This is a preprint of an article published in Periodica Mathematica Hungarica. The final authenticated version is available online:
https://doi.org/10.1007/s10998-019-00305-1

Corrigendum to: "X-coordinates of Pell equations as sums of two tribonacci numbers"

Eric F. Bravo
Departamento de Matemáticas, Universidad del Valle,
Calle 13 No 100–00, Cali, Colombia.
eric.bravo@correounivalle.edu.co

Carlos A. Gómez
Departamento de Matemáticas, Universidad del Valle,
Calle 13 No 100–00, Cali, Colombia
carlos.a.gomez@correounivalle.edu.co

Florian Luca
School of Mathematics, University of the Witwatersrand,
Private Bag X3, Wits 2050, South Africa

Max Planck Institute for Mathematics,
Vivatsgasse 7, 53111 Bonn, Germany

Department of Mathematics, Faculty of Sciences, University of Ostrava,
30 Dubna 22, 701 03 Ostrava 1, Czech Republic
florian.luca@wits.ac.za

September 10, 2019

Abstract

In this work, we correct an oversight from [1].

Key words and phrases. Pell equation, Tribonacci numbers.

2010 Mathematics Subject Classification. 11B39, 11J86.

1 Introduction

For a positive squarefree positive integer d and the Pell equation $X^2-dY^2=\pm 1$, where $X,\ Y\in\mathbb{Z}^+$, it is well–known that all its solutions (X,Y) have the form $X+Y\sqrt{d}=X_k+Y_k\sqrt{d}=(X_1+Y_1\sqrt{d})^k$ for some $k\in\mathbb{Z}^+$, where (X_1,Y_1) be its smallest positive integer solution. Let $\{T_n\}_{n\geq 0}$ be the Tribonacci sequence given by $T_0=0,\ T_1=T_2=1,\ T_{n+3}=T_{n+2}+T_{n+1}+T_n$ for all $n\geq 0$. Let $U=\{T_n+T_m : n\geq m\geq 0\}$ be the set of non-negative integers which are sums of two Tribonacci numbers. In [1], we looked at Pell equations $X^2-dY^2=\pm 1$ such that the containment $X_\ell\in U$ has at least two positive integer solutions ℓ. The following result was proved.

Theorem 1. *For each squarefree integer d, there is at most one positive integer ℓ such that $X_\ell \in U$ except for $d \in \{2, 3, 5, 15, 26\}$.*

Furthermore, for each $d \in \{2, 3, 5, 15, 26\}$, all solutions ℓ to $X_\ell \in U$ were given together with the representations of these X_ℓ's as sums of two Tribonacci numbers. Unfortunately, there was an oversight in [1], which we now correct.

The following intermediate result is Lemma 4.1 in [1].

Lemma 1. *Let (m_i, n_i, ℓ_i) be two solutions of $T_{m_i} + T_{n_i} = X_{\ell_i}$, with $0 \le m_i < n_i$ for $i = 1, 2$ and $1 \le \ell_1 < \ell_2$, then*

$$m_1 < n_1 \le 1535, \qquad \ell_1 \le 1070 \qquad \textit{and} \qquad n_2 < 2.5 \cdot 10^{42}.$$

The rest of the argument in [1] were just reductions of the above parameters. The first step of the reduction consisted in finding all the solutions to

$$X_{\ell_1} = F_{n_1} + F_{m_1}, \qquad \ell_1 \in [1, 1070] \qquad 2 \le m_1 < n_1 \le 1535.$$

Unfortunately, the case $\ell_1 = 1$ was omitted in [1]. Here, we discuss the missing case $\ell_1 = 1$.

In order to reduce the above bound on n_2 from Lemma 1, we don't consider the equation $P^{\pm}_{\ell_1}(X_1) = X_1$ since there is no polynomial equation to solve, instead, we consider each minimal solution $\delta := \delta(X_1, \epsilon)$ of Pell equation $X^2 - dY^2 = \epsilon = \pm 1$, for each $X_1 = T_{m_1} + T_{n_1}$, according to the bounds in Lemma 1. Thus, after some reductions using the Baker–Davenport method on the linear form in logarithms Γ_1 and Γ_2 from [1, inequalities 3.9 and 3.12], for $(m, n, \ell) = (m_2, n_2, \ell_2)$, one shows that the only range for the variables to be considered is

$$\ell_1 = 1, \qquad 1 \le m_1 < n_1 \le 1811, \qquad 1 \le m_2 < n_2 \le 3210, \qquad \text{and} \qquad 2 \le \ell_2 \le 2220. \tag{1}$$

Now, with this new bound on n_2, by the same procedure (LLL–algorithm and continued fractions) used on the linear form in logarithms Γ_3, Γ_4 and Γ_5 in [1, inequalities 3.15 to 3.26], we reduce again the bound on n_1 given in the Lemma 1. Then, further cycles of reductions (for n_2 with the new bound of n_1) on Γ_1 and Γ_2 yield the following result.

Lemma 2. *Let (m_i, n_i, ℓ_i) be two solutions of $T_{m_i} + T_{n_i} = X_{\ell_i}$, with $0 \le m_i < n_i$ for $i = 1, 2$. If $\ell_1 = 1$, then $1 \le m_1 < n_1 \le 160$, $1 \le m_2 < n_2 < 250$ and $2 \le \ell_2 \le 175$.*

An exhaustive search in this last range finds no new solutions. Hence, albeit the work in [1] missed one branch of computations which are described in this note, this does not affect the final result Theorem 1.

References

[1] E. F. Bravo, C. A. Gómez and F. Luca, *X–coordinates of Pell equations as sums of two Tribonacci numbers*, Period. Math. Hung. **77**(2), 175–190 (2018)