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ABSTRACT

Jordan algebras were first introduced in an effort to restructure quantum mechanics purely in terms of physical observables. In this paper, we explain why, if one attempts to reformulate the internal structure of the standard model of particle physics geometrically, one arrives naturally at a discrete internal geometry that is coordinatized by a Jordan algebra.

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I. INTRODUCTION

Ever since it was first discovered that gravity corresponds to the geometry of four dimensional spacetime, physicists have wondered if all known fundamental fields, including gravity, might be unified by interpreting them as describing the geometry of some appropriately extended spacetime. Perhaps, the most famous implementation of this strategy is that of "Kaluza–Klein theory," which imagines spacetime as a product space $M = M_4 \times M_{int}$, where M_4 is our familiar four dimensional spacetime, while M_{int} is another "internal" space that is small and compact, but which otherwise is an ordinary smooth Riemannian manifold. The idea is that the particular compactified structure of the internal dimensions determines the particle content and gauge symmetries that we detect in low energy particle experiments. This idea is beautiful, but it is unfortunately difficult to implement and, in particular, to stabilize the extra internal dimensions. At its heart, the problem with this approach is tightly linked to the assumption that the internal space should be a smooth manifold or correspondingly that it has many potentially unstable continuous deformations. In this paper, we drop this assumption and explore the possibility of replacing the smooth internal manifolds of Kaluza–Klein theory with discrete internal spaces that are non-dynamical (i.e., not Riemannian).

If a discrete internal geometry underlies the structure of the standard model of particle physics, then what should it "look" like? A hint can be found in the usual description of Weyl fermions within the familiar framework of quantum field theory. Unlike in the traditional approach to quantum mechanics, the position of a particle in quantum field theory is treated as a continuous label rather than as an operator. That is, we can think of the field $\Psi(x)$ as a spinor valued object living over the point labeled "x" in spacetime. In reality, though, we do not just have a single species of fermion. Instead, the standard model describes a collection of 48 Weyl fermions $\Psi_k(x)$ indexed by the discrete label $k = 1, \ldots, 48$. The usual approach is to think of these as 48 independent spinor fields, each of which lives over four dimensional spacetime. Note, however, that "x" and "k" are both just labels, and so it is natural to wonder why it is that we only treat "x" geometrically. An alternative approach would be to treat both labels on exactly the same geometric footing. From such a perspective, $\Psi_k(x)$ would be viewed as a single spinor field living over an extended "spacetime," which consists of two parts: an ordinary, smooth four dimensional manifold (corresponding to the label x) and an internal geometric space (corresponding to the discrete label $k = 1, \ldots, 48$).

In order to see more precisely how a discrete species index "k" might correspond to a discrete "internal" geometry, it is useful to think about how geometries are usually coordinatized. In the familiar setting of Riemannian geometry, the points of a manifold are featureless with no internal structure whatsoever. In this case, a manifold "M" is coordinatized by functions "f," each of which smoothly associates to each point $x \in M$ a single number "$f(x)$" [i.e., these are the kinds of functions that are usually denoted by "$x^\mu(x)$"]. The set of coordinate functions defined on a manifold form a coordinate algebra pointwise, which we denote as $A = C^\infty(M, \mathbb{C})$ [this means that the addition and multiplication of functions are defined locally at each point on the manifold, i.e., $(f + g)(x) = f(x) + g(x)$ and $fg(x) = f(x)g(x)$ for $f, g \in A$]. For a spin manifold (i.e., a manifold that admits a spinor bundle), one way of viewing coordinate functions is as "position-type" operators that act pointwise on spinor fields. That is, given a coordinate function $f \in A$ and an element Ψ in the Hilbert space of

square integrable Dirac spinors $H = L^2(M, S)$, one defines the following pointwise action: $(f\Psi)(x) = f(x)\Psi(x)$. The eigenvectors of the coordinate algebra are then the wavefunctions $\Psi \in H$, which are perfectly localized at any particular point "x" on the manifold. In other words, the underlying manifold is recovered as a spectrum of the coordinate algebra of "position-type" operators. The intuitive picture is one in which particles themselves are in some sense the instantiation of the points (i.e., representations) on the manifold (i.e., the coordinate algebra).

How does the above picture extend to the case in which the spinor fields $\Psi \in H$ are equipped with an additional "discrete" index "k"? To give this new index the same geometric meaning as the continuous coordinate label "x," the coordinate algebra of "position-type" observables will need to be extended to "see" the internal space. That is, if we wish to imbue the points on a manifold with some additional internal structure, then the usual coordinatization will no longer suffice because the algebra of coordinate functions must now encode information not only about the location of points but also the internal "state" of each point. A natural conjecture is that the geometry should take the form of a product space, coordinatized by an algebra of the form $A = C^\infty(M, \mathbb{C}) \otimes A_F$, where A_F is a finite dimensional (and possibly discrete) algebra that encodes the details of the internal space. That is, the coordinate algebra $A = C^\infty(M, \mathbb{C}) \otimes A_F = C^\infty(M, A_F)$ smoothly associates to each point on the manifold "M" an element of a finite dimensional algebra A_F, rather than an element of the algebra of complex numbers $\mathbb{C}$. The idea is that A_F holds internal information about the manifold at each point. Similarly, the spinor fields $\Psi_k(x)$ form a product Hilbert space denoted by $H = L^2(M, S) \otimes H_F$, where H_F is a complex vector space on which A_F is faithfully represented (and to which the index "k" is associated). In this setting, the "points" of the geometry can once again be associated with the spectrum or "states" of the coordinate algebra (or, following the GNS theorem, as the irreducible representations of the coordinate algebra).

A remarkable feature of the above picture is that it automatically presents a unified description of the four fundamental forces. In particular, general coordinate transformations of the coordinate algebra $A = C^\infty(M, A_F)$ correspond not only to diffeomorphisms along the manifold M but also to local rotations in the finite dimensional algebra A_F at each point (i.e., local "gauge" transformations). The key dream of Kaluza–Klein theory appears to be recovered, while the need for any compactification scheme is avoided. The price that is paid is that the construction necessarily sits outside the usual framework of Riemannian geometry (the coordinate algebra is something more general). In order to make complete sense of the discrete internal geometry that underlies the standard model of particle physics, a number of geometric notions beyond that of coordinate algebras and Hilbert spaces of spinors are required, including an appropriate analog of vector fields, differential forms, Dirac operators, Hochschild cohomology, Clifford representations of forms, and much more. Furthermore, in order to completely capture the structure of the standard model of particle physics, the details of the Higgs sector must be filled in, and appropriate dynamics have to be determined.

An attempt at fully constructing the discrete internal geometry corresponding to the standard model of particle physics has already been made within the framework of noncommutative differential geometry.[1] In this approach, the underlying internal topology is assumed to be coordinatized by a noncommutative matrix algebra, while metric information is encoded by a generalized Dirac operator. Remarkably, a noncommutative geometry has been found, which captures (almost) all the particle content and symmetries of the standard model, while providing new geometric meaning to a number of otherwise unexplained patterns and features observed in the experiment.[2] As a key example, the standard model Higgs and gauge fields are unified within the construction, with the Higgs gaining new meaning as a connection on the discrete internal space. Furthermore, there are a number of geometric properties that generalize naturally from Riemannian geometry and that in the noncommutative setting place strict and phenomenologically accurate constraints on the particle content of the model. As an example, because the fermions in the model arise as irreducible representations of the coordinate algebra, the matter content ends up being constrained by the representation theory for finite, noncommutative algebras. Remarkably, this alone restricts to those representations that are actually observed in the experiment (singlet and fundamental).

Despite the many intriguing features of the noncommutative geometric approach to particle theory, the noncommutative geometry that most closely captures the particle content of the standard model does suffer from a number of problems and poorly understood details. These can best be seen where the field content is added to or removed from the construction by hand in order to match phenomenology. The so-called "massless-photon" and "unimodularity" conditions,[2] for example, are employed to remove unwanted scalar and gauge bosons, respectively, while a scalar singlet has been introduced by hand in order to stabilize the Higgs mass at 125 GeV[3] (although, note that in more recent work, the massless photon condition has been replaced with the "second order condition," which does have geometric meaning[4]). Similarly, the three fermion generations remain as an unexplained input.[5] For the most part, the ad-hoc addition and removal of the particle content, while not having clear geometric justification, are not, in general, inconsistent with the rules of the underlying geometry. As explained in Ref. 6, however, this is unfortunately not always the case. More seriously, the construction suffers from what has been termed a "fermion quadrupling" problem, in which there are four times as many fermions in the model as observed experimentally.[7] These unwanted states are able to be projected out,[8] but doing so results in a space of spinors that is no longer compatible with the required noncommutative coordinate algebra, in the sense that the set of standard model fermions is not closed under the action of the noncommutative coordinate functions.[6]

The inconsistencies suffered by the noncommutative standard model do not necessarily indicate that there is a problem with the discrete internal geometric approach, in general, but rather that there are problems with the specific geometry that has been proposed to match the details of the standard model. This is the position taken in the current paper (as well as elsewhere in the literature[9,10]). When generalizing from Riemannian geometry to describe finite and discrete internal spaces, there are often multiple "good" choices with regard to which properties should be kept or extended. These choices are important when modeling physical systems and can lead to over- and

under-constraint, both of which are seen in the noncommutative standard model. A particularly interesting example of artificial over-constraint that is perhaps not so obvious is the axiomatic restriction to associative coordinate algebras.[5] Not only does this restriction automatically exclude the possibility of exploring the geometry of gauge theories with exceptional symmetry,[11,12] but it may also ultimately exclude those geometries that are of most direct interest to physics (including the standard model). In particular, while not compatible with associative, noncommutative coordinate algebras, it turns out that the standard model fermions are compatible with the representation of a certain class of nonassociative algebras known as real Jordan algebras.[6] It is natural to wonder, therefore, if the internal structure underlying the standard model of particle physics might more accurately be captured by a real "Jordan" geometry rather than a noncommutative geometry.

In this paper, we provide conceptual justification for considering internal geometries coordinatized by real Jordan algebras if one is specifically interested in constructing gauge theories (coupled to Einstein–Hilbert gravity). Of particular concern is that the reconstruction of Riemannian geometries appears to depend on C^*-algebras of complex coordinate functions $A = C^\infty(M, \mathbb{C})$ (viewed abstractly). It might therefore seem at first glance that restricting attention to real Jordan coordinate algebras would explicitly exclude Riemannian geometry and let alone any interesting extensions that are capable to accurately capturing the internal details of the standard model. Fortunately, this appears not to be the case. We explain how the differential topological information encoded by a complex, noncommutative coordinate algebra is actually contained within its maximal real Jordan subalgebra of self-adjoint elements. Furthermore, we explain how the physically relevant geometric quantities (i.e., symmetries, fermion representations, smooth structure, and distance measurements) continue to make sense when restricting attention to real Jordan coordinate algebras. We also provide an explicit construction of the differential calculi that will be required for eventually constructing the dynamics of geometries coordinatized by Jordan algebras.

This paper is organized as follows: in Sec. II, we briefly explore the noncommutative topology that most closely captures the internal fermion and gauge structure of the standard model of particle physics. We explain why such a construction is not able to capture the full content of the physics, and we observe in the experiment and explain why topological spaces coordinatized by "Jordan" algebras naturally circumvent this same impediment. In Sec. III, we explain how the physically interesting data contained in an associative coordinate algebra are fully contained within its maximal Jordan subalgebra. In particular, in Sec. III A, we explain how the full topological data of a complex C^*-algebra A are encoded entirely within the set of self-adjoint elements of A, which form a real Jordan algebra A_{sa}. In Sec. III B, we explain how distance measurements on a geometry coordinatized by a complex C^*-algebra A are able to be computed while only making reference to the maximal Jordan subalgebra A_{sa}. In Sec. III C, we explain how the symmetries of a geometry coordinatized by a complex C^*-algebra arise as (roughly) the automorphisms of the maximal real Jordan subalgebra A_{sa} of A. In Sec. III D, we explain how the differential structure encoded by an associative C^*-algebra is contained entirely within its maximal real Jordan subalgebra. We also explicitly construct the differential calculus over A_{sa} that will be required for describing the dynamics of realistic gauge theories. In Sec. IV, we conclude and explain what the next immediate steps are that will be required in order to explicitly construct the real Jordan geometry that most closely captures the full internal structure of the standard model of particle physics and explore its dynamics.

II. THE TOPOLOGY OF THE STANDARD MODEL

In theories with small extra dimensions, the usual approach is to imagine that spacetime is a product geometry, consisting of two parts $M = M_c \times M_F$. The first part M_c is the familiar four dimensional, smooth, pseudo-Riemannian manifold that we observe and live in, while the second part M_F is a manifold that is smooth and compact, but small enough that we do not yet observe it directly. If the metric g on the total space M is assumed not to depend at all on the internal part of the manifold M_F, then its components can be thought of as four dimensional fields living over the large "external" space M_c. The particular details of the geometry then encode information about gravity, as well as all the fundamental fields and symmetries of particle theory. In this paper, we follow a similar approach, with the main distinction being that we replace the internal space M_F by a geometry that is instead finite and discrete. The usual hypothesis taken in the literature is that if such a discrete internal space is to exist, then it should be coordinatized by a noncommutative C^*-algebra. Unfortunately, the noncommutative geometry that most closely captures the particle content and symmetries of the standard model runs into a key problem that appears difficult to resolve within the associative, noncommutative setting.[6] As we review in this section, this problem is neatly avoided by geometries coordinatized by Jordan algebras, which is a key motivation for this work.

The "noncommutative" geometry that most closely captures the underlying structure of the standard model of particle physics is constructed as a product geometry, coordinatized by an algebra of the form $\mathcal{A} = \mathcal{A}_c \otimes \mathcal{A}_F$, and bi-represented on a vector space of the form $\mathcal{H} = \mathcal{H}_c \otimes \mathcal{H}_F$. Here, the pair $\{\mathcal{A}_c, \mathcal{H}_c\}$ captures the differential topological data of the external 4D spin-manifold "M" in which we live. In this case, $\mathcal{A}_c = C^\infty(M, \mathbb{R})$ is the algebra of smooth functions on M, while $\mathcal{H}_c = L^2(M, S)$ is the Hilbert space of square integrable Dirac spinors defined on M. For the finite space, the topology is captured by the finite dimensional algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, where $\mathbb{C}$ is the algebra of complex numbers, $\mathbb{H}$ is the algebra of quaternions, and $M_3(\mathbb{C})$ is the algebra of 3×3 complex matrices. This internal coordinate algebra is chosen because its automorphism group corresponds (roughly) to the local gauge group of the standard model (deeper geometric motivation has subsequently been sought[5]). The algebra A_F is represented on the 96-dimensional complex Hilbert space $H_F = \mathbb{C}^{96}$, where 96 corresponds to the total number of fermionic degrees of freedom in the standard model of particle physics (after including particles and anti-particles, left and right chiralities, right-handed neutrinos, three colors, and three families).

The representation of the finite dimensional algebra A_F on H_F is chosen to reproduce the particular details of the standard model. We explain this representation now, focusing for brevity on a single generation of fermions (i.e., restricting attention to the Hilbert space $H_F = \mathbb{C}^{32}$). An element $a = \{\lambda, q, m\} \in A_F$ can be represented efficiently as an 8×8 block diagonal matrix on an element of $\Psi \in H_F$ as[6]

$$
L_a \Psi = \left(
\begin{array}{cc|cc}
\begin{matrix} q & \\ & q_\lambda \end{matrix} & & \\
\hline
 & \begin{matrix} m & \\ & \lambda \end{matrix} &
\end{array}
\right)
\left(
\begin{array}{cccc|cccc}
 & & & & u_L & u_L & u_L & \nu_L \\
 & & & & d_L & d_L & d_L & e_L \\
 & & & & u_R & u_R & u_R & \nu_R \\
 & & & & d_R & d_R & d_R & e_R \\
\hline
\overline{u}_L & \overline{d}_L & \overline{u}_R & \overline{d}_R & & & & \\
\overline{u}_L & \overline{d}_L & \overline{u}_R & \overline{d}_R & & & & \\
\overline{u}_L & \overline{d}_L & \overline{u}_R & \overline{d}_R & & & & \\
\overline{\nu}_L & \overline{e}_L & \overline{\nu}_R & \overline{e}_R & & & &
\end{array}
\right),
\tag{1}
$$

where $\lambda \in \mathbb{C}$ is a complex number, q is the standard representation of the quaternion $q \in \mathbb{H}$ as a complex 2×2 matrix, and the 2×2 block q_λ is the corresponding diagonal embedding of $\lambda \in \mathbb{C}$ in $\mathbb{H}$. The element m is given by a 3×3 complex matrix. The left- or right-representation of an element $a \in A_F$ on $h \in H_F$ is given by the left matrix product $L_a \Psi = a\Psi$ or the right matrix product $R_a \Psi = \Psi a$, respectively.

For the elements of $\Psi \in H_F$, the fermions (and anti-fermions) of a single standard model generation are expressed as an 8×8 block-off-diagonal matrix, where the 16 components in the upper-right block correspond to the 16 Weyl spinors in a single generation of fermions, while the 16 components in the lower-left block are the corresponding anti-fermions. Here, we have used red, green, and blue to indicate the three quark colors and indicate how the three quark columns (or anti-quark rows) transform into one another like a triplet (or anti-triplet) under strong $SU(3)$. This representation can also be equipped with a charge conjugation operator J_F, which maps particles to anti-particles, and is given simply by Hermitian conjugation: $J_F \Psi = \Psi^\dagger$. The left and right actions of the algebra are furthermore related by $R_a = J_F L_a^\dagger J_F$. A grading operator γ_F can similarly be constructed, which associates "+1" or "−1" to the fermions according to their usual chirality assignments. We provide an explicit representation of γ_F in Appendix A for the curious reader, although it will not play a key role in the remaining discussion.

Before discussing the symmetries of the representation given in Eq. (1), or introducing a metric, or constructing dynamics, note that the noncommutative standard model runs into an awkward problem (see Appendix B for how the symmetries of the corresponding particle representations are determined). The above construction suffers from an overcounting in fermionic degrees of freedom by a factor of 4.[7] The reason for this is that particles and anti-particles as well as left and right chiralities are accounted for both in the internal Hilbert space H_F and in the "external" Hilbert space of Dirac spinors $\mathcal{H}_c$ (this must be done in order to obtain the correct representations under both the gauge and local Lorentz symmetries). When the total Hilbert space is formed by taking the product $H = H_c \otimes H_F$, the result is an overcounting by a factor of 4. In order to solve this problem, note that the total Hilbert space can also be equipped with natural charge conjugation and grading operators $\mathcal{J} = \mathcal{J}_c \otimes J_F$ and $\Gamma = \gamma_c \otimes \gamma_F$, where J_c and γ_c are the usual charge conjugation and grading on $\mathcal{H}_c$, respectively (see Refs. 13 and 14 for further information about taking the product between geometries). The "fermion quadrupling" problem can then be solved by only considering those elements of the input Hilbert space H, which satisfy the following two requirements:[8]

$$
J\Psi = \Psi, \quad \Gamma\Psi = \Psi.
\tag{2}
$$

In other words, the over-counting problem is solved by simultaneously imposing both the Weyl and Majorana conditions on the spinor fields of the product geometry (see also Refs. 15 and 16 for alternative approaches).

Unfortunately, while projecting out the unwanted spinor fields in this way does result in the correct fermion counting and representations under the gauge and Lorentz symmetries, the spinor fields that remain are no longer compatible with the representation of the standard model coordinate algebra. The problem is that the Majorana condition amounts to imposing a "Hermiticity-like" condition on spinors.[6] Just as the product of two Hermitian matrices is not, in general, Hermitian, Majorana spinors are not, in general, closed under the action of associative matrix coordinate algebras. In particular, the space of Majorana spinors $\{\Psi \in H : J\Psi = \Psi\}$ is not closed under the action of the coordinate algebra A on H because the algebra representation does not commute with the charge conjugation operator $[L_f, J] \neq 0$ for $f \in A$. In general, this will also be true for any noncommutative, associative algebra, and what this means conceptually is that the standard model cannot correspond directly to any associative, noncommutative geometry. One might, of course, argue that it is really only the physical fields that must be compatible with the Weyl condition and not the underlying geometry itself, but this negates the whole point of the geometric construction.

A simple solution to this apparent obstruction presents itself if one allows for the possibility of nonassociative coordinate algebras. Hermitian matrices are not, in general, closed under the associative matrix product but are, instead, closed under the symmetrized "Jordan" product. That is, while the matrix product XY of two Hermitian matrices X and Y is not, in general, Hermitian, the symmetrized product

$\frac{1}{2}(XY + YX)$ will always be. Equipped with such a symmetrized product, the vector space of Hermitian $n \times n$ matrices form a special kind of nonassociative algebra known as a "Jordan" algebra. In a similar way, while the Majorana condition on spinors is not, in general, compatible with the action of associative, noncommutative coordinate algebras, it is, in general, compatible with the symmetrized action of the Hermitian elements of a coordinate algebra. This is because the symmetrized action commutes with the charge conjugation on H, i.e., $[L_f + R_f, J]$ $= [L_f + J(L_f)^\dagger J, J] = 0$ for all self-adjoint $f \in A$. It is therefore natural to wonder if phenomenologically interesting geometries might exist that are coordinatized by Jordan algebras. Indeed, in Ref. 6, the Jordan coordinate algebra and representation that most closely describes the standard model spinor content and symmetries has been found, along with a natural extension, which describes the Pati–Salam model. In what follows, we provide the conceptual justification for considering these geometries seriously and explicitly construct the first set of tools that will be required for describing the dynamics of realistic models.

III. JORDAN COORDINATE ALGEBRAS

A number of concerns might arise when considering real Jordan coordinate algebras: (i) First, Connes reconstruction theorem for Riemannian geometries relies on commutative C^*-algebras, which are complex. It is therefore natural to wonder if Jordan geometry is really able to generalize Riemannian geometry or if too much topological information is lost when restricting to the self-adjoint (i.e., real) subalgebra of a C^*-algebra. (ii) Second, Connes provides a formula for calculating distances that coincides with the usual notion of geodesic distance on Riemannian geometries, but which also continues to make sense for finite and discrete noncommutative geometries. Does this formula still make sense when restricting to the self-adjoint elements of a noncommutative coordinate algebra? (iii) Finally, do we lose information about the differential structure of the underlying geometry when we restrict to the maximal self-adjoint subalgebra of a complex C^*-algebra? In this section, we explain that each of these concerns is unfounded.

A. Topology

The first step in describing a geometry comes with topology. In the familiar setting of Riemannian geometry, the topology of a manifold M is encoded entirely by the algebra of complex functions defined over that manifold $C^\infty(M, \mathbb{C})$. This is the algebra from which the familiar coordinate functions x^μ are drawn. In particular, a famous theorem by Gelfand, Naimark, and Segal establishes that given a unital, commutative C^*-algebra A, it is always possible to build a compact space M such that A is interpreted as the algebra of continuous functions defined over M,[17,18]

$$\text{Compact topological space } M \Leftrightarrow \text{Commutative, unital } C^* - \text{algebra } A. \tag{3}$$

But how does this correspondence work? Going from left to right is easy. The complex functions defined over a manifold M form a commutative algebra A, with the following pointwise operations:

$$(f + g)(x) = f(x) + g(x), \tag{4a}$$

$$(fg)(x) = f(x)g(x) \tag{4b}$$

for $f, g \in A$, where $x \in M$. Commutativity arises from the fact that complex numbers commute. This algebra is also equipped with a natural anti-linear involution and norm

$$f^*(x) = f(\bar{x}), \tag{5}$$

$$\|f\| = \sup_{x \in M} |f(x)|, \tag{6}$$

which turn A into what is known as a complex C^*-algebra (i.e., closed in the norm topology and such that $\|f\|^2 = \|f\| \|f^*\|$). Note that this involution $* : A \to A$ satisfies the usual properties $(f^*)^* = f$ and $(fg)^* = g^* f^*$ for all $f, g \in A$. Going from right to left in Eq. (3) is less obvious but comes from the key observation that the points on a manifold $x \in M$ can be thought of as irreducible representations of the coordinate algebra A. In particular, because A is commutative and associative, all its irreducible representations will be one dimensional. What Eq. (4) is really saying, then, is that the points on a manifold provide complex irreducible representations $\pi_x(f) = f(x)$, satisfying the usual properties

$$\pi_x(f + g) = \pi_x(f) + \pi_x(g), \tag{7a}$$

$$\pi_x(fg) = \pi_x(f)\pi_x(g) \tag{7b}$$

for $f, g \in A$, $x \in M$. The maps $\pi_x : A \to \mathbb{C}$ satisfying Eq. (7) are also known as the characters of A. Given an abstract, unital, commutative, complex C^*-algebra A, its set of characters $K(A)$ form a compact topological space, hence the other half of the theorem [the set $K(A)$ of characters on a unital, commutative C^*-algebra is made into a topological space by equipping it with the topology of pointwise convergence on A[17]]. The key insight is that a point $x \in M$ can be seen as an object on which coordinate functions $f \in A$ are evaluated or, equivalently, as objects (characters) to be evaluated on functions in order to give numbers (or, in other words, to provide a one dimensional complex representation).

1. Noncommutative coordinate algebras and topology

The correspondence given in Eq. (3) allows one to think either in terms of compact topological spaces or, equivalently, in terms of commutative, unital C^*-algebras. Once this correspondence is established, however, it is natural to wonder if it might generalize to include algebras that are also noncommutative and/or nonassociative. This second option has proven less popular (although see Refs. 4, 6, 11, 19, and 20), and most attention has focused on the noncommutative case. In other words, most attention has focused on whether it is possible, starting with a noncommutative C^*-algebra A, to construct a space M, which is thought of as a noncommutative space over which the elements of A play the role of coordinate functions. What should play the role of "points" in a "noncommutative" topology? The characters of an algebra [i.e., maps from the algebra to the complex numbers satisfying (7)] are no longer the appropriate objects to work with because they lose the noncommutative information held in the algebra (because complex numbers commute). Instead, in the noncommutative setting, the appropriate tools to extract the topological information are the states of an algebra. The states of an algebra are linear maps $\pi_x : A \to \mathbb{C}$, which are positive $\pi_x(f^* f) \geq 0 \in \mathbb{R}$, $\forall f \in A$, and which satisfy $\pi_x(\mathbb{I}) = 1$, where $\mathbb{I}$ is the unit of A.

The set of states $S(A)$ of a unital C^*-algebra A is convex, which means that any state π_x can be decomposed as[18]

$$\pi_x = \lambda\phi + (1 - \lambda)\phi', \tag{8}$$

where $\phi, \phi' \in S(A)$ and $\lambda \in [0, 1]$. The extremal points of $S(A)$, i.e., the states for which the only convex combination is trivial ($\lambda = 1$), are called the pure states of A. When the coordinate algebra is commutative, its characters and pure states coincide, and so it is natural to think of pure states as the generalization of "points" in the noncommutative setting. Furthermore, just as in the commutative setting, there is a well understood correspondence between the pure states of a C^*-algebra A and the elements of irreducible representations of A (known as the GNS construction[21]).

To give a simple (but physically relevant) example, consider the C^*-algebra of 2×2 complex matrices $A = M_2(\mathbb{C})$. This algebra has an irreducible representation on the two dimensional complex Hilbert space $H = \mathbb{C}^2$. If $\langle | \rangle$ denotes the inner product on H, then the map $\rho_\Psi : A \to \mathbb{C}$ is given by

$$\rho_\Psi(f) = \langle \Psi|, f|\Psi \rangle \tag{9}$$

where $f \in A$, and where Ψ is a unit norm vector in H, acts as a pure state on A. In fact, up to an overall phase, there is a one-to-one correspondence between the pure states of A and the unit norm elements of H. If we introduce an orthonormal basis n_i, where $i = 1, 2$ on H, then a general unit norm element can be expressed as

$$\Psi = e^{i\chi}(\cos[\theta/2]n_1 + e^{i\phi} \sin[\theta/2]n_2), \tag{10}$$

where χ is an irrelevant phase and $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$. This representation is unique except for the case in which h is equal to one of the unit vectors n_1, n_2. A useful visualization of the space of pure states is then given in terms of the unit sphere in $\mathbb{R}^3$, where a given state $\vec{\psi}$ is parameterized by

$$\vec{\psi} = (\sin[\theta] \cos[\phi], \sin[\theta] \sin[\phi], \cos[\theta]). \tag{11}$$

This construction is nothing other than the familiar Bloch sphere.[22]

2. Jordan coordinate algebras and topology

As reviewed in Sec. II, the space of standard model fermions is not closed under the action of the noncommutative algebra that has been proposed to coordinatize the internal geometry of the standard model. Instead, standard model fermions are only compatible with the "symmetrized" action of the self-adjoint elements of the coordinate algebra. The self-adjoint elements of a $*$-algebra A are given by the set $A_{sa} = \{f \in A : f^* = f\}$, where $*$ is the involution on A. The self-adjoint elements of an associative $*$-algebra are themselves not, in general, closed under the basic operations of the algebra. In particular, the product fg between two self-adjoint elements $f, g \in A$ is not, in general, self-adjoint unless f and g commute. Instead, the self-adjoint elements are closed under the symmetrized product,[21,22]

$$f \circ g = \frac{1}{2}(fg + gf), \tag{12}$$

where we use juxtaposition to denote the original associative product on A. The vector space A_{sa}, when equipped with the "symmetrized" product "∘," forms what is known as a real "Jordan" algebra. A real Jordan algebra A_{sa} is a real vector space, equipped with a bilinear (abstract) product "∘" that satisfies the following two properties:[21,23]

$$f \circ g = g \circ f \qquad \text{(Commutativity)}, \tag{13a}$$
$$(f \circ g) \circ f^2 = f \circ (g \circ f^2) \qquad \text{(Jordan Identity)} \tag{13b}$$

for $f, g \in A_{sa}$, where $f^2 = f \circ f$. Note that Jordan algebras are not, in general, associative but instead satisfy the weaker "Jordan" identity. For a full classification of the finite dimensional Jordan algebras, see Ref. 24, or for a more in-depth general introduction, see Ref. 25.

Jordan algebras that can be constructed by "symmetrizing" the product on an associative algebra are known as "special." In this paper, we are primarily interested in special Jordan algebras obtained by restricting complex C^*-algebras to their self-adjoint elements (similar work focusing on the exceptional Jordan algebra is also currently being pursued[12,26,27]). A natural concern, since we are interested in reconstructing geometries, is whether important topological information is lost in the process. Fortunately, this turns out not to be the case. The self-adjoint elements of a complex C^*-algebra form a special kind of algebra known as a Jordan–Banach–Lie (JBL) algebra, and this subalgebra contains the full topological information of the C^*-algebra from which it is formed. Furthermore, a unital JBL algebra is always Jordan isomorphic to the self-adjoint part of a C^*-algebra.

A real Jordan–Lie algebra is a real Jordan algebra $\{A_{sa}, \circ\}$ that is additionally equipped with a "Lie" product "$\times$" satisfying[21]

$$f_1 \times f_2 = -f_2 \times f_1 \qquad \text{(Anti – commutativity)}, \tag{14a}$$
$$f_1 \times (f_2 \times f_3) = (f_1 \times f_2) \times f_3 + f_2 \times (f_1 \times f_3) \qquad \text{(Jacobi Identity)} \tag{14b}$$

for $f_i \in A_{sa}$. Note, in particular, that the Lie product is not, in general, associative but instead satisfies the weaker "Jacobi" identity. Furthermore, the Jordan and Lie products on a Jordan–Lie algebra are compatible in the sense that the following identities are satisfied:

$$f_1 \times (f_2 \circ f_3) = (f_1 \times f_2) \circ f_3 + f_2 \circ (f_1 \times f_3) \qquad \text{(Leibniz Identity)}, \tag{14c}$$
$$\kappa(f_3 \times f_1) \times f_2 = (f_1 \circ f_2) \circ f_3 - f_1 \circ (f_2 \circ f_3) \qquad \text{(Associator Identity)} \tag{14d}$$

for $f_i \in A_{sa}$, where κ is a positive real number. A Jordan–Lie algebra $\{A_{sa}, \circ, \times\}$ can be made into a Jordan–Banach–Lie algebra by equipping it with a norm $\|.\|$ that satisfies[21,22,28,29]

$$\|f \circ g\| \le \|f\| \|g\|, \tag{15a}$$
$$\|f \circ f\| = \|f\|^2, \tag{15b}$$
$$\|f \circ f\| \le \|f \circ f + g \circ g\| \tag{15c}$$

for all $f, g \in A_{sa}$.

Starting with a JBL algebra $\{A_{sa}, \circ, \times, \|.\|\}$, one is always able to construct from it an associative complex C^*-algebra A. In particular, an associative product can be defined on the complexification $A = A_{sa} \oplus iA_{sa}$, by making use of both the Jordan and Lie products,[21]

$$fg = f \circ g - i\sqrt{\kappa} f \times g. \tag{16}$$

Associativity follows from the Jacobi identity given in (14b) and the Leibniz identity given in (14c). When further equipped with the norm $\|f + ig\|^2 = \|f\|^2 + \|g\|^2$, the algebra A becomes a C^*-algebra whose involution is $(f + ig)^* = f - ig$. Going in the other direction, the real (or self-adjoint) part of A is precisely A_{sa}, and the norm on A induces a norm on A_{sa}. In this way, one is able to go back and forth between a C^*-algebra A and its corresponding Jordan–Banach–Lie algebra A_{sa}.

The key point of interest in this paper is that the state space $S(A)$ of a complex C^*-algebra is determined entirely by the self-adjoint part of A, which forms a JBL algebra. The space of states of a real Jordan algebra A_{sa} consists of all those real linear functionals $\pi_x : A_{sa} \to \mathbb{R}$ that are positive $\pi_x(f^2) \ge 0$ and normalized $\pi_x(\mathbb{I}) = 1$.[22,28] Just as occurs for C^*-algebras, the states of a JBL algebra form a convex set. There is a natural identification between the states $S(A_{as})$ of A_{sa} and the states $S(A)$ of the corresponding C^*-algebra A.[21] In particular, given a state π_x of A_{as}, a linear functional $\bar{\pi}_x$ can be defined on A by extending π_x by linearity: $\bar{\pi}_x(f + ig) = \pi_x(f) + i\pi_x(g)$. The converse is trivial as $\pi_x(f) = \bar{\pi}_x(f)$ for all elements $f \in A_{sa}$. In short, nothing is lost, topologically speaking, when shifting attention from C^*-algebras to JBL algebras. The full topological data of a C^*-algebra are encoded in its maximal JBL-subalgebra.

B. Measuring distances

Geometry is more than just points on a manifold. At a minimum, we would also like to be able to define a notion of distance between points. Furthermore, because we are interested in generalized notions of geometry, for which it is easier to deal with coordinate algebras than it is to deal with the topological spaces they define, it would be useful to have a definition of distance that only needs to make reference to coordinate algebras. Fortunately, an appropriate definition has already been found,[1,30] which coincides with the familiar geodesic distance on

Riemannian manifolds, but which makes no reference to any underlying manifold and which also continues to make sense in the noncommutative and discrete settings. The question, one might ask, is whether this standard definition of distance continues to make sense in the Jordan setting.

Consider a one dimensional Riemannian manifold $M = \mathbb{R}$ coordinatized by $A = C^\infty(\mathbb{R}, \mathbb{C})$ (i.e., the algebra of smooth complex coordinate functions defined over the real line). Without reference to the manifold itself, one way that the distance between two points "x" and "y" on $\mathbb{R}$ is able to be defined is as the excursion of a carefully chosen function $f \in A$ between the two points. In particular, if we select f to be the function with the maximum possible excursion subject to the condition that its derivative is never greater than one, then this recovers precisely the usual geodesic distance $|x - y|$ between the points $x, y \in \mathbb{R}$. In other words, we can define the distance as

$$d(x, y) = \sup_{f \in C^\infty(\mathbb{R})} \{|f(x) - f(y)| : \sup_{z \in \mathbb{R}} |f'(z)| \le 1\}. \tag{17}$$

This definition relies not only on the coordinate algebra but also on knowing what a derivative is. For higher dimensional manifolds, the derivative is readily replaced with a gradient, but a notion of derivation is needed that also continues to make sense in a much more general setting. Sticking with our simple example for the time being, note that if the coordinate algebra $A = C^\infty(\mathbb{R}, \mathbb{C})$ is represented on the Hilbert space of square integrable spinors $H = L^2(\mathbb{R}, S)$, then this allows us to represent the derivative of a function in terms of the Dirac operator acting on H,

$$f'(x)\Psi(x) = \frac{df(x)}{dx}\Psi(x) = \frac{d}{dx}f(x)\Psi(x) - f(x)\frac{d}{dx}\Psi(x) = [D, f(x)]\Psi(x), \tag{18}$$

where $\Psi \in H$ and D is the derivative along the manifold d/dx. Making use of Eq. (6), we have $\|[D, \pi(f)]\| = \sup_{x \in \mathbb{R}}|f'(x)|$, which allows us to re-express Eq. (17) as

$$d(x, y) = \sup_{f \in A} \{|\pi_x(f) - \pi_y(f)| : \|[D, f]\| \le 1\}. \tag{19}$$

This is Connes' distance formula,[30] and it continues to make sense for higher dimensional Riemannian manifolds, as well as for spaces coordinatized by noncommutative and discrete algebras. The main extra ingredient is the notion of a generalized Dirac operator that satisfies some appropriate properties.[1] Connes' notion of distance becomes especially interesting in situations where the usual classical definition of the distance as the length of the shortest path between two points is no longer available. In this paper, we are interested in restricting to the maximal JBL subalgebras of noncommutative C^*-algebras, and in this case, it is natural to wonder whether Connes notion of distance continues to make sense.

For a geometry coordinatized by a noncommutative C^*-algebra, consider a function f that reaches the supremum in Ref. 20 such that $\|[D, f]\| \le 1$ and $|(\pi_x - \pi_y)(f)| = \mathrm{dist}(x, y)$, and let $\theta := \mathrm{Arg}((\pi_x - \pi_y)(f))$. In this case, the supremum is also reached by the self-adjoint element $g = 1/2(fe^\theta + f^* e^{-\theta}) \in A_{sa}$ because[31]

$$\|[D, g]\| \le \frac{\|[D, f]\|}{2} + \frac{\|[D, f]\|}{2} \le 1, \tag{20}$$

$$|(\pi_x - \pi_y)(g)| = |\frac{\mathrm{dist}(x, y)}{2} + \frac{\mathrm{dist}(x, y)}{2}| = \mathrm{dist}(x, y). \tag{21}$$

In practice, this means that we can actually restrict attention in the Connes distance formula to those elements of the coordinate algebra that are self-adjoint. Not only that, but also the supremum is met for the element $k = g + \|g\|\mathbb{I} \in A^+$, and so we can really restrict attention further to the positive elements A_{sa}^+ of A (i.e., the self-adjoint elements with eigenvalues greater than or equal to zero).

Now, suppose that $\|[D, k]\| < 1$, and take $l := k/\|[D, k]\| \in A_{sa}^+$, then $\|[D, l]\| = 1$, and

$$|\pi_x(l) - \pi_y(l)| = \frac{\pi_x(k) - \pi_y(k)}{\|[D, k]\|} > |\pi_x(k) - \pi_y(k)|, \tag{22}$$

which is impossible because k was chosen to reach the supremum. We therefore have that $\|[D, k]\| = 1$ when Connes' distance formula reaches a supremum, and so Eq. (19) can be re-expressed as[31]

$$d(x, y) = \sup_{f \in A_{sa}^+} \{|\pi_x(f) - \pi_y(f)| : \|[D, f]\| = 1\}. \tag{23}$$

The distances of a noncommutative geometry are "observable" quantities. When constructing gauge theories, for example, the discrete, internal, "flat," Dirac operator D_F corresponds (roughly speaking) to the mass-matrix of the theory. The distances measured on the internal space are therefore ultimately related to the mass spectrum of the theory.[32] The take home message is that when determining "observables,"

it is the self-adjoint elements of a noncommutative coordinate algebra that are physically relevant. In other words, with regard to both the topology and distance measurements, the physically relevant data of a noncommutative, complex C^*-algebra appear to be entirely contained within its maximal JBL-subalgebra.

C. General coordinate transformations

A common theme in theories with small extra dimensions is the unification of the four known fundamental forces, with the idea being that the local gauge symmetries are contained within the diffeomorphism group of the higher dimensional total space. This same idea continues to apply when considering geometries in which the internal space is finite and discrete. The symmetries of a geometry correspond, roughly speaking, to the automorphisms of its coordinate algebra. (The symmetries of a geometry correspond more correctly to the automorphisms of the representation of the coordinate algebra A on the Hilbert space of spinors H.[20] In general, these symmetries will be slightly larger than the automorphism group of the coordinate algebra A, but we will not concern ourselves with this subtlety here. This is already a familiar feature in Riemannian spin geometry, in which the symmetries include not only the diffeomorphisms of the manifold but also local rotations in the spin bundle.) In this section, we explain how the symmetries of a noncommutative coordinate algebra are exactly the same as those defined on its maximal JBL subalgebra. Intuitively, this makes sense as the symmetries of an algebra should not map self-adjoint elements to those that are anti-selfadjoint [this is, in fact, a defining property as we outline below in Eq. (24b)].

The symmetries of an algebra are given by its automorphisms. An automorphism α is an invertible linear map from an algebra to itself, which preserves the structure of the algebra. That is, it respects the product on A,

$$\alpha(fg) = \alpha(f)\alpha(g) \tag{24a}$$

for all $f, g \in A$. If A is a $*$-algebra, then those automorphisms that also respect the involution on A are called $*$-automorphisms,

$$\alpha(f^*) = \alpha(f)^* \tag{24b}$$

for all $f \in A$. Of particular interest in this paper are those automorphisms that are continuously connected to the identity $\mathbb{I}$. An automorphism α that is infinitesimally close to the identity map can then be written as $\alpha = \mathbb{I} + \delta$, where δ is a derivation element satisfying

$$\delta(fg) = \delta(f)g + f\delta(g) \tag{25a}$$

for all $f, g \in A$. If A is also a $*$-algebra, then a $*$-derivation δ is a derivation, which respects the involution on A,

$$\delta(f^*) = \delta(f)^* \tag{25b}$$

for all $f \in A$. The derivations of A are the infinitesimal generators of the automorphisms of A; they form a Lie algebra, with a Lie product given by $\delta_1 \times \delta_2 = \delta_1\delta_2 - \delta_2\delta_1$ (where juxtaposition on the right-hand side denotes composition of operators on A).

1. Riemannian geometry

Consider the coordinate algebra $A = C^\infty(M, \mathbb{C})$ defined over a Riemannian manifold M. The diffeomorphisms on M are in one-to-one correspondence with the coordinate transformations of $C^\infty(M, \mathbb{C})$. In particular, for any diffeomorphism $\phi : M \to M$, one can construct a map $\alpha_\phi f(x) = f(\phi^{-1}x)$, where $f(x) \in C^\infty(M, \mathbb{C})$. The maps α_ϕ are known as "outer automorphisms" and satisfy all the required properties of $*$-automorphisms. In particular, they preserve the structure of the product and involution on A,

$$\begin{aligned} \alpha_\phi(f_0 f_1)(x) &= (f_0 f_1)(\phi^{-1}x) \\ &= f_0(\phi^{-1}x)f_1(\phi^{-1}x) \\ &= \alpha_\phi f_0(x)\alpha_\phi f_1(x) = (\alpha_\phi f_0 \alpha_\phi f_1)(x), \end{aligned} \tag{26a}$$

$$\begin{aligned} \alpha_\phi(f_0^*)(x) &= f_0^*(\phi^{-1}x) \\ &= \overline{f_0(\phi^{-1}x)} \\ &= \overline{\alpha_\phi f_0(x)} = \alpha_\phi(f_0)^*(x) \end{aligned} \tag{26b}$$

where $f_i \in C^\infty(M, \mathbb{C})$. If we further consider those automorphisms that are infinitesimally close to the identity, we find that their generating derivations take the form $\delta_V(f_0) = V^\mu \partial_\mu f_0$, where the $V(x)^\mu$ are real valued coefficients. Such derivations clearly satisfy the properties of

∗-derivations given in Eq. (25). In addition, they form a Lie algebra (of vector fields) with the Lie product, given by $\delta_V \times \delta_W = (V^\nu(\partial_\nu W^\mu) - W^\nu(\partial_\nu V^\mu))\partial_\mu$.

2. Noncommutative coordinate algebras and coordinate transformation

Noncommutative algebras generally have additional automorphism known as "inner automorphisms," which are constructed from elements of the algebra itself. In particular, given a unital, associative ∗-algebra A, we can define the map $\alpha_u : A \to A$,

$$\alpha_u f = u f u^{-1}, \tag{27}$$

where $f, u \in A$. It is easy to check that such maps satisfy the automorphism property,

$$\begin{aligned} \alpha_u(fg) &= u f g u^{-1} \\ &= u f u^{-1} u g u^{-1} = \alpha_u(f)\alpha_u(g) \end{aligned} \tag{28a}$$

for $f, g \in A$. Furthermore, if u is a unitary element in A (i.e., satisfying $u^* = u^{-1}$), it is easy to check that such maps α_u act as ∗-automorphisms,

$$\begin{aligned} \alpha_u(f^*) &= u f^* u^* \\ &= (u f u^*)^* = \alpha_u(f)^*. \end{aligned} \tag{28b}$$

Note that the unitary elements $\{u \in A : u^* = u^{-1}\}$ are generated by anti-Hermitian elements $\{a \in A : a^* = -a\}$ through exponentiation, which allows us to express inner automorphisms given in (27) as

$$\alpha_u f = e^a f e^{-a} = e^{L_a - R_a} f = e^{\delta_a} f, \tag{29}$$

for where we have defined $\delta_a = L_a - R_a = [a, \_]$ and where we are using the standard "left–right" notations $L_a f = af$ and $R_a f = fa$.[33] The elements δ_a act as ∗-derivations on A and form a Lie algebra when equipped with the Lie product $\delta_a \times \delta_b = \delta_a \delta_b - \delta_b \delta_a$.

As an example, if we take our coordinate algebra to be the C^*-algebra of $n \times n$ complex matrices $A = M_n(\mathbb{C})$, then all ∗-derivations are "inner" and take the form $\delta_a = (L_a - R_a)$ where $a = -a^* \in A$. The Lie algebra of inner ∗-derivations is given by $su(n)$ and generates the Lie group of inner ∗-automorphisms $SU(n)/\mathbb{Z}_n$ through exponentiation. If, further, we take the product between a canonical "Riemannian" geometry coordinatized by $A_c = C^\infty(M, \mathbb{C})$ and a finite internal geometry coordinatized by $A_F = M_n(\mathbb{C})$, then the product coordinate algebra will be given by $A = C^\infty(M, \mathbb{C}) \otimes_{\mathbb{C}} M_n(\mathbb{C}) = C^\infty(M, M_n(\mathbb{C}))$. In this case, the ∗-automorphism group will be given by $Diff(M) \ltimes SU(n)/\mathbb{Z}_n$[34] such that the product C^*-algebra can be understood as encoding the topological data of an $SU(n)$ gauge theory coupled to Einstein–Hilbert gravity.

3. Jordan coordinate algebras and coordinate transformations

Jordan algebras, in general, also have inner automorphisms. In this case, however, they are generated through exponentiation by derivation elements of the form[33]

$$\delta_{ab} = [L_a, L_b] \tag{30}$$

for $a, b \in A$. Note that, due to commutativity, these derivations can be expressed in terms of associators $\delta_{ab} = [b, , a]$ on A. In other words, C^*-algebras are associative, possibly noncommutative algebras, with inner derivations given by commutators, while Jordan algebras are possibly nonassociative, commutative algebras with inner derivations given by associators.

As an example, if we take our coordinate algebra to be the JBL-algebra of $n \times n$ complex, Hermitian matrices $A = H_n(\mathbb{C})$, then all derivations are "inner" and take the form $\delta_{ab} = [L_a, L_b] a, b \in A$. The Lie algebra of inner derivations is given by $su(n)$ and generates the Lie group of inner automorphisms $SU(n)/\mathbb{Z}_n$ through exponentiation. If, further, we take the product between a canonical "Riemannian" geometry coordinatized by $A_c = C^\infty(M, \mathbb{R})$ and a finite internal geometry coordinatized by $A_F = H_n(\mathbb{C})$, then the product coordinate algebra will be given by $A = C^\infty(M, \mathbb{R}) \otimes_{\mathbb{R}} H_n(\mathbb{C}) = C^\infty(M, H_n(\mathbb{C}))$. In this case, the automorphism group will be given by $Diff(M) \ltimes SU(n)/\mathbb{Z}_n$ such that the product Jordan algebra can be understood as encoding the topological data of an $SU(n)$ gauge theory coupled to Einstein–Hilbert gravity. In other words, nothing is lost in restricting to the maximal JBL-subalgebra of a noncommutative coordinate algebra.

D. Differential structure

Coordinate algebras hold much more information than just the topological data of a geometry. They also encode information about the *differential* structure. Consider, for instance, a *smooth* Riemannian manifold M. In order to reconstruct M from its coordinate functions, the algebra of *smooth* coordinate functions $A = C^\infty(M, \mathbb{R})$ is required (considered as an abstract JBL-algebra). If one, instead, only had access to the algebra of continuous functions $C^0(M, \mathbb{R})$, then one would lose all the smooth structure of the manifold. The key feature that distinguishes the coordinate algebra of smooth functions $A = C^\infty(M, \mathbb{R})$ from the algebra of continuous functions $C^0(M, \mathbb{R})$ is that it is equipped with an algebra of many well defined derivations, namely, the vector fields $\delta_V = V^\mu \partial_\mu$ defined over the manifold M.[35] For Riemannian manifolds, the lie algebra of vector fields coincides with the lie algebra of derivations $\mathrm{Der}(C^\infty(M, \mathbb{R}))$ on $C^\infty(M, \mathbb{R})$, and it is this interpretation that generalizes most naturally to product geometries with the internal structure.

Following previous work,[12,26] we take the view that the appropriate notion of a vector field is that of a derivation and that the analog of the differentiable structure is encoded in the Lie algebra of derivations $\mathrm{Der}(A)$ defined over the coordinate algebra A.[35] A natural concern when restricting attention to the Hermitian subalgebra of a complex C^*-algebra is whether one loses information about the differential structure of the corresponding geometry when doing so. For the coordinate algebras of interest, however, this turns out not to be the case. Consider, for example, a Riemannian geometry (M_c, g). As discussed in Sec. III C 3, the Lie algebra of $*$-derivations defined over the complex coordinate algebra $C^\infty(M_c, \mathbb{C})$ (i.e., the Lie algebra of vector fields with real coefficients) clearly coincides with the Lie algebra of derivations defined over the real coordinate algebra $C^\infty(M_c, \mathbb{R})$. The story is similar when considering coordinate algebras with inner derivations. In particular, the associator identity given in Eq. (14d) relates derivations of a Jordan algebra $\mathrm{Der}(A_{sa})$ expressed in terms of associators with $*$-derivations of the corresponding C^*-algebra $\mathrm{Der}(A)$ written in terms of the Lie product. Indeed, as shown for the case of most physical interest in Sec. III C 3, the Lie algebra of $*$-derivation defined over the C^*-algebra of $n \times n$ complex matrices $M_n(\mathbb{C})$ coincides with the Lie algebra of derivations defined over the JBL-algebra of $n \times n$ complex Hermitian matrices, both of which are given by $su(n)$. In other words, it appears as if all the physically interesting data of an associative C^*-algebra A, including the differential structure, are really captured by its maximal JBL-algebra A_{sa}.

1. Differential calculi

Following Refs. 12 and 26, we take the view that derivations are the natural generalization of vector fields and that the natural analog of the differentiable structure for a geometry coordinatized by a Jordan algebra A_{sa} is encoded in the Lie algebra of derivations $\mathrm{Der}(A_{sa})$. This point of view implies a correspondingly natural generalization of the notion of differential forms in the Jordan setting. In this subsection, we present an explicit construction of the derivation based differential graded algebra of forms defined over Jordan algebras of $n \times n$ complex Hermitian matrices.

Consider the derivation algebra $\mathrm{Der}(A_{sa})$ defined over a unital JBL-algebra A_{sa}. Given a function $f \in A_{sa}$ and a vector field $\delta_V = V^i \delta_i \in \mathrm{Der}(A_{sa})$, where δ_i form a linearly independent basis of derivations, the object $\delta_V f \in A_{sa}$ can be viewed as a linear map in two distinct ways. (We stress that the subscript "i" on the basis element $\delta_i \in \mathrm{Der}(A_{sa})$ indexes the basis and is not an element of the coordinate algebra. This should not be confused with the double subscript on the derivation elements given in Eq. (30) or the single subscript given on the derivation element just below Eq. (29), which really do correspond to elements of the coordinate algebra.). To start with, of course, we can view $\delta_V f$ as the action of the linear operator δ_V on the function f. Note however, that because derivations form a vector space, the object $\delta_V f$ is also linear in the argument "V," in the sense that

$$\delta_{V+W} f = \delta_V f + \delta_W f \tag{31}$$

for $\delta_V, \delta_W \in \mathrm{Der}(A_{sa})$. One can therefore view the object $\delta_V f$ as a linear operation being performed on the vector field δ_V. This map is denoted by $df : \mathrm{Der}(A_{sa}) \to A_{sa}$, where

$$df(\delta_V) \equiv \delta_V f \tag{32}$$

and where "d" is known as the "exterior derivative" on functions. The object df is known as an "exact" one form and is completely defined by its action on vector fields. In particular, because each vector field can be expressed in terms of an orthonormal basis $\delta_V = V^i \delta_i$, we are able to write

$$df = (\delta_i f) E^i, \tag{33}$$

where the E^i satisfy $E^i(\delta_j) = \hat{\delta}^i_j$ and where $\hat{\delta}$ denotes the Kronecker delta. If δ_i span a k-dimensional basis of $\mathrm{Der}(A_{sa})$, then E^i span a k-dimensional basis of dual vectors, which we denote by $\mathrm{Der}^*(A_{sa})$. The space of one forms is then defined as the free module $\Omega^1 A_{sa} = A_{sa} \otimes \mathrm{Der}^*(A_{sa})$ such that an arbitrary one form $\omega \in \Omega^1 A$ can be expressed as

$$\omega = \omega_i E^i \tag{34}$$

for $\omega_i \in A_{sa}$.

The notion of one forms as linear maps from $\mathrm{Der}(A_{sa})$ to A_{sa} can be generalized to that of "n-forms," which are totally anti-symmetric, multi-linear maps from $\mathrm{Der}(A_{sa})$ to A_{sa}. Given two one forms $\omega = \omega_i E^i$ and $\omega' = \omega'_j E^j$, for example, a new "two-form" can be defined by taking the generalized "wedge" product,

$$\omega \wedge \omega' \equiv \omega_i \omega'_j (E^i \otimes E^j - E^j \otimes E^i). \tag{35}$$

General n-forms can similarly be defined by taking successive wedge products between forms

$$\omega = \omega_{1,\ldots,n}\, E^1 \wedge \ldots \wedge E^n \tag{36}$$

where $\omega_{1\ldots n} \in A_{sa}$. We will almost always drop the "wedge" sign when taking the product between two or more different forms (i.e., we will usually write $\omega\omega'$ to indicate the wedge product $\omega \wedge \omega'$). Under the product given in Eq. (36), the vector space $\Omega A_{sa} = \oplus_n \Omega^n A$, where $\Omega^n A_{sa}$ is the space of n-forms with $\Omega^0 A_{sa} = A_{sa}$, forms a graded Jordan algebra that satisfies the following identities:

$$ab - (-1)^{|a\|b|} ba = 0 \qquad \text{(Graded Commutativity)}, \tag{37a}$$

$$\sum_{\{a,b,c\}} (-1)^{|a\|c|} (L_a L_{bc} - L_{ab} L_c) = 0 \qquad \text{(Graded Jordan Identity)}, \tag{37b}$$

where in the graded Jordan identity, we are summing over all even permutations of $a, b, c \in \Omega A_{sa}$ and we denote the "grading" or the "order" of an element $a \in \Omega A_{sa}$ by $|a|$, i.e., $|a| = m$ when $a \in \Omega^m A_{sa}$. Note, in particular, that at order zero, these identities reproduce the usual properties of a Jordan algebra of order zero given in Eq. (13). In particular, the more familiar form of the Jordan identity given in Eq. (13b) is easily recovered at order zero by setting $a = b = c$ in Eq. (37b) [deriving (37b) from (13b) at order zero is only slightly more involved[33,36]].

A graded Jordan algebra algebra ΩA_{sa} can be further elevated to a differential graded algebra of forms, by extending the action of the exterior derivative "d" to the whole of A such that it satisfies the following two properties:

$$d^2 = 0 \qquad \text{(Nilpotency condition)}, \tag{37c}$$

$$d(ab) = d(a)b + (-1)^{|a|} a d(b) \qquad \text{(Graded Leibniz)} \tag{37d}$$

for $a, b \in \Omega A_{sa}$.

Example: the "Canonical" differential graded algebra of forms.

Consider the canonical (Jordan) coordinate algebra $A_{sa} = C^\infty(M, \mathbb{R})$ corresponding to a Riemannian manifold M, with a local basis of coordinate functions $x^\mu \in A_{sa}$. Any vector field $\delta_V \in \mathrm{Der}(A_{sa})$ can be expanded in terms of this basis $\delta_V = V^\mu \frac{\partial}{\partial x^\mu}$. Similarly, a dual basis of forms can be constructed, $E^\mu = dx^\mu \in \Omega^1 A_{sa}$, with the action on vector fields given by $dx^\mu(\delta_V) = V^\nu \partial x^\mu / \partial x^\nu = V^\mu$. A general one form $\omega \in \Omega^1 A_{sa}$ is then given by

$$\omega = \omega_\mu\, dx^\mu \tag{38}$$

where $\omega_\mu \in A_{sa}$. In particular, we can write $df = \partial_\mu f dx^\mu$. Similarly, general n-forms are defined by taking successive wedge products between forms

$$\omega = \omega_{1,\ldots,n}\, dx^1 \wedge \ldots \wedge dx^n \tag{39}$$

where $\omega_{1\ldots n} \in A_{sa}$. Under this product, the algebra $\Omega A_{sa} = \oplus_n \Omega^n A_{sa}$ is both graded commutative and associative, which means that it trivially satisfies the conditions of a graded Jordan algebra as given in Eqs. (37a) and (37b). Furthermore, ΩA_{sa} can be elevated naturally to a differential graded algebra of forms, by extending the action of the exterior derivative "d" to arbitrary forms such that it satisfies Eqs. (37c) and (37d). Note, in particular, that the nilpotency of the exterior derivative on forms $d^2 = 0$ is automatically ensured by the graded commutativity of the wedge product, the graded Leibniz rule, and the commutativity of partial derivatives.

Example: The finite coordinate algebra $A_F = H_n(\mathbb{C})$.

Next, we explicitly construct the derivation based differential graded algebra of forms defined over the JBL-algebra $A_{sa} = H_n(\mathbb{C})$ (i.e., the Jordan algebra of $n \times n$ complex Hermitian matrices). The corresponding C^*-algebra $A = A_{sa} \oplus iA_{sa}$ is given by the matrix algebra of $n \times n$ complex matrices. Given a basis of linearly independent traceless, complex, Hermitian matrices $\lambda_i \in A$, where $i = 1, \ldots, n^2 - 1$, the associative product on A [see Eq. (16)] can be expressed as

$$\lambda_i \lambda_j = \frac{1}{2}(\underbrace{\frac{1}{n}\hat\delta_{ij}\lambda_0 + d_{ijk}\lambda_k}_{\text{Jordan}} + \underbrace{i f_{ijk}\lambda_k}_{\text{Lie}}), \tag{40}$$

where $\lambda_0 = \mathbb{I}_n$ is the identity element in A_{sa}. The d_{ijk} are real valued and completely symmetric "Jordan" structure constants, which define the Jordan product "$\circ$," while f_{ijk} are the completely anti-symmetric structure constants that define the Lie product "$\times$" [i.e., the structure constants of the $su(n)$ Lie algebra]. Note that for this class of Jordan algebras, specifically $A_{sa} = H_n(\mathbb{C})$, the dimension of the derivation algebra $\mathrm{Der}(A_{sa}) = su(n)$ coincides exactly with the number of linearly independent traceless Hermitian basis elements $\lambda_i \in A_{sa}$. This allows us to define a particularly elegant basis of $n^2 - 1$ anti-Hermitian inner derivation elements, expressed in terms of the "Lie" algebra structure constants,

$$\delta_i = \frac{4}{n} f_i^{jk}[L_{\lambda_j}, L_{\lambda_k}], \tag{41}$$

where summation is implied over repeated indices and $L_{\lambda_i} f = \frac{1}{2}(\lambda_i f + f\lambda_i) = \lambda_i \circ f$ denotes the left action under the Jordan product (note that the only distinction between the lowered and raised indices is stylistic). Making use of Eq. (40), it is easy to show that the action of a derivation $\delta_i \in \mathrm{Der}(A_{sa})$ on elements $\lambda_j \in A_{sa}$ is given by

$$\delta_i \lambda_j = -f_{ij}^{k}\lambda_k, \tag{42}$$

where the standard normalization $f^{ijk}f_{ijl} = n\hat{\delta}_l^k$ is being used. Note, in particular, that Eq. (42) is really only restating the fact that the coordinate algebra A_{sa} is a JBL-algebra with Lie product

$$\lambda_i \times \lambda_j = \delta_i \lambda_j \tag{43}$$

for $\lambda_i \in A_{sa}, i = 1, \ldots, n^2 - 1$.

Having expressed the derivation elements in a nice basis, our next task is to find a similarly nice dual basis of one forms $E^i \in \Omega^1 A_{sa}$ such that $E^i[\delta_j] = \hat{\delta}_j^i$. A little work shows that the dual elements take the form

$$E^i = \frac{16}{n} f^{jki}(\lambda_k \circ \lambda_l) \circ (\lambda_j \circ d\lambda_l), \tag{44}$$

where "d" is the order one exterior derivative. The proof that E^i form a dual basis of one forms is simple, but long, and so we provide it separately in Appendix C. Having constructed an explicit basis of derivations and dual forms, an arbitrary one form $\omega \in \Omega^1 A_{sa}$ can be expressed as

$$\omega = \omega_i E^i \tag{45}$$

where $\omega_i \in A_{sa}$. In particular, the action of the exterior derivative is given by $df = \delta_i f E^i$ for $f \in A_{sa}$. Furthermore, general n-forms are defined by taking successive wedge products between forms. Given two elements $a = a^{1\cdots n}E_1 \wedge \cdots \wedge E_n \in \Omega^n A$ and $b = b^{1\cdots m}E_1 \wedge \cdots \wedge E_m \in \Omega^m A$, their graded Jordan product is given by

$$ab = (a_{1,\ldots,n}b_{1+n,\ldots,m+n}) E^1 \wedge \ldots \wedge E^{n+m} \in A^{m+n}. \tag{46}$$

It is simple to check that the graded commutativity and graded Jordan identities given in Eqs. (37a) and (37b) are satisfied by $\Omega A_{sa} = \oplus_n \Omega^n A$. Next, the graded algebra ΩA can be promoted to a differential graded algebra of forms by extending the action of the exterior derivative "d" to higher order forms such that it satisfies the properties given in Eqs. (37c) and (37d). Following a fairly long, but straightforward calculation (which we provide in Appendix C), we find that the action of the exterior derivative on the one form basis is given by

$$dE^i = \frac{1}{2} f^i_{jk} E^j \wedge E^k. \tag{47}$$

Note that the differential graded algebra ΩA_{sa} has a number of properties that distinguish it and make it much nicer than the differential graded algebra of forms ΩA defined over the corresponding noncommutative C^*-algebra of $n \times n$ complex matrices $A = A_{sa} \oplus iA_{sa}$.[17] In particular, once one demands that the exterior derivative satisfies the nilpotency condition given in Eq. (37c), this automatically implies the graded commutativity, graded Leibniz, and graded Jordan identities given in Eq. (37). This is very much in line with the story as it occurs in the Riemannian setting and is in stark contrast to the noncommutative setting, in which the construction is much more awkward and appears somewhat contrived.

IV. CONCLUSION AND FUTURE OUTLOOK

Jordan algebras were first introduced in an effort to restructure quantum mechanics purely in terms of physical observables (i.e., the things that can actually be seen in the experiment).[23,37] In the current paper, we have similarly been drawn on physical grounds to consider geometries coordinatized by Jordan algebras. The usual assumption in the literature has been that if the patterns and features of the standard model derive from the underlying structure of a discrete internal geometry, then this geometry should be coordinatized by a noncommutative

C^*-algebra. The noncommutative geometry that most closely captures the details of the standard model, while extraordinarily beautiful and insightful, runs into a variety of problems and relies on a number of nongeometric input assumptions in order to work. In particular, it appears as though the space of standard model fermions is not compatible with the full noncommutative coordinate algebra that must been selected in order to accommodate the standard model symmetries (this remains true for extensions such as the Pati–Salam model[9]). Instead, the standard model fermions are only closed under the symmetrized action of the self-adjoint elements of the noncommutative coordinate algebra.[6] These elements form a Jordan algebra, which motivates the search for a "Jordan geometry," which is able to accommodate the full details of the standard model.

In this paper, we explain that the most important physical data captured by a complex coordinate C^*-algebra appear to be entirely contained within its maximal JBL-subalgebra. Our program is an attempt to restructure the geometric description of nature purely in terms of physically relevant quantities and furthermore to make important geometric inferences purely based on physical observables. Second, this paper presents an explicit construction of the differential calculus for the finite dimensional coordinate algebras that appear most important for constructing realistic particle theories [i.e., the Jordan algebras of $n \times n$ complex Hermitian matrices $H_n(\mathbb{C})$]. This is the first of a number of tools that will be required for describing the dynamics of physical theories. Constructing the "geometry" of a physical theory consists of roughly two parts: (i) building the underlying geometry itself (for a gravity theory, this would be the underlying Riemannian manifold and metric data) and (ii) describing the dynamics of the geometry (for a gravity theory, this might come from the familiar Einstein–Hilbert action or something more exotic). In the noncommutative geometric approach to constructing gauge theories, the "spectral action" is most often used to describe dynamics. The spectral action is introduced, in part, because it is difficult to construct dynamics by making explicit use of geometric objects such as noncommutative differential forms. While intriguing and extremely beautiful, the spectral action provides an effective description of nature and is the setting within which the incorrect Higgs mass prediction was made.[38,39] The calculus constructed in this paper presents a necessary step toward an explicit and more direct construction of dynamics.

The next immediate steps in fully constructing the internal Jordan geometries that most closely capture the underlying structure of the standard model of particle physics and viable extensions such as the Pati–Salam model are as follows:

1. **Discrete spaces**: It will be necessary to consider finite Jordan coordinate algebras, which are semisimple and, in particular, of the form

$$A_F = \underbrace{H_n(\mathbb{C}) \oplus \ldots \oplus H_n(\mathbb{C})}_{\text{`}m\text{' copies}} \tag{48}$$

with inner derivation algebra $\text{Der}(A_F) = su(n) \oplus \cdots \oplus su(n)$. For coordinate algebras of this form, we will need to generalize as follows:

- **Discrete connections**: The algebra A_F coordinatizes an "m" point (state) space with a non-trivial finite internal structure at each point, which is *a priori* completely disconnected. In order to "connect" such a space, the notion of a discrete connection that relates the "m" (identical) factors will be required. Higgs fields will arise in this way "connecting," for example, the chiral sectors of the models we are interested in.
- **Discrete calculus**: The derivation based calculus[12,26] is no longer appropriate when considering discrete geometries with disconnected sectors that are not smoothly related to one another by derivations. Connes has developed a "cochain" based generalization of differential forms appropriate for the discrete, noncommutative setting.[1] The corresponding "cyclic" cohomology only makes sense, however, for associative coordinate algebras, and so a generalization will need to be made that incorporates the benefits of both approaches. Generalizations have already been developed in very special cases, including for the octonion algebra,[40,41] and for Hom-associative algebras,[42] however, there is as yet no general construction appropriate for Jordan algebras.

2. **Clifford representation of forms**: Much of the predictive power of the noncommutative geometry that most closely resembles the standard model of particle physics derives from the geometric "rules" or axioms imposed on the finite input data. In practice, many of these rules have been "ported across" from Riemannian spin geometry, where they derive naturally when considering the Clifford action of forms on spinors. In the noncommutative setting, however, no clean analog of a Clifford action of forms has so far been developed. As a result, there is almost a complete disconnect between the symmetries, signature, and scalar representations in these theories, which limits their predictive power. A key step will be to develop a "Clifford" representation of forms appropriate for the Jordan geometries of interest. Once an appropriate notion of Clifford action is developed, it will be possible to explore whether the internal symmetries of a geometry correspond exclusively to the automorphisms of the input coordinate algebra or if there is some internal analog of local Lorentz symmetry to be accounted for, as advocated, for example, in Ref. 43 (and therefore, what, if any, relationship exists between the symmetries and the signature of an internal space). Furthermore, another key exploration will be to determine whether the "multiplicity" required in order to accommodate the internal analog of a "Clifford" representation will lead to a prediction for the number of fermion generations or whether it will provide a stronger handle on the form of the fermion mass matrices.

APPENDIX A: THE INTERNAL GRADING OF THE "NONCOMMUTATIVE" STANDARD MODEL

In Eq. (1), we introduced the representation of the internal noncommutative coordinate algebra that most closely captures the fermion and gauge content of the standard model. We further introduced a finite charge conjugation operator J_F, which acts on elements of the Hilbert space by Hermitian conjugation. Here, we introduce the finite grading operator on elements $h \in H_F$ as $\gamma_F h = XhX$, where

$$
X = \begin{pmatrix} +\mathbb{I}_2 & & & \\ & -\mathbb{I}_2 & & \\ \hline & & \mathbb{I}_2 & \\ & & & \mathbb{I}_2 \end{pmatrix} . \tag{A1}
$$

APPENDIX B: REPRESENTATIONS AND UNIMODULARITY

Let A be an algebra defined over a field $\mathbb{F}$, and let H be a vector space over $\mathbb{F}$. A bi-representation π of A on H (or, equivalently, a bi-module H over A) is a pair of $\mathbb{F}$-bilinear products $f\Psi \in H$ and $\Psi f \in H$ ($f \in A, \Psi \in H$).[4,19,20,33] This definition of a bi-representation of A on H is equivalent to the definition of a new algebra

$$
B = A \oplus H, \tag{B1}
$$

with the product between elements of $B(b = f + \Psi$ and $b' = f' + \Psi')$ given by

$$
bb' = ff' + f\Psi' + \Psi f', \tag{B2}
$$

where $ff' \in A$ is the product inherited from A, while $f\Psi' \in H$ and $\Psi f' \in H$ are the products inherited from π, and $\Psi\Psi' = 0$. We call such an algebra an "Eilenberg algebra."

Note that the above definition has not assumed anything about the associativity of either A or B. If however we assume that B is associative, then we precisely recover the traditional definition of an ordinary associative bi-representation of an associative algebra A on H.[4] If on the other hand A is a Jordan algebra, then we define its Jordan representation on H by taking B to also be a Jordan algebra.[4,11] As an example, consider the algebra $A = M_n(\mathbb{C})$ represented as matrices on $\mathbb{C}^{2n}$ by the following action:

$$
L_f \Psi = \begin{pmatrix} f & \\ & 0 \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, R_f \Psi = \begin{pmatrix} 0 & \\ & f^T \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{B3}
$$

for $f \in A$. The algebra A is naturally involutive, with the involution given by Hermitian conjugation. We can similarly turn $B = A \oplus H$ into an associative Eilenberg $*$-algebra by equipping H with the following involution:

$$
J = \begin{pmatrix} & \mathbb{I}_n \\ \mathbb{I}_n & \end{pmatrix} \circ c.c, \tag{B4}
$$

where "$c.c$" denotes complex conjugation.

The symmetries of a geometry correspond to the automorphisms of its coordinate algebra representation. In the above example, the coordinate algebra representation is expressed in terms of an associative Eilenberg $*$-algebra $B = A + H$. The inner $*$-derivations on B are then of the form $\delta_a = L_a - R_a$, where $a^* = -a \in A$ (the element a cannot be taken from H because automorphisms must be invertible). Remarkably,

one finds that the inner $*$-derivation algebra of B is larger than that of A and is given by $u(n)$ [rather than $su(n)$]. The generator $T = \delta_{i\mathbb{I}}$, while acting trivially on A, does not act trivially on H. This feature leads to an additional $U(1)$ gauge symmetry that is not observed experimentally, when constructing the noncommutative geometry that most closely captures the details of the standard model of particle physics. In order to remove the additional unwanted $U(1)$ symmetry, the so-called "unimodularity condition" is imposed, which is the statement that one should only take into account gauge generators that are traceless.

The "unimodularity condition" appears ad-hoc and poorly motivated. Remarkably, it appears as though this problem may be avoided in the Jordan setting. If in our above example one restricts to the maximal Jordan subalgebra $B_{sa} = A_{sa} + H_{sa}$, equipped with the symmetrized product given in Eq. (12), then one obtains a Jordan representation of the real Jordan algebra $A_{sa} = H_n(\mathbb{C})$ on the vector space $\mathbb{R}^{2n}$. In this case, the inner derivations on B_{sa} are of the form $\delta_{ab} = [L_a, L_b]$, where $a, b \in A_{sa}$. It is an instructive (and not too difficult) exercise to show that the inner derivation elements on B_{sa} form the Lie algebra $su(n)$. Every derivation of a semisimple algebra (that is, the direct sum of simple algebras) with a unit over a field of characteristic zero is inner, and as a result, we know that all the derivations of $A_{sa} = H_n(\mathbb{C})$ are inner [and form the Lie algebra $su(n)$]. Note, however, that the Eilenberg algebra $B_{sa} = A_{sa} \oplus H_{sa}$ is no longer semisimple, and as a consequence, it may have a larger set of symmetries. The full algebra of $*$-derivations on B_{sa} is, in fact, given by $u(n)$. In this case, however, the additional $U(1)$ generator T is explicitly distinguished from the inner $*$-derivations and appears as an outer derivation.

APPENDIX C: DIFFERENTIAL CALCULI

Our aim in this section is to show that the differential graded algebra constructed over the Jordan algebra of $n \times n$ Hermitian matrices does indeed satisfy each of the properties outlined in Sec. III D. We begin by reminding the reader of the following well-known trace identities:[44–47]

$$Tr[F_i] = 0, \qquad Tr[D_i] = 0, \qquad Tr[F_iF_j] = n\hat{\delta}_{ij},$$

$$Tr[D_iD_j] = \frac{n^2-4}{N}\delta_{ij}, \qquad Tr[F_iD_j] = 0, \qquad Tr[F_iF_jF_k] = i\frac{n}{2}f_{ijk},$$

$$Tr[D_iF_jF_k] = \frac{n}{2}d_{ijk}, \qquad Tr[D_iD_jF_k] = i\frac{n^2-4}{2n}f_{ijk}, \qquad Tr[D_iD_jD_k] = \frac{n^2-12}{2n}d_{ijk},$$

$$Tr[F_iD_jF_kD_l] = \frac{1}{2}(\hat{\delta}_{ik}\hat{\delta}_{jl} - \hat{\delta}_{ij}\hat{\delta}_{kl}) + \frac{n}{4}(d_{iln}d_{jkn} + f_{iln}f_{jkn}),$$

$$Tr[F_iF_jD_kD_l] = \frac{1}{2}(\hat{\delta}_{ij}\hat{\delta}_{kl} - \hat{\delta}_{ik}\hat{\delta}_{jl}) + \frac{n^2-8}{4n}f_{iln}f_{jkn} + \frac{n}{4}d_{iln}d_{jkn},$$

$$Tr[F_iD_jD_kD_l] = i\frac{2}{n}f_{iln}d_{jkn} + i\frac{n^2-8}{4n}f_{ijn}d_{kln} + \frac{i}{4}d_{ijn}f_{kln}$$

$$Tr[F_iF_jF_kD_l] = i\frac{n}{4}(d_{iln}f_{jkn} - f_{iln}d_{jkn}),$$

together with the identities

$$0 = f_{ije}f_{ekl} + f_{kje}f_{iel} + f_{lje}f_{ike}, \tag{C1a}$$

$$0 = f_{ijn}d_{kln} + f_{iln}d_{jkn} + f_{jkn}d_{lin}, \tag{C1b}$$

$$f_{ije}f_{jke} = \frac{2}{n}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + d_{ikn}d_{jln} - d_{jkn}d_{iln} \tag{C1c}$$

where we have defined $(F_i)_{jk} = -if_{ijk}$ and $(D_i)_{jk} = d_{ijk}$. Equation (C1a) is nothing other than the well known Jacobi identity. Our first goal will be to show that the basis of derivation elements $\delta_j \in \mathrm{Der}(A_{sa})$ and dual forms $E^i \in \Omega^1 A_{sa}$ defined in Eqs. (41) and (44), respectively, satisfy the condition $E^i[\delta_j] = \hat{\delta}^i_j$. To see this, we make use of Eq. (42) together with Jordan product expressed as the symmetrization of the matrix product given in Eq. (40) to write

$$E^i[\delta_j] = \frac{16}{n}f^{kli}(\lambda_l \circ \lambda^n) \circ (\lambda_k \circ (\delta_j\lambda_n))$$

$$= -\frac{4}{n^3}f^{kli}f_{jlk} + \frac{4}{n^2}f^{kli}(f_{ljn}d_k^{nm} + f_{njk}d_l^{nm})\lambda_m + \frac{4}{n}f^{kli}f_{njf}d_l^{nh}d_k^{fm}\lambda_h\lambda_m$$

$$= \frac{4}{n^3}Tr[F^iF_j] + \frac{4}{n^2}(Tr[F^iF_jD^k] + Tr[F^iD^kF_j])\lambda_k + \frac{4}{n}Tr[F^iD^lF_jD^k]\lambda_l\lambda_k$$

$$= \frac{4}{n^2}\hat{\delta}^i_j + \frac{4}{n}d_j^{ik}\lambda_k + \frac{2}{n}[\frac{n}{4}(d^{ikn}d_{jln} + f^{ikn}f_{ljn}) - \frac{1}{2}(\hat{\delta}^i_l\hat{\delta}^k_j - \hat{\delta}^i_j\hat{\delta}^k_l)](\frac{1}{n}\hat{\delta}^l_k + d_k^{ln}\lambda_n)$$

$$
= \frac{4}{n^2}\hat{\delta}^i_j + \frac{4}{n}d^{ik}_j\lambda_k + \frac{2}{n^2}\big[\frac{n}{4}Tr[D^iD_j - F^iF_j] - \frac{1}{2}(\hat{\delta}^i_j - \hat{\delta}^i_j\hat{\delta}^l_l)\big]
$$

$$
+ \frac{2}{n}\big[\frac{n}{4}Tr[D^iD_jD^n - F^iF_jD^n] - \frac{1}{2}d^{in}_j\big]\lambda_n
$$

$$
= \hat{\delta}^i_j \tag{C2}
$$

as required.

Our next goal will be to show that $dE^i = \frac{1}{2}f^i_{jk}E^j \wedge E^k$. Before doing so, however, we note that in deriving this result we will not make use of the anti-symmetry of the wedge product itself. To remind the reader of this fact, we explicitly replace the usual wedge product "$\wedge$" with an abstract product "$\bullet$" throughout, reinstating the wedge product only at the end. We then show that the nilpotency condition $d^2 = 0$ implies the anti-symmetry of the product between forms, just as occurs in Riemannian geometry. We begin by making use of Eq. (44) together with the symmetrization of the matrix product given in Eq. (40) to write

$$
dE^i = \frac{8}{n}f^{jki}d\big[\big(\frac{1}{n}\hat{\delta}_{kl} + d_{kln}\lambda_n\big) \circ (\lambda_j \circ d\lambda_l)\big]
$$

$$
= \frac{8}{n}f^{jki}\Big[\underbrace{d_{klp}(\delta_m\lambda_p) \circ (\lambda_j \circ \delta_n\lambda_l)}_{(A)} + \underbrace{\frac{1}{n}\delta_m\lambda_j \circ \delta_n\lambda_k}_{(B)} + \underbrace{d_{lkp}\lambda_p \circ (\delta_m\lambda_j \circ \delta_n\lambda_l)}_{(C)}\Big]E^m \bullet E^n. \tag{C3}
$$

We break down Eq. (C3) into three more manageable parts and address each separately. We begin with the first term labeled "A,"

$$
(A) = \frac{8}{n}f^{jki}d_{klp}(\delta_m\lambda_p) \circ (\lambda_j \circ \delta_n\lambda_l)E^m \bullet E^n
$$

$$
= \frac{2}{n}f^{jki}d_{klp}f_{mps}\big[\frac{2}{n}f_{nlj}\lambda_s + f_{nlt}d_{jtq}(\frac{1}{n}\hat{\delta}_{sq} + d_{sqr}\lambda_r)\big]E^m \bullet E^n
$$

$$
= \frac{2}{n}\Big[\frac{2}{n}f_{pms}Tr[F_iD_pF_n]\lambda_s + \frac{1}{n}f^{ijk}Tr[F_mD_bF_nD_k] - f_{mps}d_{sqr}Tr[F^iD_pF_nD_q]\lambda_r\Big]E^m \bullet E^n
$$

$$
= \frac{2}{n}\Big[f_{pms}d_{ipn}\lambda_s - \frac{1}{2n}f^{imn} + \frac{1}{4}f^{ijk}(f_{mce}f_{nej} + d_{mke}d_{nej})\Big]E^m \bullet E^n
$$

$$
+ \frac{2}{n}\Big[\frac{1}{2}f_{mis}d_{snr} - \frac{n}{4}(f_{iqe}f_{nep}f_{mps}d_{rsq} + f_{mps}d_{rsq}d_{iqe}d_{nep})\Big]\lambda_rE^m \bullet E^n
$$

$$
= \frac{2}{n}\Big[f_{smr}d_{isn}\lambda_r - \frac{1}{2n}f^{imn} - \frac{i}{4}Tr[F^iF_mF_n - F^iD_mD_n]\Big]E^m \bullet E^n
$$

$$
+ \frac{2}{n}\Big[\frac{1}{2}f_{mis}d_{snr} + \frac{n}{4}iTr[F_iF_nF_mD_r - F_mD_rD_iD_n]\Big]\lambda_rE^m \bullet E^n
$$

$$
= 0. \tag{C4}
$$

We next address the second term in Eq. (C3) labeled "B,"

$$
(B) = \frac{8}{n^2}f^{jki}\delta_m\lambda_j \circ \delta_n\lambda_kE^m \bullet E^n
$$

$$
= \frac{4}{n^2}f^{jki}f_{mjs}\big[\frac{1}{n}f_{nks}f_{nct}d_{stp}\lambda_p\big]E^m \bullet E^n
$$

$$
= \frac{4}{n^2}\big[\frac{i}{n}Tr[F^iF_nF_m] + iTr[F^iF_nD_pF_m]\lambda_p\big]E^m \bullet E^n
$$

$$
= \frac{1}{n}\big[\frac{2}{n}f_{imn} + (f_{mpe}d_{ine} - f_{ine}d_{mpe})\lambda_p\big]E^m \bullet E^n. \tag{C5}
$$

Finally, we address the third term in Eq. (C3) labeled "C,"

$$
(C) = \frac{8}{n}f^{jki}d_{lkp}\lambda_p \circ (\delta_m\lambda_j \circ \delta_n\lambda_l)E^m \bullet E^n
$$

$$
= \frac{4}{n}f^{jki}d_{klp}f_{mjs}\big[\frac{1}{n}f_{nls}\lambda_p + \frac{1}{2}f_{nlt}d_{stq}(\frac{1}{n}\delta_{pq} + d_{pqr}\lambda_r)\big]E^m \bullet E^n
$$

$$= \frac{2}{n}\left[\frac{2}{n}iTr[F_n F_m F^i D_p]\lambda_p + \frac{1}{n}Tr[F_m D_t D_d F^i]f_{ndt} + if^{jki}f_{mbs}Tr[F_n D_s D_r D_k]\lambda_r\right]E^m \bullet E^n$$

$$= \frac{1}{n}\left[f_{npe}d_{mie} - f_{mie}d_{npe} + \frac{1}{n}f_{mni} + \frac{n^2-8}{2n^2}f_{ile}f_{met}f_{ntl} - \frac{1}{2}d_{ile}d_{met}f_{ntl}\right]E^m \bullet E^n$$

$$+ f^{ijk}f_{mjs}\left[\frac{4}{n^2}(f_{nse}d_{rek} + d_{rse}f_{nek})\lambda_r + \frac{1}{2}(d_{nse}f_{rek} - f_{nse}d_{rek})\lambda_r\right]E^m \bullet E^n$$

$$= \frac{1}{n}\left[f_{npe}d_{mie} - f_{mie}d_{npe} + \frac{1}{n}f_{mni} - i\frac{n^2-8}{2n^2}Tr[F_i F_m F_n] - i\frac{1}{2}Tr[F_n D_i D_m]\right]E^m \bullet E^n$$

$$+ \left[\frac{4}{n^2}iTr[F^i F_m F_n D_r + F^i F_m D_r F_n]\lambda_r - \frac{i}{2}Tr[F^i F_m F_n D_r - F^i F_m D_n F_r]\lambda_r\right]E^m \bullet E^n$$

$$= \frac{1}{n}\left[f_{nre}d_{mie}\lambda_r - f_{ime}d_{nre}\lambda_r + \frac{n}{2}f_{mni} - \frac{2}{n}f_{mni}\right]E^m \bullet E^n, \tag{C6}$$

where we have made a significant use of Eq. (C1b). The right-hand side of Eq. (C3) is then given by the summation of terms "A," "B," and "C,"

$$dE^i = (A) + (B) + (C) = \frac{1}{2}f_{mni}E^m \bullet E^n. \tag{C7}$$

We may replace the abstract product "•" in Eq. (C7) by the wedge product '∧,' and indeed, the nilpotency condition $d^2 = 0$ implies that the product between two basis forms, $E^i, E^j \in \Omega^1 A_{sa}$, must be anti-symmetric. To see this, we begin by assuming an abstract product between forms denoted by "•." We then have

$$0 = d^2\lambda_i = d(\delta_j\lambda_i E^j)$$

$$= f_{jik}f_l^{kf}\lambda_f E^l \bullet E^j - f_{ji}^k\lambda_k dE^j$$

$$= \frac{1}{2}(f_{jik}f_l^{kf} - f_{lik}f_j^{kf})\lambda_f E^l \bullet E^j + \frac{1}{2}(f_{jik}f_l^{kf} + f_{lik}f_j^{kf})\lambda_f E^l \bullet E^j - f_{ji}^k\lambda_k dE^j$$

$$= f_{ji}^k\lambda_k(\frac{1}{2}f_{le}^j E^l \bullet E^e - dE^j) + \frac{1}{2}(f_{jik}f_l^{kf} + f_{lik}f_j^{kf})\lambda_f E^l \bullet E^j$$

$$= \frac{1}{2}(f_{jik}f_l^{kf} + f_{lik}f_j^{kf})\lambda_f E^l \bullet E^j, \tag{C8}$$

where in the fourth line, we have made use of the Jacobi given in Eq. (C1a) and in the last line, we have made use of the result given in Eq. (C7). Finally, the last line is identically zero if we replace the abstract product "•" with one that is anti-symmetric, such as the wedge product "∧." The wedge product together with the graded Leibniz rule given in Eq. (37d) further ensure that the exterior derivative squares to zero on all higher order forms. In particular,

$$d^2 E^i = \frac{1}{2}f_{mni}d[E^m \wedge E^n]$$

$$= \frac{1}{4}f_{mni}f_{stm}E^s \wedge E^t \wedge E^n - \frac{1}{4}f_{mna}f_{stn}E^m \wedge E^s \wedge E^t$$

$$= -\frac{1}{2}\left[\frac{2}{n}(\delta_{is}\delta_{nt} - \delta_{it}\delta_{ns}) + d_{ism}d_{ntm} - d_{nsm}d_{itm}\right]E^s \wedge E^t \wedge E^n = 0,$$

where we have made use of Eq. (C1c).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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