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$\mathcal{N}=4$ super Yang-Mills correlators without anti-commuting variables

Hermann Nicolai[a] and Jan Plefka[b]

[a] *Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut) Mühlenberg 1, D-14476 Potsdam, Germany*

[b] *Institut für Physik und IRIS Adlershof, Humboldt-Universität zu Berlin, Zum Großen Windkanal 6, D-12489 Berlin, Germany*

{nicolai@aei.mpg.de, jan.plefka@hu-berlin.de}

Abstract

Quantum correlators of pure supersymmetric Yang-Mills theories in $D = 3, 4, 6$ and 10 dimensions can be reformulated via the non-linear and non-local transformation ('Nicolai map') that maps the full functional measure of the interacting theory to that of a free bosonic theory. As a special application we show that for the maximally extended $\mathcal{N} = 4$ theory in four dimensions, and up to order $\mathcal{O}(g^2)$, all known results for scalar correlators can be recovered in this way without any use of anti-commuting variables, in terms of a purely bosonic and ghost free functional measure for the gauge fields. This includes in particular the dilatation operator yielding the anomalous dimensions of composite operators. The formalism is thus competitive with more standard perturbative techniques.

1 Introduction

Pure supersymmetric Yang-Mills theories exist in $D = 3, 4, 6$ and 10 dimensions [1]. As is well known, the corresponding extended super-Yang-Mills theories in lower dimensions can be obtained from these by dimensional reduction. Among the supersymmetric Yang-Mills theories, the maximally extended $\mathcal{N} = 4$ theory in four dimensions stands out for several reasons, especially in connection with the AdS/CFT correspondence, as a result of which there now exists an enormous variety and wealth of results (indeed, too many to list here!). In this paper we want to take a new and different look at this theory, exploiting the existence of a non-local and non-linear transformation T_g ('Nicolai map') that maps the full functional measure of the interacting Yang-Mills theory to the one of a theory of $\dim G$ free (Maxwell) vector fields, where G is the gauge group in question (usually $G = SU(N)$). The existence of this map for the $\mathcal{N} = 1, D = 4$ theory was established long ago [2, 3], and a detailed prescription for its iterative construction was presented in [4–7] and [3]. It was, however, only very recently that these constructions were extended to other dimensions, and in particular to the maximally extended $D = 10$ and $\mathcal{N} = 4, D = 4$ theories [8]. The existence of the map T_g opens very different perspectives on the quantization of supersymmetric Yang-Mills theories, in terms of a ghost and fermion free formalism and with a purely bosonic functional measure. This concerns especially the computation of quantum correlators. Previous work in this direction remains somewhat scattered: in [5, 6] several perturbative results for the $\mathcal{N} = 1, D = 4$ theory (for instance, wave function renormalization factors and the β-function to order g^2) were recovered in a perturbative approach. Non-perturbative aspects were studied in [9] where it was shown in particular that there exists a *local* expression for T_g in the light-cone gauge. This result was subsequently used to recalculate 2-gluon and 3-gluon Green's functions up to one loop [10]. The $\mathcal{N} = 1, D = 4$ Yang-Mills theory can also be investigated in terms of anti-selfdual variables, yielding (amongst other results) a non-perturbative derivation of the β-function [11]. However, as far as we are aware, [12] is the only attempt towards understanding *extended*, and more specifically, half-maximal (*i.e.* $D = 6$ or $\mathcal{N} = 2, D = 4$) super-Yang-Mills theories in this framework, with an intriguing proposal for a closed form expression of T_g. Yet, to the best of our knowledge, no results in this direction have been available so far for the maximally extended $\mathcal{N} = 4$ theory, which from many points of view is by far the most interesting. This is the main issue we want to (begin to) address in this paper.

Accordingly, we wish to investigate certain quantum correlators, and more specifically scalar correlation functions of the $\mathcal{N} = 4$ theory in terms of the map T_g, and to show that several known results can be easily recovered with this formalism and in terms of the map T_g, at least to the extent that it has been worked out. It should, however, be understood that these results – being confined to the perturbative domain – constitute only a very first step. Ultimately, we would hope that this formalism can provide essentially new insights on the $\mathcal{N} = 4$ theory. Amongst other things, these include prospects for a non-perturbative regularization of the $\mathcal{N} = 4$ theory, especially in conjunction with its conjectured integrability properties [13].

The non-linear and non-local transformation (which more generally exists for all rigidly supersymmetric theories with Lagrangians quadratic in the fermions)

$$T_g[A]^a_\mu(x) \equiv A'^a_\mu(x, g; A) \tag{1.1}$$

is characterized by the following properties:

1. Substitution of $A'(A)$ into the free Maxwell action (or rather: sum of Maxwell actions) yields the interacting theory, *viz.*

$$\mathcal{S}_0[A'(A)] = \mathcal{S}_g[A] \equiv \frac{1}{4}\int d^D x\, F^a_{\mu\nu}F^a_{\mu\nu} \tag{1.2}$$

   where

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{1.3}$$

   is the Yang-Mills field strength [with fully antisymmetric structure constants f^{abc} for the chosen gauge group, usually SU(N)], and $\mathcal{S}_0$ is the free Maxwell action

$$\mathcal{S}_0[A'] \equiv \frac{1}{4}\int d^D x\, (\partial_\mu A'^a_\nu - \partial_\nu A'^a_\mu)^2\,, \tag{1.4}$$

   i.e. $\mathcal{S}_g$ for $g = 0$.

2. T_g preserves the gauge condition

$$T_g[G^a(A)] = G^a(A) \tag{1.5}$$

3. The Jacobian of the transformation equals the product of the Matthews-Salam-Seiler (MSS) determinant (or Pfaffian) [14] obtained by integrating out the gauginos, and the Faddeev-Popov (FP) determinant [15] (obtained by integrating out the ghost fields $C^a, \bar{C}^a$),

$$\det\left(\frac{\delta A'^a_\mu(x,g;A)}{\delta A^b_\nu(y)}\right) = \Delta_{MSS}[A]\,\Delta_{FP}[A] \tag{1.6}$$

   at least in the sense of formal power series.

The existence of the map T_g allows for a ghost free and fermion free quantization of supersymmetric theories, and can thus provide a completely different perspective also on super-Yang-Mills theories. The main advance of the present work consists in applying these techniques to the computation of simple correlators for the maximal $\mathcal{N} = 4$ theory, and in showing that the calculational effort with this formalism is comparable to the usual one, thus providing a proof of principle for its workability and demonstrating its competitiveness with more standard perturbative techniques. Of course, to push these computations further one must determine the map T_g to higher orders. Ultimately, the main goal would be to go beyond the perturbative framework, by exploiting as yet unknown properties of the map T_g, presumably related to the maximally extended superconformal symmetry of the $\mathcal{N} = 4$ theory. Certainly it would be fascinating to make a connection between the map T_g and the integrable properties of the $\mathcal{N} = 4$ theory (see e.g. [13] for a review) – after all the image of the map T_g is a free field theory which is certainly integrable. A distinctive feature of the map T_g is that it works for finite N in the $SU(N)$ gauge theory, in contradistinction to integrability, which is tied to the planar ($N \to \infty$) limit. Indeed, while it appears unlikely that there exists a closed form expression for T_g (as is the case for some special theories, like supersymmetric quantum mechanics and the $\mathcal{N} = 2, D = 2$ Wess-Zumino model, see [9, 16–18]) there could be an underlying integrable structure. Likewise, it would be interesting to find a link with the conformal bootstrap program (see *e.g.* [19] for a review), where again the $\mathcal{N} = 4$ theory appears to play a distinguished role [20] (see also [21] and references therein for more recent work) and to elucidate the role of the conformal and dual-conformal symmetries in this context.

2 Preliminaries

Let us briefly summarize our conventions. We use the Euclidean metric; this is not essential, as analogous results can be derived with Lorentzian signature (as in [4–6]). The scalar propagator is (with the Laplacian $\Box \equiv \partial^\mu \partial_\mu$)

$$C(x) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{ikx}}{k^2} \quad \Rightarrow \quad -\Box C(x) = \delta(x) \tag{2.1}$$

where $\delta(x) \equiv \delta^{(D)}(x)$ is the D-dimensional δ-function. For the relevant dimensions we have

$$\begin{aligned} C(x) &= \frac{1}{4\pi} \cdot \frac{1}{|x|} && \text{for } D = 3 \\ C(x) &= \frac{1}{4\pi^2} \cdot \frac{1}{x^2} && \text{for } D = 4 \\ C(x) &= \frac{1}{4\pi^3} \cdot \frac{1}{(x^2)^2} && \text{for } D = 6 \\ C(x) &= \frac{3}{2\pi^5} \cdot \frac{1}{(x^2)^4} && \text{for } D = 10 \end{aligned} \tag{2.2}$$

For all dimensions the free fermionic propagator is

$$\gamma^\mu \partial_\mu S_0(x) = \delta(x) \quad \Rightarrow \quad S_0(x) = -\gamma^\mu \partial_\mu C(x) \tag{2.3}$$

The number r_D of spinor components depends on D, and we here restrict attention to those values for which supersymmetric Yang-Mills theories exist, *viz.*

$$D = 3, 4, 6, 10 \qquad \Longleftrightarrow \qquad r_D = 2, 4, 8, 16 \tag{2.4}$$

For $D = 4$ this corresponds to a Majorana spinor, for $D = 6$ to a Weyl spinor, while for $D = 10$ we get an extra factor of $\frac{1}{2}$ because of the Majorana-Weyl condition (otherwise we would have $r = 32$). Here and in other formulas below we usually suppress spinor indices. To derive the extended theories in four dimensions we will consider *dimensional reduction* of the corresponding theories to $D = 4$ such that all integrals will be performed *in four dimensions* (or rather, $D = 4 - 2\varepsilon$ for the regularized theory).

Covariant derivatives are only needed for the adjoint representation:

$$D_\mu V^a \equiv \partial_\mu V^a + g f^{abc} A_\mu^b V^c \quad \Rightarrow \quad [D_\mu, D_\nu] V^a = g f^{abc} F_{\mu\nu}^b V^c \tag{2.5}$$

Although results also hold for other gauges, we will here stick with the Landau gauge fixing function

$$G^a[A_\mu] = \partial^\mu A_\mu^a \tag{2.6}$$

For the map T_g there is a systematic construction via its inverse T_g^{-1} in terms of its infinitesimal generator [3–7]. The latter is realized by the so-called $\mathcal{R}$-operator, such that

$$(T_g^{-1}A)^a_\mu(x) \;=\; A^a_\mu(x) \;+\; \sum_{n=1}^{\infty}\frac{1}{n!}\,g^n\left(\mathcal{R}^n\big[A\big]^a_\mu(x)\right)_{g=0} \tag{2.7}$$

As we will see it is also the *inverse* map that is needed for the computation of quantum correlation functions. For the Landau gauge the $\mathcal{R}$-operator is compactly represented by the (functional) differential operator

$$\mathcal{R} = \frac{d}{dg} - \frac{1}{2r_D}\int dx\,du\,dv\;\Pi_{\mu\nu}(x-u)\mathrm{Tr}\left(\gamma_\nu\gamma^{\rho\sigma}S^{ba}(v-u)\right)f^{bcd}A^c_\rho(v)A^d_\sigma(v)\frac{\delta}{\delta A^a_\mu(x)} \tag{2.8}$$

with the transversal projector

$$\Pi_{\mu\nu}(x-y) \;\equiv\; \left(\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\Box}\right)\delta(x-y) \;\cong\; \delta_{\mu\nu}\delta(x-y)+\partial_\mu C(x-y)\partial_\nu \tag{2.9}$$

where "$\cong$" means equality in the sense of distributions. Note that in the above we write $du = d^Du$ for short and that space-time derivatives on propagators are to be understood as $\partial_\mu C(x-y) := \frac{\partial}{\partial x^\mu}C(x-y)$, *i.e.* as acting always on the first argument. $S^{ab}(x,y;A)$ is the full fermionic propagator in the gauge field dependent background with $A^a_\mu(x)$, and thus defined by

$$\gamma^\mu\big[\delta^{ac}\partial_\mu - gf^{acd}A^d_\mu(x)\big]\,S^{cb}(x,y;A) = \delta^{ab}\delta(x-y) \tag{2.10}$$

The $\mathcal{R}$ operator acts distributively,

$$\mathcal{R}\big[A^a_\mu(x)A^b_\nu(y)\cdots\big] \;= \mathcal{R}\big[A^a_\mu(x)\big]A^b_\nu(y)\cdots \;+\; A^a_\mu(x)\mathcal{R}\big[A^b_\nu(y)\big]\cdots \;+\;\cdots \tag{2.11}$$

Specializing the action of $\mathcal{R}$ to the gauge field A^a_μ, we get

$$\mathcal{R}[A]^a_\mu(x) \;\equiv\; -\frac{1}{2r_D}\int dudv\,\Pi_{\mu\nu}(x-u)\mathrm{Tr}\left(\gamma_\nu\gamma^{\rho\sigma}S^{ba}(v-u)\right)f^{bcd}A^c_\rho(v)A^d_\sigma(v) \tag{2.12}$$

From (2.12) it follows immediately that the $\mathcal{R}$ operation preserves the Landau gauge

$$\partial^\mu\,\mathcal{R}\big[A^a_\mu(x)\big] = 0 \tag{2.13}$$

This will guarantee that the equality

$$\partial^\mu(T_g(A)^a_\mu)(x) = \partial^\mu A^a_\mu(x) \tag{2.14}$$

holds for all values of the Yang-Mills coupling constant g. Once we have the result for T_g^{-1} the map T_g itself can be obtained by perturbatively inverting the power series (2.15) (in principle, there is also a direct construction of T_g [18]).

To order $\mathcal{O}(g^2)$ a double application of the $\mathcal{R}$-operator leads to [8]

$$
\begin{aligned}
(T_g^{-1}A)^a_\mu(x) \;=\; & A^a_\mu(x) \;-\; g f^{abc}\int du\, \partial_\lambda C(x-u) A^b_\mu(u) A^c_\lambda(u) \\
& + \frac{1}{2} g^2 f^{abc} f^{bde} \int dv dw \Big[-\partial_\rho C(x-v) A^c_\sigma(v) \partial_\sigma C(v-w) A^d_\rho(w) A^e_\mu(w) \\
& \qquad + \partial_\rho C(x-v) A^c_\sigma(v) \partial_\rho C(v-w) A^d_\sigma(w) A^e_\mu(w) \\
& \qquad - \partial_\rho C(x-v) A^c_\sigma(v) \partial_\mu C(v-w) A^d_\sigma(w) A^e_\rho(w) \\
& \qquad + 2\, \partial_\rho C(x-v) A^c_\mu(v) \partial_\sigma C(v-w) A^d_\sigma(w) A^e_\rho(w) \\
& \qquad - 2\, \partial_\rho C(x-v) A^c_\rho(v) \partial_\sigma C(v-w) A^d_\sigma(w) A^e_\mu(w) \Big] \;+\; \mathcal{O}(g^3)
\end{aligned}
\tag{2.15}
$$

The map T_g itself is obtained by inverting up to second order

$$
\begin{aligned}
(T_g A)^a_\mu(x) \quad = \quad & A^a_\mu(x) \;+\; g f^{abc} \int du\, \partial_\lambda C(x-u) A^b_\mu(u) A^c_\lambda(u) \qquad (2.16) \\
& + \frac{3}{2}\, g^2 f^{abc} f^{bde} \int du dv\, \partial_\rho C(x-u) A^c_\lambda(u) \partial_{[\mu} C(u-v) A^d_\lambda(v) A^e_{\rho]}(v) \;+\; \mathcal{O}(g^3)
\end{aligned}
$$

thus reproducing the old result from [2]. These formulas are valid in all dimensions where pure supersymmetric Yang-Mills theories exist. While our main interest is in the maximally extended $\mathcal{N}=4$ theory in four dimensions, we will keep D general in the following section, and consider the dimensional reduction to $D=4$ in later sections.

3 Correlation functions

For all admissible dimensions, and for any n-point correlator of *bosonic* operators $\mathcal{O}_j(x_j)$ our basic relation is

$$
\big\langle\!\big\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n) \big\rangle\!\big\rangle \;=\; \big\langle T_g^{-1}[\mathcal{O}_1](x_1)\cdots T_g^{-1}[\mathcal{O}_n](x_n) \big\rangle_0 \tag{3.1}
$$

where $\mathcal{O}_j(x_j)$ are either elementary or composite bosonic fields. Here $\langle\!\langle \cdots \rangle\!\rangle$ denotes the full expectation value of the interacting supersymmetric Yang-Mills theory (with fermions, ghosts and all interactions), while $\langle \cdots \rangle_0$ denotes the free field expectation value of the purely bosonic non-interacting gauge theory where one integrates only over the bosonic fields (with the notation from [5, 6]). More precisely, we have

$$
\begin{aligned}
\big\langle\!\big\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n) \big\rangle\!\big\rangle \quad \equiv \quad & \int \mathcal{D}A\mathcal{D}\chi\mathcal{D}C\mathcal{D}\bar{C} \prod_{x,a} \delta\big(\partial^\mu A^a_\mu(x)\big)\, e^{-S[A,\chi,C,\bar{C}]} \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n) \\
= \quad & \int \mathcal{D}_g[A] \prod_{x,a} \delta\big(\partial^\mu A^a_\mu(x)\big)\, \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)
\end{aligned}
\tag{3.2}
$$

where $S[A,\chi,C,\bar{C}]$ is the full supersymmetric action (with gauginos χ^a and ghost fields $\{C^a, \bar{C}^a\}$), while $\mathcal{D}_g[A]$ denotes the (non-local) bosonic functional measure of the interacting theory obtained *after integrating out the gauginos and the ghosts.* Likewise

$$\big\langle T_g^{-1}[O_1](x_1)\cdots T_g^{-1}[\mathcal{O}_n](x_n)\big\rangle_0 \;\equiv\; \int \mathcal{D}_0[A] \prod_{x,a} \delta\big(\partial^\mu A_\mu^a(x)\big)\, T_g^{-1}[O_1](x_1)\cdots T_g^{-1}[\mathcal{O}_n](x_n) \tag{3.3}$$

with the free measure $\mathcal{D}_0[A]$ (where the fermionic and ghost determinants become trivial). Importantly, the gauge fixing function is not affected by the transformation since $\mathcal{R}(\partial^\mu A_\mu^a) = 0$ hence $\partial^\mu A_\mu'^a = \partial^\mu A_\mu^a$ to any given order. Due to the presence of the gauge fixing δ-functional in (3.2) the vector propagator is

$$\langle A_\mu^a(x) A_\nu^b(y)\rangle_0 \;=\; C_{\mu\nu}^\perp(x-y) \;\equiv\; \delta^{ab}\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right) C(x-y) \tag{3.4}$$

For both the interacting and the free theory one can make use of the 't Hooft trick of shifting the argument of the δ-functional by c^a and integrating with a Gaussian weight over the dummy variable c^a to remove the δ-functional, and implement the gauge condition via the Gaussian factor $\propto \exp\left(-\frac{1}{2\xi}\int(\partial\cdot A)^2\right)$ in the functional integral, thereby introducing the gauge parameter ξ. While the Landau gauge corresponds to $\xi = 0$ we shall work here in the Feynman-gauge ($\xi = 1$) for which the propagator takes the more convenient form

$$\big\langle A_\mu^a(x)\, A_\nu^b(y)\big\rangle_0 = \delta^{ab}\,\delta_{\mu\nu}\, C(x-y)\,. \tag{3.5}$$

Equivalently, we can ignore the longitudinal contributions as they will drop out in all gauge invariant expressions. Below we will therefore use the propagator in the form (3.5).

Let us also note that with either choice, the free measure is already properly normalized for supersymmetric theories because

$$\begin{aligned} \int \mathcal{D}A\, e^{\frac{1}{2}\int A\Box A} \quad &\sim \quad [\det(-\Box)]^{-D/2}\,, \\ \int \mathcal{D}C\mathcal{D}\bar{C}\, e^{\int \bar{C}\Box C} \quad &\sim \quad \det(-\Box)\,, \\ \int \mathcal{D}\chi\, e^{\frac{1}{2}\int \bar{\chi}\partial\!\!\!/\chi} \quad &\sim \quad [(\det(-\Box)]^{r_D/4} \end{aligned} \tag{3.6}$$

if $r_D = 2(D-2)$, which implies that bosonic and fermionic degrees of freedom match on shell.

In summary, by means of (3.1) we are able to express any bosonic correlator of the fully supersymmetric theory as a *purely bosonic* correlator with a purely bosonic functional measure. In fact, the same statement also applies to fermionic correlators if we replace the fermionic two-point functions by the full propagator $S(x,y;A)$ in a gauge field background (and a $2n$-point correlator by the corresponding Wick product). Alternatively, one may invoke supersymmetry to reduce fermionic correlators to bosonic ones via superconformal Ward identities [20]. Analogous relations also hold for composite operators, as we will illustrate below.

As a simple example we compute the 2-point function to second order

$$\big\langle\!\big\langle A_\mu^a(x) A_\nu^b(y)\big\rangle\!\big\rangle \;=\; \big\langle T_g^{-1}[A_\mu^a](x) T_g^{-1}[A_\nu^b](y)\big\rangle_0 \tag{3.7}$$

At $\mathcal{O}(g^2)$ there are two contributions from (2.15), namely the free Wick-contractions of the product of two $\mathcal{O}(g)$ terms

$$
\begin{aligned}
& g^2 f^{acd} f^{bmn} \int dudv\, \partial_\lambda C(x-u) \partial_\rho C(y-v) \langle A^c_\mu(u) A^d_\lambda(u) A^m_\nu(v) A^n_\rho(v) \rangle_0 \\
&= g^2 N \delta^{ab} \int dudv\, \partial_\lambda C(x-u) \partial_\rho C(y-v) \Big(C^\perp_{\mu\nu}(u-v) C^\perp_{\lambda\rho}(u-v) - C^\perp_{\mu\rho}(u-v) C^\perp_{\lambda\nu}(u-v) \Big)
\end{aligned}
\tag{3.8}
$$

as well as the contractions emerging from the product of the leading order term with the order $\mathcal{O}(g^2)$ terms

$$
\begin{aligned}
\frac{1}{2} g^2 N \delta^{ab} \int dudv \Big[& - \partial_\rho C(x-u) C^\perp_{\sigma\rho}(u-v) \partial_\sigma C(u-v) C^\perp_{\mu\nu}(v-y) \\
& + \partial_\rho C(x-u) C^\perp_{\sigma\mu}(u-v) \partial_\sigma C(u-v) C^\perp_{\rho\nu}(v-y) \\
& + \partial_\rho C(x-u) C^\perp_{\sigma\sigma}(u-v) \partial_\rho C(u-v) C^\perp_{\mu\nu}(v-y) \\
& - \partial_\rho C(x-u) C^\perp_{\sigma\mu}(u-v) \partial_\rho C(u-v) C^\perp_{\sigma\nu}(v-y) \\
& - \partial_\rho C(x-u) C^\perp_{\sigma\sigma}(u-v) \partial_\mu C(u-v) C^\perp_{\rho\nu}(v-y) \\
& + \partial_\rho C(x-u) C^\perp_{\sigma\rho}(u-v) \partial_\mu C(u-v) C^\perp_{\sigma\nu}(v-y) \\
& + 2\, \partial_\rho C(x-u) C^\perp_{\mu\sigma}(u-v) \partial_\sigma C(u-v) C^\perp_{\rho\nu}(v-y) \\
& - 2\, \partial_\rho C(x-u) C^\perp_{\mu\rho}(u-v) \partial_\sigma C(u-v) C^\perp_{\sigma\nu}(v-y) \\
& - 2\, \partial_\rho C(x-u) C^\perp_{\rho\sigma}(u-v) \partial_\sigma C(u-v) C^\perp_{\mu\nu}(v-y) \\
& + 2\, \partial_\rho C(x-u) C^\perp_{\rho\mu}(u-v) \partial_\sigma C(u-v) C^\perp_{\sigma\nu}(v-y) \Big] \; + \; (x \leftrightarrow y)
\end{aligned}
\tag{3.9}
$$

For the reasons explained above we can neglect longitudinal contributions, and thus replace the transversal propagator $C^\perp_{\mu\nu}(x)$ by the simpler expression $\delta_{\mu\nu} C(x)$ in (3.5). Then a straightforward calculation gives

$$
\begin{aligned}
\langle\!\langle A^a_\mu(x) A^b_\nu(y) \rangle\!\rangle \;=\; & = \delta^{ab}\, \delta_{\mu\nu} \left[C(x-y) + g^2 N \, \frac{6-D}{2} \int du\, C(x-u)\, C(y-u)^2 \right] \\
& - \delta^{ab}\, g^2 N \, \frac{6-D}{2} \, \frac{\partial}{\partial x^\mu} \, \frac{\partial}{\partial y^\nu} \int dudv\, C(x-u)\, C(y-v)\, C(u-v)^2 \,,
\end{aligned}
\tag{3.10}
$$

where all that was used were suitable partial integrations. Recall that du is a shorthand notation for the D-dimensional measure $d^D u$. Inspecting the integrands near four dimensions reveals a logarithmic divergence in both integrals when the argument of the squared Green's function $C(z)$ vanishes. Therefore, the two-point function exhibits a divergence, illustrating the (known) fact that 'finiteness' of the theory does not mean that every correlator is finite. Curiously, the $D = 6$ theory, and thus also the $\mathcal{N} = 2$ theory in $D = 4$, has a vanishing next-to-leading order contribution.

Similarly one may obtain the three-point function at the leading perturbative order upon expanding T_g^{-1} to $\mathcal{O}(g)$ in each term

$$\langle\!\langle A_{\mu_1}^{a_1}(x_1)A_{\mu_2}^{a_2}(x_2)A_{\mu_3}^{a_3}(x_3)\rangle\!\rangle = \big\langle T_g^{-1}[A_{\mu_1}^{a_1}](x_1)T_g^{-1}[A_{\mu_2}^{a_2}](x_2)T_g^{-1}[A_{\mu_3}^{a_3}](x_3)\big\rangle_0 = \tag{3.11}$$
$$f^{a_1a_2a_3}\, g\Big[\delta_{\mu_1\mu_2}\left(\frac{\partial}{\partial x_2^{\mu_3}} - \frac{\partial}{\partial x_1^{\mu_3}}\right) + \delta_{\mu_2\mu_3}\left(\frac{\partial}{\partial x_3^{\mu_1}} - \frac{\partial}{\partial x_2^{\mu_1}}\right)$$
$$+\,\delta_{\mu_3\mu_1}\left(\frac{\partial}{\partial x_1^{\mu_2}} - \frac{\partial}{\partial x_3^{\mu_2}}\right)\int du\, C(x_1-u)\, C(x_2-u)\, C(x_3-u)\Big],$$

reproducing the standard three-gluon vertex of Yang-Mills theory. In order to compute the one-loop correction to this result we would need to know the map T_g^{-1} to cubic order $\mathcal{O}(g^3)$.

4 Scalar correlation functions at one loop in the $\mathcal{N}=4$ theory

Next we turn to the extended theories in *four* dimensions which can be obtained by dimensional reduction. To this aim we split the indices as $\mu \to \{\mu, i\}$ where $\mu, \nu, ... = 1, ..., 4$ and $i, j, ...$ label the remaining internal dimensions. Likewise we decompose the coordinates as $x^\mu \to \{x^\mu, y^i\}$ and the fields and indices in (2.15) in an analogous fashion:

$$A_\mu^a(x,y) \quad\longrightarrow\quad \big\{A_\mu^a(x)\,,\, \phi_i^a(x)\big\} \tag{4.1}$$

The dependence on the internal coordinates y^i is dropped for the dimensionally reduced theory. We then proceed to compute the scalar two and four-point functions up to the next-to-leading perturbative order in the gauge coupling constant g. In the remainder we shall focus on the $\mathcal{N}=4$ super Yang-Mills theory, for which the internal indices run over six dimensions: $i, j, \cdots = 1, \ldots, 6$. In the reduced and regulated theory all loop integrals are performed in $D = 4 - 2\varepsilon$ dimensions while the number of scalars is $S = 6 + 2\varepsilon$. This prescription maintains the balance of fermionic and bosonic degrees of freedom in the original supersymmetric theory and is known as dimensional regularization by dimensional reduction [23][1].

For the computation of correlation functions of the scalar fields $\phi_i^a(x)$ to next-to-leading order, i.e. $\mathcal{O}(g^2)$, we need to consider the inverse map T_g^{-1} of the interacting scalar fields $\phi_i^a(x)$ to the free-field correlators

$$\langle\!\langle \phi_{i_1}^{a_1}(x_1)\phi_{i_2}^{a_2}(x_2)\rangle\!\rangle \;=\; \big\langle T_g^{-1}[\phi_{i_1}^{a_1}](x_1)T_g^{-1}[\phi_{i_2}^{a_2}](x_2)\big\rangle_0 \tag{4.2}$$

as well as

$$\langle\!\langle \phi_{i_1}^{a_1}(x_1)\phi_{i_2}^{a_2}(x_2)\phi_{i_3}^{a_3}(x_3)\phi_{i_4}^{a_4}(x_4)\rangle\!\rangle \quad = \tag{4.3}$$
$$= \big\langle T_g^{-1}[\phi_{i_1}^{a_1}](x_1)T_g^{-1}[\phi_{i_2}^{a_2}](x_2)T_g^{-1}[\phi_{i_3}^{a_3}](x_3)T_g^{-1}[\phi_{i_4}^{a_4}](x_4)\big\rangle_0\,.$$

For $\mathcal{N}=4$ super Yang-Mills theory, the action of inverse map T_g^{-1} on the vector fields A_μ^a and the six scalar fields ϕ_i^a is easily derived by applying the split (4.1) to the formula (2.15).

[1] Note that for the $D=4$ and $\mathcal{N}=1$ or $\mathcal{N}=2$ super Yang-Mills theories this would amount to include $S=2\varepsilon$ and $S=2+2\varepsilon$ scalars respectively.

For the gauge field this gives

$$
\begin{aligned}
(T_g^{-1}A)^a_\mu(x) \;=\; & A^a_\mu(x) \;-\; g f^{abc}\int du\,\partial_\lambda C(x-u)A^b_\mu(u)A^c_\lambda(u) \\
& + \frac{1}{2}g^2 f^{abc}f^{bde}\int dvdw\Big[-\partial_\rho C(x-v)A^c_\sigma(v)\partial_\sigma C(v-w)A^d_\rho(w)A^e_\mu(w) \\
& \qquad + \partial_\rho C(x-v)A^c_\sigma(v)\partial_\rho C(v-w)A^d_\sigma(w)A^e_\mu(w) \\
& \qquad + \partial_\rho C(x-v)\phi^c_j(v)\partial_\rho C(v-w)\phi^d_j(w)A^e_\mu(w) \\
& \qquad - \partial_\rho C(x-v)A^c_\sigma(v)\partial_\mu C(v-w)A^d_\sigma(w)A^e_\rho(w) \\
& \qquad - \partial_\rho C(x-v)\phi^c_j(v)\partial_\mu C(v-w)\phi^d_j(w)A^e_\rho(w) \\
& \qquad + 2\,\partial_\rho C(x-v)A^c_\mu(v)\partial_\sigma C(v-w)A^d_\sigma(w)A^e_\rho(w) \\
& \qquad - 2\,\partial_\rho C(x-v)A^c_\rho(v)\partial_\sigma C(v-w)A^d_\sigma(w)A^e_\mu(w)\Big] \;+\; \mathcal{O}(g^3)
\end{aligned}
\tag{4.4}
$$

while for the scalar fields we obtain

$$
\begin{aligned}
(T_g^{-1}\phi)^a_i(x) \;=\; & \phi^a_i(x) \;-\; g f^{abc}\int d^Du\,\partial_\lambda C(x-u)\phi^b_i(u)A^c_\lambda(u) \\
+ \frac{g^2}{2}f^{abc}f^{bde}\int d^Dud^Dv\Big[& -\partial_\rho C(x-u)A^c_\lambda(u)\partial_\lambda C(u-v)A^d_\rho(v)\phi^e_i(v) \\
& + \partial_\rho C(x-u)A^c_\lambda(u)\partial_\rho C(u-v)A^d_\lambda(v)\phi^e_i(v) \\
& + \partial_\rho C(x-u)\phi^c_j(u)\partial_\rho C(u-v)\phi^d_j(v)\phi^e_i(v) \\
& + 2\partial_\rho C(x-u)\phi^c_i(u)\partial_\lambda C(u-v)A^d_\lambda(v)A^e_\rho(v) \\
& - 2\partial_\rho C(x-u)A^c_\rho(u)\partial_\lambda C(u-v)A^d_\lambda(v)\phi^e_i(v)\Big] \;+\; \mathcal{O}(g^3)\,.
\end{aligned}
\tag{4.5}
$$

In the calculation we will apply regularization by dimensional reduction with

$$
D = \delta^\mu_\mu = 4-2\epsilon \quad , \quad S = \delta^i_i = 6+2\epsilon \tag{4.6}
$$

This implies $D+S=10$ which in fact is the combination always arising in our computations up to one-loop order. The D-dimensional scalar propagator in position space reads

$$
C(x) := \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}}\frac{1}{p^2}e^{ip\cdot x} = \frac{\Gamma(\omega-1)}{4\pi^\omega}\,[x^2]^{1-\omega}\,. \tag{4.7}
$$

In particular, we set the self-contraction $C(0)=0$ as a consequence of dimensional regularization for scale-less integrals.

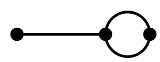
4.1 Two-point function

To compute the scalar two-point function to $\mathcal{O}(g^2)$ we insert the expansion (4.5), retaining only terms of order $\mathcal{O}(g^2)$ (the first order contributions vanish trivially), and then perform the necessary Wick contractions. Just as before in the computation of the gauge fields in (3.7) there are two contributions to this correlator: The contractions of the $\mathcal{O}(g)$ terms emerging from each operator $T_g^{-1}[\phi_i^a]$ as well as the contractions of the $\mathcal{O}(g^2)$ terms of one $T_g^{-1}[\phi_i^a]$ with the leading ϕ_i^a term of the other. A straightforward calculation gives the result

$$\big\langle T_g^{-1}[\phi_{i_1}^{a_1}](x) T_g^{-1}[\phi_{i_2}^{a_2}](y)\big\rangle_0 = \delta_{i_1 i_2}\,\delta^{a_1 a_2}\,C(x-y)$$
$$+g^2 N\,\delta_{i_1 i_2}\,\delta^{a_1 a_2}\int d^D u\, d^D v \Big\{ \Big(\frac{D+S}{2}-2\Big)\,\big[\,C(x-u)C(u-v)\partial_\rho C(y-v)\partial_\rho C(v-u) + (x\leftrightarrow y)\,\big]$$
$$+ C(u-v)^2\,\partial_\rho C(x-u)\,\partial_\rho C(y-v)\Big\} + \mathcal{O}(g^4)\,. \tag{4.8}$$

where the $\mathcal{O}(1)\times\mathcal{O}(g^2)$ contractions yield the term proportional to $(\frac{1}{2}(D+S)-2)$ while the $\mathcal{O}(g)\times\mathcal{O}(g)$ contractions yield the second term in the above. Importantly, in the course of performing the $\mathcal{O}(1)\times\mathcal{O}(g^2)$ Wick-contractions one also takes into account the self-contractions of the $\mathcal{O}(g^2)$ terms, i.e. the operator insertions $T_g^{-1}[\phi_i^a]$ are not to be understood as normal ordered.

All integrals appearing in (4.8) may in fact be reduced to the bubble integral



$$= \int d^D u\, C(x-u)\, C(y-u)^2 = I_{xy} \tag{4.9}$$

which is symmetric in $(x\leftrightarrow y)$ and which appeared already in (3.10). To see the symmetry, we integrate by parts, using $\Box C(x) = -\delta(x)$ to obtain the integral relations

$$\int d^D u\, d^D v\, C(x-u)C(u-v)\partial_\rho C(y-v)\partial_\rho C(v-u) = -\frac{1}{2}\, I_{xy}$$
$$\int d^D u\, d^D v\, C(u-v)^2\,\partial_\rho C(x-u)\,\partial_\rho C(y-v) = I_{xy}\,. \tag{4.10}$$

The complete result for the two-point scalar correlation function in $\mathcal{N}=4$ super Yang-Mills theory up to one-loop accuracy therefore reads

$$\big\langle\!\big\langle \phi_{i_1}^{a_1}(x_1)\phi_{i_2}^{a_2}(x_2)\big\rangle\!\big\rangle \;=\; \delta_{i_1 i_2}\,\delta^{a_1 a_2}\,C(x-y)\,\Big(1 - 2\,g^2 N\,\frac{I_{xy}}{C(x-y)}\Big) + \mathcal{O}(g^4)\,, \tag{4.11}$$

reproducing established results in the literature, see e.g. [24, 25]. We note that

$$\frac{I_{xy}}{C(x-y)} = \frac{1}{16\pi^\omega(2-\omega)}\,[(x-y)^2]^{2-\omega} \quad \text{with } D = 2\omega\,, \tag{4.12}$$

yielding the expected logarithmic divergence near $D=4$.

4.2 Four-point functions

For the computation of the four-point function

$$\Big\langle T_g^{-1}[\phi_{i_1}^{a_1}](x_1) T_g^{-1}[\phi_{i_2}^{a_2}](x_2) T_g^{-1}[\phi_{i_3}^{a_3}](x_3) T_g^{-1}[\phi_{i_4}^{a_4}](x_4)\Big\rangle_0 \tag{4.13}$$

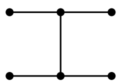
we proceed from (4.5). When expanding out this formula to $\mathcal{O}(g^2)$ the combinatorics of Wick contractions grows considerably. Again one inserts $\mathcal{O}(g)$ terms twice or $\mathcal{O}(g^2)$ terms once next to the leading terms in the other slots. Here there are two types of color structures emerging: the connected terms are proportional to two structure constants $f^{a_i a_j e} f^{a_k a_l e}$ while the disconnected terms are proportional to $\delta^{a_i a_j} \delta^{a_k a_l}$.

Gathering the connected terms one encounters the following integral identities

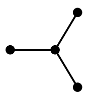
$$\int du\, dv\, C(x_1 - u)\, C(x_2 - u)\, C(u - v)\, \partial_\mu C(x_3 - v)\, \partial^\mu C(x_4 - v) = \partial_3 \cdot \partial_4 H_{12;34} \tag{4.14}$$

$$\int du\, dv\, C(x_1 - u)\, C(x_3 - v)\, C(x_4 - v)\, \partial_\mu C(x_2 - u)\, \partial^\mu C(u - v) = -C_{12} Y_{234} + \partial_1 \cdot \partial_2 H_{12;34}$$

where we defined the H and Y-functions



$$= \int du\, dv\, C(x_1 - u) C(x_2 - u) C(u - v) C(x_3 - v) C(x_4 - v) \;\equiv\; H_{12,34}\,,$$



$$= \int du\, C(x_1 - u) C(x_2 - u) C(x_3 - u) \;\equiv\; Y_{123}\,. \tag{4.15}$$

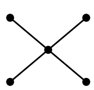
The connected part may then be brought into the form (with $C_{12} \equiv C(x_1 - x_2)$)

$$\begin{aligned}
(4.13)_{\text{connected}} = f^{a_1 a_2 e} f^{a_3 a_4 e} \Big[& \delta_{i_1 i_2} \delta_{i_3 i_4} \left(\partial_1 - \partial_2\right) \cdot \left(\partial_3 - \partial_4\right) H_{12;34} \\
& + \left(\delta_{i_1 i_3} \delta_{i_2 i_4} - \delta_{i_1 i_4} \delta_{i_2 i_3}\right) \Big\{ (\partial_1 \cdot \partial_2 + \partial_3 \cdot \partial_4) H_{12;34} - \frac{1}{2} C_{12} (Y_{134} + Y_{234}) - \frac{1}{2} C_{34} (Y_{123} + Y_{124}) \Big\} \Big] \\
& + \text{permutations}
\end{aligned} \tag{4.16}$$

The integrals in the last line may be reduced upon noting that

$$\partial_1 \cdot \partial_2 H_{12;34} \;=\; \frac{1}{2} C_{12} (Y_{134} + Y_{234}) \;-\; \frac{1}{2} X_{1234} \tag{4.17}$$

using partial integrations and where we introduced the X integral



$$= \int du\, C(x_1 - u) C(x_2 - u) C(x_3 - u) C(x_4 - u) = X_{1234}\,. \tag{4.18}$$

The H, Y and X integrals are also known analytically, *cf.* [29]. Putting everything together we obtain the connected part of four-point function up to $\mathcal{O}(g^2)$

$$\begin{aligned}
& \langle\!\langle \phi^{a_1}_{i_1}(x_1) \phi^{a_2}_{i_2}(x_2) \phi^{a_3}_{i_3}(x_3) \phi^{a_4}_{i_4}(x_4) \rangle\!\rangle_{\text{connected}} = \\
& \qquad g^2 f^{a_1 a_2 e}\, f^{a_3 a_4 e}\, \left[\delta_{i_1 i_2} \delta_{i_3 i_4} (\partial_1 - \partial_2) \cdot (\partial_3 - \partial_4)\, H_{12,34} - (\delta_{i_1 i_3} \delta_{i_2 i_4} - \delta_{i_1 i_4} \delta_{i_2 i_3})\, X_{1234}\right] \\
& \qquad + g^2 f^{a_1 a_3 e}\, f^{a_2 a_4 e}\, \left[\delta_{i_1 i_3} \delta_{i_2 i_4} (\partial_1 - \partial_3) \cdot (\partial_2 - \partial_4)\, H_{13,24} - (\delta_{i_1 i_2} \delta_{i_3 i_4} - \delta_{i_1 i_4} \delta_{i_2 i_3})\, X_{1234}\right] \\
& \qquad + g^2 f^{a_1 a_4 e}\, f^{a_2 a_3 e}\, \left[\delta_{i_1 i_4} \delta_{i_2 i_3} (\partial_1 - \partial_4) \cdot (\partial_2 - \partial_3)\, H_{14,23} - (\delta_{i_1 i_2} \delta_{i_3 i_4} - \delta_{i_1 i_3} \delta_{i_2 i_4})\, X_{1234}\right] \\
& \qquad + \mathcal{O}(g^4)\,.
\end{aligned} \tag{4.19}$$

For the disconnected terms one consistently finds pairwise appearances of two-point function contractions

$$\begin{aligned}
& \langle\!\langle \phi^{a_1}_{i_1}(x_1) \phi^{a_2}_{i_2}(x_2) \phi^{a_3}_{i_3}(x_3) \phi^{a_4}_{i_4}(x_4) \rangle\!\rangle_{\text{disconnected}} = \\
& \quad \delta^{a_1 a_2} \delta^{a_3 a_4} \delta_{i_1 i_2} \delta_{i_3 i_4} \big(C(x_1 - x_2) - 2\, g^2 N I_{12}\big)\big(C(x_3 - x_4) - 2\, g^2 N I_{34}\big) + \text{permutations} + \mathcal{O}(g^4)\,,
\end{aligned} \tag{4.20}$$

using the identical integral relations of (4.10). These results of (4.19) and (4.20) reproduce the known results obtained in the standard perturbative computation in $\mathcal{N}=4$ super Yang-Mills theory, see *e.g.* [25]. In consequence, *any* n-point scalar field correlation function will be reproduced up to the $\mathcal{O}(g^2)$ order using the inverse non-local map T_g^{-1} to the free gauge theory. This is due to the fact that the connected part of n-point scalar correlators are of order $\mathcal{O}(g^{n-2})$. Hence, at order $\mathcal{O}(g^2)$ accuracy only the disconnected parts will contribute for $n>4$.

5 Deriving the one-loop dilatation operator

A central class of gauge invariant observables in $\mathcal{N}=4$ super Yang-Mills theory are the anomalous scaling dimensions of composite operators. They have been subject to intense studies and remarkable results were produced in the AdS/CFT integrability program, including exact results to all orders in g^2N in the planar $N\to\infty$ limit of the $SU(N)$ gauge theory [13]. Focusing on the class of composite operators built from scalar fields, these are constructed as traces of the scalar fields $\phi_i(x)\equiv t^a\phi_i^a(x)$ at a common space-time point (with the $SU(N)$ generators t^a). These take the schematic form

$$\mathcal{O}_\alpha = \mathrm{Tr}\left(\phi_i\phi_j\phi_k\ldots\right)\mathrm{Tr}\left(\phi_l\phi_m\phi_n\ldots\right)\cdots \tag{5.1}$$

where α is a superindex labeling all possible compositions. As a consequence of the conformal symmetry the two-point functions of these (renormalized) operators take canonical form

$$\left\langle\!\!\left\langle \mathcal{O}_\alpha^{\mathrm{ren}}(x)\,\mathcal{O}_\beta^{\mathrm{ren}}(0)\right\rangle\!\!\right\rangle = \frac{\delta_{\alpha\beta}}{[x^2]^{\Delta_\alpha(g^2,N)}}\,, \tag{5.2}$$

where the scaling dimensions receive perturbative corrections in an expansion in g^2 starting out with the the naive classical (tree level) scaling dimension obtained by standard power counting. In order to achieve this the operator mixing problem needs to be resolved. A superior tool for doing this (and thereby finding the $\Delta_n(g^2,N)$) is the construction of the dilatation operator $\hat{D}$ as developed in [26], following initial results at one-loop in [27,28]. The dilatation operator $\hat{D}$ acts on states at the origin of space-time (in a radial quantization scheme) – its eigenvalues correspond to the anomalous dimensions

$$\hat{D}\,\mathcal{O}_\alpha = \Delta_\alpha(g^2,N)\,\mathcal{O}_\alpha\,. \tag{5.3}$$

We now wish to extract the dilatation operator from our inverse map T_g^{-1} to order g^2 by taking the two-point limit of the four-point results in section 4.2. For this we need to establish some technology following [26]. To begin with, we distinguish the fields at points x and 0 by the superscript $\pm$

$$\Phi_i^+ \equiv t^a\,\phi_i^a(x)\,,\quad \Phi_i^- \equiv t^a\,\phi_i^a(0)\,. \tag{5.4}$$

The tree-level two point function of composite operators at these points may then formally be written as

$$\langle\!\langle \mathcal{O}_\alpha^+\,\mathcal{O}_\beta^-\rangle\!\rangle_{\mathrm{tree}} = \exp[W_0(x,\check{\Phi}^+,\check{\Phi}^-)]\,\mathcal{O}_\alpha^+\mathcal{O}_\beta^+\Big|_{\Phi=0} \tag{5.5}$$

with the field derivatives

$$\check{\Phi}_i^+ := t^a\frac{\delta}{\delta\phi_i^a(x)}\,,\quad \check{\Phi}_i^- := t^a\frac{\delta}{\delta\phi_i^a(0)}\,, \tag{5.6}$$

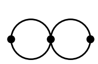
and the tree level generator W_0 inserting free field scalar propagators in between Φ_i^+ and Φ_i^-

$$W_0(x, \check{\Phi}^+, \check{\Phi}^-) = C(x)\,\mathrm{Tr}\;\check{\Phi}_i^+ \check{\Phi}_i^- \,. \tag{5.7}$$

The exponentiated W_0 in equation 5.5 performs free Wick contractions between all constituent fields of the $\mathcal{O}_\alpha^+$ and $\mathcal{O}_\beta^-$ and thus computes the tree-level correlator. The one-loop correction to this two-point correlator then takes the form

$$\langle\!\langle \mathcal{O}_\alpha^+\, \mathcal{O}_\beta^- \rangle\!\rangle_{\text{one-loop}} = \exp[W_0(x, \check{\Phi}^+, \check{\Phi}^-)]\, \left(1 + g^2\, W_2(x, \check{\Phi}^+, \check{\Phi}^-)\right) \mathcal{O}_\alpha^+ \mathcal{O}_\beta^+ \Big|_{\Phi=0} . \tag{5.8}$$

Let us now extract $W_2(x, \check{\Phi}^+, \check{\Phi}^-)$ from the pinching limits of our four-point functions (4.19) and (4.20) by taking $x_{1,2} \to x$ and $x_{3,4} \to 0$. In order to do this we need the following integral identities obtained by standard one-loop Feynman integral techniques in dimensional regularization

$$X_{00xx} =$$  $$= \int du\, C(x-u)^2 C(u)^2 = 2\,\Omega(x)\, C(x)^2 + \mathcal{O}(\epsilon)$$

$$Y_{00x} = Y_{0xx} = I_{x0} = \int du\, C(x-u)^2 C(u) = \Omega(x)\, C(x)^2 + \mathcal{O}(\epsilon) \tag{5.9}$$

with the common divergent factor

$$\Omega(x) = \frac{1}{16\pi^\omega} \frac{1}{2-\omega} [x^2]^{2-\omega} \,, \qquad D = 2\omega = 4 - 2\epsilon \tag{5.10}$$

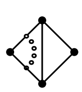
and the scalar propagator $C(x) = \frac{\Gamma(\omega-1)}{4\pi^\omega}[x^2]^{1-\omega}$. More subtle are the pinching limits of the derivatives of the H functions appearing in (4.15) which amount to two-loop Feynman integrals. Defining the relevant integral as

$$\tilde{H}_{12;34} = (\partial_1 - \partial_2)\cdot(\partial_3 - \partial_4) H_{12;34} \tag{5.11}$$

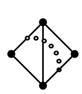
the key identities we found are

$$\begin{aligned} \tilde{H}_{00;xx} &= 0 \\ \tilde{H}_{0x;0x} &= 2\,\Omega(x)\, C(x)^2 + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{5.12}$$

Whereas the first relation is easy to see, reaching the second relation we made use of the `Tarcer` package [30]. We also cross checked our result with the results of [26] in Appendix B. Concretely the identities established are

 $$= \int du\, dv\, \partial_\mu C(x-u)\, \partial_\mu C(x-v)\, C(u-v)\, C(y-u)\, C(y-v)$$

$$= \Omega(x-y)\, C(x-y)^2 + \mathcal{O}(\epsilon)$$

 $$= \int du\, dv\, \partial_\mu C(x-u)\, C(x-v)\, C(u-v)\, C(y-u) \partial_\mu C(y-v)$$

$$= (2-\omega)\,\Omega(x-y)\, C(x-y)^2 + \mathcal{O}(\epsilon^2)\,, \tag{5.13}$$

where the dots on the graphs indicate the contracted indices of the derivatives. Importantly the second integral is not divergent and does not contribute to the anomalous dimensions.

Using theses results the connected part of the pinched four point function of (4.19) in the limit $x_{1,2} \to x$ and $x_{3,4} \to 0$ takes the form

$$
\begin{aligned}
&\langle\!\langle \phi^{a_1}_{i_1}(x)\phi^{a_2}_{i_2}(x)\phi^{a_3}_{i_3}(0)\phi^{a_4}_{i_4}(0)\rangle\!\rangle_{\text{connected}} = \\
&= 2\,\Omega(x)\,C(x)^2\,g^2\Big(f^{a_1a_2e}\,f^{a_3a_4e}\,[\delta_{i_1i_4}\delta_{i_2i_3} - \delta_{i_1i_3}\delta_{i_2i_4}] \\
&\qquad\qquad + f^{a_1a_3e}\,f^{a_2a_4e}\,[\delta_{i_1i_3}\delta_{i_2i_4} - \delta_{i_1i_2}\delta_{i_3i_4} + \delta_{i_1i_4}\delta_{i_2i_3}] \\
&\qquad\qquad + f^{a_1a_4e}\,f^{a_2a_3e}\,[\delta_{i_1i_4}\delta_{i_2i_3} - \delta_{i_1i_2}\delta_{i_3i_4} + \delta_{i_1i_3}\delta_{i_2i_4}]\Big)\,.
\end{aligned} \tag{5.14}
$$

Using the matrix variation notation introduced above this correlator may be translated to the operator[2]

$$
\begin{aligned}
W_2^C = \Omega(x)\,C(x)^2\Big\{&\operatorname{Tr}[\check\Phi^+_i,\check\Phi^+_j]\,[\check\Phi^-_i,\check\Phi^-_j] + \frac{1}{2}\operatorname{Tr}[\check\Phi^+_i,\check\Phi^-_j]\,[\check\Phi^+_i,\check\Phi^-_j] \\
&- 2\operatorname{Tr}[\check\Phi^+_i,\check\Phi^+_i]\,[\check\Phi^-_j,\check\Phi^-_j]\Big\}
\end{aligned} \tag{5.15}
$$

which acts on the two operators $\mathcal{O}^+_\alpha\mathcal{O}^-_\beta$ located at x and 0. It yields the correlation function in the sense of (5.8). The disconnected contribution arises from (4.20) and takes the form

$$
\begin{aligned}
&\langle\!\langle \phi^{a_1}_{i_1}(x)\phi^{a_2}_{i_2}(x)\phi^{a_3}_{i_3}(0)\phi^{a_4}_{i_4}(0)\rangle\!\rangle_{\text{disconnected}} = \\
&\qquad (\delta_{i_1i_3}\delta_{i_2i_4}\delta^{a_1a_3}\delta^{a_2a_4} + \delta_{i_1i_4}\delta_{i_2i_3}\delta^{a_1a_4}\delta^{a_2a_3})\Big\{1 - 4\,g^2N\,\Omega(x)\Big\}\,,.
\end{aligned} \tag{5.16}
$$

Translating the one-loop $\mathcal{O}(g^2)$ contribution to the matrix variation notation yields

$$
W_2^D = 2\,\Omega(x)\,C(x)^2\operatorname{Tr}[t^a,\check\Phi^+_i]\,[t^a,\check\Phi^-_i]\,. \tag{5.17}
$$

In order to extract the dilatation operator from these results we return to (5.8) and now change the argument of W_2 from $\check\Phi^+$ to $C(x)^{-1}\Phi^-$ at the cost of normal ordering the W_2

$$
\langle\mathcal{O}^+_\alpha\,\mathcal{O}^-_\beta\rangle_{\text{one-loop}} = \exp[W_0(x,\check\Phi^+,\check\Phi^-)]\;\big(1 + g^2\,V_2(x)\big)\,\mathcal{O}^+_\alpha\mathcal{O}^+_\beta\Big|_{\Phi=0}\,. \tag{5.18}
$$

with the normal ordered one loop effective vertex [26]

$$
V_2(x) =: W_2(x, C(x)^{-1}\Phi^-,\Phi^-):\,. \tag{5.19}
$$

This replacement may be done, as the result $\langle\mathcal{O}_\alpha\mathcal{O}_\beta\rangle$ vanishes unless every Φ^- is contracted with a Φ^+ before the fields are set to zero. Here, the only possibility is to contract with a term in W_0 which effectively changes the argument back to $\check\Phi^+$. Normal ordering $::$ secures that no new contractions are introduced within W_2. Operator renormalization is then performed via

$$
\mathcal{O}^{\text{ren}} = \big(1 - \tfrac{1}{2}g^2V_2(x_0)\big)\,\mathcal{O} \tag{5.20}
$$

with an arbitrary reference point x_0. The resulting two-point function is finite

$$
\langle\mathcal{O}^{\text{ren}\,+}_\alpha\,\mathcal{O}^{\text{ren}\,-}_\beta\rangle_{\text{one-loop}} = \exp[W_0(x,\check\Phi^+,\check\Phi^-)]\;\big(1 + g^2\,V_2(x) - g^2\,V_2(x_0)\big)\,\mathcal{O}^+_\alpha\mathcal{O}^+_\beta\Big|_{\Phi=0}\,. \tag{5.21}
$$

[2]Our conventions are $[t^a,t^b] = if^{abc}t^c$ and $\operatorname{Tr}\,t^at^b = \delta^{ab}$, we also note $f^{aef}f^{bef} = N\delta^{ab}$.

The dilatation operator D_2 is now extracted upon sending the regulator to zero

$$\lim_{\epsilon\to 0} \big(V_2(x) - V_2(x_0)\big) = \log(x_0^2/x^2)\, \hat{D}_2 \tag{5.22}$$

with

$$\hat{D}_2 = -\lim_{\epsilon\to 0} \epsilon\, V_2(x) \tag{5.23}$$

as the $\log x^2$ contribution to $V_2(x)$ is always paired with the $1/\epsilon$ pole in dimensional regularization. The final answer for the renormalized two-point function then reads

$$\langle \mathcal{O}^{\mathrm{ren}\,+}_\alpha\, \mathcal{O}^{\mathrm{ren}\,-}_\beta \rangle_{\text{one-loop}} = \exp[W_0(x,\check{\Phi}^+,\check{\Phi}^-)]\, \exp[g^2 \log(x_0^2/x^2)\, D_2] \mathcal{O}^+_\alpha \mathcal{O}^+_\beta \Big|_{\Phi=0} . \tag{5.24}$$

Applying this rational to our results (5.15) and (5.17) we find

$$\hat{D}_2 = -\frac{1}{8\pi^2} \big(: \mathrm{Tr}\,[\Phi_i,\Phi_j]\,[\check{\Phi}_i,\check{\Phi}_j] : -\tfrac{1}{2}\ : \mathrm{Tr}\,[\Phi_i,\check{\Phi}_j]\,[\Phi_i,\check{\Phi}_j] : \big) + \tfrac{1}{8\pi^2}\, \hat{V}_D \tag{5.25}$$

where we separated off the piece V_D which turns out to just amount to a gauge transformation generated by $\hat{G}^a = \mathrm{Tr}\,[t^a,\Phi_i]\,\check{\Phi}_i$ as

$$\begin{aligned} \hat{V}_D &=: \mathrm{Tr}\,[\Phi_i,\check{\Phi}_i]\,[\Phi_j,\check{\Phi}_j] : + \mathrm{Tr}\,[\Phi_i,t^a]\,[t^a,\check{\Phi}_i] \\ &= \mathrm{Tr}\,[\Phi_i,\check{\Phi}_i]\,[\Phi_j,\check{\Phi}_j] = \mathrm{Tr}\, \big([t^a,\Phi_i]\check{\Phi}_i\big)\ \mathrm{Tr}\, \big([t^a,\Phi_j]\check{\Phi}_j\big) = \hat{G}^a\, \hat{G}^a . \end{aligned} \tag{5.26}$$

Hence V_D vanishes on gauge invariant composite operators and the one-loop dilatation operator in the scalar sector reads

$$\hat{D}_2 = -\tfrac{1}{8\pi^2} \big(: \mathrm{Tr}\,[\Phi_i,\Phi_j]\,[\check{\Phi}_i,\check{\Phi}_j] : -\tfrac{1}{2}\ : \mathrm{Tr}\,[\Phi_i,\check{\Phi}_j]\,[\Phi_i,\check{\Phi}_j] : \big) . \tag{5.27}$$

It precisely coincides with the dilatation operator established in [26] upon adapting the conventions for the gauge coupling constants. As a consequence *all* scaling dimensions in the scalar $SO(6)$ sector of $\mathcal{N}=4$ SYM are reproduced with the map T_g up to the g^2 order.

6 Outlook

In principle there are now many topics to explore in terms of the map T_g. One especially interesting question is how the present formalism applies to the computation of the Wilson loop integral

$$\langle\!\langle W(\mathcal{C}) \rangle\!\rangle \equiv \left\langle\!\!\left\langle \mathcal{P} \exp\left(ig \oint_{\mathcal{C}} A^a_\mu t^a dx^\mu \right) \right\rangle\!\!\right\rangle \tag{6.1}$$

with $\mathcal{C}$ a closed curve in $\mathbb{R}^4$ and fundamental $SU(N)$ generators t^a. In principle we can evaluate this by considering

$$\left\langle \mathcal{P} \exp\left(ig \oint_{\mathcal{C}} (T_g^{-1} A)^a_\mu t^a dx^\mu \right) \right\rangle_0 \tag{6.2}$$

which again can be determined up to $\mathcal{O}(g^2)$ for special cases of interest, making use of the results of the previous chapters. As the n-point correlators agree to $\mathcal{O}(g^2)$, as was shown, the perturbative evaluation of the Wilson loop (6.2) using the inverse map T_g^{-1} is guaranteed to reproduce the original expectation value (6.1) using standard perturbation theory to that order.

An interesting extension of the above lies in the study of supersymmetric Maldacena-Wilson loops [22]. Here the path couples to the gauge fields and the scalars, i.e. the loop exponent takes the form

$$W_S(\mathcal{C}) = \mathcal{P} \exp\left(ig \int_0^1 (A_\mu^a \dot{x}^\mu + i\phi_i^a |\dot{x}|\, \theta^i) t^a ds \right) \quad \text{with} \quad \theta^i\theta^i = 1\,, \tag{6.3}$$

where we have parametrized the loop $\mathcal{C}$ by $x^\mu = x^\mu(s)$ with $0 \leqslant s \leqslant 1$. For special curves such as a straight line or a circle the Maldacena-Wilson loop expectation value $\langle\!\langle W_S(\mathcal{C}) \rangle\!\rangle$ *does not* receive contributions from bulk interactions [31,32] in a Feynman diagrammatic evaluation. Put differently it is equal to the same Wilson loop operator in the *free* gauge theory

$$\left\langle\!\!\left\langle W_S\left(\mathcal{C}\right)\right\rangle\!\!\right\rangle = \left\langle W_S\left(\mathcal{C}\right)\right\rangle_0 . \tag{6.4}$$

Hence from the perspective of this work for these special geometries the Maldacena-Wilson loop operator should be *invariant* under the $\mathcal{R}$ map. Evaluating the $\mathcal{R}$ on (6.1) we deduce with (2.8) (now interpreting the gauge fields as 10 dimensional)

$$\begin{aligned} \mathcal{R} \left\langle\!\!\left\langle W\left(\mathcal{C}\right)\right\rangle\!\!\right\rangle &= \left\langle\!\!\left\langle W\left(\mathcal{C}\right)\right\rangle\!\!\right\rangle - \\ - \frac{1}{2r_D} \int_0^1 ds & \int du\, dv\, \Pi_{\mu\nu}(x(s)-u) \mathrm{Tr}\left(\gamma_\nu \gamma^{\rho\sigma} S^{ba}(v-u)\right) f^{bcd} A_\rho^c(v) A_\sigma^d(v) \times \\ \times \left\langle\!\!\left\langle \left(\mathcal{P}\exp\left(ig \int_0^s dt A_\mu(x(t)) \dot{x}^\mu(t) dt \right)\right) \right.\right. & \left.\left. t^a \dot{x}^\mu(s) \left(\mathcal{P}\exp\left(ig \int_s^1 dt A_\mu(x(t)) \dot{x}^\mu(t) \right)\right) \right\rangle\!\!\right\rangle . \end{aligned} \tag{6.5}$$

Note that the first term on the r.h.s. (resulting from the application of d/dg) is again the Wilson loop operator. Invariance under $\mathcal{R}$ thus amounts to the vanishing of the remaining expressions for special contours. Working this out in detail is left to future work.

A central question concerns the existence of the $\mathcal{N}=4$ theory *beyond* perturbation theory. As is well known, the construction of an interacting quantum field theory in *four* space-time dimensions obeying the Wightman axioms remains an outstanding problem of quantum field theory (see *e.g.* [33]). In that framework, the non-triviality of the theory would be ensured by ascertaining the non-triviality of the S-matrix. Among all rigidly supersymmetric theories the $\mathcal{N}=4$ theory would seem to come closest to realizing this quantum field theorist's dream. However, being an exactly conformal theory without asymptotic one-particle states, it has no S-matrix in the usual sense. Hence standard arguments do not apply; rather, it appears that the Wightman axioms of ordinary quantum field theory must be replaced by the axioms of the conformal bootstrap program [19]. A major goal of the present approach (and still a dream) would be to exploit the existence of the map T_g and its properties towards a completely rigorous construction of $\mathcal{N}=4$ Yang-Mills theory at the non-perturbative level.

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