on $\mathcal{M}$ in some basis x^α and some other set of coordinates on the same $\mathcal{M}$ in another basis x'^β, we can transform operators living on $\mathcal{M}$ from one basis to another using the Jacobian matrix $\Lambda^\alpha{}_\beta = \frac{\partial x^\alpha}{\partial x'^\beta}$ or its inverse $\Lambda^\beta{}_\alpha$. We also require that these coordinate transformations be differentiable, in which case they are known as diffeomorphisms. An object that transforms covariantly under diffeomorphisms is called a tensor. For a rank-1 tensor this looks like

$$V'^\alpha = \Lambda^\alpha{}_\beta V^\beta \,. \tag{1.9}$$

While some basis may be more convenient than another when it comes to calculations, we require that physical observables be independent of the choice of coordinates. All physical laws that we derive using this framework must be written in terms of invariant quantities that do not change under a change of coordinate system. Now, if we assign unit vectors $\vec{e}_\alpha$ to each coordinate in some basis (basis vectors), then the standard definition of a vector can be written as $\vec{V} \equiv V^\alpha \vec{e}_\alpha$.

This is a convenient point to introduce one of the most important objects in GR - the symmetric metric tensor $g_{\mu\nu}$. The metric is a rank-2 tensor defined as the inner product of basis vectors, and as such, it allows us compute the inner product of general vectors.

$$g_{\mu\nu} = g(\vec{e}_\mu, \vec{e}_\nu) \equiv \vec{e}_\mu \cdot \vec{e}_\nu \qquad \vec{A} \cdot \vec{B} = \left(A^\mu B^\nu\right)\left(\vec{e}_\mu \cdot \vec{e}_\nu\right) = A^\mu B^\nu g_{\mu\nu} \tag{1.10}$$

The metric (and its inverse $g^{\mu\nu}$) is a very powerful object because its components fully categorize a given basis and it allows us to lower (and raise) indices.

$$A_\mu = g_{\mu\nu} A^\nu \tag{1.11}$$

We define the inner product of two vectors in a given basis as the contraction of indices using the metric that describes that basis.

$$g_{\mu\nu} A^\mu B^\nu = A_\nu B^\nu = \vec{A} \cdot \vec{B} \tag{1.12}$$

Manifolds that are equipped with a metric are known as pseudo-Riemannian manifolds and they are completely described by the metric itself. In the language of differential forms, we say that the metric is a $(0,2)$ tensor, or 2-form, which means that it serves as a function that maps two vectors onto the real number line. It is a covariant tensor while it's inverse $g^{\mu\nu}$ is a contravariant $(2,0)$ tensor that serves as the inverse map. With the metric and inner product we can construct quantities that are basis independent i.e. invariant under coordinate transformations. A particularly important example of an invariant quantity is the line element ds^2 which describes the length of the infinitesimal interval of spacetime. If we include special relativity in the picture, the line element for flat Minkowski space is

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2 && x^\lambda = \{t, x, y, z\}\,, \\ &= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g_{\mu\nu} \Lambda^\alpha{}_\rho dx^\rho \Lambda^\beta{}_\sigma dx^\sigma = g'_{\mu\nu} dx'^\mu dx'^\nu && x'^\lambda = \{t, r, \theta, \phi\}\,, \\ &= dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2\,. \end{aligned} \tag{1.13}$$

In the first line above we expressed the metric and coordinates in the Cartesian basis and in the subsequent lines we performed coordinate transformations on the metric and coordinates to the spherical basis. The transformation matrices commute and drop out since $\Lambda^\mu{}_\alpha \Lambda^\alpha{}_\rho = \delta^\mu_\rho$, so ds^2 is unaffected. This is easy to anticipate if we notice that ds^2 has no free un-contracted indices. It is a scalar quantity and has no indices to be primed or un-primed so it is unaffected by the coordinate transformation. Any operator that has no free indices is invariant under a change of coordinates and is known as a Lorentz scalar.

Moving forward, we investigate the fact that whenever we take the derivative of a vector the derivative of the basis vectors appears as a result of the chain rule.

$$\frac{\partial \vec{V}}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha} V^\beta \vec{e}_\beta = \frac{\partial V^\beta}{\partial x^\alpha} \vec{e}_\beta + V^\beta \frac{\partial \vec{e}_\beta}{\partial x^\alpha} \tag{1.14}$$

The derivative of a basis vector can be nonzero and is itself a vector. For example, consider polar coordinates where $\alpha = \{r, \theta\}$. It is easy to check that

$$\frac{\partial \vec{e}_r}{\partial r} = 0\,, \qquad \frac{\partial \vec{e}_\theta}{\partial \theta} = -r\vec{e}_r\,, \qquad \frac{\partial \vec{e}_r}{\partial \theta} = \frac{1}{r}\vec{e}_\theta\,, \qquad \frac{\partial \vec{e}_\theta}{\partial r} = \frac{1}{r}\vec{e}_\theta\,. \tag{1.15}$$

We write this in a general form as

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\mu{}_{\alpha\beta} \vec{e}_\mu\,, \tag{1.16}$$

where $\Gamma^\mu{}_{\alpha\beta}$ are called the Christoffel symbols. This object does not transform covariantly so it is not a tensor. It represents the μ component of the vector $\partial_\alpha \vec{e}_\beta$. We can use the Christoffel symbols to define a very important operation known as the covariant derivative which gives us the components of the $(1,1)$ tensor $\frac{\partial \vec{V}}{\partial x^\alpha}$.

$$\nabla_\alpha V^\beta \equiv \partial_\alpha V^\beta + V^\mu \Gamma^\beta{}_{\mu\alpha} \tag{1.17}$$

$\nabla_\alpha V^\beta$ transforms as a tensor as a result of its Christoffel term accounting for the non-covariant part of $\partial_\alpha V^\beta$. To be more specific, the Christoffel symbols serve as the connection coefficients that define the Levi-Cevita connection. This is the special torsion-free case of the more general affine connection that exists on any tangent manifold. In the next section we will dig a bit deeper into the concept of a connection. For now it is sufficient to understand that the covariant derivative/Levi-Cevita connection generalizes to tensors of any rank and allows us to create more invariant quantities that contain gradients, divergences, etc. In GR we assume the Christoffel symbols to be metric-compatible which means that

$$\nabla_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - \Gamma^\nu{}_{\alpha\mu} g_{\nu\beta} - \Gamma^\nu{}_{\beta\mu} g_{\alpha\nu} = 0 \,. \tag{1.18}$$

This is also known as the metric-compatibility condition. It is straightforward to derive this result using the definitions provided above, but it is also possible to define a theory of gravity via more general means that is not metric-compatible. In fact, we will see just such a theory in Chapter 3.

It is actually possible to write the Christoffel symbols entirely in terms of the metric if we also assume that the Christoffel symbols are symmetric in their lower indices i.e. assume that $\Gamma^\mu{}_{\alpha\beta} - \Gamma^\mu{}_{\beta\alpha} = 0$. This is known as the torsion-free condition, and just like the metric-compatibility condition, it is merely an assumption that we make in order to simplify calculations. So far, these assumptions fit our observable data but there are plenty of interesting alternative theories of gravity that have metric-incompatible and/or torsionful Christoffel symbols. Using these assumptions it is possible to show that

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu} \left(\partial_\mu g_{\lambda\alpha} + \partial_\alpha g_{\lambda\beta} - \partial_\lambda g_{\alpha\beta} \right) . \tag{1.19}$$

In theories with torsion, $\Gamma^\mu{}_{\alpha\beta}$ and $g_{\mu\nu}$ are independent of each other, but in GR we can write the Christoffel symbols as functions of the metric. Now that we have constructed all of the important mathematical ingredients, let's use them to describe curved spacetime and the motion of objects in a gravitational field. To do this we first need the notion of parallel transport.

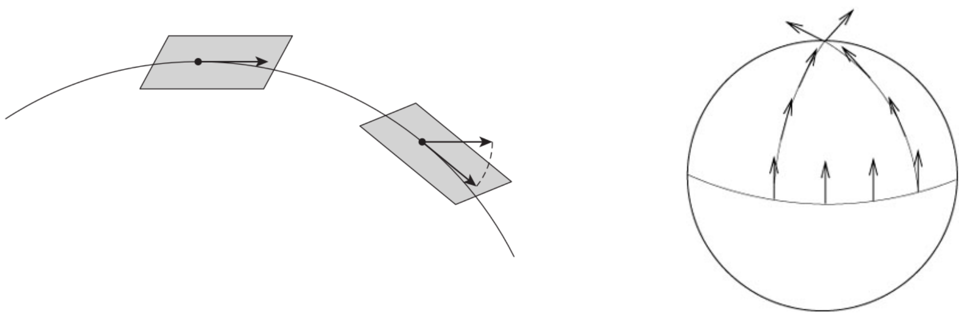

Figure 1.1: Parallel transport of a vector tangent to a curve and the deviation that arises when a vector is transported along a closed path embedded in a manifold with intrinsic curvature [Zee, 2013] [Carroll, 1997]

Parallel transport is the action of continuously transporting a vector along a curve embedded in some manifold in such a way that all of its components remain constant in a locally inertial (flat) frame. An important postulate of GR is that there always exists a transformation to a locally inertial coordinate system. In other words, we can always zoom in on a curved space until the local patch appears to be flat even though globally the manifold may have complicated curvature. On a flat manifold, a vector that is parallel-transported along a closed path will obviously not be affected when it returns to its original position, but this is not the case when the closed path is embedded in a manifold with intrinsic curvature. In this case intrinsic refers to the curvature being an inherent quality of the manifold meaning we don't need to refer to a higher-dimensional space to describe it. If we choose to parallel transport a vector that is tangent to the manifold, then it traces out a particular curve called a geodesic. This is the equivalent of a "straight" line on a curved manifold and is precisely the line in which the tangent vectors are all parallel to each other. This curve is special because, by definition, it represents the shortest path between two points on a manifold. If the manifold is flat then the geodesic is globally a straight line, but it will be something more complicated on a curved manifold.

The concepts of parallel transport and the geodesic equation are important to understand on a deeper level if one wants to do a full treatment of gravitational physics, but we will not be dealing with them directly in this work so we refer the curious reader to the literature mentioned at the beginning of this section for more details. Now, what does all of this have to do with the motion of objects under the influence of gravity? We know from the Principle of Least Action that objects in motion will always choose the unique path that minimizes the difference of their kinetic and potential energy - it turns out that this path is precisely the geodesic on the manifold the object is moving on. Thus we identify spacetime itself as our manifold in question and allow it to have curvature, which allows us to describe the motion of objects under the influence of gravity as motion along a geodesic in an intrinsically curved spacetime. In this language, objects move under the force of gravity in the way they do because it is energetically favorable to move along the geodesic determined by the curvature of spacetime.

At this point we need to quantify the notion of curvature, specifically the curvature at any local point on a manifold. Fortunately for us, there is an object that does precisely this - the Riemann tensor $R^{\alpha}{}_{\beta\mu\nu}$. Forgoing a rigorous derivation for the sake of brevity, we follow Carroll [1997] and simply ask, what precisely do we expect this object to measure? We know that a vector that is parallel-transported around a closed loop on a curved manifold will have altered components. We also know that the covariant derivative of a vector in some direction is a measure of the infinitesimal change to that vector relative to the change of the same vector after being parallel transported. Now, if we take the difference of covariant derivatives of a vector that is parallel transported around an infinitesimal closed loop in both directions, then the result should be a value that depends entirely on the local curvature and is independent of the derivative's direction. What we have just described is nothing more than the commutator of two covariant derivatives. Applying this idea to an arbitrary test vector V^{α}, we define the Riemann tensor as follows.

$$\begin{aligned}[\nabla_\mu, \nabla_\nu]V^\alpha &= \nabla_\mu\nabla_\nu V^\alpha - \nabla_\nu\nabla_\mu V^\alpha \\ &= \left(\partial_\mu\Gamma^\alpha{}_{\beta\nu} - \partial_\nu\Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\lambda\mu}\Gamma^\lambda{}_{\nu\beta} - \Gamma^\alpha{}_{\nu\lambda}\Gamma^\lambda{}_{\beta\mu}\right)V^\beta \end{aligned} \tag{1.20}$$

$$R^{\alpha}{}_{\beta\mu\nu} \equiv \partial_\mu \Gamma^{\alpha}{}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}{}_{\beta\mu} + \Gamma^{\alpha}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\nu\beta} - \Gamma^{\alpha}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\beta\mu} \tag{1.21}$$

The Riemann tensor provides a description of the local intrinsic curvature and $R^{\alpha}{}_{\beta\mu\nu} = 0$ is a sufficient condition for confirming that a given manifold is flat. Since it is defined entirely in terms of the Christoffel symbols, we can also write it in terms of metric which we know is symmetric in its indices. In this formulation is straight forward to derive the following symmetries[2] which often come in handy when doing calculations.

$$R_{\alpha\beta\mu\nu} = R_{\nu\mu\beta\alpha} \qquad R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\nu\mu} \qquad R_{\alpha(\beta\mu\nu)} = 0 \tag{1.22}$$

The Riemann tensor is also subject to the Bianchi identities

$$\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\alpha R_{\beta\lambda\mu\nu} + \nabla_\beta R_{\lambda\alpha\mu\nu} = 0\,, \tag{1.23}$$

which actually apply to curvature defined on any type of smooth manifold, not just the pseudo-Riemannian case we are investigating here. Contracting the first and third indices of the Riemann tensor defines a convenient rank-two symmetric measure of curvature called the Ricci tensor.

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu} \qquad R_{[\mu\nu]} = 0 \tag{1.24}$$

If we perform a contraction of the Ricci tensor using the metric, then we arrive at R, the Ricci scalar.

$$R \equiv g^{\mu\nu} R_{\mu\nu} \tag{1.25}$$

This object is a Lorentz scalar built from the Riemann tensor and metric and it serves as a coordinate invariant measure of curvature all on its own.

Finally we have all the machinery we need to arrive at an expression describing gravitational dynamics. In modern physics, we use the Principle of Least Action to determine equations of motion. Einstein and his contemporaries chose the simplest action that can be created from curvature invariants, which refer to as the Einstein-Hilbert action. Defined in a canonical fashion where M_{pl} is the Planck mass, it looks like

$$S_{EH} = -\int d^4x \sqrt{|g|}\,\frac{M_{pl}}{2} R\,. \tag{1.26}$$

Of course we are free to add an arbitrary constant Λ to this expression as Einstein did - this is the infamous cosmological constant Λ. If we also include an arbitrary matter Lagrangian $\mathcal{L}_m$ then the full action for General Relativity is

$$S_{GR} = \int d^4x \sqrt{|g|}\Big(-\frac{M_{pl}}{2} R + \Lambda + \mathcal{L}_{matter}\Big)\,. \tag{1.27}$$

[2] We will use standard index symmetry notation throughout this work where parentheses (...) indicate total symmetrization and brackets [...] indicate total anti-symmetrization. For example, $T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$ and $T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$.

As usual, setting $\delta S_{GR} = 0$ yields the equations of motion via Noether's theorem and if we define the energy momentum tensor as

$$T_{\mu\nu} \equiv \frac{-2}{M_{pl}\sqrt{|g|}} \frac{\partial\big(\sqrt{|g|}\mathcal{L}_{matter}\big)}{\partial g^{\mu\nu}}\,, \tag{1.28}$$

then we finally arrive at the celebrated Einstein equations.

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \tag{1.29}$$

For a given energy-momentum tensor, solutions to these equations are nothing more than explicit forms of the metric, or in other words, complete descriptions of the spacetime manifold. On the left side of (1.29) we have the Einstein tensor $G_{\mu\nu}$ which describes the energy stored in curvature and the intrinsic energy of spacetime (Λ), while on the right we see the energy contributions from the matter sector encoded in the energy momentum tensor $T_{\mu\nu}$. In the words of Wheeler, *Space tells matter how to move; matter tells space how to curve.*

The two most important solutions to the Einstein equations that we will discuss here describe the effects of gravity around a localized mass and on large scales in an expanding universe. The first of these is known as the Schwarzschild solution.

$$ds^2 = g^{(S)}_{\mu\nu}dx^\mu dx^\nu = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big) \tag{1.30}$$

This metric describes the gravitational effects of a slowly rotating, uncharged mass M for distant observers and can be used to reproduce the kinematics of orbital motion experienced by planets, stars, and even galaxies. It also predicts the existence of black holes and the singularity that appears as $r \to 0$. From this metric we can define the notion of a Schwarzchild radius $r_s \equiv 2M$ where this metric clearly becomes ill-defined. The spherical shell defined at $r = r_s$ is called an event horizon and it marks where any object whose mass is concentrated entirely inside of it will inevitably experience gravitational collapse into a black hole. While it is possible to remove the apparent singularity at $r = r_s$ with a change of coordinates, this point still marks where the spatial coordinates become time-like and the time coordinate becomes space-like, leading to some very interesting physics. More details on black holes and gravitational collapse can be found in any textbook on GR, for example, the classic book by Wald [1984].

There also exists a metric solution to Einstein's equations that forms the backbone of modern cosmology; the Friedmann–Lemaître–Robertson–Walker (FLRW) metric

$$ds^2 = g^{(FLRW)}_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big)\right), \tag{1.31}$$

where $a(t)$ is the global scale factor at time t and k is the curvature at $a(t) = 1$. This metric describes the gravitational influence of a homogeneous and isotropic perfect fluid, which has the well-known energy momentum tensor

$$T_{\mu\nu} = \left(\rho + p\right)U_\mu U_\nu + p g_{\mu\nu}\,, \tag{1.32}$$

where ρ is the energy density, p is the pressure, and U_μ is the four-velocity of the fluid. Inserting the previous two equations into (1.29) gives us the Friedmann equations, which can be conveniently formulated as

$$\left(\frac{H}{H_0}\right)^2 = \Omega_\Lambda + \Omega_k a^{-2} + \Omega_M a^{-3} + \Omega_R a^{-4}\,, \tag{1.33}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, H_0 is the present day Hubble parameter, and Ω_Λ, Ω_k, Ω_M, and Ω_R are the current energy densities of the vacuum energy, spatial curvature, matter (dark and baryonic), and radiation respectively, such that $1 = \Omega_\Lambda + \Omega_k + \Omega_M + \Omega_R$. From this equation cosmologists construct the ΛCDM model, which predicts/describes the Big Bang, cosmic inflation (for $\Omega_\Lambda < 0$), structure formation, and many other important features of the large scale universe. We will get into some more specifics of cosmology with respect to other theories of gravity later on, but we recommend the textbook by Weinberg [2008] for more details on the standard treatment of cosmology.

1.3 Overview of Yang-Mills gauge theory

Modern theoretical physics is built around the formal notion of symmetry, written in the language of group theory and differential geometry. In this section we will give a short overview of how we use symmetry groups and their defining algebras to understand the behavior and interactions of fields, whether they be the quantum fields of the Standard Model, or the classical fields that describe gravitational interactions. This will come in very handy later on because it is only through the lens of these mathematical tools that we are able to discuss theories of gravitation and particle interactions on the same footing. We follow the works by Weigand and Fuchs and Schweigert [2003].

In particle physics, the fields that appear in an action or equation of motion always transform as representations of a Lie algebra. Take for example some generic fermionic matter field $\psi(x)$ whose dynamics are described by the Lagrangian

$$\mathcal{L} = \bar{\psi}(x)\left(i\partial\!\!\!/ - m_\psi\right)\psi(x)\,. \tag{1.34}$$

This Lagrangian is invariant under the transformation $\psi(x) \to U\psi(x)$ where $U = e^{-iq\alpha}$ is an element of the Lie group H, $q \in \mathbb{R}$, and α is a member of the Lie algebra $\mathfrak{h}$. If we define $\alpha \equiv \alpha^a T_a$ where α^a are constant parameters and T_a are the generators of $\mathfrak{h}$, then we say that $\mathcal{L}$ has a global H symmetry. As we discussed in Section 1.1, we expect that all symmetries of a theory containing gravity should be local, so our next step is to promote $\alpha^a \to \alpha^a(x)$. $\mathcal{L}$ as it stands is not invariant under this local version of the symmetry since

$$\partial_\mu \psi(x) \;\; \to \;\; \partial_\mu\big(U(x)\psi(x)\big) = U(x)\big(\partial_\mu\psi(x) - iq\partial_\mu\alpha(x)\big)\,. \tag{1.35}$$

If we enforce invariance then we must also introduce a field that transforms in such a way that it cancels out the $\partial_\mu\alpha(x)$ term in the equation above. This is accomplished by introducing new degrees of freedom in the form of an $\mathfrak{h}$-valued connection and the ensuing covariant derivative

$$\partial_\mu\psi(x) \;\; \to \;\; D_\mu\psi(x) = \big(\partial_\mu + iqA_\mu(x)\big)\psi(x)\,, \tag{1.36}$$

where $A_\mu(x) = A_\mu{}^a(x)T_a$ is a gauge potential that transforms as

$$\begin{aligned} A_\mu(x) \;\; &\to U(x)A_\mu(x)U^{-1}(x) + \frac{i}{q}\partial_\mu U(x)U^{-1}(x) \\ &\qquad\qquad = A_\mu(x) + \partial_\mu\alpha^a(x)T_a - iq\alpha^a(x)\big[T_a, A_\mu(x)\big] + \mathcal{O}\big(\alpha(x)^2\big)\,. \end{aligned} \tag{1.37}$$

With the introduction of this new field it is easy to check that, for small $\alpha^a(x)$, $D_\mu\psi(x) \to U(x)D_\mu\psi(x)$ and thus, $\bar\psi(x)\big(i\not{D} - m_\psi\big)\psi(x)$ is locally invariant under the gauge group H. The $\mathfrak{h}$-valued gauge potential A_μ defines the connection D_μ just as the Christoffel symbols defined the connection ∇_μ that we saw in the last section. So, in the same way that we defined the Riemann tensor as the curvature associated with diffeomorphism invariance in (1.21), we can define the curvature (often called the field strength) associated with local H invariance using the a commutator of connections.

$$F_{\mu\nu} = F_{\mu\nu}{}^aT_a \equiv \frac{1}{iq}\big[D_\mu, D_\nu\big] = \partial_\mu A_\nu - \partial_\nu A_\mu + iq\big[A_\mu, A_\nu\big] \tag{1.38}$$

$F_{\mu\nu}$ transforms in the adjoint representation of H i.e. $F_{\mu\nu} \to UF_{\mu\nu}U^{-1}$, so we can construct a new invariant operator by squaring it, which yields a Lorentz scalar, and by taking the trace whose cyclicity allows the transformations to drop out. Taking the trace also removes the generators since $Tr[T_aT_b] = \frac{1}{2}\delta_{ab}$.

$$\begin{aligned} &Tr\big[F_{\mu\nu}F^{\mu\nu}\big] = Tr\big[F_{\mu\nu}{}^aT_aF^{\mu\nu b}T_b\big] \\ &\to \;\; Tr\big[UF_{\mu\nu}{}^aT_aU^{-1}UF^{\mu\nu b}T_bU^{-1}\big] = \frac{1}{2}F_{\mu\nu}{}^aF^{\mu\nu}{}_a = Tr\big[F_{\mu\nu}F^{\mu\nu}\big] \end{aligned} \tag{1.39}$$

After a canonical normalization we can add this term to $\mathcal{L}$ where it serves as a kinetic term for the gauge fields $A_\mu{}^a$ and yields our complete gauge-invariant Lagrangian.

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}Tr\big[F_{\mu\nu}F^{\mu\nu}\big] + \bar\psi\big(i\not{D} - m_\psi\big)\psi \\ &= -\frac{1}{4}F_{\mu\nu}{}^aF^{\mu\nu}{}_a + \bar\psi\big(i\not{\partial} - m_\psi\big)\psi - q\bar\psi\not{A}^aT_a\psi \end{aligned} \tag{1.40}$$

This is precisely the Yang-Mills-plus-matter Lagrangian that describes the interactions of all the particles and forces in the Standard Model. In this way, the Standard Model is a gauge theory based on the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ where the interactions of quarks and leptons are mediated by dynamical gauge bosons associated with the generators of these Lie groups. Using this same logic it is also possible to go in the other direction. Without considering matter contributions from the start, we can arrive at a Lagrangian that describes the dynamics of a force-carrying boson by specifying its symmetry group, assigning a connection/gauge field to each of the generators, and computing the curvature as a commutator of the connections. Thus if we know a priori, or at least have an educated guess, what the defining symmetry of a force should be then we can describe its dynamics with an action composed solely from the associated curvature.

The strong and electro-weak forces are very well described in these terms, but things get significantly more complicated when we try to include gravity in the same framework. The issue with viewing gravity as a gauge theory is due to the fact that its structure group H is directly related to the spacetime manifold. The structure groups of the standard model are "internal", they are compact unitary groups that we add on after the spacetime manifold has been established. Standard Model gauge theories are formulated in a flat spacetime which is unaffected by their connection, and the curvature ${F_{\mu\nu}}^a$ that describes their dynamics is the curvature on an abstract manifold defined by H. However, since gravity is described via the curvature of spacetime, the structure group corresponds to the tangent space of $\mathcal{M}$. In this case $\mathfrak{h}$ is non-compact and is already defined as soon as we define $\mathcal{M}$; dependence on this type of symmetry group is referred to as a spacetime or "external" symmetry. Gravitational curvature $R_{\mu\nu\rho\sigma}$ is the curvature on $\mathcal{M}$, which is not a separate abstract manifold, but physical spacetime itself.

This self-referenital nature of gravity makes viewing it in the gauge theory picture tricky, but not impossible. Kibble [1961] was among the first to show that one can arrive at a theory of gravity that is at least very similar to General Relativity by gauging the Poincaré group. This is a natural choice for a structure group since it dictates the symmetries of flat Minkowski space that underlies all theories which respect Special Relativity. The generators of the Poincaré algebra are J_{ab} and P_a which correspond to Lorentz transformations and translations i.e. the full set of operations that leave Minkowski space invariant. In order to assign gauge fields to these generators like we did in (1.37), it becomes necessary to change up our formalism a bit since the fields describing gravity in our current formulation, $g_{\mu\nu}$ and ${\Gamma^\lambda}_{\mu\nu}$, do not carry any structure group indices. We replace them with the vierbein (sometimes called the tetrad) ${e^a}_\mu$ and spin connection ${\chi^{ab}}_\mu$ via the following relationships.

$$g_{\mu\nu} = {e_\mu}^a {e_\nu}^b \eta_{ab} \qquad \sqrt{|g|} = e \qquad {\Gamma^\lambda}_{\mu\nu} = {e^\lambda}_a \partial_\mu {e_\nu}^a + {e^\lambda}_a {e_\nu}^b {\chi_\mu}^a{}_b \tag{1.41}$$

The use of these Yang-Mills-style fields is known as the vierbein formalism and we refer the reader to appendix A for more details on its construction. The vierbein formalism allows us to describe gravity via the interaction of the Poincaré gauge fields ${e_\mu}^a$ and ${\chi_\mu}^{ab}$, but we will not go through a full derivation of the resulting theory here since many of the details will be repeated in the next chapter. Suffice to say that Poincaré-gauged theories of gravity reduce to GR in the appropriate limits, though they can deviate when coupled to fermions for example [Karananas, 2016]. However, this is not always considered to be a problem and PGT theories have been studied in great detail. Indeed, many may consider it a mark against GR that it resists a "clean" gauge-theoretical representation. Let us now see if we can use the gauge theory principles discussed here to describe gravitation in a consistent manner with the added requirement that our theory also be conformally invariant.

2 Conformal Gravity

As we discussed in Section 1.1, there is good reason to believe that gravity should be conformally symmetric, at least at the classical level. Since we are now well equipped to describe physical forces as dynamical gauge bosons that correspond to Lie-algebra-valued connections, let's see if we can apply this knowledge to describe gravity. There are actually a few different ways to arrive at the conformal gravity action, but a direct gauging of the Lie group corresponding to conformal symmetry is the most enlightening route in our present context.

2.1 Gauge-theoretical derivation

We begin this section by writing the infinitesimal generators ξ^μ defined in (1.4) as differential operators.

$$\begin{aligned} P_a &= \partial_a & J_{ab} &= \big(x_a\partial_b - x_b\partial_a\big) \\ D &= x_a\partial^a & K_a &= \big(\eta_{ab}x^2 - 2x_a x_b\big)\partial^b \end{aligned} \tag{2.1}$$

Here, P_a corresponds to translations, J_{ab} to Lorentz transformations (rotations), D to dilations, and K_a to special conformal transformations. We use Latin indices to indicate that x^a is a coordinate on the flat tangent manifold. These operators generate the conformal algebra as defined by the following commutation relations.

$$\begin{aligned} \big[P_a, P_b\big] &= 0 & \big[J_{ab}, J_{cd}\big] &= -\eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{bc}J_{ad} + \eta_{bc}J_{ad} \\ \big[D, D\big] &= 0 & \big[K_a, K_b\big] &= 0 \\ \big[P_a, J_{cd}\big] &= \eta_{ab}P_c - \eta_{ac}P_b & \big[P_a, D\big] &= P_a \\ \big[P_a, K_b\big] &= -2\big(J_{ab} + \eta_{ab}D\big) & \big[J_{ab}, D\big] &= 0 \\ \big[J_{ab}, K_c\big] &= -\eta_{ac}K_b + \eta_{bc}K_a & \big[D, K_a\big] &= K_a \end{aligned} \tag{2.2}$$

With this, we may proceed with the Yang-Mills-style procedure presented in the last chapter. To start, we must assign gauge fields to to each generator so that we may define a gauge potential and the resulting curvature. In line with standard Poincaré gauge theory we let the vierbein ${e_\mu}^a$ and spin connection ${\chi_\mu}^{ab}$ account for translations and rotations respectively. Next we define the new fields κ_μ and ${\zeta_\mu}^a$ which will accompany the dilation and SCT generators. With this, the potential for our conformal gauge theory is

$$\mathcal{A}_\mu = {e_\mu}^a P_a + \frac{1}{2}{\chi_\mu}^{ab}J_{ab} + \kappa_\mu D + {\zeta_\mu}^a K_a\,, \tag{2.3}$$

where the $\frac{1}{2}$ is inserted as a normalization convention. Now we can define our conformal curvature in the usual way.

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i\big[\mathcal{A}_\mu, \mathcal{A}_\nu\big] \tag{2.4}$$

Expanding this expression and grouping the result in terms of the generators yields

$$\mathcal{F}_{\mu\nu} \equiv \mathcal{P}_{\mu\nu}{}^a P_a + \frac{1}{2}\mathcal{J}_{\mu\nu}{}^{ab} J_{ab} + \mathcal{D}_{\mu\nu} D + \mathcal{K}_{\mu\nu}{}^a K_a\,, \tag{2.5}$$

where we have defined the following curvatures corresponding to each generator [Manolakos et al., 2019].

$$\mathcal{P}_{\mu\nu}{}^a = -\big(\partial_\mu e_\nu{}^a - \chi_{\mu b}{}^a e_\nu{}^b\big) + \big(\partial_\nu e_\mu{}^a - \chi_{\nu b}{}^a e_\mu{}^b\big) + \big(e_\mu{}^a \kappa_\nu - e_\nu{}^a \kappa_\mu\big) \tag{2.6}$$

$$\begin{aligned} \mathcal{J}_{\mu\nu}{}^{ab} &= -\partial_\mu \chi_\nu{}^{ab} + \partial_\nu \chi_\mu{}^{ab} + \chi_{\mu c}{}^a \chi_\nu{}^{cb} - \chi_{\nu c}{}^a \chi_\mu{}^{cb} \\ &\qquad - 2\big(e_\mu{}^a \zeta_\nu{}^b - e_\mu{}^b \zeta_\nu{}^a\big) + 2\big(e_\nu{}^a \zeta_\mu{}^b - e_\nu{}^b \zeta_\mu{}^a\big) \end{aligned} \tag{2.7}$$

$$\mathcal{D}_{\mu\nu} = -\partial_\mu \kappa_\nu + \partial_\nu \kappa_\mu + 2\big(e_{\mu a}\zeta_\nu{}^a - e_{\nu a}\zeta_\mu{}^a\big) \tag{2.8}$$

$$\mathcal{K}_{\mu\nu}{}^a = -\big(\partial_\mu \zeta_\nu{}^a - \chi_{\mu b}{}^a \zeta_\nu{}^b\big) + \big(\partial_\nu \zeta_\mu{}^a - \chi_{\nu b}{}^a \zeta_\mu{}^b\big) + \big(\zeta_\mu{}^a \kappa_\nu - \zeta_\nu{}^a \kappa_\mu\big) \tag{2.9}$$

Now the procedure dictates that we construct a Lagrangian from Lorentz scalars built from of these curvature tensors. We must restrict ourselves to quadratic terms in order to avoid introducing any dimensionful coupling constants (which would break the global symmetry), and if we also enforce parity conservation by neglecting the pseudo-tensor options [Kaku et al., 1977], then it turns out that the only possible action is

$$S_{CG} = \frac{\alpha}{4}\int d^4x|e|\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abcd}\mathcal{J}_{\mu\nu}{}^{ab}\mathcal{J}_{\rho\sigma}{}^{cd}\,, \tag{2.10}$$

where α is a dimensionless constant and ε is the totally anti-symmetric Levi-Cevita symbol. We can recognize the first line of (2.7) is just the Riemann tensor written in the vierbein formalism [Yepez, 2011] which we define here as $\mathcal{R}_{\mu\nu}{}^{ab} \equiv \partial_\mu \chi_\nu{}^{ab} - \partial_\nu \chi_\mu{}^{ab} + \chi_{\mu c}{}^a \chi_\nu{}^{cb} - \chi_{\nu c}{}^a \chi_\mu{}^{cb}$. After expanding out all the terms and dropping a total derivative[1], we arrive at the following.

$$\begin{aligned} S_{CG} = 16\alpha \int d^4x|e|\Big(&-\mathcal{R}^{\mu\lambda ab}e_{\lambda a}\big(e_{\nu b}\zeta_{\mu c}e^{\nu c} - \frac{1}{2}e_{\mu b}\zeta_{\nu c}e^{\nu c}\big) \\ &+ 2\zeta_{\mu a}\big(e^{\mu a}\zeta_{\nu b}e^{\nu b} - e^{\nu a}\zeta_{\nu b}e^{\mu b}\big)\Big) \\ = 16\alpha \int d^4x|e|\Big(&-\mathcal{R}^{\mu\nu}\zeta_{\mu\nu} - \frac{1}{2}\mathcal{R}\zeta_\nu{}^\nu + 2\big(\zeta_\mu{}^\mu \zeta_\nu{}^\nu - \zeta_{\mu\nu}\zeta^{\mu\nu}\big)\Big) \end{aligned} \tag{2.11}$$

[1] This term is in fact the Gauss-Bonnet invariant. See Section 3.1 for more details.

This expression is a bit unattractive, but we can simplify it by enforcing invariance under the gauge transformations. Kaku et al. [1977] showed that we can only achieve invariance under the full conformal group if we are able write ${\chi_\mu}^{ab}$ in terms of ${e_\mu}^a$ and κ_μ by setting ${\mathcal{P}_{\mu\nu}}^a = 0$. Solving this constraint for ${\chi_\mu}^{ab}$ yields

$$\begin{aligned}{\chi_\mu}^{ab} =& -\frac{1}{2}\Big(e^{\nu a}\big(\partial_\mu {e_\nu}^b - \partial_\nu {e_\mu}^b\big) + e^{\lambda a}e^{\nu b}(\partial_\nu e_{\lambda c}){e_\mu}^c\Big) \\ &-\frac{1}{2}\Big(a \leftrightarrow b\Big) - \big({e_\mu}^a\kappa^b - {e_\mu}^b\kappa^a\big)\,. \end{aligned} \tag{2.12}$$

Setting the translation curvature ${\mathcal{P}_{\mu\nu}}^a$ to zero is well motivated by considering other gauge theories of gravity where it appears naturally as the equation of motion for ${\chi_\mu}^{ab}$. It makes sense to consider this a necessary condition if our theory is to resemble Einstein gravity in the low energy limit. Next, we note that while ${e_\mu}^a$ has a kinetic term buried inside of ${\chi_\mu}^{ab}$, there is no kinetic term for ${\zeta_\mu}^a$ present in the action. No derivatives of it were introduced because quadratic ${\mathcal{K}_{\mu\nu}}^a$ terms generate pseudo-scalars which do not respect parity. In reality, these constraints perhaps make even more sense if we follow the logic in the opposite direction. This theory would presumably have a hard time reducing to Einstein gravity in any limit if it contained an extra gauge boson in the spectrum. For this reason we could assume that ${\zeta_\mu}^a$ is non-dynamical and pick up parity invariance as a bonus. In any case, this all boils down to the fact that in this theory we can solve for ${\zeta_\mu}^a$ algebraically using its equation of motion and eliminate it from the action.

$$\delta_\zeta S_{CG} = \zeta_{\mu a} + \frac{1}{8}\big(\mathcal{R}_{\mu\nu ba}e^{\nu b} - \frac{1}{6}\mathcal{R}_{\nu\lambda bc}e^{\lambda b}e^{\nu c}e_{\mu a}\big) = 0 \tag{2.13}$$

$$\Rightarrow\ {\zeta_\mu}^a = -\frac{1}{8}\big({\mathcal{R}_\mu}^a - \frac{1}{6}\mathcal{R}{e_\mu}^a\big) \tag{2.14}$$

If we integrate ${\zeta_\mu}^a$ out by inserting this expression back into the action (2.11), we see a dramatic simplification.

$$\begin{aligned} S_{CG} &= 2\alpha\int d^4x|e|\Big(\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} - 4\big(\partial_\mu\kappa_\nu\partial^\mu\kappa^\nu - \partial_\mu\kappa_\nu\partial^\nu\kappa^\mu\big) - \frac{1}{3}\mathcal{R}^2\Big) \\ &= 2\alpha\int d^4x|e|\Big(\mathcal{R}_{\mu\nu}\mathcal{R}^{\nu\mu} - \frac{1}{3}\mathcal{R}^2\Big) \\ &= \frac{\alpha}{4}\int d^4x|e|\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abcd}{\mathcal{C}_{\mu\nu}}^{ab}{\mathcal{C}_{\rho\sigma}}^{cd} \end{aligned} \tag{2.15}$$

Here we have defined ${\mathcal{C}_{\mu\nu}}^{ab}$ to be the vierbein-language-version of the famous Weyl tensor $C_{\mu\nu\rho\sigma}$ which is defined in the metric formalism as

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \big(g_{\mu[\rho}R_{\nu]\sigma} - g_{\mu[\sigma}R_{\nu]\rho}\big) + \frac{1}{3}\big(g_{\mu[\rho}g_{\nu]\sigma}\big)R\,. \tag{2.16}$$

The Weyl tensor is essentially just the Riemann tensor with all of its contractions subtracted out. It has all the same index symmetries as the Riemann tensor, but its trace with respect to any index vanishes. Most importantly though, it is the only combination of the standard curvature tensors that is totally invariant under conformal transformations, thus it should not be a surprise to see it appear in our theory.

Now, the $\mathcal{R}_{\mu\nu}$ in (2.15) is obviously not symmetric, but we can make it so with the convenient gauge fixing condition $\kappa_\mu = 0$ [Manolakos et al., 2019]. If we also assume a torsionless connection, then it is straightforward to show that not only does $\mathcal{R}_{\mu\nu}$ become symmetric, but also that $\mathcal{R}_{\mu\nu}{}^{ab} = R_{\mu\nu\rho\sigma}e^{\rho a}e^{\sigma b}$, $\mathcal{R}_{\mu}{}^{a} = R_{\mu\nu}e^{\nu a}$, $\mathcal{R} = R$, and $\mathcal{C}_{\mu\nu}{}^{ab} = C_{\mu\nu\rho\sigma}e^{\rho a}e^{\sigma b}$. Putting everything together back in the standard metric formulation, we see that our gauge theory of the conformal group is described simply by the square of the Weyl tensor.

$$
\begin{aligned}
S_{CG} &= \frac{\alpha}{4}\int d^4x|e|\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}C_{\mu\nu\alpha\beta}C_{\rho\sigma\gamma\delta} \\
&= -\alpha\int d^4x\sqrt{|g|}C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}
\end{aligned} \tag{2.17}
$$

We have finally succeeded in creating a conformally (and thus also Weyl) invariant description of gravity using gauge theory and the mathematical notion of symmetry, but since we are physicists and not mathematicians, the job is not done. We must now investigate the physical implications of our theory.

2.2 Phenomenology of CG

We begin this section by considering the cosmological ramifications that arise from coupling matter to our theory of conformal gravity. Assuming we do not wish to include and non-Standard Model fields, the matter portion of our total action $S_{Total} = S_{CG} + S_M$ must look like

$$
\begin{aligned}
S_M = \int d^4x\sqrt{|g|}\Big(&-\frac{1}{2}\nabla_\mu\Phi^\dagger\nabla^\mu\Phi + \frac{1}{12}\Phi^\dagger\Phi R - \lambda\big(\Phi^\dagger\Phi\big)^2 \\
&- \bar{\psi}_i\big(\delta_{ij}\not{D} - Y_{ij}\Phi\big)\psi_j\Big),
\end{aligned} \tag{2.18}
$$

where Φ is the Higgs field, R is the Ricci scalar, λ is the Higgs self-coupling, ψ_i is a fermion of generation i, and Y_{ij} are the Yukawa couplings. There is no Higgs-Ricci scalar coupling constant because the exact factor of $\frac{1}{12}$ is necessary to ensure conformal invariance. Indeed, the whole action S_{Total} is conformally invariant.

Now, in order to make a comparison with standard cosmology, we need the equivalent of the Einstein equations for conformal gravity. In this case the variation of the gravitational sector yields not the Einstein tensor, but the traceless and conformally invariant Bach tensor $B_{\mu\nu}$.

$$\frac{1}{\sqrt{|g|}}\frac{\delta S_{CG}}{\delta g^{\mu\nu}} = 4\alpha\Big(\nabla^\alpha\nabla^\beta C_{\mu\alpha\nu\beta} + \frac{1}{2}C_{\mu\alpha\nu\beta}R^{\alpha\beta}\Big) \equiv 4\alpha B_{\mu\nu} \tag{2.19}$$

Just as we do in GR, we define the energy momentum tensor as the variation of the matter action

$$\begin{aligned} T_{\mu\nu} &\equiv -\frac{1}{\sqrt{|g|}}\frac{\delta S_M}{\delta g^{\mu\nu}} \\ &= \frac{2}{3}\nabla_\mu\Phi^\dagger\nabla_\nu\Phi - \frac{1}{3}\Phi^\dagger\nabla_\mu\nabla_\nu\Phi - \frac{1}{6}g_{\mu\nu}\nabla_\lambda\Phi^\dagger\nabla^\lambda\Phi + \frac{1}{3}g_{\mu\nu}\Phi^\dagger\nabla_\lambda\nabla^\lambda\Phi \\ &\quad - \frac{1}{6}\Phi^\dagger\Phi\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\Big) - \lambda g_{\mu\nu}\big(\Phi^\dagger\Phi\big)^2 + i\bar{\psi}_i\gamma_\mu D_\nu\psi_i\,, \end{aligned} \tag{2.20}$$

where we have eliminated the Yukawa term by inserting the fermion equation of motion $\not{D}\psi_i = Y_{ij}\Phi\psi_j$ in the same way as Roberts et al. [2017]. This expression can be greatly simplified by a conformal transformation to unitary gauge where $\Phi = \langle\Phi\rangle = const$ and by recognizing the Einstein tensor $G_{\mu\nu}$ in the second line.

$$T_{\mu\nu} = -\frac{1}{6}\langle\Phi\rangle^2 G_{\mu\nu} - g_{\mu\nu}\lambda\langle\Phi\rangle^4 + i\bar{\psi}_i\gamma_\mu D_\nu\psi_i \tag{2.21}$$

Finally, we arrive at the Bach equation by using the fact that $\delta S_{Total} = 0$.

$$4\alpha B_{\mu\nu} = T_{\mu\nu} \tag{2.22}$$

Now, if we want to use this equation to build a cosmological model, then we need to make the crucial assumption that the fermionic matter can be modeled as a perfect fluid on cosmological scales. Mannheim [1990] presents a convincing argument as to why this is justified. It essentially boils down to the requirement that fermion masses are only generated dynamically via spontaneous symmetry breaking, which we already know to be a feature of the Standard Model. This allows us to perform an incoherent averaging over the directions of the fermion momentum and exchange the fermion term for the energy-momentum of a perfect fluid as in (1.32) and write

$$4\alpha B_{\mu\nu} = -\frac{1}{6}\langle\Phi\rangle^2 G_{\mu\nu} - \lambda\langle\Phi\rangle^4 g_{\mu\nu} + \big(\rho + p\big)U_\mu U_\nu + p g_{\mu\nu}\,. \tag{2.23}$$

Since we now have a homogeneous and isotropic universe on our hands, we would like to implement the FLRW metric as a solution. It is in fact easy to check that it is a solution by noting that $g_{\mu\nu}^{(FLRW)}$ is a conformally flat metric. Under the coordinate transformation

$$t \to \tau(t) = \int \frac{dt}{a(t)}\,, \tag{2.24}$$

we get $dt = a(t)d\tau$ (provided that $a(t)$ is well-behaved) and

$$g'^{(FLRW)}_{\mu\nu} dx'^{\mu} dx'^{\nu} = a^2(\tau)\left(d\tau^2 - \frac{dr^2}{1-kr^2} - r^2\left(d\theta^2 - \sin^2\theta d\phi^2\right)\right). \tag{2.25}$$

Here, $a(\tau)$ serves as the conformal factor $\Omega(x)$ proving that $g^{(FLRW)}_{\mu\nu}$ is indeed conformally flat. Since the Bach tensor is conformally invariant, it disappears when computed with a conformally flat metric $B_{\mu\nu}(g^{(FLRW)}_{\mu\nu}) = 0$. This allows us to reformulate the Bach equations as

$$G_{\mu\nu} = \frac{6}{\langle\Phi\rangle^2}\left((\rho+p)U_\mu U_\nu + \left(p - \lambda\langle\Phi\rangle^4\right)g_{\mu\nu}\right), \tag{2.26}$$

which is nothing more than a modified version of the Einstein equations with $\Lambda = \lambda\langle\Phi\rangle^4$ and $6/\langle\Phi\rangle^2$ as a gravitational coupling constant. There is no need to insert a cosmological constant by hand as in GR - it has been generated directly from the Higgs vacuum energy. However, even though we get Λ for free in this cosmology (after including the Higgs), we are unfortunately not exempt from a fine-tuning problem. Instead of setting $\Lambda \approx 10^{-120}$ by hand order to fit known data, here we are forced to set $\lambda \approx -10^{-176}$ [Roberts et al., 2017].

Our next step is to insert $G_{\mu\nu}(g^{(FLRW)}_{\mu\nu})$ and find the corresponding equivalent to the Friedmann equations. After normalizing them to the standard Friedmann equations, we find

$$\left(\frac{H}{H_0}\right)^2 = -\frac{3}{4\pi G\langle\Phi\rangle^2}\left(\Omega_{\Phi_0} + \Omega_M a^{-3} + \Omega_R a^{-4}\right) + \Omega_k a^{-2}. \tag{2.27}$$

This equation shares many of the same features as (1.33), but has some key differences. In this cosmological model, the gravitational effects of matter and radiation become repulsive at large scales and essentially remove the need to invoke dark energy by hand. Additionally, if we solve this equation for $a(t)$ and trace it backwards, we find that the scale factor has a minimum non-zero (though very small) value, indicating that there is no initial singularity present at $t = 0$ [Mannheim, 2006].

Another very attractive feature of conformal gravity is that it appears to negate the need for dark matter, at least on the level of galaxies. The reason for this stems from the fact that the Bach equation (2.22) contains four derivatives of the metric instead of just the two present in the Einstein equation. While the Bach equation admits the standard Schwarzschild solution (1.30), this is actually just a simplified limit in a broader class of solutions for the case where $T_{\mu\nu}$ represents a static, spherically symmetric (and possibly slowly rotating) matter source.

$$ds^2 = B(r)dt^2 - B^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.28}$$

$B(r)$ is in general a complicated function that depends on the matter distribution $\rho(r)$ as

$$\nabla^4 B(r) = \frac{3}{4\alpha B(r)}\left(T^0{}_0 - T^r{}_r\right) \equiv \rho(r). \tag{2.29}$$

When we solve this differential equation for $r > R$ where R is the radius of our matter distribution, we find

$$B(r > R) = 1 - \frac{\beta}{r} + \gamma r \ , \qquad \beta = \frac{1}{6}\int_0^R dr' r'^4 \rho(r') \quad \gamma = -\frac{1}{2}\int_0^R dr' r'^2 \rho(r') \, , \tag{2.30}$$

and so the gravitational potential in conformal gravity picks up an additional term [Mannheim and O'Brien, 2013].

$$V_{CG}(r > R) = -\frac{MG}{r} + \frac{\gamma r}{2} \tag{2.31}$$

This extra term means that Newton's shell theorem is no longer valid in conformal gravity. An object in orbit around a spherical mass M now experiences different gravitational fields for different distributions of M. The effects of external homogeneous gravitational sources also no longer cancel out, which means that the energy distribution of the whole universe now contributes to the gravitational field felt by orbiting objects. When r is small, the linear term becomes negligible and we recover the Schwarzschild solution, Newtonian potential, and all of our familiar solar-system-scale orbital mechanics. However, for processes such as galactic structure formation where r is large, this term becomes very important.

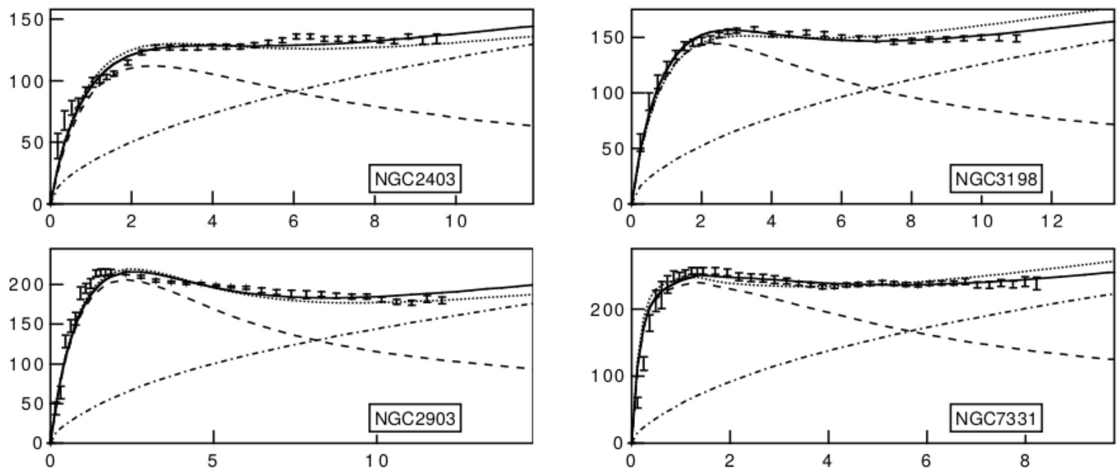

Figure 2.1: Rotation curves for four different galaxies with velocity (km/s) on the y-axis and distance from the galactic center (R/R_0 normalized for each galaxy) on the x. The solid line represents the full conformal gravity prediction, while the dashed and dot-dashed lines represent the Newtonian and linear correction terms respectively [Mannheim, 1997].

In standard cosmology, we are usually forced to invoke the existence of dark matter in order to match experimental data with the predicted rotation curves for stars in orbit around a galactic center. The extra mass present in a dark matter halo is a nice way to explain the higher-than-expected orbital velocity of stars that would normally be moving too quickly to be constrained by conventional gravity. However, when we fit experimental data to the predictions of conformal gravity, we find that the second term in equation (2.31) ends up performing the same function as dark matter, without the need to introduce any new particles. There have been many studies on conformal gravity's potential to replace our need for dark matter and dark energy. Indeed, it seems that the type of fits shown in Figure 2.1 tend to hold for most, if not all, of the available experimental data [O'Brien et al., 2018]. However, there has also been criticism regarding conformal cosmology. Apparently, the model presented here does not always outperform ΛCDM when it comes to cosmic expansion [Roberts et al., 2017]. Additionally, it is not clear at this point what the conformal gravity has to say regarding gravitational waves, the CMB power spectrum, or nucleosynthesis. Many aspects of this theory are still up for debate; this is a very active field of research and there is a lot more work that needs to be done.

Now, enough about cosmology, how does conformal gravity fit into the particle physics framework? The short answer is, despite some long-held reservations it has recently begun to show extreme promise. Due to its lack of dimensionful coupling constants, conformal gravity appears to be power-counting renormalizable, just like the Standard Model. This of course means that even though superficially divergent diagrams can be drawn at any loop-order, they can always be canceled by a finite amount of counter terms. When attempting to quantize Einstein gravity we run into a power series of the metric in powers of the dimensionful gravitational constant, which inevitably leads to an infinite amount of divergent diagrams at every order. In conformal gravity, we can choose to instead expand the metric in powers of $\hbar$, just as we do for quantum fields in the Standard Model. In fact, it is thought to be the quantization process itself that leads to the spontaneous breaking of conformal symmetry. There is no way to define curvature in a theory of free conformal gravity, since all of the metric solutions are conformally flat; there are no length scales to define a curvature with. Length scales are inevitably introduced by commutation relations of the form $[\phi(x,t), \pi(x',t)] = i\hbar\delta^3(x-x')$, and so the presence of such a commutator naturally generates commutators of a similar form on the gravitational side, which in turn allows for the presence of non-zero curvature. In the conformal gravity picture, only through the coupling to quantized matter fields does the curvature become non-zero [Mannheim, 2006]. As we might expect from the well-known conformal anomaly, only the classical theory is conformally invariant - the quantized version cannot be. Now, while all of this seems to indicate that conformal gravity could serve as a UV-complete theory of quantum gravity, there are also some significant theoretical challenges that arise when we attempt to quantize a theory that contains fourth-order derivatives.

2.2.1 The ghost problem

It is well-known that theories with higher time derivative (such as conformal gravity) will propagate more than the two degrees of freedom (DOFs) corresponding to the massless spin-2 graviton in standard Einstein gravity. There are a few ways to go about counting the extra DOFs in a higher derivative theory, but the following method laid out by Riegert [1984] is particularly enlightening.

We begin by writing the metric in terms of small perturbation $h_{\mu\nu}$ on a flat background $\eta_{\mu\nu}$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{2.32}$$

in order to arrive at the linearized conformal gravity Lagrangian

$$\begin{aligned}\mathcal{L}_{CG} = \alpha\Big(&-\frac{2}{3}h^{\mu\nu}\partial_\mu\partial_\nu\partial_\rho\partial_\sigma h^{\rho\sigma} + 2h^{\mu\nu}\partial_\nu\partial_\rho\partial_\sigma\partial^\sigma h_\mu{}^\rho - \frac{2}{3}h^\mu{}_\mu\partial_\nu\partial_\rho\partial_\sigma\partial^\sigma h^{\nu\rho} \\ &- h^{\mu\nu}\partial_\rho\partial^\rho\partial_\sigma\partial^\sigma h_{\mu\nu} + \frac{1}{3}h^\mu{}_\mu\partial_\rho\partial^\rho\partial_\sigma\partial^\sigma h^\nu{}_\nu\Big)\,.\end{aligned} \tag{2.33}$$

Next, we make another decomposition and write $h_{\mu\nu}$ in terms of its trace $h_\mu{}^\mu$ and traceless part $\bar{h}_{\mu\nu}$.

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{4}h_\mu{}^\mu\ , \qquad \bar{h}_\mu{}^\mu = 0 \tag{2.34}$$

Interestingly, when we compute the variation of $\mathcal{L}_{CG}$, it turns out that all of the $h_\mu{}^\mu$ terms end up canceling.

$$\delta S_{CG} = \partial^4\bar{h}_{\mu\nu} + 2\partial_{(\mu}V_{\nu)} - \frac{1}{2}\eta_{\mu\nu}\partial_\lambda V^\lambda = 0\ , \quad V_\mu \equiv \frac{1}{3}\partial_\mu\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} - \partial^2\partial^\lambda\bar{h}_{\mu\lambda} \tag{2.35}$$

It is not difficult to show that this whole equation of motion is traceless, which is in fact a feature of any conformal field theory at the classical level. We can simplify this expression by exploiting its invariance under diffeomorphisms $x^\mu \to x^\mu + \xi^\mu(x)$. Performing this transformation on V^μ yields

$$V^\mu \ \rightarrow\ V'^\mu = V^\mu - \partial^4\xi^\mu\,, \tag{2.36}$$

and if we fix our gauge freedom by selecting ξ^μ such that $V'^\mu = 0$, our equation of motion becomes simply

$$\partial^4\bar{h}'_{\mu\nu} = 0\,. \tag{2.37}$$

The choice $V'^\mu = 0$ is known as the conformal gauge and it is essentially just the higher-derivative analogue of the harmonic gauge, which is the standard choice used when describing gravitational radiation in Einstein gravity. From here on out we will drop the primes in our notation for the sake of clarity, but we should remember that we are in a gauge-fixed scenario.

The general solution to (2.37) is

$$\bar{h}_{\mu\nu} = \big(A_{\mu\nu} + B_{\mu\nu} n_\alpha x^\alpha\big) e^{ik_\beta x^\beta} + \text{c.c.}\,, \tag{2.38}$$

where $A_{\mu\nu}$ and $B_{\mu\nu}$ are polarization tensors, and n_α and k_β are a time-like unit vector and a null momentum vector respectively. Performing the analogous calculation for Einstein gravity yields the Poisson equation $\partial^2 h_{\mu\nu} = 0$ which has the well-known plane wave solutions corresponding to $A_{\mu\nu}$. Here we have arrived at a higher-derivative version of the Poisson equation which has the additional $B_{\mu\nu}$ solutions; this is the origin of our extra degrees of freedom. $A_{\mu\nu}$ and $B_{\mu\nu}$ are both symmetric and traceless which gives eighteen total DOFs. By imposing our gauge constraints on (2.38), we can show that only six of these are independent. By further enforcing that the polarization tensors are transverse, the rather lengthy calculation performed by Riegert [1984] shows that their components are arranged as

$$A = \begin{pmatrix} 0 & a_3 & a_4 & 0 \\ a_3 & a_1 & a_2 & 0 \\ a_4 & a_2 & -a_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_1 & b_2 & 0 \\ 0 & b_2 & -b_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,. \tag{2.39}$$

In this form it is easy to read off that the $A_{\mu\nu}$ solution propagates the familiar spin-2 state from GR $(a_1 \pm ia_2)e^{i\vec{k}\cdot\vec{x}}$ as well as an additional spin-1 state $(a_3 \pm ia_4)e^{i\vec{k}\cdot\vec{x}}$, while the $B_{\mu\nu}$ solution propagates just the spin-2 state $(b_1 \pm ib_2)\vec{n}\cdot\vec{x}e^{i\vec{k}\cdot\vec{x}}$. The $A_{\mu\nu}$ waves fit the standard well-behaved form that we expect from massless radiation, but the same cannot be said for the the $B_{\mu\nu}$ wave. The factor of x it carries in front of the exponential will cause the wave amplitude to grow linearly with time or distance from the origin. A wave that gets stronger the longer it propagates all on its own is obviously unphysical; we call this type of poorly-behaved solution a ghost.

It is very interesting that the plane wave of radiation in conformal gravity corresponds not to a single massless particle, but to three - a spin-2 graviton, a spin-1 boson, and a spin-2 ghost. However, the presence of a ghost is a major theoretical issue for a quantum theory, since in general this type of solution corresponds to states with negative energy in the Hamiltonian. A state with negative energy means that the Hamiltonian can no longer be bounded from below, thus spoiling the unitarity of the theory. This effect has been known for some time and is known as the Ostrogradsky instability. To see how it arises, we consider a general Lagrangian $L(q, \dot{q}, \ddot{q})$ that depends on second derivatives of time (as ours does). The Euler-Lagrange equation of such a theory is then

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0\,. \tag{2.40}$$

Now as usual, we choose canonical variables $Q_1 = q$ and $Q_2 = \dot{q}$, define the conjugate momenta as

$$P_1 = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} \qquad P_2 = \frac{\partial L}{\partial \ddot{q}}\,, \tag{2.41}$$

and perform the usual Legendre transform to arrive at the Hamiltonian.

$$H = \sum_i P_i \dot{Q}_i - L = P_1 Q_2 + P_2 \ddot{q}(Q_1, Q_2, P_2) - L(Q_1, Q_2, \ddot{q}) \tag{2.42}$$

We should immediately recognize that the first term is atypical when compared to theories with only first time derivatives in their Lagrangian. P_1 can take any value and be arbitrarily negative, and since here H is only linearly dependent on P_1, it can also be arbitrarily negative and is thus unbounded from below [Salvio, 2018]. Since H represents the total energy of our theory this would seem to imply that there is always available energy to create new positive energy states and this represents an unstable vacuum for any quantum theory.

The apparent presence of Ostrogradsky ghosts is what many would consider to be the main theoretical issue with conformal gravity, in fact this is the main reason why conformal gravity has not historically been considered a viable physical theory. However, even though Ostrogradsky's theorem holds in general, there appear to be clever ways around it. It has been shown only very recently that fourth-order theories whose Hamiltonians are anti-Hermitian *and* PT-symmetric produce strictly positive real eigenvalues and thus do not suffer from Ostrogradsky instabilities. By carefully defining the dressed propagator and associated Feynman rules, it is apparently possible to show that all unstable resonances associated with negative energy quantum states disappear from the asymptotic spectrum [Mannheim, 2018],[Donoghue and Menezes, 2019]. It should be noted that the proofs in these papers are complicated and not yet fully accepted by the community. However, this is the norm in physics; any new theory must be appropriately tested and needs to stand up to intense scrutiny. At the very least, we can say that conformal gravity is showing renewed theoretical promise and certainly warrants further study, especially due its potential to unify gravity with the other fundamental forces.

3 Weyl Quadratic Gravity

It is also possible to arrive at a locally scale-invariant theory of gravity without starting from scratch and gauging the full conformal group like we did to derive the action for conformal gravity in the last chapter. If we begin instead with a theory of gravity derived in the style of GR that is globally Weyl-invariant, we should be able to gauge just this global symmetry to arrive at a theory with our desired property of local Weyl (and thus also conformal) invariance. To see how this might work, we begin by taking a small detour to discuss a class of theories known as quadratic gravity.

3.1 Standard quadratic gravity

Quadratic gravity has been popular among theorists for quite some time. It is power-counting renormalizable due to its lack of dimensionful couplings, and so it serves as a candidate for a combined theory of gravity and particle interactions that is valid to infinite energies. Of course this is only true if the gravitational part of the theory also flows to something UV-complete (such as conformal gravity). We will not discuss the phenomenology here, but rather refer the reader to the reviews on quadratic gravity by Alvarez-Gaume et al. [2016] and Salvio [2018], among others.

While traditional general relativity has only one power of curvature in the action (1.27), it has been known for some time that higher powers of curvature appear as quantum corrections in any quantum theory that is coupled to gravity, even if gravity itself is not quantized [Utiyama and DeWitt, 1962]. Since it is not possible to avoid these terms in a relativistic quantum theory, it makes sense to insert them at the classical level to study their effects directly. Besides R^2, we must also consider the full basis of Lorentz-invariant combinations which also includes $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. If λ_i are arbitrary dimensionless constants, the most general action for a theory of quadratic gravity given by

$$S_{QG} = \int d^4x\sqrt{|g|}\Big(\lambda_1 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \lambda_2 R_{\mu\nu}R^{\mu\nu} + \lambda_3 R^2 - \frac{M_{pl}^2}{2}R - \Lambda\Big)\,. \tag{3.1}$$

This action is also well-motivated by gauge theory [Benisty et al., 2018], but we will not go through the details of another gauge-theoretical derivation here. Instead we will take this as a starting point and proceed with other calculations.

We can simplify S_{QG} by taking advantage of the existence of the well-known

Gauss-Bonnet term, which is a topological invariant in four dimensions.

$$
\begin{aligned}
I_{GB} &= \int d^4x\sqrt{|g|}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\delta}R_{\mu\nu}{}^{\alpha\beta}R_{\rho\sigma}{}^{\gamma\delta} \\
&= -2\int d^4x\sqrt{|g|}\Big(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\Big) \;\;\rightarrow\;\; \text{TD}
\end{aligned}
\tag{3.2}
$$

This term can be written as a total derivative so, provided that we do not consider any non-trivial topological solutions, it will not contribute to the equations of motion in four dimensions. This means that we are free to set it to zero at the level of the action and use it to eliminate the Riemann tensor term. Doing this yields

$$
S_{QG} = \int d^4x\sqrt{|g|}\Big((\lambda_2 + 4\lambda_3)R_{\mu\nu}R^{\mu\nu} + (\lambda_1 - \lambda_3)R^2 - \frac{M_{pl}^2}{2}R - \Lambda\Big)\,. \tag{3.3}
$$

In anticipation of the work to come, we can also rewrite the Ricci tensor term using the Weyl tensor defined in (2.16) as $R_{\mu\nu}R^{\mu\nu} = \frac{1}{2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{1}{3}R^2$, where we have again eliminated a Riemann tensor term using the Gauss-Bonnet invariant. After introducing the Weyl tensor and reparameterizing in terms of the *positive* arbitrary constants α and β, we have

$$
S_{QG} = \int d^4x\sqrt{|g|}\Big(-\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2 - \frac{M_{pl}^2}{2}R - \Lambda\Big)\,. \tag{3.4}
$$

We can immediately recognize that the last two terms in this action are not scale-invariant, and that the R^2 term is not conformally invariant. So, let us now see if we can modify this theory of quadratic gravity so that it achieves our goal of a locally scale-symmetric theory of gravity. To do so, we need to gauge the global scale symmetry and define the notion of a Weyl-invariant spacetime.

3.2 Weyl space

Simply put, Weyl-invariant spacetime (or Weyl space) is the same as the pseudo-Riemannian space familiar from GR, but with an added gauge field that ensures the covariance of the curvature tensors under a Weyl transformation. This inherent Weyl symmetry means that a Weyl manifold is actually an equivalence class of conformally related pseudo-Riemannian manifolds, just like the setup we saw in conformal gravity. It is equipped with the same metric tensor $g_{\mu\nu}$ that we see in GR as well the familiar gauge boson κ_μ associated with the generator of dilation symmetry.

As we saw in Section 1.1.1, a Weyl transformation is defined as the local scale transformation of the general field $\Psi(x)$ and, in cases where we have gauged the symmetry, the gauge transformation of the Weyl gauge field[1] $\kappa_\mu(x)$.

$$\Psi(x) \;\to\; e^{q[\Psi]\omega(x)}\Psi(x) \qquad\qquad \kappa_\mu(x) \;\to\; \kappa_\mu(x) - \partial_\mu\omega(x) \tag{3.5}$$

Here, $\omega(x)$ is the local scale factor and $q[\Psi]$ is the Weyl weight of the field Ψ. Similarly to a standard Abelian gauge group, the covariant derivative associated with this symmetry is defined as

$$D_\mu\Psi = \big(\partial_\mu + q[\Psi]\kappa_\mu\big)\Psi\,. \tag{3.6}$$

Of course, $D_\mu\Psi$ transforms covariantly. One may notice that there is no factor of i in the covariant derivative or in the exponent of the group element like there would be for a $U(1)$ symmetry for example. This is related to the fact that Weyl symmetry is a spacetime symmetry - its group of generators is non-compact. As a spacetime symmetry, it naturally has an effect on the affine connection present in standard Riemannian geometry. Curvature in Weyl space is defined using the familiar symmetric metric tensor $g_{\mu\nu}$ and the symmetric (i.e. torsionless) Weyl connection $\tilde{\Gamma}^\lambda{}_{\mu\nu}$.

$$\tilde{\Gamma}^\lambda{}_{\mu\nu} \equiv \frac{1}{2}g^{\lambda\rho}\big(D_\mu g_{\rho\nu} + D_\nu g_{\mu\rho} - D_\rho g_{\mu\nu}\big) = \Gamma^\lambda{}_{\mu\nu} + \kappa_\mu\delta^\lambda_\nu + \kappa_\nu\delta^\lambda_\mu - \kappa^\lambda g_{\mu\nu}\,, \tag{3.7}$$

where $\Gamma^\lambda{}_{\mu\nu}$ is the standard Christoffel connection defined with the metric that we see in GR. One can easily see here how the addition of a new gauge field has contributed extra terms to the connection. These additions to the Christoffel symbols lead to one of the most notable differences between Weyl and Riemannian geometry. If ∇_λ is the covariant derivative in Riemannian space and $\tilde{\nabla}_\lambda$ is the covariant derivative in Weyl space, then we have

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho{}_{\lambda\mu}g_{\rho\nu} - \Gamma^\rho{}_{\lambda\nu}g_{\mu\rho} = 0 \tag{3.8}$$

$$\tilde{\nabla}_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \tilde{\Gamma}^\rho{}_{\lambda\mu}g_{\rho\nu} - \tilde{\Gamma}^\rho{}_{\lambda\nu}g_{\mu\rho} = -2\kappa_\lambda g_{\mu\nu}\,. \tag{3.9}$$

This lack of metric-compatibility that we find in Weyl space is a direct consequence of the κ_μ terms we have introduced into the connection.

Now we use the Weyl connection to define the Weyl space analogues of the Riemann, Ricci, and Weyl tensors. Starting with the Riemann tensor defined as a field

[1] The conventions for defining κ_μ and its transformation rule differ throughout the literature and we have chosen to follow Iorio et al. [1997] and Oda [2019]. One might also include a coupling constant, but we have set it to 1 here for brevity.

strength in the usual way, we have

$$\tilde{R}_{\mu\nu\rho}{}^{\sigma} \equiv \partial_\nu \tilde{\Gamma}^{\sigma}{}_{\mu\rho} - \partial_\mu \tilde{\Gamma}^{\sigma}{}_{\nu\rho} + \tilde{\Gamma}^{\alpha}{}_{\mu\rho}\tilde{\Gamma}^{\sigma}{}_{\alpha\nu} - \tilde{\Gamma}^{\alpha}{}_{\nu\rho}\tilde{\Gamma}^{\sigma}{}_{\alpha\mu}$$
$$= R_{\mu\nu\rho}{}^{\sigma} + \delta^{\sigma}_{[\mu}\nabla_{\nu]}\kappa_\rho - \delta^{\sigma}_{\rho}\nabla_{[\mu}\kappa_{\nu]} - g_{\rho[\mu}\nabla_{\nu]}\kappa^{\sigma}$$
$$+ \kappa_{[\mu}\delta^{\sigma}_{\nu]}\kappa_\rho - \kappa_{[\mu}g_{\nu]\rho}\kappa^{\sigma} + \delta^{\sigma}_{[\mu}g_{\nu]\rho}\kappa_\alpha\kappa^{\alpha}\,, \tag{3.10}$$

$$\tilde{R}_{\mu\nu} \equiv \tilde{R}_{\mu\alpha\nu}{}^{\alpha}$$
$$= R_{\mu\nu} - 2\nabla_\mu\kappa_\nu - F_{\mu\nu} - g_{\mu\nu}\nabla_\alpha\kappa^{\alpha} + 2\left(\kappa_\mu\kappa_\nu - g_{\mu\nu}\kappa_\alpha\kappa^{\alpha}\right), \tag{3.11}$$

$$\tilde{R} \equiv \tilde{R}_{\mu}{}^{\mu}$$
$$= R - 6\nabla_\mu\kappa^{\mu} - 6\kappa_\mu\kappa^{\mu}\,, \tag{3.12}$$

where $F_{\mu\nu} \equiv \nabla_\mu\kappa_\nu - \nabla_\nu\kappa_\mu = \partial_\mu\kappa_\nu - \partial_\nu\kappa_\mu$ is the field strength tensor for κ_μ. The Weyl tensor is defined using these three tensors in the same way that it is defined in Riemann space.

$$\tilde{C}_{\mu\nu\rho\sigma} \equiv \tilde{R}_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu\rho}\tilde{R}_{\nu\sigma} + g_{\nu\sigma}\tilde{R}_{\mu\rho} - g_{\mu\sigma}\tilde{R}_{\nu\rho} - g_{\nu\rho}\tilde{R}_{\mu\sigma}\right)$$
$$+ \frac{1}{6}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)\tilde{R}$$
$$= C_{\mu\nu\rho\sigma} - g_{\rho\sigma}F_{\mu\nu} + \frac{1}{2}\left(g_{\mu\rho}F_{\nu\sigma} + g_{\nu\sigma}F_{\mu\rho} - g_{\mu\sigma}F_{\nu\rho} - g_{\nu\rho}F_{\mu\sigma}\right) \tag{3.13}$$

Luckily for us, its square simplifies quite nicely.

$$\tilde{C}_{\mu\nu\rho\sigma}\tilde{C}^{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + 3F_{\mu\nu}F^{\mu\nu} \tag{3.14}$$

Using Table 1.1 and (3.10 - 3.13) it is easy to derive the following behaviors under a Weyl transformation.

$$\begin{aligned} \tilde{C}_{\mu\nu\rho\sigma} &\to e^{2\omega}\tilde{C}_{\mu\nu\rho\sigma} & \qquad \tilde{R}_{\mu\nu\rho\sigma} &\to e^{2\omega}\tilde{R}_{\mu\nu\rho\sigma} \\ \tilde{R}_{\mu\nu} &\to \tilde{R}_{\mu\nu} & \qquad \tilde{\Gamma}^{\lambda}{}_{\mu\nu} &\to \tilde{\Gamma}^{\lambda}{}_{\mu\nu} \\ \tilde{R} &\to e^{-2\omega}\tilde{R} \end{aligned} \tag{3.15}$$

These tensors also enjoy some of the same symmetries as their Riemann space counterparts.

$$\begin{aligned} &\tilde{C}_{\mu\nu\rho\sigma} = -\tilde{C}_{\nu\mu\rho\sigma} \qquad \tilde{C}_{\mu\nu\rho}{}^{\nu} = 0 \\ &\tilde{R}_{\mu\nu\rho}{}^{\sigma} = -\tilde{R}_{\nu\mu\rho}{}^{\sigma} \qquad \tilde{R}_{[\mu\nu\rho]}{}^{\sigma} = 0 \qquad \tilde{\nabla}_{[\lambda}\tilde{R}_{\mu\nu]\rho}{}^{\sigma} = 0 \end{aligned} \tag{3.16}$$

However, a couple of their index symmetries show the effects of including κ_μ terms in the connection.

$$\tilde{C}_{\mu\nu\rho}{}^{\rho} = -4F_{\mu\nu} \qquad \tilde{R}_{[\mu\nu]} = -4F_{\mu\nu} \tag{3.17}$$

3.3 Weyl-gauged quadratic gravity

We can use the same logic as we did in Section 3.1 to construct a quadratic Lagrangian in Weyl space that has all the benefits of standard quadratic gravity with the added benefit of being invariant under a local Weyl transformation. This time we use the Weyl space version of the Gauss-Bonnet invariant [Wheeler, 2018]

$$
\begin{aligned}
I_{GB} &= -\frac{1}{32\pi^2}\int d^4x\sqrt{|g|}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\delta}\tilde{R}_{\mu\nu}{}^{\alpha\beta}\tilde{R}_{\rho\sigma}{}^{\gamma\delta} \\
&= -\frac{1}{32\pi^2}\int d^4x\sqrt{|g|}\Big(\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\rho\sigma\mu\nu} - 4\tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu} + \tilde{R}^2\Big) \quad \rightarrow \quad \mathrm{TD}\,, \qquad (3.18)
\end{aligned}
$$

to simplify a general action constructed from the squares of our Weyl space tensors (3.10 - 3.12).

$$
\begin{aligned}
S_{WQG} &= \int d^4x\sqrt{|g|}\Big(\lambda_1\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + \lambda_2\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + \lambda_3\tilde{R}^2\Big) \\
&= \int d^4x\sqrt{|g|}\Big(-\alpha\tilde{C}_{\mu\nu\rho\sigma}\tilde{C}^{\mu\nu\rho\sigma} + \beta\tilde{R}^2\Big) \qquad (3.19)
\end{aligned}
$$

We are forced to drop the Einstein gravity and cosmological constant terms present in standard quadratic gravity because they are not globally scale-symmetric and cannot be Weyl-gauged. It is well known (see for example [Quiros, 2019]) that the $\tilde{R}^2$ term in this type of setup propagates an extra scalar degree of freedom and we can extract it via the introduction of an auxiliary dilaton we will call ϕ. To see how this works, consider the Lagrangian $\mathcal{L} = -\sqrt{|g|}(2\phi^2\tilde{R} + \phi^4)$ where the equation of motion for ϕ is $\phi^2 = -\tilde{R}$. We can integrate ϕ out of this Lagrangian using this equation of motion which takes us back to the original $\tilde{R}^2$. This tells us that (3.19) is classically equivalent to the following, provided of course that the new ϕ terms obey all of the necessary symmetries and ϕ does not appear anywhere else in the Lagrangian.

$$
\mathcal{L}_{WQG} = \sqrt{|g|}\Big[-\alpha\tilde{C}_{\mu\nu\rho\sigma}\tilde{C}^{\mu\nu\rho\sigma} - \beta\Big(2\phi^2\tilde{R} + \phi^4\Big)\Big] \qquad (3.20)
$$

Our next step is to convert this expression to the Riemannian picture using the relations we derived in (3.10 - 3.14). These conversions yield

$$
\begin{aligned}
\mathcal{L}_{WQG} = \sqrt{|g|}\Big[&-\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - 3\alpha F_{\mu\nu}F^{\mu\nu} \\
&- \beta\Big(2\phi^2 R - 12\phi^2\left(\nabla_\mu\kappa^\mu + \kappa_\mu\kappa^\mu\right) + \phi^4\Big)\Big]\,. \qquad (3.21)
\end{aligned}
$$

It may not be obvious at this stage, but the dilaton is in fact a dynamical field [Ghilencea, 2019b]. To see this we use the fact that $\sqrt{|g|}\nabla_\mu\kappa^\mu = \partial_\mu(\sqrt{|g|}\kappa^\mu)$ and we drop a total derivative after integrating by parts, adding $\partial_\mu\phi\partial^\mu\phi - \partial_\mu\phi\partial^\mu\phi = 0$, and completing a square.

$$\begin{aligned}\sqrt{|g|}\phi^2\big(\nabla_\mu\kappa^\mu + \kappa_\mu\kappa^\mu\big) &= \phi^2\partial_\mu\big(\sqrt{|g|}\kappa^\mu\big) + \sqrt{|g|}\phi^2\kappa_\mu\kappa^\mu \\ &= \sqrt{|g|}\big(- \kappa^\mu\partial_\mu\phi^2 + \phi^2\kappa_\mu\kappa^\mu + \partial_\mu\phi\partial^\mu\phi - \partial_\mu\phi\partial^\mu\phi\big) \\ &= \sqrt{|g|}\big(- 2\kappa^\mu\phi^2\partial_\mu \ln\phi + \phi^2\kappa_\mu\kappa^\mu - \partial_\mu\phi\partial^\mu\phi + \phi^2\partial_\mu\ln\phi\partial^\mu\ln\phi\big) \\ &= \sqrt{|g|}\big(- \partial_\mu\phi\partial^\mu\phi + \phi^2(\kappa_\mu - \partial_\mu\ln\phi)(\kappa^\mu - \partial^\mu\ln\phi)\big) \end{aligned} \tag{3.22}$$

Now we make the canonically normalizing redefinitions

$$\kappa'_\mu = \sqrt{12\alpha}\big(\kappa_\mu - \partial_\mu\ln\phi\big) \qquad \phi' = \sqrt{24\beta}\phi \qquad \lambda = \frac{1}{24\beta}\,, \tag{3.23}$$

while noting that $F_{\mu\nu} = \frac{1}{\sqrt{12\alpha}}F'_{\mu\nu}$. Finally, we write the action for Weyl quadratic gravity as

$$\begin{aligned} S_{WQG} = \int d^4x\sqrt{|g|}\bigg(&- \alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - \frac{\phi'^2}{12}R - \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \lambda\frac{\phi'^4}{4!} \\ &- \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{\lambda}{12\alpha}\phi'^2\kappa'_\mu\kappa'^\mu\bigg)\,. \end{aligned} \tag{3.24}$$

It is important to note that the kinetic term for the dilaton has appeared as a natural consequence of its interactions with the Weyl gauge field κ_μ which indicates that ϕ' is actually a physical propagating degree of freedom. It is possible to introduce a dilaton via its equations of motion in standard quadratic gravity, but no kinetic term comes about for free like it does in this version. The dilaton in standard quadratic gravity is non-dynamical in the Jordan frame. However, we should immediately notice that ϕ' is a ghost since its kinetic term has the same sign as its interaction term. This is direct consequence of how we extracted ϕ via its equation of motion and it is unavoidable. As we saw in the last section, this would normally be a cause for concern, but we will see shortly that our ghost will be eaten up as a result of the Stückelberg mechanism, so we have nothing to fear.

One can easily check that the theory described by this action (3.24) is Weyl-invariant and so we have succeeded in deriving a locally scale-invariant theory by gauging global scale symmetry alone. Additionally, due to its quadratic dependence on the Weyl tensor, WQG is power-counting renormalizable for the same reasons that conformal gravity and standard quadratic gravity are. This means it also has the potential to serve as a theory of quantum gravity. However, this model contains even more interesting features, and to see them we must first make a short detour into the world of particle physics.

3.3.1 Dimensional transmutation in WQG

WQG as presented in (3.24) is classically Weyl-invariant, but that does not guarantee that it is also invariant at the quantum level. In fact, it is common for classical theories to exhibit spontaneous symmetry breaking upon the inclusion of quantum corrections [Weinberg, 1973]. To see how this works, let us consider a dynamical scalar σ with classical action $S[\sigma]$. We can expand σ as small quantum fluctuations φ around a classical background

$$\sigma \quad \rightarrow \quad \sigma_c + \varphi\,, \tag{3.25}$$

and employ the the path integral formalism to define the effective action $\Gamma[\sigma_c]$.

$$\Gamma[\sigma_c] = S[\sigma_c] + \hbar K[\sigma_c] \equiv S[\sigma_c] - i\hbar\Bigg(\int \mathcal{D}\varphi e^{\frac{i}{\hbar}\left(\frac{1}{2}\varphi\cdot\frac{\delta^2 S}{\delta\sigma_c^2}\cdot\varphi - \hbar\frac{\delta K}{\delta\sigma_c}\cdot\varphi + \mathcal{O}(\varphi^3)\right)}\Bigg) \tag{3.26}$$

$\Gamma[\sigma_c]$ represents contributions from the classical action with the addition of quantum effects in $K[\sigma_c]$ and describes the full quantum theory. $K[\sigma_c]$ encodes all of the loop-order interactions in powers of $\hbar$ and must be solved for perturbatively. If we assume a constant background $\sigma_c = const$, then we can compute the integral and define an effective potential V_{eff} [Weigand].

$$\Gamma[\sigma_c] = -\int d^4x V_{eff}(\sigma_c) = -\int d^4x\Big(V_c(\sigma_c) + V_{eff}^{(\text{1-loop})}(\sigma_c) + ...\Big) \tag{3.27}$$

This tells us how quantum effects contribute to the classical theory with tree-level potential V_c on an order-by-order basis. Given only the Lagrangian for some quantum field theory, one can use this formalism to calculate the Coleman-Weinberg effective potential V_{eff} to any loop order. Of course with the addition of loop-order contributions, this effective potential will have a different shape than its classical counterpart. When this new shape has a local minimum at $\sigma \neq 0$, σ picks up a non-zero vacuum expectation value ($\langle\sigma\rangle$) which can cause a symmetry of the theory to be violated at energies below $\langle\sigma\rangle$. This process by which a classically symmetric theory exhibits spontaneous symmetry breaking after the inclusion of quantum corrections is known as the Coleman-Weinberg mechanism.

An interesting consequence of this mechanism occurs when we apply it in the context of renormalization. When we include renormalizing counter-terms in the action, the effective potential comes with dimensionless constants that need to be solved for using the renormalization conditions. These conditions usually look like

$$V_{eff}^{(\text{n-loop})}(\sigma_c)\Bigg|_{\sigma_c=\langle\sigma\rangle} = \partial^2_{\sigma_c} V_{eff}^{(\text{n-loop})}\Bigg|_{\sigma_c=\langle\sigma\rangle} = \partial^4_{\sigma_c} V_{eff}^{(\text{n-loop})}\Bigg|_{\sigma_c=\langle\sigma\rangle} = 0\,, \tag{3.28}$$

and when we see dependencies like $V_{eff}^{(\text{1-loop})}(\sigma_c) \propto \ln\frac{\sigma_c^2}{\langle\sigma\rangle^2}$ we cannot evaluate the conditions at $\langle\sigma\rangle = 0$ since the log would be undefined. Thus, we must introduce the dimensionful $\langle\sigma\rangle \neq 0$ when we solve for the dimensionless renormalization constants. We are forced to trade dependence on a dimensionless parameter for a dimensionful one; a phenomenon known as dimensional transmutation [Peskin and Schroeder, 1995]. This is of particular importance in theories with scale-invariance (such as ours) because the introduction of a scale via the renormalization conditions introduces effective terms with dimensionful parameters that violate scale invariance. In short, the inclusion of quantum effects via the Coleman-Weinberg potential can lead to the unavoidable introduction of dimensionful parameters to a classically scale-invariant theory which causes a spontaneous breakdown of scale symmetry at the quantum level.

This process turns out to be crucial for our theory because, as was recently shown by Oda [2019], the Weyl symmetry in Weyl quadratic gravity breaks down spontaneously as a result of the Coleman-Weinberg mechanism to produce a non-zero vacuum expectation value (VEV) for ϕ. We refer the reader to Oda's paper for the details, but in summary, after expanding the dilaton and metric in the action (3.21) as small quantum fluctuations around their classical background fields

$$\phi \quad\rightarrow\quad \phi+\varphi \qquad\qquad g_{\mu\nu} \quad\rightarrow\quad \eta_{\mu\nu}+h_{\mu\nu}\,, \tag{3.29}$$

we can compute the one-loop effective potential for ϕ by integrating out φ and $h_{\mu\nu}$. With the inclusion of renormalizing counter-terms, we find that the effective potential takes the form

$$V_{eff}^{(1\text{-loop})}(\phi)=\xi\phi^4\ln\left(\frac{\phi^2}{\langle\phi\rangle^2}-\frac{1}{2}\right). \tag{3.30}$$

Here, ξ is a positive constant and there is an obvious non-zero minimum at $\phi=\langle\phi\rangle$. After renormalization, the dependence on the dimensionless coupling λ that we see in (3.24) is replaced by dependence on the dimensionful $\langle\phi\rangle$. Weyl-invariance is lost as a result of dimensional transmutation. If we match constants in this setup so that Einstein gravity is reproduced at low energies we find that

$$\langle\phi\rangle^2=\mathcal{O}(M_{pl}^2)\,. \tag{3.31}$$

The fact that ϕ picks up non-zero VEV turns out to be very important. To see why, we return to (3.21) and note that it is written in the Jordan frame characterized by $\mathcal{L}^{\mathcal{J}}\supset(\phi R, g_{\mu\nu})$. There always exists conformal transformations from the Jordan frame to the Einstein frame, which is characterized by $\mathcal{L}^{\mathcal{E}}\supset(\hat{R},\phi\hat{g}_{\mu\nu})$, and it will be instructive to see what form our theory takes there. Since, by construction, our Lagrangian is Weyl-symmetric, a conformal transformation will not affect the physics. We should mention that there is not a unanimous consensus among experts on the physical equivalence of these two frames in general. However, the controversy arises only in specific examples where one can show that an additional interaction term appears when certain theories that are coupled to matter are rotated between frames. We will not go into further details here since it does not affect our present discussion, but this is an interesting problem in its own right and we refer the reader to publications such as the review by Quiros [2019] for more details.

Turning back the matter at hand, we select the following Weyl transformation which will take us to the Einstein frame.

$$\omega=\ln\Omega\,, \qquad\qquad \Omega\equiv\frac{2\sqrt{\beta}}{\langle\phi\rangle}\phi \tag{3.32}$$

Obviously this is only well-defined for a nonzero $\langle\phi\rangle$. It acts on our fields as

$$g_{\mu\nu} \;\to\; \hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \qquad \phi \;\to\; \hat{\phi} = \frac{\phi}{\Omega} \qquad \kappa_\mu \;\to\; \hat{\kappa}_\mu = \kappa_\mu - \partial_\mu \ln\Omega\,, \tag{3.33}$$

and so under this particular Weyl transformation, (3.21) becomes

$$\mathcal{L}^{(\mathcal{E})}_{WQG} = \sqrt{|\hat{g}|}\Big(-\alpha \hat{C}_{\mu\nu\rho\sigma}\hat{C}^{\mu\nu\rho\sigma} - \frac{\langle\phi\rangle^2}{2}\hat{R} - \frac{\langle\phi\rangle^4}{16\beta^2} - 3\alpha \hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - 12\beta\hat{\kappa}_\mu\partial^\mu\Big(\frac{\phi}{\Omega}\Big)^2 + \frac{\langle\phi\rangle^2}{4\beta}\hat{\kappa}_\mu\hat{\kappa}^\mu\Big). \tag{3.34}$$

Here, $\hat{R}$ and $\hat{C}_{\mu\nu\rho\sigma}$ are computed with the conformally transformed metric $\hat{g}_{\mu\nu}$. With a bit of algebra it is easy to show that the second to last term above drops out after expanding the derivative and inserting our choice of Ω.

$$12\beta\hat{\kappa}_\mu\partial^\mu\Big(\frac{\phi}{\Omega}\Big)^2 = 6\hat{\kappa}_\mu\Big(\langle\phi\rangle^2\partial_\mu\ln\phi - \langle\phi\rangle^2\partial_\mu\ln\Omega\Big) = 0 \tag{3.35}$$

As in (3.23) we are still free to perform a canonical normalizing field redefinition; $\hat{\kappa}_\mu \to \sqrt{12\alpha}\hat{\kappa}_\mu$. We are only free to do this field redefinition since we have yet to fix the magnitude of $\hat{\kappa}_\mu$ relative to the other fields. Doing this redefinition is nothing more than introducing a coupling constant for $\hat{\kappa}_\mu$ and solving for it relative to α all in one step. With all of this in mind, our final expression for the Lagrangian of Weyl quadratic gravity in the Einstein frame is the following.

$$S^{(\mathcal{E})}_{WQG} = \int d^4x\sqrt{|\hat{g}|}\Big(-\alpha \hat{C}_{\mu\nu\rho\sigma}\hat{C}^{\mu\nu\rho\sigma} - \frac{\langle\phi\rangle^2}{2}\hat{R} - \frac{\langle\phi\rangle^4}{16\beta^2} - \frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \frac{\langle\phi\rangle^2}{48\alpha\beta}\hat{\kappa}_\mu\hat{\kappa}^\mu\Big) \tag{3.36}$$

This rotation to Einstein frame yields some very interesting features. First off, we see that our theory contains the Proca action for a spin-1 boson with mass $m^2_{\hat{\kappa}} = \frac{\langle\phi\rangle^2}{24\alpha\beta}$, while the field ϕ no longer appears anywhere. Selecting (3.32) in particular has caused the massless κ_μ to swallow the dynamical ghost degree of freedom ϕ and become the massive $\hat{\kappa}_\mu$ as a side effect of dimensional transmutation in the scalar sector [Ghilencea, 2019a].

$$\hat{\kappa}_\mu = \kappa_\mu - \partial_\mu\ln\Omega = \kappa_\mu - \partial_\mu\ln\phi \tag{3.37}$$

The phenomenon by which a massless Abelian gauge boson acquires a mass from interactions with a scalar that picks up a non-zero VEV is known as the Stückelberg mechanism and is well known in the world of particle physics [Ruegg and Ruiz-Altaba, 2004]. In addition to the massive dynamical $\hat{\kappa}_\mu$, we also see the Einstein action (with a negative cosmological constant) as well as the squared Weyl tensor action familiar to us from conformal gravity.

The fact that κ_μ acquires a large mass is extremely significant because it dispels the long-held reservations about theories with a dynamical Weyl boson. Weyl's original formulation of the theory [Weyl, 1929] was an attempt to identify the massless κ_μ as the photon and provide a purely geometric derivation of electromagnetism, but it ran into some unavoidable contradictions. Among others, Einstein himself pointed out that, due to the conformal origin of such a boson, parallel-transported spacetime vectors in this setup would change not only their direction, but also their length. An interesting corollary of this effect is what is known as the "second clock problem" where it is possible to show that a traveler who visits some distant galaxy and returns to Earth finds that not only has he aged differently than his comrades on Earth (as predicted by GR), but also that his clock is now running a different rate than theirs. In general, the very notion of length becomes path-dependent in a setup with unbroken conformal symmetry, and we know from countless experiments that this is not physical. There have been many attempts to get around this issue by, for example, assuming that κ_μ is "pure-gauge" so that it can be written as the divergence of a scalar field $\kappa_\mu = \partial_\mu \varphi$. This renders κ_μ non-dynamical since $F_{\mu\nu} = \partial_\mu \partial_\nu \varphi - \partial_\nu \partial_\mu \varphi = 0$. Our setup is in a way the best of both worlds, since $\hat{\kappa}_\mu$ is dynamical, but decoupled at low energies due to its large mass. This means that the second clock problem only shows up at energies above the level of dynamical Weyl symmetry breaking, which is far out of the reach of modern experiments. As we will see, having a massive gauge boson in the spectrum can lead to some interesting phenomena. Indeed, it is about time we investigate the physical implications of Weyl quadratic gravity.

3.4 Phenomenology of WQG

WQG presented in this form is a relatively new theory and though it has not been studied to the degree that conformal gravity has, some very interesting work has recently been published on the topic. Due to the presence of the Weyl tensor term, we can expect many of the same cosmological features of conformal gravity to be present in WQG as well. For this reason, we will focus on the effects of including ϕ and κ_μ in the spectrum.

The first interesting consequence of κ_μ picking up a large mass, is that it has the potential to serve as a dark matter candidate. Beltran Jimenez and Koivisto [2014] have completed a nice preliminary study of this possibility in a similar model that does not include the Weyl tensor term. They were able to show that if we consider $\hat{\kappa}_\mu$ as normal matter and move its contribution into the energy-momentum tensor, then in a FLRW universe, we arrive at the following equations constraining $\hat{\kappa}_\mu$

$$\hat{\kappa}_0 = 0 \qquad \ddot{\hat{\kappa}}_i + H\dot{\hat{\kappa}}_i + m_{\hat{\kappa}}^2 \hat{\kappa}_i = 0\,, \tag{3.38}$$

where the subscript i indicates only the spatial components of $\hat{\kappa}_\mu$. This second equation is precisely that of a harmonic oscillator with frequency $\omega = m_{\hat{\kappa}} = \langle\phi\rangle/\sqrt{24\alpha\beta}$. Thus, provided that $m_{\hat{\kappa}}$ is large enough ($m_{\hat{\kappa}}^2 \gtrsim H^2$), $\hat{\kappa}_\mu$ will oscillate rapidly in the early universe and has the potential to contribute a significant fraction of the observed dark matter, provided of course that $\hat{\kappa}_\mu$ receives a small primordial amplitude which the authors argue may arise from quantum fluctuations. Now one should wonder, why do we care about possible dark matter contributions when we have a conformal gravity term that has the potential to negate the need for dark matter entirely? The answer is of course that the observed effects of dark matter may be a result of the effects from both the linear potential term in conformal gravity *and* from un-observable particles. Rather than enforcing all of the constraints from observations on one or the other contribution, they can both "share the load" in this model. It may be that one or the other is dominant, but in any case, having both options broadens the parameter space for cosmologies that employ WQG. In principle, we should be able to draw up a model that takes both phenomena into account and allows us to put constraints on the constants α and β. To our knowledge, no such study has yet been performed in depth, and we look forward to pursuing this option in the future.

Next, let us consider in detail the ramifications of dimensional transmutation in WQG. As we saw in the previous section, this theory contains all of the necessary ingredients to generate scale dependence all on its own without the addition of any other fields into the spectrum. This is in contrast to conformal gravity where we must consider interactions with the Higgs (or in principle any other scalar) in order to spontaneously break conformal symmetry. In WQG however, the scalar degree of freedom comes directly from the metric when we make the replacement $\tilde{R}^2 \to -2\phi^2\tilde{R} - \phi^4$. In essence, here we have a theory of gravity that is Weyl-invariant at high energies that, through self-interactions alone, dynamically breaks that symmetry and generates the very notion of physical scale. While this is quite remarkable in its own right, the generation of the Planck scale through dimensional transmutation appears to have even farther-reaching consequences. If we also consider interactions with the Higgs Φ, then the effective potential in equation (3.30) becomes

$$V_{eff}(\phi,\Phi) = \xi\phi^4 \ln\left(\frac{\phi^2}{\langle\phi\rangle^2} - \frac{1}{2}\right) + \lambda_{\Phi\phi}\left(\Phi^\dagger\Phi\right)\phi^2 + \lambda_\Phi\left(\Phi^\dagger\Phi\right)^2, \tag{3.39}$$

where $\lambda_{\Phi\phi}$ is a portal coupling constant and λ_Φ is the Higgs self-coupling [Oda, 2019]. If we now insert $\phi = \langle\phi\rangle$ and do some algebra this expression becomes

$$V_{eff}(\langle\phi\rangle,\Phi) = \lambda_\Phi\Big(\Phi^\dagger\Phi + \frac{2\lambda_{\Phi\phi}}{\lambda_\Phi}\langle\phi\rangle^2\Big)^2 - \frac{1}{4}\Big(\frac{\lambda_{\Phi\phi}^2}{\lambda_\Phi} + \xi\Big)\langle\phi\rangle^4\,. \tag{3.40}$$

Provided that $\lambda_\Phi > 0$, this exactly fits the form of the Higgs potential in the Standard Model. So, if we take it to be such and proceed with the standard Higgs mechanism procedure of going to unitary gauge, etc., we find that the Higgs mass is given by

$$m_\Phi^2 = 2|\lambda_{\Phi\phi}|\langle\phi\rangle^2\,. \tag{3.41}$$

Thus, the dimensional transmutation that occurs in Weyl quadratic gravity can generate not only the Planck scale, but also the electro-weak scale after we include the Higgs in our spectrum and properly tune $\lambda_{\Phi\phi}$ [Oda, 2019].

Now, we must mention that despite all of the very attractive features listed here, WQG also suffers from the same theoretical difficulties as conformal gravity i.e. the higher derivatives still give us reason to be concerned about ghost states. However, the ghosts in WQG take a slightly different form.

3.4.1 Degrees of freedom

In order to see exactly what type of ghost we are dealing with in WQG, we need to analyze the propagating degrees of freedom. Instead of using the same method that we used to look at conformal gravity, we will take another route by first considering the DOFs in standard quadratic gravity. Transforming the action (3.4) from the Jordan to Einstein frame, we have

$$\begin{aligned} S_{QG}^{(\mathcal{J})} &= \int d^4x\sqrt{|g|}\Big(-\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2 - \frac{M_{pl}^2}{2}R - \Lambda\Big) \\ \rightarrow\ S_{QG}^{(\mathcal{E})} &= \int d^4x\sqrt{|g|}\Big(-\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{1}{2}\partial_\mu S\partial^\mu S - V(S) - \frac{M_{pl}^2}{2}R - \Lambda\Big)\,, \end{aligned} \tag{3.42}$$

where S is a scalar field with potential $V(S)$. It has been shown by Julve and Tonin [1978], among others, that this action has eight propagating degrees of freedom - a spin-2 massless graviton, a spin-2 massive ghost, and a spin-0 massive boson. With this, we note that our action (3.36) has a similar form except that it lacks the scalar DOF and contains the additional Proca terms for $\hat{\kappa_\mu}$. Thus, a cursory glance tells us that Weyl quadratic gravity propagates ten degrees of freedom $(8-1+3=10)$.

This statement is hardly a proof, so we will confirm our count of the degrees of freedom and go a step further by directly diagonalizing the linear perturbations of the metric. This type of calculation, in the style of Stelle [1978], is called an oscillator variable decomposition. We will only go over the theoretical ideas and conclusion here; the details are shown in appendix B. The general idea is that we introduce an auxiliary spin-2 field $X_{\mu\nu}$ via an ansatz Lagrangian that, after being integrated out, leaves us with two separate spin-2 fields with canonical second-order kinetic terms corresponding to the graviton and massive ghost that we expect to be hidden inside of the metric perturbation $h_{\mu\nu}$.

We begin by making the separation $\mathcal{L}_{WQG} = \mathcal{L}_{grav} + \mathcal{L}_{matter}$ since it will end up being much simpler to consider $\hat{\kappa}_\mu$ and $\langle\phi\rangle$ as part of a separate matter sector, despite their geometric origin.

$$\mathcal{L}_{grav} = \sqrt{|\hat{g}|}\bigg(-\alpha\hat{C}_{\mu\nu\rho\sigma}\hat{C}^{\mu\nu\rho\sigma} - \frac{\langle\phi\rangle^2}{2}\hat{R}\bigg) \tag{3.43}$$

$$\mathcal{L}_{matter} = \sqrt{|\hat{g}|}\bigg(-\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \frac{\langle\phi\rangle^2}{48\alpha\beta}\hat{\kappa}_\mu\hat{\kappa}^\mu - \frac{\langle\phi\rangle^4}{16\beta^2}\bigg) \tag{3.44}$$

The equation of motion here is of course $\delta\mathcal{L}_{WQG} = \delta\mathcal{L}_{grav} + \delta\mathcal{L}_{matter} = 0$. We can recast this in the canonical "field coupled to a source" form by first defining

$$\begin{aligned} T_{\mu\nu} \equiv -\frac{1}{\sqrt{|g|}}\frac{\partial\mathcal{L}_{matter}}{\partial g^{\mu\nu}} &= -\frac{1}{2}\hat{F}_{\mu\rho}\hat{F}_\nu{}^\rho + \frac{1}{8}g_{\mu\nu}\hat{F}_{\rho\sigma}\hat{F}^{\rho\sigma} + g_{\mu\nu}\frac{\langle\phi\rangle^4}{32\beta^2} \\ &\quad + \frac{\langle\phi\rangle^2}{48\alpha\beta}\Big(\hat{\kappa}_\mu\hat{\kappa}_\nu - 2g_{\mu\nu}\hat{\kappa}_\rho\hat{\kappa}^\rho\Big)\,, \end{aligned} \tag{3.45}$$

so that the equation of motion is $\delta\mathcal{L}_{grav} = T_{\mu\nu}$. Now when we expand the metric as small perturbations[2] around a flat background

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{3.46}$$

we can write the whole linearized Lagrangian as

$$\mathcal{L}_{WQG}[h_{\mu\nu}] = \mathcal{L}_{grav}[h_{\mu\nu}] + \mathcal{L}_{matter}[\eta_{\mu\nu}] + h_{\mu\nu}T_{\mu\nu}\,, \tag{3.47}$$

where $\mathcal{L}_{matter}[\eta_{\mu\nu}]$ is the same matter Lagrangian after sending $g_{\mu\nu} \to \eta_{\mu\nu}$.

Turning back to $\mathcal{L}_{grav}$, it is convenient to write the linearized curvature tensors in this type of setup in terms of the rank-4 spin projectors

$$P^{(2)}_{\mu\nu,\rho\sigma} = \frac{1}{2}\big(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}\big) - P^{(0,s)}_{\mu\nu,\rho\sigma} \qquad P^{(0,s)}_{\mu\nu,\rho\sigma} = \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}\,, \tag{3.48}$$

where $\theta_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}$. There is a whole set of these projectors that form a complete basis on the spin decomposition of generic symmetric rank-2 tensor fields [Van Nieuwenhuizen, 1973], but we will not list them all here. It is straightforward to check by direct computation that they allow us to write

$$\begin{aligned} \sqrt{|g|}\hat{C}_{\mu\nu\rho\sigma}\hat{C}^{\mu\nu\rho\sigma} &= h^{\mu\nu}P^{(2)}_{\mu\nu,\rho\sigma}\partial^4 h^{\rho\sigma} \\ \sqrt{|g|}\hat{R} &= \frac{1}{2}h^{\mu\nu}\Big(P^{(2)}_{\mu\nu,\rho\sigma} - 2P^{(0,s)}_{\mu\nu,\rho\sigma}\Big)\partial^2 h^{\rho\sigma}\,, \end{aligned} \tag{3.49}$$

which means we can express our linearized gravitational Lagrangian in the following compact form.

$$\mathcal{L}_{grav}[h_{\mu\nu}] = -\alpha h^{\mu\nu}P^{(2)}_{\mu\nu,\rho\sigma}\partial^4 h^{\rho\sigma} - \frac{\langle\phi\rangle^2}{4}h^{\mu\nu}\big(P^{(2)}_{\mu\nu,\rho\sigma} - 2P^{(0,s)}_{\mu\nu,\rho\sigma}\big)\partial^2 h^{\rho\sigma} \tag{3.50}$$

[2] One could also define this expansion in terms of a small dimensionful parameter ξ as $g_{\mu\nu} = \eta_{\mu\nu} + \xi h_{\mu\nu}$ so that $h_{\mu\nu}$ picks up a mass dimension, if one so desired. The natural choice in this case would be $\xi = \langle\phi\rangle^{-1}$. For our purposes we will simply assume that $h_{\mu\nu}$ is small.

Our next step is to define the auxiliary field $X_{\mu\nu}$ as the diagonal counterpart to $h_{\mu\nu}$, and write them both in terms of the physically propagating oscillator variables, $\sigma_{\mu\nu}$ and $\Sigma_{\mu\nu}$.

$$h_{\mu\nu} = \frac{1}{2}\Big(\sigma_{\mu\nu} + \Sigma_{\mu\nu}\Big) \qquad X_{\mu\nu} = \frac{1}{2}\Big(\sigma_{\mu\nu} - \Sigma_{\mu\nu}\Big) \tag{3.51}$$

To see how these oscillator variables separate when plugged into (3.50), we must first consider the following ansatz for the auxiliary Lagrangian.

$$\begin{aligned} \mathcal{L}_X[h_{\mu\nu}, X_{\mu\nu}] = \; & 2X_{\mu\nu}\partial^\nu\partial_\rho h^{\mu\rho} + \frac{1}{6}\big(-3 + i\sqrt{3}\big)X_\rho{}^\rho\partial_\mu\partial_\nu h^{\mu\nu} - X_{\mu\nu}\partial_\rho\partial^\rho h^{\mu\nu} \\ & - X_{\mu\nu}\partial^\mu\partial^\nu h_\rho{}^\rho + \frac{1}{6}\big(3 + i\sqrt{3}\big)X_\mu{}^\mu\partial_\rho\partial^\rho h_\nu{}^\nu + \frac{\langle\phi\rangle^2}{\alpha}\Bigg(\frac{1}{4}h_{\mu\nu}h^{\mu\nu} \\ & + \frac{1}{4}X_{\mu\nu}X^{\mu\nu} - \frac{1}{2}X_{\mu\nu}h^{\mu\nu} - \frac{1}{4}h_\mu{}^\mu h_\nu{}^\nu + \frac{1}{8}\big(1 + i\sqrt{3}\big)X_\mu{}^\mu h_\nu{}^\nu\Bigg) \end{aligned} \tag{3.52}$$

The prefactors in front of each term here have been precisely chosen so that we get our original Lagrangian back when $X_{\mu\nu}$ is integrated out using its equation of motion.

$$\mathcal{L}_X[h_{\mu\nu}] = \mathcal{L}_{grav}[h_{\mu\nu}] \tag{3.53}$$

At this point we also fix our gauge freedom by imposing the standard transverse-traceless conditions on our gravitational radiation.

$$\partial^\mu\sigma_{\mu\nu} = \partial^\mu\Sigma_{\mu\nu} = 0 \qquad \sigma_\mu{}^\mu = \Sigma_\mu{}^\mu = 0 \tag{3.54}$$

When we plug (3.51) into (3.52) with these gauge conditions, we see that all of the degrees of freedom buried inside $h_{\mu\nu}$ separate nicely into two canonical spin-2 fields.

$$\begin{aligned} \mathcal{L}_X[\sigma_{\mu\nu}, \Sigma_{\mu\nu}] &= \mathcal{L}_{grav}[\sigma_{\mu\nu}, \Sigma_{\mu\nu}] \\ &= -\frac{1}{2}\sigma_{\mu\nu}\partial_\rho\partial^\rho\sigma^{\mu\nu} + \frac{1}{2}\Sigma_{\mu\nu}\partial_\rho\partial^\rho\Sigma^{\mu\nu} + \frac{\langle\phi\rangle^2}{16\alpha}\Sigma_{\mu\nu}\Sigma^{\mu\nu} \end{aligned} \tag{3.55}$$

As expected, we have a massless spin-2 particle $\sigma_{\mu\nu}$ that we can identify as the graviton, along with a massive spin-2 particle $\Sigma_{\mu\nu}$ whose kinetic term has the wrong sign, making it a ghost. With this we can finally express the entire linearized Lagrangian for Weyl quadratic gravity in canonical form.

$$\begin{aligned} \mathcal{L}_{WQG} = & -\frac{1}{2}\sigma_{\mu\nu}\partial_\rho\partial^\rho\sigma^{\mu\nu} + \frac{1}{2}\Sigma_{\mu\nu}\partial_\rho\partial^\rho\Sigma^{\mu\nu} + \frac{\langle\phi\rangle^2}{16\alpha}\Sigma_{\mu\nu}\Sigma^{\mu\nu} \\ & -\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \frac{\langle\phi\rangle^2}{48\alpha\beta}\hat{\kappa}_\mu\hat{\kappa}^\mu - \frac{\langle\phi\rangle^4}{16\beta^2} + \frac{1}{2}\Big(\sigma_{\mu\nu} + \Sigma_{\mu\nu}\Big)T^{\mu\nu} \end{aligned} \tag{3.56}$$

From this it is easy to count ten total propagating DOFs - two graviton, five massive ghost, and three massive gauge boson. We also note that the mass of the ghost is dependent on both $\langle \phi \rangle$ and the conformal coupling constant α.

At first glance it would appear that the ghost problem is even worse in Weyl quadratic gravity than it is in conformal gravity. There we have only the two spin-2 ghost DOFs while here we have five, all of them of the same Ostrogradsky nature. Of course if the proponents of conformal gravity are correct in their assertions that this type of ghost disappears from the asymptotic spectrum, then we have nothing to worry about here either. However, as we will see in the next part of this work, there may be another way around the ghost problem in Weyl quadratic gravity.

Part II

Connections

4 Conformal Gravity vs. Weyl Quadratic Gravity

A natural question to ask at this point would be, what is the connection between CG and WQG? We have seen that both theories are defined on a conformal class of manifolds, quadratic in curvature, torsionless, and locally Weyl-invariant. There is an important difference however in the fact that WQG propagates four more degrees of freedom than CG, and this difference is directly related to the role that κ_μ plays in each theory. In the conformal gravity derivation, it is possible to set $\kappa_\mu = 0$ from the start in what is essentially a gauge fixing. The constraints we imposed in Section 2.1 and the subsequent mixing of gauge fields between the curvatures lead to cancellations that preserve the invariance of the theory. In fact, even if we don't fix the gauge in this way, the terms containing κ_μ end up dropping out anyway as a result of the anti-symmetry of the $\mathcal{R}_{\mu\nu}$ tensor. In Weyl quadratic gravity however, setting $\kappa_\mu = 0$ just brings us back to standard quadratic gravity, which we know to explicitly break local Weyl symmetry. What makes conformal gravity special in that it can maintain conformal invariance without explicit κ_μ dependence? If our end goal is the same before deriving both theories, then why do we pick up additional degrees of freedom by taking the WQG route as opposed to the CG route? In this chapter we present a resolution to these puzzling questions by making contact between CG and WQG using a process known as "Ricci gauging".

4.1 Ricci-gauging

Ricci gauging is a clever process that was first laid out by Iorio et al. [1997] and expanded on by Karananas [2016] in an effort to show that a theory with global Weyl invariance can be made locally invariant without introducing extra degrees of freedom into the spectrum. The general idea is to find a relation between the standard curvature tensors and the gauge field κ_μ so that κ_μ can be removed from the action and replaced with functions of the metric. We could expect that such a relationship may exist given that κ_μ is associated with a gauged spacetime symmetry. Indeed, we can already see a relation in the non-metricity condition $\tilde{\nabla}_\lambda g_{\mu\nu} = -2\kappa_\lambda g_{\mu\nu}$.

Specifically, we choose to search for a relation between κ_μ and the Ricci curvature $R_{\mu\nu}$. To this end, we need all possible local rank-two tensors made out of κ_μ. We can construct three such objects: $\nabla_\mu \kappa_\nu$, $\kappa_\mu \kappa_\nu$, and $g_{\mu\nu}\kappa_\lambda\kappa^\lambda$. Their variations under a Weyl transformation are

$$\begin{aligned}
\delta\big(\nabla_\mu\kappa_\nu\big) &= \nabla_\mu\omega_\nu - g_{\mu\nu}\big(\kappa_\lambda\omega^\lambda - \omega_\lambda\omega^\lambda\big) + 2\big(\omega_\mu\omega_\nu - \kappa_{(\mu}\omega_{\nu)}\big)\,, \\
\delta\big(\kappa_\mu\kappa_\nu\big) &= \omega_\mu\omega_\nu - 2\kappa_{(\mu}\omega_{\nu)}\,, \\
\delta\big(g_{\mu\nu}\kappa_\lambda\kappa^\lambda\big) &= g_{\mu\nu}\big(\omega_\lambda\omega^\lambda - 2\kappa_\lambda\omega^\lambda\big)\,,
\end{aligned} \tag{4.1}$$

where ω is the local scale factor defined as part of the Weyl transformation in (3.5) and we have defined $\omega_\mu \equiv \partial_\mu\omega$. From these variations we can construct a combination of the three tensors whose variation is totally independent of κ_μ. We call it $\Theta_{\mu\nu}$.

$$\Theta_{\mu\nu} \equiv \nabla_\mu\kappa_\nu - \kappa_\mu\kappa_\nu + \frac{1}{2}g_{\mu\nu}\kappa_\lambda\kappa^\lambda \tag{4.2}$$

$$\delta\Theta_{\mu\nu} = -\nabla_\mu\omega_\nu + \omega_\mu\omega_\nu - \frac{1}{2}g_{\mu\nu}\omega_\lambda\omega^\lambda \tag{4.3}$$

ω is the parameter of a spacetime transformation, so we can expect that some combination of curvature tensors transforms under a Weyl transformation in the same way as $\Theta_{\mu\nu}$. To find this combination, we first calculate the variation of the Ricci tensor

$$\delta R_{\mu\nu} = g_{\mu\nu}\nabla_\lambda\nabla^\lambda\omega + (n-2)\big(\nabla_\mu\omega_\nu - \omega_\mu\omega_\nu + g_{\mu\nu}\omega_\lambda\omega^\lambda\big)\,, \tag{4.4}$$

where n is the spacetime dimension. This is clearly not equal to $\delta\Theta_{\mu\nu}$, however, we can also calculate that

$$\begin{aligned}
\delta\big(g_{\mu\nu}R\big) &= (n-1)\Big(2g_{\mu\nu}\nabla_\lambda\nabla^\lambda\omega + (n-2)\big(4\nabla_\mu\omega_\nu - 4\omega_\mu\omega_\nu + 3g_{\mu\nu}\omega_\lambda\omega^\lambda\big)\Big) \\
&= 2(n-1)\Big(\delta R_{\mu\nu} - (n-2)\delta\Theta_{\mu\nu}\Big)\,,
\end{aligned} \tag{4.5}$$

and so it turns out that the correct combination of $R_{\mu\nu}$ and R is none other than the well-known Schouten tensor $S_{\mu\nu}$.

$$S_{\mu\nu} \equiv \frac{1}{(n-2)}\Big(R_{\mu\nu} - \frac{1}{2(n-1)}g_{\mu\nu}R\Big) \tag{4.6}$$

Thus, we have arrived at our desired relation.

$$\delta S_{\mu\nu} = \delta\Theta_{\mu\nu} \tag{4.7}$$

This is a significant result. It tells us that as long as κ_μ appears in the action only in the combination $\Theta_{\mu\nu}$ or ${\Theta_\lambda}^\lambda$, it can be replaced by terms proportional to $S_{\mu\nu}$ and ${S_\mu}^\mu$ without spoiling gauge invariance. Making this substitution is what we call "Ricci gauging". One might think that this only applies to very specific actions where κ_μ appears only in the exact combination $\Theta_{\mu\nu}$. However, it turns out that in any theory that is conformally invariant in the flat space limit, one can write the κ_μ dependence entirely in terms of $\Theta_{\mu\nu}$ and $\Theta_\lambda{}^\lambda$ [Iorio et al., 1997]. These types of Weyl-invariant actions then also admit Ricci gauging and allow us to make the replacements

$$\Theta_{\mu\nu} \;\to\; S_{\mu\nu} \qquad\qquad \Theta_\lambda{}^\lambda \;\to\; S_\lambda{}^\lambda = \frac{R}{3(n-2)} \,. \tag{4.8}$$

Of course this is not valid for $n = 2$ spacetime dimensions. We should also note that the reverse replacement is not valid since $S_{\mu\nu}$ is subject to the Bianchi identity

$$\nabla^\mu S_{\mu\nu} - \nabla_\nu S_\mu{}^\mu = 0 \,, \tag{4.9}$$

which $\Theta_{\mu\nu}$ does not satisfy.

To see how Ricci gauging works in practice, let us consider the globally scale-invariant Lagrangian describing a massless scalar in curved four dimensional space.

$$\mathcal{L}_{global} = \frac{1}{2}\sqrt{|g|}\partial_\mu\phi\partial^\mu\phi \tag{4.10}$$

To Weyl gauge this expression and make it locally Weyl-invariant, we use the Weyl connection and promote the partial derivatives to covariant derivatives.

$$\begin{aligned}\mathcal{L}_{local} &= \frac{1}{2}\sqrt{|g|}\tilde{\nabla}_\mu\phi\tilde{\nabla}^\mu\phi = \frac{1}{2}\sqrt{|g|}\left(\nabla_\mu + \kappa_\mu\right)\phi\left(\nabla^\mu + \kappa^\mu\right)\phi \\ &= \frac{1}{2}\sqrt{|g|}\Big(\partial_\mu\phi\partial^\mu\phi - \big(\nabla_\mu\kappa^\mu \; - \kappa_\mu\kappa^\mu\big)\phi^2\Big)\end{aligned} \tag{4.11}$$

In the second line above, we have performed an integration by parts and used that $\nabla_\mu\phi = \partial_\mu\phi$, which is true even in curved space[1]. This theory is now invariant under local Weyl transformations thanks to the κ_μ terms whose transformation properties compensate for the extra terms that arise when locally transforming $\partial_\mu\phi\partial^\mu\phi$. Ricci gauging tells us that we can replace the κ_μ compensator with a different compensator containing only functions of the metric. Writing our Lagrangian in terms of (4.2) and performing a Ricci gauging we find the following.

$$\begin{aligned}\mathcal{L}_{local} &= \frac{1}{2}\sqrt{|g|}\Big(\partial_\mu\phi\partial^\mu\phi - \Theta_\mu{}^\mu\phi^2\Big) \\ &\to\; \frac{1}{2}\sqrt{|g|}\Big(\partial_\mu\phi\partial^\mu\phi - \frac{1}{6}R\phi^2\Big)\end{aligned} \tag{4.12}$$

This is precisely the well known Lagrangian for a "conformally coupled" scalar. It is locally Weyl-invariant just like its Weyl-gauged counterpart, even though it does not contain any κ_μ's. The Ricci scalar term transforms in just the right way to serve as a compensator.

[1] This is just the statement that scalar fields carry no direction (they are not vectors), so they are always invariant under parallel transport. However, they do change under scale transformations since they have a magnitude, so the κ_μ part of the Weyl connection does not drop out.

4.1.1 Ricci-gauged Weyl space

We now have a way to replace dependence on κ_μ with dependence on $g_{\mu\nu}$ while still maintaining local Weyl invariance, so let's see if we can apply this knowledge to WQG by Ricci gauging the Weyl space tensors defined in (3.10) - (3.13). First, we need to write these tensors in terms of $\Theta_{\mu\nu}$. As expected, we can express all of their κ_μ dependence in this way if we note that the field strength $F_{\mu\nu}$ is proportional to the anti-symmetric part of $\Theta_{\mu\nu}$.

$$F_{\mu\nu} = 2\Theta_{[\mu\nu]} \tag{4.13}$$

$$\tilde{R} = R - 6\Theta_\lambda{}^\lambda \tag{4.14}$$

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - 3\Theta_{\mu\nu} + \Theta_{\nu\mu} - g_{\mu\nu}\Theta_\lambda{}^\lambda \tag{4.15}$$

$$\tilde{R}_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + g_{\mu\sigma}\Theta_{\nu\rho} + g_{\nu\rho}\Theta_{\mu\sigma} - 2g_{\rho\sigma}\Theta_{[\mu\nu]} - g_{\mu\rho}\Theta_{\nu\sigma} - g_{\nu\sigma}\Theta_{\mu\rho} \tag{4.16}$$

$$\tilde{C}_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} - 2g_{\rho\sigma}\Theta_{[\mu\nu]} + g_{\mu\rho}\Theta_{[\nu\sigma]} + g_{\nu\sigma}\Theta_{[\mu\rho]} - g_{\mu\sigma}\Theta_{[\nu\rho]} - g_{\nu\rho}\Theta_{[\mu\sigma]} \tag{4.17}$$

Something very interesting occurs when we perform a Ricci gauging of these tensors. Setting $n = 4$, we have the following.

$$F_{\mu\nu} \rightarrow 0 \tag{4.18}$$

$$\tilde{R} \rightarrow 0 \tag{4.19}$$

$$\tilde{R}_{\mu\nu} \rightarrow 0 \tag{4.20}$$

$$\tilde{R}_{\mu\nu\rho\sigma} \rightarrow C_{\mu\nu\rho\sigma} \tag{4.21}$$

$$\tilde{C}_{\mu\nu\rho\sigma} \rightarrow C_{\mu\nu\rho\sigma} \tag{4.22}$$

We see here that employing Ricci gauging in Weyl space renders all measures of curvature equal to zero or to the standard Weyl tensor and so we are left with the following conclusion.

If λ_i are arbitrary dimensionless constants that parameterize (globally scale-invariant) quadratic gravity with the action

$$S_{QG} = \int d^4x\sqrt{|g|}\Big(\lambda_1 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \lambda_2 R_{\mu\nu}R^{\mu\nu} + \lambda_3 R^2\Big)\,, \tag{4.23}$$

then we can make this theory Weyl-invariant by introducing κ_μ via the Weyl connection, which promotes this action to that of Weyl-quadratic gravity.

$$
\begin{aligned}
S_{WQG} &= \int d^4x\sqrt{|g|}\Big(\lambda_1\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + \lambda_2\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + \lambda_3\tilde{R}^2\Big) \\
&= \int d^4x\sqrt{|g|}\Big(\lambda_1 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \lambda_2 R_{\mu\nu}R^{\mu\nu} + \lambda_3 R^2 \\
&\qquad\qquad - \big(4\lambda_2 + 8\lambda_3\big)R_{\mu\nu}\Theta^{\mu\nu} - \big(12\lambda_1 + 2\lambda_2\big)R\Theta_\mu{}^\mu \\
&\qquad\qquad + \big(10\lambda_2 + 16\lambda_3\big)\Theta_{\mu\nu}\Theta^{\mu\nu} - \big(6\lambda_2 + 8\lambda_3\big)\Theta_{\mu\nu}\Theta^{\nu\mu} \\
&\qquad\qquad + \big(36\lambda_1 + 8\lambda_2 + 4\lambda_3\big)\Theta_\mu{}^\mu\Theta_\nu{}^\nu\Big)
\end{aligned}
\tag{4.24}
$$

Using Ricci gauging, we can trade dependence on $\Theta_{\mu\nu}$ in this action for dependence on $S_{\mu\nu}$. Thus, in $n = 4$ spacetime dimensions, this seemingly complicated action reduces to none other than the action for conformal gravity[2].

$$
\begin{aligned}
S_{WQG} \;\; \rightarrow & \;\; \int d^4x\sqrt{|g|}\Big(\lambda_1 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \lambda_2 R_{\mu\nu}R^{\mu\nu} + \lambda_3 R^2 \\
&\qquad\qquad - \big(\lambda_2 + \lambda_3\big)R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}\big(3\lambda_1 - \lambda_3\big)R^2\Big) \\
&= \int d^4x\sqrt{|g|}\lambda_1\Big(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\Big) \\
&= \int d^4x\sqrt{|g|}\lambda_1 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = S_{CG}
\end{aligned}
\tag{4.25}
$$

It appears that the two Weyl-invariant theories we studied in the last section are intimately related after all. However, it is perhaps not immediately clear how to interpret the nature of their relationship. If we had found an algebraic expression of the form $\Theta_{\mu\nu} = S_{\mu\nu}$, that would be one thing; we could simply say that the two theories are equivalent. This is not the case however, since it is the *variations* of these objects that are equivalent. To better understand what is going on here, we need to take a step back and investigate what is actually happening from the gauge theory perspective.

4.2 Gauge fixing Weyl quadratic gravity

Let us put Ricci gauging aside for a moment and recast the relationship between S_{WQG} and S_{CG} in a more canonical light. The whole point of introducing compensating terms in a gauge theory is so that, for any local gauge parameter $\omega(x)$,

[2] There is no need to exploit the Gauss-Bonnet invariant in this calculation, it works in general.

our theory remains invariant. This means we can arrive at a relation between the compensator and $\omega(x)$ by equating $\mathcal{L}'_{QG}$ and $\mathcal{L}_{WQG}$.

$$\begin{aligned}\mathcal{L}_{QG}[g_{\mu\nu}] &\rightarrow \mathcal{L}'_{QG}[g_{\mu\nu}] \\ &\qquad = \mathcal{L}_{QG}[g_{\mu\nu}] + \delta\mathcal{L}_{QG}[g_{\mu\nu},\omega] \overset{!}{=} \mathcal{L}_{QG}[g_{\mu\nu}] + K[g_{\mu\nu},\kappa_\mu] = \mathcal{L}_{WQG} \\ \delta\mathcal{L}_{QG}[g_{\mu\nu},\omega] &\overset{!}{=} K[g_{\mu\nu},\kappa_\mu]\,, \end{aligned} \tag{4.26}$$

where $K[g_{\mu\nu},\kappa_\mu]$ is the compensator familiar from equation (4.24).

$$\begin{aligned} K[g_{\mu\nu},\kappa_\mu] \equiv \sqrt{|g|}\Big(&-\big(4\lambda_2+8\lambda_3\big)R_{\mu\nu}\Theta^{\mu\nu} - \big(12\lambda_1+2\lambda_2\big)R\Theta_\mu{}^\mu \\ &+ \big(10\lambda_2+16\lambda_3\big)\Theta_{\mu\nu}\Theta^{\mu\nu} - \big(6\lambda_2+8\lambda_3\big)\Theta_{\mu\nu}\Theta^{\nu\mu} \\ &+ \big(36\lambda_1+8\lambda_2+4\lambda_3\big)\Theta_\mu{}^\mu\Theta_\nu{}^\nu\Big) \end{aligned} \tag{4.27}$$

$\delta\mathcal{L}_{QG}[g_{\mu\nu},\omega]$ is very lengthy when expanded out and its form is not particularly enlightening, so we do not show it here. Suffice to say that after some simplifications, (4.26) simplifies to the following.

$$\nabla_\mu\partial^\mu\omega + \partial_\mu\omega\partial^\mu\omega \overset{!}{=} \Theta_\mu{}^\mu \tag{4.28}$$

This equation tells us exactly how the extra degrees of freedom, κ_μ, must relate to the transformation parameter ω so that our theory always remains locally Weyl-invariant.

Now, as an unfixed gauge theory, $\mathcal{L}_{WQG}$ necessarily contains redundant degrees of freedom. We introduced them in order to make Weyl symmetry local. In order to get any interesting physical predictions out of a gauge theory, we must choose a gauge in order to select a particular physical configuration out of the whole equivalence class of theories characterized by our gauge field. For example, in electrodynamics we often select the Coulomb gauge $\nabla\cdot A(t,\vec{r}) = 0$ in order to reduce the full field equations to $A(t,\vec{r}) = \frac{1}{2}\vec{B}\times\vec{r}$ and identify the photon as electromagnetic radiation. We have a large amount of freedom in how to select a gauge, but selecting one that actually simplifies calculations is not a trivial task. However, we can make an educated guess as to what a useful gauge choice will be based on the identical transformations of $\Theta_{\mu\nu}$ and $S_{\mu\nu}$. So, let us select a particular configuration of $\mathcal{L}_{WQG}$ by selecting the gauge $\Theta_\mu{}^\mu = S_\mu{}^\mu = \frac{R}{6}$.

$$\begin{aligned} &\mathcal{L}_{QG}[g_{\mu\nu}] + \delta\mathcal{L}_{QG}[g_{\mu\nu},\omega]\Big|_{\nabla_\mu\partial^\mu\omega+\partial_\mu\omega\partial^\mu\omega=\frac{R}{6}} \\ &= \mathcal{L}_{QG}[g_{\mu\nu}] - \sqrt{|g|}\Big(\big(\lambda_2+\lambda_3\big)R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}\big(3\lambda_1-\lambda_3\big)R^2\Big) \\ &= \int d^4x\sqrt{|g|}\lambda_1\Big(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\Big) = \mathcal{L}_{CG}[g_{\mu\nu}] \end{aligned} \tag{4.29}$$

This is precisely the end result of Ricci gauging, thus, it is easy to see the whole process is nothing more than selecting a particular gauge where the resulting gauge-fixed theory (conformal gravity) happens to maintain all of the same symmetry as its un-gauged predecessor (Weyl quadratic gravity). Fundamentally, fixing a gauge means that we impose a constraint on the redundant gauge boson degrees of freedom that we introduced and this is precisely what we have done by setting the particular combination $\Theta_{\mu\nu}[\kappa_\mu]$ equal to the combination $S_{\mu\nu}[g_{\mu\nu}]$, which is composed of fields that we already had in our spectrum ($g_{\mu\nu}$). The reverse operation does not constitute a gauge fixing since we increase degrees of freedom when we go from conformal gravity to Weyl quadratic gravity.

5 Discussion and Prospects

The connection between conformal and Weyl quadratic gravity that we discovered in the last chapter still needs some interpretation. Traditionally, if we want to promote a theory from globally to locally invariant, we must introduce extra degrees of freedom in the form of gauge fields. We have seen that this also works in our case; we can make standard quadratic gravity locally Weyl-invariant by using the Weyl connection and introducing κ_μ into the covariant derivative. This amounts to adding the compensator $K[g_{\mu\nu}, \kappa_\mu]$ to the Lagrangian, which gives us two additional degrees of freedom. However, there is in fact another procedure for making quadratic gravity locally Weyl-invariant. Instead of adding degrees of freedom, we can also *remove* them - this is the end result of Ricci gauging. This type of behavior does not appear in more traditional gauge theories. For example, there is no way to make a globally $U(1)$-invariant theory locally invariant by removing fermionic degrees of freedom. It is the combination of fourth order derivatives and the fact that we are dealing with an external symmetry that makes our theory a special case.

As we demonstrated in equations (4.10) - (4.12), Ricci gauging in the form presented by its author is meant to allow one to replace dependence on κ_μ with dependence on $g_{\mu\nu}$ when gauging Weyl invariance. There is no difference between the number of degrees of freedom added to a matter theory when it is Weyl-gauged or Ricci-gauged, the two procedures simply present the choice between using the graviton or κ_μ as a gauge field. What we have done is use Ricci gauging in a situation that it was not originally designed for. Our theory already contained graviton degrees of freedom, and replacing κ_μ with fields that were already present in our theory led to cancellations and the appearance of the conformal gravity action. Rather than swapping dependence on different gauge fields as Ricci gauging was intended to do, we instead used it to exploit the fact that $\delta S_{\mu\nu} = \delta\Theta_{\mu\nu}$ and identify a consistent gauge choice that brought us from Weyl quadratic to conformal gravity. We can view this whole process as another derivation of the conformal gravity action. Conformal gravity is not only the unique theory of the gauged conformal group (as claimed by Kaku et al. [1977]), it is also a particular physical configuration of Weyl quadratic gravity.

It is interesting that, as confirmed by direct computation, conformal gravity is still locally Weyl-invariant. No symmetry has been lost by gauge fixing WQG, yet we were able to remove degrees of freedom. This is just another indication that WQG contains redundant degrees of freedom, but it also tells us that our gauge choice is "incomplete". This is not a problem, in fact this is a common feature of more traditional gauge theories. If one fixes the Lorenz gauge in electrodynamics, the resulting field equations are still Lorentz-invariant for example.

The fact that we have redundant degrees of freedom is significant, because it means that we still have the freedom to select a gauge that corresponds to a different convenient physical configuration. This is particularly interesting in the case of Weyl quadratic gravity. We have seen that it is possible to maintain local Weyl-invariance in a theory through the inclusion of κ_μ or $g_{\mu\nu}$, and WQG already contains both fields. We have just found a gauge that allowed us to drop our dependence on κ_μ, so in principle, it should be possible to identify a different gauge that removes the extra spin-2 degrees of freedom instead. The resulting theory would essentially serve as a dual to conformal gravity.

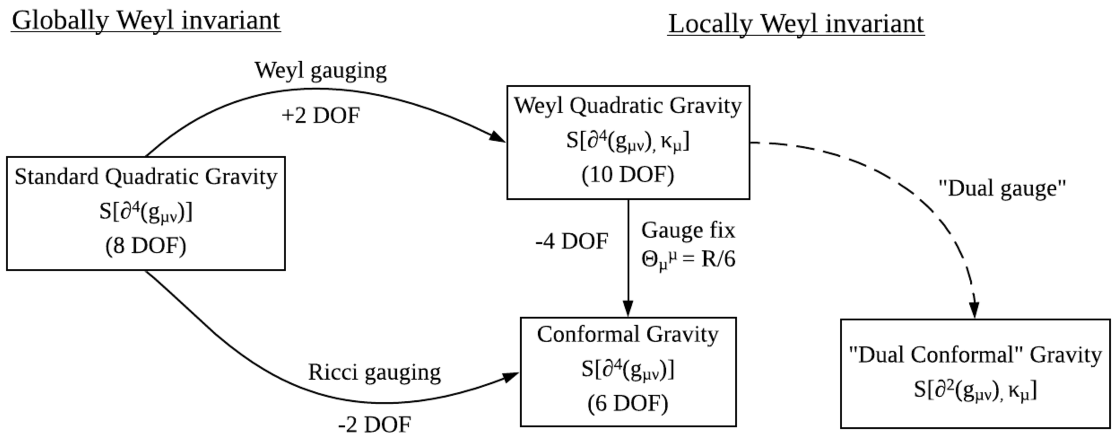

Figure 5.1: The paths to local Weyl invariance

This theoretical "dual gauge" is attractive for a few reasons. First, by removing spin-2 two degrees of freedom, we would necessarily remove some of the ghost states from our spectrum, without the need to rely on the PT/anti-Hermitian proofs that are used in conformal gravity. Of course it may turn out that these different ways around the ghost problem are related in some way as well. Additionally, assuming that we can still use the same type of tricks with the dilaton that we used in Section 3.3, the "dual conformal" theory would have all of the benefits of having a massive $\hat{\kappa}_\mu$ in the spectrum that we saw in Section 3.4. Most notably, it would have the ability to dynamically generate the Planck mass and electro-weak scale, as well as put forth a dark matter candidate. It is in this sense that we call this theory a dual. It corresponds nicely to the low-energy regime after the spontaneous breakdown of scale symmetry, while regular conformal gravity represents the high-energy regime with different degrees of freedom and intact conformal symmetry.

It is no trivial task to identify such a dual gauge however, and even the existence of such an alluring theory is not guaranteed. Physics is notorious for shooting down even the most promising theories. At the end of the day, it is Nature who makes the final call on what is correct and physical, not us. However, this author is very excited about the prospects of the Weyl-symmetric setups that we have presented here and looks forward to pursuing them in much greater detail.

6 Conclusion

The main focus of this thesis has been to investigate modifications to our current understanding of gravitation, under the guiding principle of local scale invariance, so that it might better fit into our modern understanding of physics as described by quantum field theory. Despite the fact that we do not see scale invariance in our everyday low-energy regime, there is still very good reason to expect that physics becomes insensitive to changes in scale as the energies involved move into the UV. In line with standard theories of gravity that are symmetric under the Poincaré group, we chose to consider its natural extension to the conformal group. This group includes the well-tested set of Poincaré spacetime symmetries as well as scale and special conformal symmetry, so it serves as a perfect way to formalize our desired goal of scale invariance. It is also the largest symmetry group of the light cone $ds^2 = 0$ and of Maxwell's equations, which makes it an even more aesthetic choice for the foundation of a theory of gravity. Finally, by also using the fact that we can express conformal transformations as Weyl transformations, which are transformations that act directly on the fields, we established the starting point for our search.

With a brief review of Einstein's General Relativity and Yang-Mills gauge theory, we acquired the necessary tools to construct the type of theory we were after. The first such theory was based on a formal gauging of the conformal group, and is known as conformal gravity. CG has many attractive features including the ability to account for dark energy and dark matter without the addition of extra matter fields, but perhaps most notably, it has the potential to serve as a theory of quantum gravity due to the fact that it is UV-complete and power-counting renormalizable. However, we also saw that this theory may contain Ostrogradsky ghosts and that more research needs to be done before we can determine how large of an issue this actually presents.

Next, we adopted a slightly different procedure to show that a locally Weyl-invariant theory of quadratic gravity can be constructed by gauging the global Weyl symmetry present in the well-studied theories of quadratic gravity. By introducing compensating terms containing the gauge boson κ_μ into the spacetime connection, we were able to define the notion of Weyl space and the theory of Weyl quadratic gravity. We saw that this theory shares many features with conformal gravity, including renormalizability and the fourth order derivatives that lead to ghost states. However, unlike conformal gravity, Weyl quadratic gravity also exhibits spontaneous symmetry breaking as a result of dimensional transmutation and we saw that this dynamical breaking can generate the very notion of scale itself.

In the final part of this thesis, we chose to search for a connection between our two theories. After all, given that Weyl invariance always implies conformal invariance, it only seems natural that a gauge theory of the conformal group would be intimately related to a theory based on gauged Weyl symmetry. The Ricci gauging procedure ended up providing exactly the type of relationship that we were looking for and we used its key result, $\delta\Theta_{\mu\nu} = \delta S_{\mu\nu}$, to find that conformal gravity can be viewed as a gauge-fixed version of Weyl quadratic gravity. By spending some redundant gauge degrees of freedom, we showed that our first theory was just a particular physical configuration of the second.

The nature of the relationship between conformal gravity and Weyl quadratic gravity has some exciting implications. We speculate that there exists a different physical configuration of Weyl quadratic gravity that could serve as a dual to conformal gravity. This is based on the fact that either the metric $g_{\mu\nu}$ or the gauge boson κ_μ can be utilized to ensure the Weyl invariance of a theory. Weyl quadratic gravity contains both of these, and we saw that by gauging away the bosons we arrived at conformal gravity. In principle then, we should instead be able to gauge away redundant metric degrees of freedom in Weyl quadratic gravity, to land at a new theory containing κ_μ and only second derivatives of $g_{\mu\nu}$. It is too early to say for sure, but the existence of such a theory could help to dispel the currently known issues with both conformal and Weyl quadratic gravity.

Part III

Appendix

A The vierbein formalism

While theories of curved space are often formulated in the metric formalism that we introduced in Section 1.2, this is not the only option. The vierbein formalism that we used to define gravitational gauge fields in Section 2.1 is perhaps even more powerful, though often less convenient than the metric formalism when doing a classical treatment of gravity. In this chapter we present a short overview of the vierbein formulation of curved space and its ability to consistently describe the behavior of spinor fields on a curved background. We follow the works by Yepez [2011] and Shapiro [2016].

Given a curved Riemannian manifold $\mathcal{M}$ with metric $g_{\mu\nu}(x)$, we begin by assigning an additional flat Minkowski metric η_{ab} to every point $x \in \mathcal{M}$. As usual, this flat metric is defined as the inner product of basis vectors $\eta_{ab} = \vec{e}_a \cdot \vec{e}_b$. These vectors parameterize the tangent manifold $\mathcal{T}_x\mathcal{M}$ at each point to which we assign the coordinates ξ^a. Since the curved space metric is defined as an inner product of basis vectors on $\mathcal{M}$, $g_{\mu\nu}(x) = \vec{e}_\mu(x) \cdot \vec{e}_\nu(x)$, we now define the vierbein ${e_\mu}^a(x)$ as a function that translates between the local basis on $\mathcal{M}$ and the basis on the tangent manifold $\mathcal{T}_x\mathcal{M}$ at each point x.

$$\vec{e}_\mu(x) = {e_\mu}^a(x)\vec{e}_a \tag{A.1}$$

In this sense, the vierbein and its inverse are nothing more than a change of basis[1].

$$ {e_\mu}^a = \frac{\partial \xi^a}{\partial x^\mu} \qquad {e^\mu}_a = \frac{\partial x^\mu}{\partial \xi^a} \tag{A.2}$$

Just as we use the metric to raise or lower indices, we can now use the vierbein to convert indices back and forth from flat space (Latin indices) to spacetime (Greek indices). This allows us to derive the following relations.

$$\begin{gathered} g_{\mu\nu} = {e_\mu}^a {e_\nu}^b \eta_{ab} = {e_\mu}^a e_{\nu a} \qquad \eta_{ab} = {e^\mu}_a {e^\nu}_b g_{\mu\nu} = {e^\mu}_a e_{\mu b} \\ \delta^\mu_\nu = {e^\mu}_a {e_\nu}^a \qquad \delta^a_b = {e_\mu}^a {e^\mu}_b \qquad \sqrt{|g|} = \det({e^\mu}_a) \equiv |e| \end{gathered} \tag{A.3}$$

Any vector $\vec{V}$ can be written in terms of either basis as $\vec{V} = V^a\vec{e}_a = V^\mu\vec{e}_\mu$, and we can use the vierbein to relate the different components.

$$V^a = {e_\mu}^a V^\mu \qquad V^\mu = {e^\mu}_a V^a \tag{A.4}$$

[1] From here on out we suppress dependence on x^μ to reduce clutter, but we should always remember that $g_{\mu\nu}(x)$ and ${e_\mu}^a(x)$ are functions that depend on spacetime coordinates, while η_{ab} is a constant.

As we have seen throughout this work, we often run across objects that carry both types of indices. We are not able to use the standard covariant derivative on these type of objects, since the standard Christoffel symbols only carry spacetime indices, so we need some new kind of connection. The new connection coefficients need to fill the same role as the Christoffel symbols, namely, if the standard covariant derivative looks like $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda} V^\lambda$, then our new coefficients $\chi_\mu{}^a{}_b$ must satisfy

$$\nabla_\mu V^a = \partial_\mu V^a + \chi_\mu{}^a{}_b V^b \,. \tag{A.5}$$

Unfortunately, finding an expression for these coefficients in terms of known quantities is not as simple as multiplying by a few vierbeins since the chain rule will come into play. This is directly related to the fact that the Christoffel symbols do not transform covariantly. However, we can solve for $\chi_\mu{}^a{}_b$ by matching the covariant derivative of a vector $\vec{V}$ in the mixed and non-mixed bases.

$$\begin{aligned}
\nabla\vec{V} &= \Big(\nabla_\mu V^a\Big) dx^\mu \otimes \vec{e}_a \\
&= \Big(\partial_\mu V^a + \chi_\mu{}^a{}_b V^b\Big) dx^\mu \otimes \vec{e}_a \\
&= \Big(\partial_\mu\big(e_\lambda{}^a V^\lambda\big) + \chi_\mu{}^a{}_b e_\sigma{}^b V^\sigma\Big) dx^\mu \otimes e^\nu{}_a \vec{e}_\nu \\
&= \Big(e^\nu{}_a V^\lambda \partial_\mu e_\lambda{}^a + e^\nu{}_a e_\lambda{}^a \partial_\mu V^\lambda + e^\nu{}_a \chi_\mu{}^a{}_\sigma V^\sigma\Big) dx^\mu \otimes \vec{e}_\nu \\
&= \Big(\partial_\mu V^\nu + \big(e^\nu{}_a \partial_\mu e_\lambda{}^a + \chi_\mu{}^\nu{}_\lambda\big) V^\lambda\Big) dx^\mu \otimes \vec{e}_\nu
\end{aligned} \tag{A.6}$$

$$\nabla\vec{V} = \Big(\partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda} V^\lambda\Big) dx^\mu \otimes \vec{e}_\nu \tag{A.7}$$

While the three-index coefficients do not match up directly, we can easily spot the relation

$$\begin{aligned}
\Gamma^\nu{}_{\mu\lambda} &= e^\nu{}_a \partial_\mu e_\lambda{}^a + \chi_\mu{}^\nu{}_\lambda \\
&= e^\nu{}_a \partial_\mu e_\lambda{}^a + e^\nu{}_a e_\lambda{}^b \chi_\mu{}^a{}_b \,,
\end{aligned} \tag{A.8}$$

and solve for the connection coefficients.

$$\chi_\mu{}^a{}_b = e_\nu{}^a e^\lambda{}_b \Gamma^\nu{}_{\mu\lambda} - e^\lambda{}_b \partial_\mu e_\lambda{}^a \tag{A.9}$$

These new coefficients define what we call the spin connection, and we use it to calculate the covariant derivative of tensors with mixed indices in the following way.

$$\nabla_\mu V_\nu{}^a = \partial_\mu V_\nu{}^a - \Gamma^\lambda{}_{\mu\nu} V_\lambda{}^a + \chi_\mu{}^a{}_b V_\nu{}^b \tag{A.10}$$

If we use this to take the covariant derivative of the vierbein and insert (A.8), we arrive at the vierbein postulate.

$$\begin{aligned}
\nabla_\mu e_\nu{}^a &= \partial_\mu e_\nu{}^a - e_\lambda{}^a \Gamma^\lambda{}_{\mu\nu} + e_\nu{}^b \chi_\mu{}^a{}_b \\
&= \partial_\mu e_\nu{}^a - \Big(\delta^a_b \partial_\mu e_\nu{}^b + \delta^a_b e_\nu{}^b \chi_\mu{}^a{}_b\Big) + e_\nu{}^b \chi_\mu{}^a{}_b = 0
\end{aligned} \tag{A.11}$$

It is easy to show that this is equivalent to the metricity condition we find in GR.

$$\begin{aligned}\nabla_\lambda g_{\mu\nu} &= \nabla_\lambda\big(e_\mu{}^a e_\nu{}^b \eta_{ab}\big)\\ &= \eta_{ab}\big(e_\nu{}^b\nabla_\lambda e_\mu{}^a + e_\mu{}^a\nabla_\lambda e_\nu{}^b\big) = 0\end{aligned} \tag{A.12}$$

In another parallel to GR, we can write the spin connection coefficients as a function of the vierbein if we enforce that the connection is torsion-free.

$$\chi_\mu{}^{ab} = e^{\nu a}\partial_{[\mu}e_{\nu]}{}^b - e^{\nu b}\partial_{[\mu}e_{\nu]}{}^a - e^{\rho a}e^{\sigma b}e_{\mu c}\partial_{[\rho}e_{\sigma]}{}^c \tag{A.13}$$

With a bit of algebra one can show that this is just the same as saying we can write the Christoffel symbols as a functional of the metric given the same condition.

The Riemann tensor (and its contractions) transform covariantly, so writing its mixed-index counterpart is as simple as applying a few vierbeins.

$$R_{\mu\nu}{}^{ab} = e^{\rho a}e^{\sigma b}R_{\mu\nu\rho\sigma} \tag{A.14}$$

This is simple enough to prove using the Levi-Cevita definition of the Riemann tensor, equation (A.5), and a bunch of algebra. The enlightening part of this calculation is that the spin connection coefficients follow the exact same pattern as the Christoffel symbols when defining their respective versions of the Riemann tensor.

$$\begin{aligned}\big[\nabla_\mu,\nabla_\nu\big]V^\alpha &= \big(\partial_\mu\Gamma^\alpha{}_{\beta\nu} - \partial_\nu\Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\lambda\mu}\Gamma^\lambda{}_{\nu\beta} - \Gamma^\alpha{}_{\nu\lambda}\Gamma^\lambda{}_{\beta\mu}\big)V^\beta\\ &= R^\alpha{}_{\beta\mu\nu}V^\beta\end{aligned} \tag{A.15}$$

$$\begin{aligned}\big[\nabla_\mu,\nabla_\nu\big]V^a &= \big(\partial_\mu\chi_\nu{}^a{}_b - \partial_\nu\chi_\mu{}^a{}_b + \chi_{\mu c}{}^a\chi_\nu{}^c{}_b - \chi_{\nu c}{}^a\chi_\mu{}^c{}_b\big)V^b\\ &= R_{\mu\nu}{}^a{}_b V^b\end{aligned} \tag{A.16}$$

The vierbein formalism becomes particularly useful when gravity is formulated as a gauge theory because it allows us to decompose the full affine connection corresponding to the Poincaré group into its translation and Lorentz-symmetric pieces. We consider the vierbein and spin connection as separate potentials, each belonging to their own symmetry; translations and spacetime rotations (Lorentz transformations) respectively. This is useful because, as we saw in Section 2.1, we often see constraints that correspond to only one or the other symmetry. These constraints allow us to find algebraic relationships between our gauge fields and eliminate the non-propagating fields from the action. This would not be possible in the standard metric formalism where translation and Lorentz potentials are mixed up together in the affine connection.

In order to extend all of the results above to Weyl space, we just need to promote all of the partial derivatives as we did in Section 3.2 [Drechsler and Tann, 1999].

$$\partial_\mu e_\nu{}^a \quad\rightarrow\quad D_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \kappa_\mu e_\nu{}^a \tag{A.17}$$

We define the full Weyl space connection as $\tilde{\nabla}_\mu \Psi = \big(\nabla_\mu + q[\Psi]\kappa_\mu\big)\Psi$ and use it to derive the Weyl space non-metricity condition

$$\tilde{\nabla}_\mu e_\nu{}^a = \big(\nabla_\mu + \kappa_\mu\big) e_\nu{}^a = \kappa_\mu e_\nu{}^a \,, \tag{A.18}$$

the Weyl space spin connection

$$\tilde{\chi}_\mu{}^a{}_b = \chi_\mu{}^a{}_b + \big(e^{\lambda a} e_{\mu b} - e^\lambda{}_b e_\mu{}^a\big)\kappa_\lambda + \delta^a_b \kappa_\mu \,, \tag{A.19}$$

and the Weyl space Riemann tensor

$$\tilde{R}_{\mu\nu}{}^{ab} = \partial_\mu \tilde{\chi}_\nu{}^{ab} - \partial_\nu \tilde{\chi}_\mu{}^{ab} + \tilde{\chi}_{\mu c}{}^a \tilde{\chi}_\nu{}^{cb} - \tilde{\chi}_{\nu c}{}^a \tilde{\chi}_\mu{}^{cb} \,. \tag{A.20}$$

Of course this all fits nicely into the gauge theory picture as well since can view κ_μ as the gauge potential associated with the new Weyl symmetry.

A.1 Fermions in curved space

The vierbein formalism allows us to describe gravity as a gauge theory, but is also necessary if we want to properly describe the coupling between fermions and curvature. As we know, a consistent theory of gravity must remain invariant under diffeomorphisms, which is most generally described by the symmetry group $GL(4)$. However, the $\mathfrak{gl}(4)$ algebra does not admit any spinor representations, so there is no way to couple metric-formulated-gravity to fermions without making additional assumptions[2] [Brill and Wheeler, 1957]. In other words, we run into a problem when we try to compute the metric version of the covariant derivative of a spinor because the affine connection is that of the whole $GL(4)$ group, of which there are no spinor representations. We can get around this issue by using the spin connection instead. Since it is the connection on only the tangent manifold $\mathcal{TM} \to SO(1,3)$, it can couple to spinors since the Lorentz group does allow for spinor representations. To put this in another light, we observe that while the flat space Dirac equation is symmetric under the Lorentz group, it is not symmetric under the full $GL(4)$. Thus when we promote the partial derivative to a covariant derivative, the associated connection must also be related to just the Lorentz group. Of course this where the name "spin connection" comes from; it is the spacetime connection used for spinors.

Let us proceed by promoting the flat space Lagrangian for a free fermion to its curved space counterpart.

$$\mathcal{L}_\psi = \bar{\psi}\big(i\gamma^a \partial_a + m\big)\psi \quad \to \quad \sqrt{|g|}\,\bar{\psi}\big(i\gamma^\mu \nabla_\mu + m\big)\psi \tag{A.21}$$

Here, $\gamma^\mu = \gamma^a e^\mu{}_a$ is the curved space version of the standard Dirac gamma matrices which satisfies a curved space version of the Clifford algebra, $\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}$. Now

[2] An example of such an assumption is given in an interesting paper by Woodard [1984]. Here it is shown that the metric formulation is sufficient provided that we fix the Lorentz gauge and stick to perturbation theory.

we follow Shapiro [2016] and consider the relation

$$\nabla_\mu\big(\bar{\psi}\gamma^\nu\psi\big) = \partial_\mu\big(\bar{\psi}\gamma^\nu\psi\big) + \Gamma^\nu{}_{\mu\lambda}\bar{\psi}\gamma\psi\,. \tag{A.22}$$

A few applications of the chain rule along and clever use of some γ matrix identities allows us to isolate the covariant derivative of a single fermion in the vierbein language.

$$\nabla_\mu\psi = \partial_\mu\psi + \frac{i}{2}\chi_\mu{}^{ab}\Sigma_{ab}\psi \;, \qquad \Sigma_{ab} = \frac{i}{2}[\gamma_a,\gamma_b]\,, \tag{A.23}$$

where Σ_{ab} is a representation of the Lorentz algebra that arises naturally as part of the calculation. With this covariant derivative, we finally have all the machinery we need to describe spinors, that necessarily live in flat space, on a curved background.

Finally, we conclude this section by examining how a fermion couples to curvature in Weyl space after κ_μ comes into play. After gauging Weyl symmetry ($\nabla_\mu \to \tilde{\nabla}_\mu$), (A.21) becomes

$$\tilde{\mathcal{L}}_\psi = |e|\bar{\psi}\Big[i\gamma^a e^\mu{}_a\Big(\partial_\mu + \frac{i}{2}\tilde{\chi}_\mu{}^{ab}\Sigma_{ab} + \frac{3}{2}\kappa_\mu\Big) + m\Big]\psi\,. \tag{A.24}$$

If we plug (A.19) into this expression in order to re-express things in the Riemannian language, we see that something quite interesting occurs[3].

$$\begin{aligned}
\frac{i}{2}\tilde{\chi}_\mu{}^{ab}\Sigma_{ab} &= \frac{i}{2}\Big(\chi_\mu{}^{ab} + \big(e^{\nu a}e_\mu{}^b - e^{\nu b}e_\mu{}^a\big)\kappa_\nu + \eta^{ab}\kappa_\mu\Big)\frac{i}{2}[\gamma_a,\gamma_b] \\
&= \frac{i}{2}\chi_\mu{}^{ab}\Sigma_{ab} + \frac{1}{4}\Big(\big(e_\mu{}^a - \gamma^a\gamma_\mu\big)e^\lambda{}_a + \big(e_\mu{}^b - \gamma^b\gamma_\mu\big)e^\lambda{}_b\Big)\kappa_\lambda \\
&= \frac{i}{2}\chi_\mu{}^{ab}\Sigma_{ab} + \frac{1}{2}\big(\delta^\lambda_\mu - \gamma^c\gamma_\mu e^\lambda{}_c\big)\kappa_\lambda \\
&= \frac{i}{2}\chi_\mu{}^{ab}\Sigma_{ab} - \frac{3}{2}\kappa_\mu
\end{aligned} \tag{A.25}$$

The Weyl space spin connection $\tilde{\chi}_\mu{}^{ab}$ contributes an extra κ_μ term that perfectly cancels the other term introduced as part of the full Weyl connection. Practically, this means that the κ_μ-ψ minimal coupling drops out and we need only consider the standard affine spin connection term [de Cesare et al., 2017]. We conclude that fermions are essentially unaffected by the gauging of Weyl invariance (at least at tree level) and that the Weyl space Lagrangian $\tilde{\mathcal{L}}_\psi$ is exactly equivalent to the Riemannian space version in (A.21).

[3] The identities $\{\gamma^a,\gamma^b\} = 2\eta^{ab}, \gamma_a\gamma^a = 4,$ and $\gamma_b\gamma^a\gamma^b = -2\gamma^a$ are invaluable here.

B Oscillator variable decomposition

Here we present the details of the oscillator variable decomposition calculations described in Section 3.4.1. The (metric) gravitational part of the Einstein frame WQG Lagrangian is

$$\mathcal{L}_{grav} = -\sqrt{|g|}\Big(\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{\langle\phi\rangle^2}{2}R\Big)\,. \tag{B.1}$$

We can expand this expression up to second order in terms of the metric perturbations $h_{\mu\nu}$ by writing

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{B.2}$$

and using the spin projectors

$$P^{(2)}_{\mu\nu,\rho\sigma} = \frac{1}{2}\big(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}\big) - P^{(0,s)}_{\mu\nu,\rho\sigma} \qquad\qquad P^{(0,s)}_{\mu\nu,\rho\sigma} = \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}\,, \tag{B.3}$$

where $\theta_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}$. This yields

$$\begin{aligned}\mathcal{L}_{grav}[h_{\mu\nu}] &= -\alpha h^{\mu\nu}P^{(2)}_{\mu\nu,\rho\sigma}\partial^4 h^{\rho\sigma} - \frac{\langle\phi\rangle^2}{4}h^{\mu\nu}\big(P^{(2)}_{\mu\nu,\rho\sigma} - 2P^{(0,s)}_{\mu\nu,\rho\sigma}\big)\partial^2 h^{\rho\sigma}\\ &= \alpha\Big(-\frac{2}{3}h^{\mu\nu}\partial_\mu\partial_\nu\partial_\rho\partial_\sigma h^{\rho\sigma} + 2h^{\mu\nu}\partial_\nu\partial_\rho\partial_\sigma\partial^\sigma h_\mu{}^\rho\\ &\qquad - \frac{2}{3}h^\mu{}_\mu\partial_\nu\partial_\rho\partial_\sigma\partial^\sigma h^{\nu\rho} - h^{\mu\nu}\partial_\rho\partial^\rho\partial_\sigma\partial^\sigma h_{\mu\nu}\\ &\qquad + \frac{1}{3}h^\mu{}_\mu\partial_\rho\partial^\rho\partial_\sigma\partial^\sigma h^\nu{}_\nu\Big) + \frac{\langle\phi\rangle}{4}\Big(2h^{\mu\nu}\partial_\nu\partial_\rho h_\mu{}^\rho - 2h^\mu{}_\mu\partial_\nu\partial_\rho h^{\nu\rho}\\ &\qquad - h^{\mu\nu}\partial_\rho\partial^\rho h_{\mu\nu} + h^\mu{}_\mu\partial_\rho\partial^\rho h^\nu{}_\nu\Big)\,. \end{aligned}\tag{B.4}$$

Now we define the auxiliary field $X_{\mu\nu}$ and make the following ansatz for its Lagrangian.

$$\begin{aligned}\mathcal{L}_X[h_{\mu\nu}, X_{\mu\nu}] =& \lambda_1 X_{\mu\nu}\partial^\nu\partial_\rho h^{\mu\rho} + \lambda_2 X_\rho{}^\rho\partial_\mu\partial_\nu h^{\mu\nu} + \lambda_3 X_{\mu\nu}\partial_\rho\partial^\rho h^{\mu\nu}\\ &+ \lambda_4 X_{\mu\nu}\partial^\mu\partial^\nu h_\rho{}^\rho + \lambda_5 X_\mu{}^\mu\partial_\rho\partial^\rho h_\nu{}^\nu + \frac{\langle\phi\rangle^2}{\alpha}\Big(\lambda_6 h_{\mu\nu}h^{\mu\nu}\\ &+ \lambda_7 X_{\mu\nu}X^{\mu\nu} + \lambda_8 X_{\mu\nu}h^{\mu\nu} + \lambda_9 h_\mu{}^\mu h_\nu{}^\nu + \lambda_{10}X_\mu{}^\mu h_\nu{}^\nu\Big)\end{aligned}\tag{B.5}$$

$X_{\mu\nu}$'s equation of motion is then $\delta\mathcal{L}_X = 0$, which gives us the algebraic expression

$$X_{\mu\nu} = -\frac{1}{2\lambda_7}\left[\frac{\alpha}{4\langle\phi\rangle}\left(\frac{1}{2}\lambda_1\big(\partial_\mu\partial^\rho h_{\nu\rho} + \partial_\nu\partial^\rho h_{\mu\rho}\big) + \lambda_2\eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \lambda_3\partial_\rho\partial^\rho h_{\mu\nu}\right.\right.$$
$$\left.\left. + \lambda_4\partial_\mu\partial_\nu h_\rho{}^\rho + \lambda_5\eta_{\mu\nu}\partial_\rho\partial^\rho h_\sigma{}^\sigma\right) + \lambda_8 h_{\mu\nu} + \lambda_{10}\eta_{\mu\nu}h_\mu{}^\mu\right]. \tag{B.6}$$

Next, we plug this definition for $X_{\mu\nu}$ back into (B.5) and set $\mathcal{L}_X = \mathcal{L}_{grav}$. By matching the coefficients in front of each term in $\mathcal{L}_X$ with the correct coefficients in $\mathcal{L}_{grav}$, we arrive at the following system of equations

$$\begin{aligned}
&\lambda_6 - \frac{\lambda_8^2}{4\lambda_7} = 0 \qquad \frac{1}{\lambda_7}\left(\lambda_{10}^2 + \frac{1}{2}\lambda_{10}\lambda_8 - \lambda_7\lambda_9\right) = 0\\
&-\frac{\lambda_1\lambda_8}{2\lambda_7} = 2 \qquad -\frac{1}{2\lambda_7}\left(\lambda_1\lambda_{10} + 4\lambda_{10}\lambda_2 + \lambda_8\big(\lambda_2 + \lambda_4\big)\right) = -2\\
&-\frac{\lambda_3\lambda_8}{2\lambda_7} = -1 \qquad -\frac{1}{2\lambda_7}\left(\lambda_{10}\big(\lambda_3 + \lambda_4 + 4\lambda_5\big) + \lambda_5\lambda_8\right) = 1\\
&-\frac{1}{8\lambda_7}\left(\lambda_1^2 + 4\lambda_1\lambda_2 + 8\lambda_2^2\right) = -\frac{2}{3} \qquad -\frac{1}{8\lambda_7}\left(4\lambda_1\lambda_3 + \lambda_1^2\right) = 2\\
&-\frac{1}{2\lambda_7}\left(\lambda_4\big(\lambda_1 + \lambda_2 + \lambda_3\big) + \lambda_2\lambda_3 + \lambda_1\lambda_5 + 4\lambda_2\lambda_5\right) = -\frac{2}{3}\\
&-\frac{\lambda_3^2}{4\lambda_7} = -1 \qquad -\frac{1}{2\lambda_7}\left(\frac{\lambda_4^2}{2} + \lambda_5\big(\lambda_3 + \lambda_4 + 2\lambda_5\big)\right) = \frac{1}{3},
\end{aligned} \tag{B.7}$$

which has the two solutions

$$\lambda_i = \left\{\lambda_1\ ,\ \frac{\lambda_1}{12}\Big(-3 \pm i\sqrt{3}\Big)\ ,\ -\frac{\lambda_1}{2}\ ,\ -\frac{\lambda_1}{2}\ ,\ \frac{\lambda_1}{12}\Big(3 \pm i\sqrt{3}\Big)\ ,\right.$$
$$\left.\frac{1}{4}\ ,\ \frac{\lambda_1^2}{16}\ ,\ -\frac{\lambda_1}{4}\ ,\ -\frac{1}{4}\ ,\ \frac{\lambda_1}{16}\Big(1 \pm i\sqrt{3}\Big)\right\}. \tag{B.8}$$

Interestingly, it appears that we still have the freedom to choose λ_1 as we like. Now that we have justified our ansatz for $\mathcal{L}_X$ given this λ_i, we can perform the diagonalizing change of variables

$$h_{\mu\nu} = \rho\big(\sigma_{\mu\nu} + \Sigma_{\mu\nu}\big) \qquad X_{\mu\nu} = \rho\big(\sigma_{\mu\nu} - \Sigma_{\mu\nu}\big)\,, \tag{B.9}$$

where $\sigma_{\mu\nu}$ and $\Sigma_{\mu\nu}$ will be our physically propagating spin-2 fields and ρ is an arbitrary normalization constant. We also fix our gauge freedom by imposing the transverse-traceless conditions

$$\partial^\mu\sigma_{\mu\nu} = \partial^\mu\Sigma_{\mu\nu} = 0 \qquad \sigma_\mu{}^\mu = \Sigma_\mu{}^\mu = 0\,. \tag{B.10}$$

Plugging (B.9) back into $\mathcal{L}_X$ with these conditions and either the plus or minus version of λ_i yields the same result.

$$\begin{aligned}\mathcal{L}_X[\sigma_{\mu\nu}, \Sigma_{\mu\nu}] &= \mathcal{L}_{grav}[\sigma_{\mu\nu}, \Sigma_{\mu\nu}] \\ &= \rho^2\lambda_1\Big(- \sigma_{\mu\nu}\partial_\rho\partial^\rho\sigma^{\mu\nu} + \Sigma_{\mu\nu}\partial_\rho\partial^\rho\Sigma^{\mu\nu}\Big) \\ &\quad + \frac{\rho^2\langle\phi\rangle}{64\alpha}\Big(2\big(4-\lambda_1^2\big)\sigma_{\mu\nu}\Sigma^{\mu\nu} + \big(\lambda_1 - 2\big)^2\sigma_{\mu\nu}\sigma^{\mu\nu} + \big(\lambda_1 + 2\big)^2\Sigma_{\mu\nu}\Sigma^{\mu\nu}\Big)\end{aligned} \tag{B.11}$$

We still have the freedom to choose λ_1 and ρ, so we set $\lambda_1 = 2$ in order to kill the cross term and $\rho = 1/2$ to normalize, which gives us our final decomposed gravitational Lagrangian for Weyl quadratic gravity in the Einstein frame.

$$\mathcal{L}_{grav}[\sigma_{\mu\nu}, \Sigma_{\mu\nu}] = -\frac{1}{2}\sigma_{\mu\nu}\partial_\rho\partial^\rho\sigma^{\mu\nu} + \frac{1}{2}\Sigma_{\mu\nu}\partial_\rho\partial^\rho\Sigma^{\mu\nu} + \frac{\langle\phi\rangle^2}{16\alpha}\Sigma_{\mu\nu}\Sigma^{\mu\nu} \tag{B.12}$$

Note that while we could have also chosen $\lambda_1 = -2$, the particle content would have been the same with only the names reversed.

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den September 26, 2019 ....