Published for SISSA by Springer

Received: *April 2, 2020*
Revised: *May 21, 2020*
Accepted: *June 3, 2020*
Published: *June 23, 2020*

Proof of dispersion relations for the amplitude in theories with a compactified space dimension

Jnanadeva Maharana[1]

Institute of Physics,
Bhubaneswar 751005, India
Max-Planck Institute for Gravitational Physics, Albert Einstein Institute,
Golm, Germany

E-mail: maharana@iopb.res.in

Abstract: The analyticity properties of the scattering amplitude in the nonforward direction are investigated for a field theory in the manifold $R^{3,1} \otimes S^1$. The theory is obtained from a massive, neutral scalar field theory of mass m_0 defined in flat five dimensional spacetime upon compactification on a circle, S^1. The resulting theory is endowed with a massive scalar field which has the lowest mass, m_0, as of the original five dimensional theory and a tower of massive Kaluza-Klein states. We derive nonforward dispersion relations for scattering of the excited Kaluza-Klein states in the Lehmann-Symanzik-Zimmermann formulation of the theory. In order to accomplish this object, first we generalize the Jost-Lehmann-Dyson theorem for a relativistic field theory with a compact spatial dimension. Next, we show the existence of the Lehmann-Martin ellipse inside which the partial wave expansion converges. The scattering amplitude satisfies fixed-t dispersion relations when $|t|$ lies within the Lehmann-Martin ellipse.

Keywords: Field Theories in Higher Dimensions, Large Extra Dimensions

ArXiv ePrint: 2003.14330

[1]Adjunct Professor, NISER, Bhubaneswar.

https://doi.org/10.1007/JHEP06(2020)139

Contents

1 Introduction

This article is a continuation of our investigation of the analyticity properties of scattering amplitude in scalar field theory defined in a manifold $R^{3,1} \otimes S^1$. First we consider a neutral, massive scalar field theory of mass m_0 in a flat five dimensional Minkowski space. Subsequently, one spatial coordinate is compactified on a circle of radius R. The spectrum of the resulting theory consists of a neutral scalar of mass m_0 (same as the mass of the original uncompactified theory) and a tower of massive Kaluza-Klein (KK) states carrying the KK charges. We adopt the Lehmann-Symanzik-Zimmermann (LSZ) [1] formalism to construct the amplitude and to study the analyticity property of the scattering amplitude. We had proved the forward dispersion relation for scattering of KK states in an earlier paper [2] (henceforth referred to as I). The present investigation brings our programme to a completion.

The analyticity properties of scattering amplitude plays a very important role in our understanding of collisions of relativistic particle in the frame works of general field theories without appealing to any specific model. The scattering amplitude, $F(s,t)$, is an analytic function of the center of mass energy squared, s, for fixed momentum transferred squared, t. The fixed-t dispersion relations in s have been proved when $|t|$ lies within the Lehmann ellipse in the axiomatic approach in the case of $D = 4$ field theory, mostly for a single neutral massive field. These results are derived from the general field theories (axiomatic field theories) in the axiomatic approach of Lehmann-Symanzik-Zimmenmann (LSZ) [1] and in the more general frameworks of axiomatic formulation of field theories [3–9, 11–16]. We recall that some of the fundamental principles of such formulations are locality, mircocausality, Lorentz invariance to mention a few. There are very strong reasons to believe that if the dispersion relations are violated then the validity of some of the axioms of these generalized relativistic field theories might be in question. The subsequent progress in this field has led to several rigorous theorems which impose constraints on experimentally observable parameters, generally stated as bounds. These bounds have been put to tests in high energy collision experiments and there is so far no evidence of the violation of these bounds. Notable among them is the Froissart-Martin bound [17–19] that restricts the growth of total cross sections at asymptotic energies: $\sigma_t \leq \frac{4\pi}{t_0}(logs)^2$ where t_0 is determined from the first principles for a given scattering process. The experimental data respect this upper bound for diverse scattering processes over a wide energy range. In the event of any experimental violations of the bound, we shall be compelled to reexamine some of the axioms of the general theories.

The scattering amplitude in nonrelativistic potential scattering exhibit certain analyticity properties in energy k for a large class of potentials as is known for a very long time [20–22]. We recall that the analyticity of scattering amplitude in QFT enjoys a very intimate relationship with the principle of microcausality. In contrast, however, in the context of potential scattering, there is no such deep reason which leads to analyticity of the corresponding amplitude. We recall that the nonrelativistic theory is invariant only under Galilean transformations whereas QFT's are required to be Lorentz invariant. Khuri [27] encountered a situation, in a nonrelativistic potential model, where the amplitude does not satisfy analyticity in momentum k. The consequences of such a violation of analyticity would not the so serious. Whereas, if the amplitude constructed in the frame works of general field theories based on LSZ or Wightman axioms, does not exhibit analyticity then it will raise serious concerns.

The roles played by higher spacetime dimensional field theories ($D > 4$) has become increasingly important. One of the primary reasons is that our quest to construct unified fundamental theories have led physicists to explore consistent theories in higher spacetime dimensions so that the physical phenomena understood in four spacetime dimensions are through effective theories. It is worth while to recall, in this context, supersymmetric theories, supergravity theories and the string theories which have been investigated intensively over past several decades, are consistently defined in higher spacetime dimensions. In order to understand the physics in four dimensions, we adopt the ideas of Kaluza-Klein compactifications in the modern perspective. Thus it is invoked that some of the extra spatial dimensions are compactified in order to facilitate construction of four dimensional theories enabling us to comprehend physical phenomena observed in the present accessible energies. There are a large class of effective four dimensional theories arising from various compactification schemes. Moreover, there are proposals, the so called large radius compactification schemes where the signatures of the extra spatial dimensions might be observed in current high energy colliders [23, 24]. As a consequence, there has been a lot of phenomenological studies to investigate and build models for possible experimental observations of the decompactified dimensions at the present high energy accelerators such as LHC. Indeed, the scale of the extra compact dimensions is extracted from the LHC experiments and it puts the compactification scale to be more than 2 TeV [25, 26]. The signatures of models of large radius compactification and the number of extra compactified dimensions envisaged in a model, go into getting the experimental limits. In some cases, even the limit could be higher than 2 TeV and we refer the readers to the two papers cited here.

JHEP06(2020)139

The large radius compactification ideas motivated Khuri [27], in order to investigate the analyticity properties of scattering amplitude in a nonrelativistic potential model. He identified a model where the potential is spherically symmetric as a function of noncompact coordinates and is of short range, on the other hand one extra spatial coordinate is compctified on S^1. Khuri [27] discovered that, under certain circumstances, the amplitude does not always satisfy the analyticity properties. He also recalled that the analyticity properties of amplitudes were investigated earlier [21, 22] with noncompact spatial coordinates (for $d = 3$ case); (when there was no S^1 compactification) the amplitude satisfied the dispersion relations. Khuri [27] provided counter examples for a model with the S^1 compactification to demonstrate how the analyticity of the forward scattering amplitude beaks down in the presence of S^1 compactification. This result is based on perturbation theoretic approach to nonrelativistic potential scattering. We shall very briefly summarize Khuri's result in the next section.

It was shown in I that the forward scattering amplitude in a relativistic quantum field theory (QFT), with a compact spatial coordinate, satisfies forward dispersion relation unlike what Khuri had has concluded in his potential model [27]. We had considered a five dimensional massive, neutral scalar field theory in five dimensional Minkowski (flat) space to start with. Subsequently, one spatial coordinate was compactified on S^1. The LSZ formalism was adopted to derive the scattering amplitude. As mentioned earlier, if dispersion relations were violated in such a theory then foundations of general relativistic quantum field theories would be questioned. However, the proof of dispersion relations in the forward direction does not provide a complete study of analyticity properties of the theory. It is necessary to prove the nonforward dispersion relations for a general relativistic QFT. We had discussed the requisite steps necessary in order to accomplish this goal in I. The purpose of this article is to bring to completion the investigation of the analyticity of the four point amplitude.

We briefly recall our previous work [28] on study of analyticity in higher dimensional theories as those results will be quite useful for the continuation to the present investigation. We proceeded as follows to study high energy behaviors and analyticity of higher dimensional theories. It was shown, in the LSZ formalism, that the scattering amplitude has desired attributes in the following sense: (i) We proved the generalization of the Jost-Lehmann-Dyson theorem for the retarded function [29, 30] for the $D > 4$ case [31]. (ii) Subsequently, we showed the existence of the Lehmann-Martin ellipse for such a theory. (iii) Thus a dispersion relation can be written in s for fixed t when the momentum transfer squared lies inside Lehmann-Martin ellipse [32, 33]. (iv) The analog of Martin's theorem can be derived in the sense that the scattering amplitude is analytic the product domain $D_s \otimes D_t$ where D_s is the cut s-plane and D_t is a domain in the t-plane such that the scattering amplitude is analytic inside a disk, $|t| < \tilde{R}$, $\tilde{R}$ is radius of the disk and it is independent of s. Thus the partial wave expansion converges inside this bigger domain. (v) We also derived the analog of Jin-Martin [37] upper bound on the scattering amplitude which states that the fixed t dispersion relation in s does not require more than two subtractions. (vi) Therefore, a generalized Froissart-Martin bound was be proved.

In order to accomplish our goal for a $D = 4$ theory which arises from S^1 compactification of a $D = 5$ theory i.e. to prove nonforward dispersion relations, we have to establish the results (i) to (iv) for this theory. It is important to point out, at this juncture, that (to be elaborated in sequel) the spectrum of the theory consists of a massive particle of the original five dimensional theory and a tower of Kaluza-Klein states. Thus the requisite results (i)-(iv) are to obtained in this context in contrast to the results of the D-dimensional theory with a single massive neutral scalar field.

The paper is organized as follows. In the next section (section 2) we recapitulate the main results of Khuri's work [27] without details. The interested reader might consult the original paper of Khuri or section 2 of paper I. This section also contains essential aspects of the LSZ formulation which are utilized to prove the dispersion relations. The third section is devoted to investigation of the analyticity of the scattering amplitude. Our first step is to obtain the Jost-Lehmann-Dyson representation. Consequently, we would obtain the domain free from singularity in t-plane. Next, we shall outline the derivation of the Lehmann ellipses in the present context. The derivation needs to account for the fact that, unlike the case of the usual derivation for single scalar theory, there are the KK towers and theirs presence is to be considered. Subsequently, we are in a position to write the nonforward dispersion relations. The spectral representations of retarded function, advanced function and the causal function play an important role where we have to sum over complete set of physical intermediate states. A theory with the KK tower is endowed with an infinite sum (we shall explain this point later). It is natural to ask how to deal with this problem. We shall argue that as long as s is finite, may be very large, the contributions of the number of intermediate KK states to the sum is finite once the unitarity constraint is imposed. One of our important results is that we prove the analog of Martin's theorem where the unitarity and positivity properties are invoked. Moreover, Martin's theorem leads to constrain the growth properties of partial wave amplitudes. We also derive a version of the Froissart-Martin bound for a field theory with S^1 compactification. Another important question is to find out how many subtractions are required to write the fixed-t dispersion relation. This issue is intimately related to the proof of Jin-Martin bound. We prove that the scattering amplitude requires at most two subtractions. We summarize and discuss our results in section 5.

2 Analyticity property of scattering amplitude and compact spatial dimension

In this section, we shall briefly present some of the results which motivated the present investigation. We enlist important axioms and the relevant kinematical variables. First we summarize essential results of Khuri's work [27]. The interested reader on this topic may go through his paper for details.

2.1 Scattering in nonrelativistic quantum mechanics with a compact dimension

Khuri [27] studied analyticity property of scattering amplitude in a nonrelativistic potential model with a compact spatial dimension. The theory is defined as follows: the potential is $V(r,\Phi)$, where r is the radial coordinate, $|\mathbf{r}| = r$, of the three dimensional space and Φ is compact coordinate; $\Phi + 2\pi R = \Phi$. The radius of compactification, R, is taken to be very small, $R \ll 1$, compared to the scale available in the potential theory (there is no Planck scale here). The perturbative Greens function technique is adopted. The scattering amplitude depends on three variables: the momentum $\mathbf{k}$, the scattering angle and an integer associated with the periodicity of Φ. The free Greens function satisfies the free Schrödinger equation:

$$\left[\nabla^2 + \frac{1}{R^2}\frac{\partial^2}{\partial \mathbf{\Phi}^2} + K^2\right] G_0(\mathbf{K};\mathbf{x},\Phi : \mathbf{x}',\Phi') = \delta^3(\mathbf{x}-\mathbf{x}')\delta(\Phi-\Phi') \tag{2.1}$$

The plane wave solution to the Schrödinger equation is $\Psi_0(\mathbf{x},\Phi) = \frac{1}{(2\pi)^2} e^{i\mathbf{k}.\mathbf{x}} e^{in\Phi}$, $n \in \mathbf{Z}$ and $K^2 = k^2 + (n^2/R^2)$. The closed form expression for the free Greens function has been derived in [27]. A notable feature is that for $(n^2/R^2) > K^2$ the Greens function is exponentially damped as $e^{-\sqrt{n^2/R^2-K^2}}$. The expression for the scattering amplitude is extracted from the large $|\mathbf{x}|$ limit when one looks at the asymptotic behavior of the wave function,

$$\Psi_{\mathbf{k},n} \to e^{\mathbf{k}.\mathbf{x}} e^{in\Phi} + \sum_{m=-[KR]}^{+[KR]} T(\mathbf{k}', m : \mathbf{k}, n) \frac{e^{ik'_{mn}|\mathbf{x}|}}{|\mathbf{x}|} e^{im\Phi} \tag{2.2}$$

where $[KR]$ is the largest integer less than KR and

$$k'_{mn} = \sqrt{k^2 + \frac{n^2}{R^2} - \frac{m^2}{R^2}} \tag{2.3}$$

Khuri [27] identifies a conservation rule: $K^2 = k^2 + (n^2/R^2) = k'^2 + (m^2/R^2)$. Moreover, it is argued that the scattered wave has only $(2[KR]+1)$ components and those states with $(m^2/(R^2) > k^2 + (n^2/R^2)$ are exponentially damped for large $|\mathbf{x}|$ and consequently these do not appear in the scattered wave. Now the scattering amplitude is extracted by Khuri using the standard prescriptions. It takes the following form

$$T(\mathbf{k}', n'; \mathbf{k}, n) = -\frac{1}{8\pi^2} \int d^3\mathbf{x}' \int_0^{2\pi} d\Phi' e^{-\mathbf{k}'.\mathbf{x}'} e^{-in'\Phi'} V(\mathbf{x}',\Phi')\Psi_{\mathbf{k},n}(\mathbf{x}',\Phi') \tag{2.4}$$

Note that the condition, $k'^2 + n'^2/R^2 = k^2 + n^2/R^2$ is to be satisfied. Thus the scattering amplitude describes the process where incoming wave $|\mathbf{k}, n>$ is scattered to final state $|\mathbf{k}', n'>$ with the above constraint.

Khuri proceeds further to extract the scattering amplitude starting from the full Greens function. It satisfied the Schrödinger equation in the presence of the potential. The equation assumes the following form

$$\begin{aligned} T(\mathbf{k}', n'; \mathbf{k}, n) - T_B = -\frac{1}{8\pi^2} \int \dots \int d^3\mathbf{x} d^3\mathbf{x}' d\Phi d\Phi' e^{-i(\mathbf{k}'.\mathbf{x}'+n'\Phi')} V(\mathbf{x}',\Phi') \\ G(\mathbf{K};\mathbf{x}',\mathbf{x};\Phi',\Phi)V(\mathbf{x},\Phi)e^{i(\mathbf{k}.\mathbf{x}+n\Phi)} \end{aligned} \tag{2.5}$$

Here T_B is the Born term given by

$$T_B = -\frac{1}{8\pi^2} \int d^3x \int_0^{2\pi} e^{i(\mathbf{k}-\mathbf{k}').\mathbf{x}} V(x,\Phi) e^{i(n-n')\Phi} d\Phi \tag{2.6}$$

The perturbative Greens function technique is utilized to extract the scattering amplitude order by order. The crucial observation of Khuri [27] is that when he considers the forward amplitude for the case of $n = 1$, to second order, the amplitude does not satisfy analyticity property in k, whereas for $n = 0$ he does not encounter any such problem. He had considered a general class of potentials of the type

$$V(r, \Phi) = u_0(r) + 2 \sum_{m=1}^{N} u_m(r) cos(m\Phi) \tag{2.7}$$

where $u_m(r) = \lambda_m \frac{e^{-\mu r}}{r}$ and the potential is short range in nature. Khuri drew attention to an important fact that in absence of any compactified coordinates, when analyticity of scattering amplitude was investigated in a theory in the 3-dimensional space with same type of potential as above the amplitude did respect analyticity [21, 22].

Remarks.

(i) Khuri [27] noted that, in the context of large radius compactification scenario, if the amplitude exhibits such a nonanalytic behavior in k, there will be serious implications for the physics at LHC energies.

(ii) Moreover, it is to be noted that in the frameworks of nonrelativistic quantum mechanics, the analyticity of scattering amplitude is not so intimately connected with causality compared to a close relationship between the two as in relativistic quantum field theory. In other words, the analyticity of the scattering amplitude in nonrelativistic quantum mechanics is not so sacred as in QFT since analyticity is deeply related with a fundamental principle like microcausality. Recall that the nonrelativistic theory is only invariant under Galilean transformations i.e. they are not required to be Poincaré invariant. The relativistic quantum field theories (QFT) are Poincaré invariant. The principle of microcausality plays a very crucial role in local field theories. Furthermore, microcausality and analyticity are very intimately related. Thus the proof of dispersion relations in QFT very critically depends on microcausality. A violation of dispersion relation would necessarily lead to questioning the foundations of general quantum field theories.

(iii) In view of above remarks, we are led to investigate the analyticity property of scattering amplitude in a quantum field theory with a compactified spatial dimension.

2.2 Quantum field theory with compact spatial dimensions

We have shown in I that the forward scattering amplitude of a theory, defined on the manifold $R^{3,1} \otimes S^1$, satisfied dispersion relations. This result was obtained in the frame works of the LSZ formalism. We summarize, in this subsection, the starting points of I as stated below.

We considered a neutral, scalar field theory with mass m_0 in flat five dimensional Minkowski space $R^{4,1}$. It is assumed that the particle is stable and there are no bound states. The notation is that the spacetime coordinates are, $\hat{x}$, and all operators are denoted with a *hat* when they are defined in the five dimensional space where the spatial coordinates are noncompact.The LSZ axioms are [1]:

A1. The states of the system are represented in a Hilbert space, $\hat{\mathcal{H}}$. All the physical observables are self-adjoint operators in the Hilbert space, $\hat{\mathcal{H}}$.

A2. The theory is invariant under inhomogeneous Lorentz transformations.

A3. The energy-momentum of the states are defined. It follows from the requirements of Lorentz and translation invariance that we can construct a representation of the orthochronous Lorentz group. The representation corresponds to unitary operators, $\hat{U}(\hat{a}, \hat{\Lambda})$, and the theory is invariant under these transformations. Thus there are Hermitian operators corresponding to spacetime translations, denoted as $\hat{P}_{\hat{\mu}}$, with $\hat{\mu} = 0, 1, 2, 3, 4$ which have following properties:

$$\left[\hat{P}_{\hat{\mu}}, \hat{P}_{\hat{\nu}}\right] = 0 \tag{2.8}$$

If $\hat{\mathcal{F}}(\hat{x})$ is any Heisenberg operator then its commutator with $\hat{P}_{\hat{\mu}}$ is

$$\left[\hat{P}_{\hat{\mu}}, \hat{\mathcal{F}}(\hat{x})\right] = i\hat{\partial}_{\hat{\mu}}\hat{\mathcal{F}}(\hat{x}) \tag{2.9}$$

It is assumed that the operator does not explicitly depend on spacetime coordinates. If we choose a representation where the translation operators, $\hat{P}_{\hat{\mu}}$, are diagonal and the basis vectors $|\hat{p}, \hat{\alpha} >$ span the Hilbert space, $\hat{\mathcal{H}}$,

$$\hat{P}_{\hat{\mu}}|\hat{p}, \hat{\alpha} >= \hat{p}_{\hat{\mu}}|\hat{p}, \hat{\alpha} > \tag{2.10}$$

then we are in a position to make more precise statements:

- Existence of the vacuum: there is a unique invariant vacuum state $|0>$ which has the property

  $$\hat{U}(\hat{a}, \hat{\Lambda})|0 >= |0 > \tag{2.11}$$

  The vacuum is unique and is Poincaré invariant.

- The eigenvalue of $\hat{P}_{\hat{\mu}}$, $\hat{p}_{\hat{\mu}}$, is light-like, with $\hat{p}_0 > 0$. We are concerned only with massive stated in this discussion. If we implement infinitesimal Poincaré transformation on the vacuum state then

  $$\hat{P}_{\hat{\mu}}|0 >= 0, \quad \text{and} \quad \hat{M}_{\hat{\mu}\hat{\nu}}|0 >= 0 \tag{2.12}$$

  from above postulates and note that $\hat{M}_{\hat{\mu}\hat{\nu}}$ are the generators of Lorentz transformations.

A4. The locality of theory implies that a (bosonic) local operator at spacetime point $\hat{x}^{\hat{\mu}}$ commutes with another (bosonic) local operator at $\hat{x}'^{\hat{\mu}}$ when their separation is spacelike i.e. if $(\hat{x} - \hat{x}')^2 < 0$. Our Minkowski metric convention is as follows: the inner product of two 5-vectors is given by $\hat{x}.\hat{y} = \hat{x}^0\hat{y}^0 - \hat{x}^1\hat{y}^1 - \ldots - \hat{x}^4\hat{y}^4$. Since we are dealing with a

JHEP06(2020)139

neutral scalar field, for the field operator $\hat{\phi}(\hat{x})$: $\hat{\phi}(\hat{x})^{\dagger} = \hat{\phi}(\hat{x})$ i.e. $\hat{\phi}(\hat{x})$ is Hermitian. By definition it transforms as a scalar under inhomogeneous Lorentz transformations

$$\hat{U}(\hat{a},\hat{\Lambda})\hat{\phi}(\hat{x})\hat{U}(\hat{a},\hat{\Lambda})^{-1} = \hat{\phi}(\hat{\Lambda}\hat{x}+\hat{a}) \tag{2.13}$$

The micro causality, for two local field operators, is stated to be

$$\left[\hat{\phi}(\hat{x}),\hat{\phi}(\hat{x}')\right] = 0, \qquad for \quad (\hat{x}-\hat{x}')^2 < 0 \tag{2.14}$$

It is well known that, in the LSZ formalism, we are concerned with vacuum expectation values of time ordered products of operators as well as with the retarded product of fields. The requirements of the above listed axioms lead to certain relationship, for example, between vacuum expectation values of R-products of operators. Such a set of relations are termed as the *linear relations* and the importance of the above listed axioms is manifested through these relations. In contrast, unitarity imposes *nonlinear* constraints on amplitude. For example, if we expand an amplitude in partial waves, unitarity demands certain positivity conditions to be satisfied by the partial wave amplitudes.

We summarize below some of the important aspects of LSZ formalism as we utilize them through out the present investigation. Moreover, the conventions and definitions of I will be followed for the conveniences of the reader.

(i) The asymptotic condition: according to LSZ the field theory accounts for the asymptotic observables. These correspond to particles of definite mass, charge and spin etc. $\hat{\phi}^{in}(\hat{x})$ represents the free field and a Fock space is generated by the field operator. The physical observable can be expressed in terms of these fields.

(ii) $\hat{\phi}(\hat{x})$ is the interacting field. LSZ technique incorporates a prescription to relate the interacting field, $\hat{\phi}(\hat{x})$, with $\hat{\phi}^{in}(\hat{x})$; consequently, the asymptotic fields are defined with a suitable limiting procedure. Thus we introduce the notion of the adiabatic switching off of the interaction. A cutoff adiabatic function is postulated such that this function controls the interactions. It is **1** at finite interval of time and it has a smooth limit of passing to zero as $|t| \to \infty$. It is argued that when adiabatic switching is removed we can define the physical observables.

(iii) The fields $\hat{\phi}^{in}(\hat{x})$ and $\hat{\phi}(\hat{x})$ are related as follows:

$$\hat{x}_0 \to -\infty \qquad \hat{\phi}(\hat{x}) \to \hat{Z}^{1/2}\hat{\phi}^{in}(\hat{x}) \tag{2.15}$$

By the first postulate, $\hat{\phi}^{in}(\hat{x})$ creates free particle states. However, in general $\hat{\phi}(\hat{x})$ will create multi particle states besides the single particle one since it is the interacting field. Moreover, $< 1|\hat{\phi}^{in}(\hat{x})|0 >$ and $< 1|\hat{\phi}(\hat{x})|0 >$ carry same functional dependence in $\hat{x}$. If the factor of $\hat{Z}$ were not the scaling relation between the two fields (2.15), then canonical commutation relation for each of the two fields (i.e. $\hat{\phi}^{in}(\hat{x})$ and $\hat{\phi}(\hat{x})$) will be the same. Thus in the absence of $\hat{Z}$ the two theories will be identical. Moreover, the postulate of asymptotic condition states that in the remote future

$$\hat{x}_0 \to \infty \qquad \hat{\phi}(\hat{x}) \to \hat{Z}^{1/2}\hat{\phi}^{out}(\hat{x}). \tag{2.16}$$

JHEP06(2020)139

We may as well construct a Fock space utilizing $\hat{\phi}^{out}(\hat{x})$ as we could with $\hat{\phi}(\hat{x})^{in}$. Furthermore, the vacuum is unique for $\hat{\phi}^{in}$, $\hat{\phi}^{out}$ and $\hat{\phi}(\hat{x})$. The normalizable single particle states are the same i.e. $\hat{\phi}^{in}|0>=\hat{\phi}^{out}|0>$. We do not display $\hat{Z}$ from now on. If at all any need arises, $\hat{Z}$ can be introduced in the relevant expressions.

We define creation and annihilation operators for $\hat{\phi}^{in}$, $\hat{\phi}^{out}$. We recall that $\hat{\phi}(\hat{x})$ is not a free field. Whereas the fields $\hat{\phi}^{in,out}(\hat{x})$ satisfy the free field equations $[\Box_5+m_0^2]\hat{\phi}^{in,out}(\hat{x})=0$, the interacting field satisfies an equation of motion which is endowed with a source current: $[\Box_5+m_0^2]\hat{\phi}(\hat{x})=\hat{j}(\hat{x})$. We may use the plane wave basis for simplicity in certain computations; however, in a more formal approach, it is desirable to use wave packets.

The relevant vacuum expectation values of the products of operators in LSZ formalism are either the time ordered products (the T-products) or the retarded products (the R-products). We shall mostly use the R-products and we use them extensively throughout this investigation. It is defined as

$$R\ \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1)\dots\hat{\phi}_n(\hat{x}_n)=(-1)^n\sum_P\theta(\hat{x}_0-\hat{x}_{10})\theta(\hat{x}_{10}-\hat{x}_{20})\dots\theta(\hat{x}_{n-10}-\hat{x}_{n0})$$
$$[[\dots[\hat{\phi}(\hat{x}),\hat{\phi}_{i_1}(\hat{x}_{i_1})],\hat{\phi}_{i_2}(\hat{x}_{i_2})]\dots],\hat{\phi}_{i_n}(\hat{x}_{i_n})] \tag{2.17}$$

note that $R\hat{\phi}(\hat{x})=\hat{\phi}(\hat{x})$ and P stands for all the permutations $i_1,\dots i_n$ of $1,2\dots n$. The R-product is hermitian for hermitian fields $\hat{\phi}_i(\hat{x}_i)$ and the product is symmetric under exchange of any fields $\hat{\phi}_1(\hat{x}_1)\dots\hat{\phi}_n(\hat{x}_n)$. Notice that the field $\hat{\phi}(\hat{x})$ is kept where it is located in its position. We list below some of the important properties of the R-product for future use [6]:

(i) $R\ \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1)\dots\hat{\phi}_n(\hat{x}_n)\neq 0$ only if $\hat{x}_0>\ \max\ \{\hat{x}_{10},\dots\hat{x}_{n0}\}$.

(ii) Another important property of the R-product is that

$$R\ \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1)\dots\hat{\phi}_n(\hat{x}_n)=0 \tag{2.18}$$

whenever the time component $\hat{x}_0$, appearing in the argument of $\hat{\phi}(\hat{x})$ whose position is held fix, is less than time component of any of the four vectors $(\hat{x}_1,\dots\hat{x}_n)$ appearing in the arguments of $\hat{\phi}(\hat{x}_1)\dots\hat{\phi}(\hat{x}_n)$.

(iii) We recall that

$$\hat{\phi}(\hat{x}_i)\to\hat{\phi}(\hat{\Lambda}\hat{x}_i)=\hat{U}(\hat{\Lambda},0)\hat{\phi}(\hat{x}_i)\hat{U}(\hat{\Lambda},0)^{-1} \tag{2.19}$$

Under Lorentz transformation $\hat{U}(\hat{\Lambda},0)$. Therefore,

$$R\ \hat{\phi}(\hat{\Lambda}\hat{x})\hat{\phi}(\hat{\Lambda}\hat{x}_i)\dots\hat{\phi}_n(\hat{\Lambda}\hat{x}_n)=\hat{U}(\hat{\Lambda},0)R\ \phi(x)\phi_1(x_1)\dots\phi_n(x_n)U(\Lambda,0)^{-1} \tag{2.20}$$

And

$$\hat{\phi}_i(\hat{x}_i)\to\hat{\phi}_i(\hat{x}_i+\hat{a})=e^{i\hat{a}.\hat{P}}\hat{\phi}_i(\hat{x}_i)e^{-i\hat{a}.\hat{P}} \tag{2.21}$$

under spacetime translations. Consequently,

$$R\ \hat{\phi}(\hat{x}+\hat{a})\hat{\phi}(\hat{x}_i+\hat{a})\dots\hat{\phi}_n(\hat{x}_n+\hat{a})=e^{i\hat{a}.\hat{P}}R\ \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1)\dots\hat{\phi}_n(\hat{x}_n)e^{-i\hat{a}.\hat{P}} \tag{2.22}$$

JHEP06(2020)139

Therefore, the vacuum expectation value of the R-product dependents only on difference between pair of coordinates: in other words it depends on the following set of coordinate differences: $\hat{\xi}_1 = \hat{x}_1 - \hat{x}, \hat{\xi}_2 = \hat{x}_2 - \hat{x}_1 \dots \hat{\xi}_n = \hat{x}_{n-1} - \hat{x}_n$ as a consequence of translational invariance.

We may define 'in' and 'out' states in terms of the creation operators associated with 'in' and 'out' fields as follows

$$|\hat{k}_1, \hat{k}_2, \dots \hat{k}_n \; in > = \hat{a}^\dagger_{in}(\hat{\mathbf{k}}_1)\hat{a}^\dagger_{in}(\hat{\mathbf{k}}_2)\dots\hat{a}^\dagger_{in}(\hat{\mathbf{k}}_n)|0> \tag{2.23}$$

$$|\hat{k}_1, \hat{k}_2, \dots \hat{k}_n \; out > = \hat{a}^\dagger_{out}(\hat{\mathbf{k}}_1)\hat{a}^\dagger_{out}(\hat{\mathbf{k}}_2)\dots\hat{a}^\dagger_{out}(\hat{\mathbf{k}}_n)|0> \tag{2.24}$$

We can construct a complete set of states either starting from 'in' field operators or the 'out' field operators and each complete set will span the Hilbert space, $\hat{\mathcal{H}}$. Therefore, a unitary operator will relate the two sets of states in this Hilbert space. This is a heuristic way of introducing the concept of the S-matrix. We shall define S-matrix elements through LSZ reduction technique in subsequent section.

We shall not distinguish between notations like $\hat{\phi}^{out,in}$ or $\hat{\phi}_{out,in}$ and therefore, there might be use of the sloppy notation in this regard.

We record the following important remark *en passant*. The generic matrix element $< \hat{\alpha}|\hat{\phi}(\hat{x}_1)\hat{\phi}(\hat{x}_2)\dots|\hat{\beta} >$ is not an ordinary function but a distribution. Thus it is to be always understood as smeared with a Schwarz type test function $f \in \mathcal{S}$. The test function is infinitely differentiable and it goes to zero along with all its derivatives faster than any power of its argument. We shall formally derive expressions for scattering amplitudes and the absorptive parts by employing the LSZ technique. It is to be understood that these are generalized functions and such matrix elements are properly defined with smeared out test functions.

We are in a position to study several attributes of scattering amplitudes in the five dimensional theory such as proving existence of the Lehmann-Martin ellipse, give a proof of fixed t dispersion relation to mention a few. However, these properties have been derived in a general setting recently [28] for D-dimensional theories. The purpose of incorporating the expression for the VEV of the commutator of two fields in the 5-dimensional theory is to provide a prelude to the modification of similar expressions when we compactify the theory on S^1 as we shall see in the next section.

The compactification of scalar field theory: $R^{4,1} \to R^{3,1} \otimes S^1$. We compile below the relevant materials necessary to proceed further in order to prove the fixed-t dispersion relations. The details are presented in I. One spatial dimension of the 5-dimensional theory is compactified on S^1. If y is the compact coordinate and x^μ are spacetime coordinates, defined on $R^{3,1}$, then $\hat{x}^{\hat{\mu}} = (x^\mu, y)$. The asymptotic field in $D = 5$ satisfy the free field equation $[\Box_5 + m_0^2]\hat{\phi}^{in,out}(\hat{x}) = 0$. Due to the periodicity of y, $y + 2\pi R = y$, R being the radius of S^1, $\hat{\phi}^{in,out}(\hat{x})$ admit KK mode expansion.

$$\hat{\phi}^{in,out}(\hat{x}) = \hat{\phi}^{in,out}(x,y) = \phi_0^{in,out}(x) + \sum_{n=-\infty, n\neq 0}^{+n=\infty} \phi_n^{in,out}(x) e^{\frac{iny}{R}} \tag{2.25}$$

The equation of motion is

$$[\Box_4 - \frac{\partial}{\partial y^2} + m_n^2]\phi_n^{in,out}(x,y) = 0 \tag{2.26}$$

where $\phi_n^{in,out}(x,y) = \phi_n^{in,out} e^{\frac{iny}{R}}$ and $n = 0$ term has no y-dependence being denoted as $\phi_0(x)$; from now on $\Box_4 = \Box$ here and everywhere; and $m_n^2 = m_o^2 + \frac{n^2}{R^2}$. Thus we have tower of massive states. The momentum associated in the y-direction is $q_n = n/R$ and is quantized in the units of $1/R$. It is an additive conserved quantum number and designated as the Kaluza-Klein (KK) charge. The interacting field $\hat{\phi}(\hat{x})$, admits a similar KK expansion; however, the equation motion contains a source current, $\hat{j}(x,y)$. This current is expanded in KK modes: $\{j_0(x), J_n(x)e^{iny/R}\}$ where n takes values from $-\infty$ to $+\infty$ and $n = 0$ is excluded from J_n. Note that $j_0(x)$ and $J_n(x)e^{iny/R}$ are source associated with ϕ_0 and $\phi_n(x)e^{iny/R}$ respectively. The mode expansion, analogous to (2.25) is

$$\hat{j}(x,y) = j_0(x) + \sum_{n=-\infty, n\neq 0}^{n=+\infty} J_n(x)e^{iny/R} \tag{2.27}$$

and each current carries the KK charge n.

Let us consider the set of asymptotic fields, $\{\phi_0^{in}, \phi_n^{in}\}$. We can construct the Fock spaces associated with each of these fields from their corresponding creation operators. Let $a^\dagger(\mathbf{k})$ and $A^\dagger(p, q_n)$ be creation operators for $\phi_0(x)$ and $\phi_n(x)$ respectively. The latter is endowed with the KK charge q_n. Now each set of operator will create Hilbert spaces $\mathcal{H}_n, n = 0, \pm 1, \ldots$. The same construction can be carried out with the set of *out* field. The spectrum of the compactified theory is: a field of mass m_0, associated with ϕ_0 and tower of Kaluza-Klein (KK) states characterized by mass and discrete 'charge', $(m_n^2 = m_o^2 + \frac{n^2}{R^2}, q_n)$, respectively. The Hilbert space, $\hat{\mathcal{H}}$, of $D = 5$ theory is decomposed as a direct sum of Hilbert spaces where each one is characterized by its quantum number q_n

$$\hat{\mathcal{H}} = \sum \oplus \mathcal{H}_n \tag{2.28}$$

Thus $\mathcal{H}_0$ is the Hilbert space constructed from ϕ_0^{in} with $q_n = 0$ and $\mathcal{H}_n$ are constructed from the KK fields. Therefore, vectors belong to two spaces with different KK charges are orthogonal to one another

$$< \mathbf{p}, q_n | \mathbf{p}', q_{n'} >= \delta^3(\mathbf{p} - \mathbf{p}')\delta_{n,n'} \tag{2.29}$$

Remark. Note also that $n \in \mathbf{Z}$; if there is an additional 'parity' invariance under $y \to -y$ we need only to sum over positive set of integers $\{n\}$ in the KK expansions. We introduce the notion of an antiparticle. If a 'particle' carried charge $q_n > 0$ its 'anti-particle' has negative charge $-q_n$; however, it must have positive energy. Thus an intermediate state $|q_n, q_{-n} >$ has vacuum quantum number and so on.

2.3 Definitions and kinematical variables

In order to investigate the analyticity of an amplitude and determine its analyticity domains we have to define kinematical variables and mass thresholds. There are three different class

JHEP06(2020)139

of scattering processes: (i) scattering of states with $q_n = 0$, i.e. scattering of zero modes. (ii) The scattering of a zero mode state with a KK state i.e. $q_n \neq 0$. The reactions (i) and (ii) have been dealt in I for forward scattering. The study of the nonforward scattering of reactions (i) and (ii) is a straight forward generalization. It is omitted in this work.

The states carrying $q_n \neq 0$ are denoted by χ_n (from now on a state carrying charge is defined with a subscript n and momenta carried by external particles are denoted as $p_a, p_b, \ldots$). Moreover, we shall consider elastic scattering of states carrying equal charge; the elastic scattering of unequal charge particles is just the elastic scattering of unequal mass states due to mass-charge relationship for the KK states.

Let us consider a generic 4-body reaction (all states carry non-zero n)

$$a + b \to c + d \tag{2.30}$$

The particles (a, b, c, d) (the corresponding fields being $\chi_a, \chi_b, \chi_c, \chi_d$) respectively carrying momenta p_a, p_b, p_c, p_d. The Lorentz invariant Mandelstam variables are

$$s = (p_a + p_b)^2 = (p_c + p_d)^2, \ t = (p_a - p_d)^2 = (p_b - p_c)^2, \ u = (p_a - p_c)^2 = (p_b - p_d)^2 \tag{2.31}$$

and $\sum p_a^2 + p_b^2 + p_c^2 + p_d^2 = m_a^2 + m_b^2 + m_c^2 + m_d^2$. The independent identities of the four particles will facilitate the computation of the amplitude so that we keep track of the fields reduced using LSZ procedure. We list below some relevant (kinematic) variables which will be required in future

$$M_a^2, \quad M_b^2, \quad M_c^2, \quad M_d^2 \tag{2.32}$$

These correspond to lowest mass two or more particle states which carry the same quantum number as that of particle a, b, c and d respectively. We define below six more variables

$$(M_{ab}, M_{cd}), \quad (M_{ac}, M_{bd}), \quad (M_{ad}, M_{bc}) \tag{2.33}$$

The variable M_{ab} carries the same quantum number as (a *and* b) and it corresponds to two or more particle states. Similar definition holds for the other five variables introduced above. We define two types of thresholds: (i) the physical threshold, s_{phys}, and s_{thr}. In absence of anomalous thresholds (and equal mass scattering) $s_{thr} = s_{phys}$. Similarly, we may define u_{phys} and u_{thr} which will be useful when we discuss dispersion relations. We assume from now on that $s_{thr} = s_{phys}$ and $u_{thr} = u_{phys}$. Now we outline the derivation of the expression for a four point function in the LSZ formalism. We start with $|p_d, p_c \ out>$ and $|p_b, p_a \ in>$ and considers the matrix element $< p_d, p_c \ out|p_b, p_a \ in>$. Next we subtract out the matrix element $< p_d, p_c \ in|p_b, p_a \ in>$ to define the S-matrix element.

$$\begin{aligned} < p_d, p_c \ out|p_b, p_a \ in> = 4p_u^0 p_b^0 \delta^3(\mathbf{p}_d - \mathbf{p}_b)\delta^3(\mathbf{p}_a - \mathbf{p}_c) - \\ \frac{i}{(2\pi)^3}\int d^4x \int d^4x' e^{-i(p_a.x - p_c.x')} \times \\ \tilde{K}_x \tilde{K}_{x'} < p_d \ out|\bar{R}(x';x)|p_b \ in> \end{aligned} \tag{2.34}$$

$$R(x, x') = -i\theta(x_0 - x'_o)[\chi_a(x), \chi_c(x')] \tag{2.35}$$

JHEP06(2020)139

We have reduced fields associated with a and c in (2.34). In the next step we may reduce all the four fields and in such a reduction we shall get VEV of the R-product of four fields which will be operated upon by four K-G operators. However, the latter form of LSZ reduction (when all fields are reduced) is not very useful when we want to investigate the analyticity property of the amplitude in the present context. In particular our intent is to write the nonforward dispersion relation. Thus we abandon the idea of reducing all the four fields.

Remark. Note that on the right hand side of the equation (2.34) the operators act on $R\chi_a(x)\chi(x')_c$ and there is a θ-function in the definition of the R-product. Consequently, the action of $K_x K_{x'}$ on $R\chi_a(x)\chi_c(x')$ will produce a term $RJ_a(x)J_c(x')$. In addition the operation of the two K-G operators will give rise to δ-functions and derivatives of δ-functions and some equal time commutators i.e. there will terms whose coefficients are $\delta(x_0 - x'_0)$. When we consider Fourier transforms of the derivatives of these δ-functions they will be transformed to momentum variables. However, the amplitude is a function of Lorentz invariant quantities. Thus one will get only finite polynomials of such variables, as has been argued by Symanzik [38]. His arguments is that in a local quantum field theory only finite number of derivatives of δ-functions can appear. Moreover, in addition, there are some equal time commutators and many of them vanish when we invoke locality arguments. Therefore, we shall use the relation

$$K_x K_{x'} R\chi(x)\chi_c(x') = RJ_a(x)J_c(x') \tag{2.36}$$

keeping in mind that there are derivatives of δ-functions and some equal time commutation relations which might be present. Moreover, since the derivative terms give rise to polynomials in Lorentz invariant variables, the analyticity properties of the amplitude are not affected due to the presence of such terms. This will be understood whenever we write an equation like (2.36).

3 Nonforward elastic scatting of $n \neq 0$ Kaluza-Klein states

We envisage elastic scattering of two equal mass, $m_n^2 = m_0^2 + \frac{n^2}{R^2}$, hence equal charge KK particles and we take n positive. Our first step is to define the scattering amplitude for this reaction (see (2.34))

$$\begin{aligned} < p_d, p_c\ out|p_b, p_a\ in > = 4p_u^0 p_b^0 \delta^3(\mathbf{p}_d - \mathbf{p}_b)\delta^3(\mathbf{p}_a - \mathbf{p}_c) - \\ \frac{i}{(2\pi)^3}\int d^4x \int d^4x' e^{-i(p_a.x - p_c.x')} \times \\ \tilde{K}_x \tilde{K}_{x'} < p_d\ out|\bar{R}(x';x)|p_b\ in > \end{aligned} \tag{3.1}$$

where

$$\bar{R}(x';x) = -i\theta(x_0 - x'_0)[\chi_a(x), \chi_c(x')] \tag{3.2}$$

and $\tilde{K}_x = (\Box + m_n^2)$. We let the two KG operators act on $\bar{R}(x;x')$ in the VEV and resulting equation is

$$< p_d, p_c \; out|p_b, p_a \; in >=< p_d, p_c \; in|p_b, p_a \; in > -\frac{1}{(2\pi)^3}\int d^4x \int d^4x' e^{-i(p_a.x - p_c.x')} \times$$
$$< p_d|\theta(x'_0 - x_0)[J_c(x'), J_a(x)]|p_b > \tag{3.3}$$

Here $J_a(x)$ and $J_c(x')$ are the source currents associated with the fields $\chi_a(x)$ and $\chi_b(x')$ respectively. We arrive at (3.3) from (3.1) with the understanding that the r.h.s. of (3.3) contains additional terms; however, these terms do not affect the study of the analyticity properties of the amplitude as alluded to earlier. We shall define three distributions which are matrix elements of the product of current. The importance of these functions will be evident in sequel

$$F_R(q) = \int_{\infty}^{+\infty} d^4z e^{iq.z}\theta(z_0) < Q_f|[J_a(z/2), J_c(-z/2)]|Q_i > \tag{3.4}$$

$$F_A(q) = -\int_{\infty}^{+\infty} d^4z e^{iq.z}\theta(-z_0) < Q_f|J_a(z/2), J_c(-z/2)]|Q_i > \tag{3.5}$$

and

$$F_C(q) = \int_{-\infty}^{+\infty} d^4z e^{iq.z} < Q_f|[J_a(z/2), J_c(-z/2)]|Q_i > \tag{3.6}$$

Moreover,

$$F_C(q) = F_R(q) - FA(q) \tag{3.7}$$

$|Q_i >$ and $|Q_f >$ are states which carry four momenta; these momenta are held fixed and treated as fixed parameter. Let us focus attention on the matrix element of the causal commutator defined in (3.6). The prescription is to open up the commutator of the currents and introduce a complete set of physical states. Let us assign KK charge n to initial and final states. Thus the conservation of KK charge dictates which intermediate physical states are permitted consistent under the KK charge conservation law. The complete set of physical states are: $\sum_n |\mathcal{P}_n\tilde{\alpha}_n >< \mathcal{P}_n\tilde{\alpha}_n| = \mathbf{1}$ and $\sum_{n'} |\bar{\mathcal{P}}_{n'}\tilde{\beta}_{n'} >< \bar{\mathcal{P}}_{n'}\tilde{\beta}_{n'}| = \mathbf{1}$. Here $\{\tilde{\alpha}_n, \tilde{\beta}_{n'}\}$ stand for quantum numbers that are permitted for the intermediate states. The momenta $\mathcal{P}_n, \bar{\mathcal{P}}_{n'} \in V^+$; V^+ is the forward light cone. The matrix element defining $F_C(q)$, (3.7), assumes the following form

$$\int d^4z e^{iq.z}\Big[\sum_n \left(\int d^4\mathcal{P}_n < Q_f|J_a\left(\frac{z}{2}\right)|\mathcal{P}_n\tilde{\alpha}_n >< \mathcal{P}_n\tilde{\alpha}_n|J_c\left(-\frac{z}{2}\right)|Q_i >\right)$$
$$-\sum_{n'}\left(\int d^4\bar{\mathcal{P}}_{n'} < Q_f|J_c\left(-\frac{z}{2}\right)|\bar{\mathcal{P}}_{n'}\tilde{\beta}_{n'} >< \bar{\mathcal{P}}_{n'}\tilde{\beta}_{n'}|J_a\left(\frac{z}{2}\right)|Q_i >\right)\Big] \tag{3.8}$$

In order to derive the spectral representation for (3.8) following steps are used. We implement judicious translation operations to get rid of the z-dependence of the currents. Then carry out the integration $\int d^4z$ which leads to a δ-function. The details of the derivations are given in I. Finally, $F_C(q)$ is expressed as

$$F_C(q) = \frac{1}{2}(A_u(q) - A_s(q)) \tag{3.9}$$

JHEP06(2020)139

where

$$2A_s(q) = \sum_{n'} \Bigg(< Q_f|j(0)_a|\bar{\mathcal{P}}_{n'} = \frac{(Q_i+Q_f)}{2} + q, \tilde{\beta}_{n'} > \times \\ < \tilde{\beta}'_n, \bar{\tilde{P}}_n = \frac{(Q_i+Q_f)}{2} + q|j_c(0)|Q_i > \Bigg) \tag{3.10}$$

and

$$2A_u = \sum_{n} \Bigg(< Q_f|j_c(0)|\mathcal{P}_n = \frac{(Q_i+Q_f)}{2} - q, \tilde{\alpha}_n > \times \\ < \tilde{\alpha}_n, \mathcal{P}_n = \frac{(Q_i+Q_f)}{2} - q|j_l(0)|Q_i > \Bigg) \tag{3.11}$$

Consequences of microcausality. The Fourier transform of $F_C(q), \bar{F}_C(z)$, vanishes outside the light cone. Moreover, $F_C(q)$ will also vanish as function of q wherever, both $A_s(q)$ and $A_u(q)$ vanish simultaneously. Furthermore, since $(\frac{Q_i+Q_f}{2}+q) \in V^+$ and $(\frac{Q_i+Q_f}{2}-q) \in V^+$ we arrive at the following conclusions

$$\left(\frac{Q_i+Q_f}{2}+q\right)^2 \geq 0, \quad \left(\frac{Q_i+Q_f}{2}\right)_0 + q_0 \geq 0 \tag{3.12}$$

and

$$\left(\frac{Q_i+Q_f}{2}-q\right)^2 \geq 0, \quad \left(\frac{Q_i+Q_f}{2}\right)_0 - q_0 \geq 0 \tag{3.13}$$

The above two conditions, for nonvanishing of $A_u(q)$ and $A_s(q)$ implies the existence of the minimum mass parameters for the nonvanishing of $A_u(q)$ and $A_s(q)$: (i) $(\frac{Q_i+Q_f}{2}+q)^2 \geq {\mathcal{M}_+}^2$ and (ii) $(\frac{Q_i+Q_f}{2}-q)^2 \geq {\mathcal{M}_-}^2$.

Let us discuss how the theory with KK states differs from the one with a single scalar field. In the spectral represents of $A_u(q)$ and $A_s(q)$ we sum over all physical intermediate states which means the sum includes the KK states as long as their quantum numbers are such that KK charge conservation is satisfied (depending on the charges of $|Q_i>$ and $|Q_f>$). On the other hand, for theory with a single scalar field the intermediate states correspond to physical multiparticle states. Naturally, it begs an answer to the question whether the entire KK tower (infinite number of them) contributes. This issue cannot be resolved in the 'linear program' to study analyticity in the frameworks of axiomatic field theory. We shall return to this question and provide a resolution in the next section.

In order to derive a fixed-t dispersion relation we have to identify a domain which is free from singularities in the t-plane. The first step is to obtain the Jost-lehmann-Dyson representation for the causal commutator, $F_C(q)$, for the case of equal mass elastic processes with $n \neq 0$. Therefore, the technique of Jost and Lehmann [29] is quite adequate. We do not have to resort to more elegant and general approach of Dyson [30]. We present the results concisely and refer to [28] for details. As noted in (3.12) and (3.13), $F_C(q)$ is nonvanishing in those domains. We designate this region as $\bar{\mathbf{R}}$,

$$\bar{\mathbf{R}} : \left\{ (Q+q)^2 \geq {\mathcal{M}_+}^2, Q+q \in V^+ \text{ and } (Q-q)^2 \geq {\mathcal{M}_-}^2, Q-q \in V^+ \right\} \tag{3.14}$$

JHEP06(2020)139

where $Q = \frac{Q_i+Q_f}{2}$ and V^+ being the future light cone. There is no need to repeat the derivation of Jost-Lehmann representation here. The present case differs from the single-field case in the following way. Here we are looking for the nearest singularity to determine the singularity free region. For the case at hand, the presence of the towers of KK states is to be envisaged in the following perspective. Since we consider equal mass scattering the location of nearest singularity will be decided by the lowest values of $\mathcal{M}_+$ and $\mathcal{M}_-$. Let us elaborate this point. We recall that there is the tower of KK states appearing as intermediate states (see (3.10) and (3.11)). Thus each new threshold could create region of singularity of $F_C(q)$. We are concerned about the identification of the singularity free domain. Thus the lowest threshold of two particle intermediate state, consistent with desired constraints, control the determination of this domain of analyticity. Therefore, for the equal mass case, the Jost-Lehmann representation for $F_C(q)$ is such that it is nonzero in the region $\bar{\mathbf{R}}$,

$$F_C(q) = \int_S d^4u \int_0^\infty d\chi^2 \epsilon(q_0 - u_0)\delta[(q-u)^2 - \chi^2)]\Phi(u, Q.\chi^2) \tag{3.15}$$

Note that u is also a 4-dimensional vector (*not the Mandelstam variable* u). The domain of integration of u is the region S specified below

$$\mathbf{S} : \left\{ Q+u \in V^+,\ Q-u \in V^+,\ Max\,[0, \mathcal{M}_+ - \sqrt{(Q+u)^2}, \mathcal{M}_- - \sqrt{(Q-u)^2}] \leq \chi \right\} \tag{3.16}$$

and $\Phi(u, Q.\chi^2)$ arbitrary. Here χ^2 is to be interpreted like a mass parameter. Moreover, recall the assumptions about the features of the causal function stated above. Since the retarded commutator involves a θ-function, if we use integral representation for it (see [29]) we derive an expression for the retarded function,

$$F_R(q) = \frac{i}{2\pi} \int d^4q' \delta^3(\mathbf{q}' - \mathbf{q}) \frac{1}{q'_0 - q_0} F_C(q'),\ Im\ q_0 > 0 \tag{3.17}$$

Moreover, for the retarded function, $F_R(q)$, the corresponding Jost-Lehmann representation reads [29]

$$F_R(q) = \frac{i}{2\pi} \int_S d^4u \int_0^\infty d\chi^2 \frac{\Phi(u, Q, \chi^2)}{(q-u)^2 - \chi^2} \tag{3.18}$$

Note that these integral representations are written for the case where the integral converges. It is well known, in the LSZ framework, that the integrand will, at most, have polynomial growth. It follows from the fact that the matrix elements are tempered distributions. In any case, the aforementioned properties of the integrand in the representation does not affect the analyticity of $F_R(q)$. One important observation is that the singularities lie in the complex q-plane.[1] We provide below a short and transparent discussion for the sake of completeness. The locations of the singularities are found by examining where the denominator (3.18) vanishes,

$$(q_0 - u_0)^2 - (q_1 - u_1)^2 - (q_2 - u_2)^2 - (q_3 - u_3)^2 = \chi^2 \tag{3.19}$$

[1] See Itzykson and Zuber [5] and Sommer [8] for elaborate discussions.

We conclude that the singularities lie on the hyperboloid give by (3.19) and those points are in domain $\mathbf{S}$ as defined in (3.16). There are points in the hyperboloid which belong to the domain $\mathbf{S}$. These are called admissible. Moreover, according to our earlier definition, the domain $\bar{\mathbf{R}}$ is where $F_C(q)$ is nonvanishing (see (3.14)). Then there is a domain which contains a set of real points where $F_C(q)$ vanishes, call it $\mathbf{R}$ and this is compliment to real elements of $\bar{\mathbf{R}}$. From the above arguments, we arrive at the conclusion that $F_C(q) = 0$ for every real point belonging to $\mathbf{R}$ (the compliment of $\bar{\mathbf{R}}$). Thus these are the real points in the q-plane where $F_R(q) = F_A(q)$ since $F_C(q) = 0$ there. Recall the definition of $\bar{\mathbf{R}}$, (3.14). A border is defined by the upper branch of the parabola given by the equation $(Q+q)^2 = {\mathcal{M}_+}^2$ and the other one is given by the equation for another parabola $(Q-q)^2 = {\mathcal{M}_-}^2$. Now we identify the *coincidence region* to be the domain bordered by the two parabolae. It is obvious from the above discussions that the set $\mathbf{S}$ is defined by the range of values u and χ^2 assume in the admissible parabola. Now we see that those set of values belong to a subset of (u, χ^2) of all parabolas (recall equation (3.19)) [8] and [29, 30]. In order to transparently discuss the location of a singularity, let us go through a few short steps as the prescription to illustrate essential points. We discussed about the identification of admissible parabola. The amplitude is function of Lorentz invariant kinematical variables; therefore, it is desirable to express the constraints and equations in terms of those variables eventually. Let us focus on $Q \in V^+$ and choose a Lorentz frame such that four vector $Q = (Q_0, \mathbf{0})$ where $\mathbf{0}$ stands for the *three* spatial components of Q. Next step is to choose four vector q appropriately to exhibit the location of singularity in a simple way. This is achieved as follows: choose one spatial component of q in order to identify the position of the singularity in this variable and treat q_0 and the rest of the components of q as parameters and hold them fixed [8]. We remind the reader that all the variables appearing in the Jost-Lehmann representation for $F_C(q)$ and $F_R(q)$ are Lorentz invariant objects. Thus going to a specific frame will not alter the general attributes of the generalized functions. If we solve for q_1^2 in (3.19) after obtaining an expression for q_1^2

$$q_1 = u_1 \pm i\sqrt{\chi^2_{min}(u) - (q_0 - u_0)^2 + (q_2 - u_2)^2 + (q_3 - u_3)^2 + \rho}, \rho > 0 \tag{3.20}$$

We remind that the set of points $\{u_0, u_1, u_2, u_3; \chi^2_{min} = min\ \chi^2\}$ lie in $\mathbf{S}$. The above exercise has enabled us to identify the domain where the singularities might lie with the choice for the variables Q and u we have made. We are dealing with the equal mass case and note that the location of the singularities are symmetric with respect to the real axis. We now examine a further simplified scenario where the coincidence region is bounded by two branches of hyperboloids so that $\mathcal{M}_+^2 = \mathcal{M}_-^2 = \mathcal{M}^2$. Now the singular points are

$$q_1 = u_1 \pm i\sqrt{Min\ [\chi^2_{min} - u_0^2 + u_2^2 + u_3^2] + \rho}, \rho > 0 \tag{3.21}$$

For the case under considerations: $(Q+q)^2 = (Q-q)^2 = \mathcal{M}^2$, and

$$q_1 = u_1 \pm i\sqrt{(\mathcal{M} - \sqrt{Q^2 - u_1^2})^2 + \rho}, \rho > 0 \tag{3.22}$$

The above result paves the way to prove the existence of the Lehmann ellipses. It is important to recognize the essential difference between the present investigation (i.e. the presence of the KK towers) and the results derived for a single massive scalar field. We have to deal with the appearance of several thresholds for identification of the coincidence regions. These thresholds are the multiparticle states in various channels as discussed earlier and introduced earlier in this section through the two equations (2.32) and (2.33). Their relevance is already reflected in the spectral representations, (3.10) and (3.11), when we introduced complete set of intermediate states. We remark that the presence of the excited KK states do not shrink the singularity free regions. Therefore, the domain we have obtained is the smallest domain of analyticity; nevertheless, we feel that in order to arrive at this conclusion the entire issue had to be examined *ab initio*.

The Lehmann Ellipses. Our goal is to derive fixed-t dispersion relations. Noted that as $s \to s_{thr}$, $\cos\theta$ goes out of the physical region $-1 \leq cos\theta \leq +1$, ($\theta$ being the *c.m.* scattering angle) when we wish to hold t fixed. We choose the following kinematical configuration in order to derive the Lehmann ellipse for the case at hand i.e. elastic scattering of equal (nonzero) charge KK states, hence particles of equal mass. Here (a, b) and (c, d) are respectively the incoming and outgoing particles. They are assigned the following energies and momenta in the *c.m.* frame:

$$p_a = (E_a,\ \mathbf{k}),\quad p_b = (E_b, -\mathbf{k}),\quad p_c = (E_c,\ \mathbf{k}'),\quad p_d = (E_d, -\mathbf{k}') \tag{3.23}$$

$\mathbf{k}$ is the *c.m.* momentum, $|\mathbf{k}| = |\mathbf{k}'|$, $E_a = \sqrt{(m_a^2 + \mathbf{k}^2)}$, $E_b = \sqrt{(m_c^2 + \mathbf{k}^2)}$, $E_c = \sqrt{(m_c^2 + \mathbf{k}'^2)}$ and $E_d = \sqrt{(m_d^2 + \mathbf{k}'^2)}$. Although all the particles, (a, b, c, d), are identical, we keep labeling them as individual one for the purpose which will be clear shortly. Thus $E_a = E_b$ and $E_c = E_d$ and $\hat{\mathbf{k}}.\hat{\mathbf{k}}' = cos\theta$. It is convenient to choose the following coordinate frame for the ensuing discussions.

$$p_a = (\sqrt{s}, +\mathbf{k},\ \mathbf{0}),\quad p_b = (\sqrt{s}, -\mathbf{k},\ \mathbf{0}) \tag{3.24}$$

$\mathbf{0}$ is the two spatial components of vector $\mathbf{k}$ and

$$p_c = (\sqrt{s}, +kcos\theta, +ksin\theta, 0)\ \ p_d = (\sqrt{s}, -kcos\theta, -ksin\theta, 0) \tag{3.25}$$

with $k = |\mathbf{k}| = |\mathbf{k}'|$. Thus, $s = (p_a + p_b)^2 = (p_c + p_d)^2$

$$q = \frac{1}{2}(p_d - p_c) = (0, -kcos\theta, -ksin\theta, 0),\quad P = \frac{1}{2}(p_a + p_b) = (\sqrt{s}, 0, 0, 0) \tag{3.26}$$

With these definitions of q and P, when we examine the conditions for nonvanishing of the spectral representations of A_s and A_u we arrive at

$$(P+q)^2 > {\mathcal{M}_+}^2,\ \text{for}\ A_s \neq 0,\quad (P-q)^2 > {\mathcal{M}_-}^2,\ \text{for}\ A_u \neq 0 \tag{3.27}$$

Thus the coincidence region is given by the condition

$$(P+q)^2 < {\mathcal{M}_+}^2,\quad (P-q)^2 < {\mathcal{M}_-}^2 \tag{3.28}$$

We are dealing with the equal mass case; therefore, ${\mathcal{M}_+}^2 = {\mathcal{M}_-}^2 = \mathcal{M}^2$. We conclude from the energy momentum conservation constraints (use the expressions for P and q)

that $p_c^2 = (P-q)^2 < M_c^2$ and $p_d^2 = (P+q)^2 < M_d^2$ in this region. Moreover, $(p_a - p_c)^2 = (P-q-p_a)^2 < {M_{ac}}^2$ and $(p_a + p_d)^2 = (P-q-p_a)^2 < {M_{ad}}^2$. We also note that $(P-q) \in V^+$ and $(P+q) \in V^+$. The admissible hyperboloid is $(q-u)^2 = \chi^2_{\rm min} + \rho, \rho > 0$ with $(\frac{p_a+p_b}{2} \pm u) \in V^+$. χ^2_{min} assumes the following form for the equal mass case,

$$\chi^2_{min} = Max\left\{0, \mathcal{M} - \sqrt{\left(\frac{(p_a+p_b)}{2} + u\right)^2}, \mathcal{M} - \sqrt{\left(\frac{(p_a+p_b)}{2} - u\right)^2}\right\} \tag{3.29}$$

Notice that $\mathcal{M}$ appearing in the second term of the curly in (3.29) is the mass of two or more multiparticle states carrying the quantum numbers of particle c; whereas $\mathcal{M}$ appearing in the third term inside the curly bracket is the mass of two or more multiparticle states carrying the quantum numbers of particle d. In the present case $\mathcal{M}$ has the same quantum number as that of the incoming state carrying KK charge n. Thus, in this sector, we can proceed to show the existence of the small Lehmann Ellips (SLE). It is not necessary to present the entire derivation here. The extremum of the ellipse is given by

$$cos\theta_0 = \left(1 + \frac{(M_c^2 - m_c^2)(M_d^2 - m_d^2)}{k^2(s - M_c^2 - M_d^2)}\right)^{1/2} \tag{3.30}$$

We note that $M_c = \sqrt{m_n^2 + m_0^2}$ is the mass of the lowest multiparticle state (one particle with KK charge n and another with KK charge *zero*); moreover, $M_c = M_d$. Thus the denominator is $k^2 s$.

$$cos\theta_0 = \left(1 + \frac{9m_0^4}{k^2 s}\right)^{1/2} \tag{3.31}$$

It will be a straightforward work to derive the properties of the large Lehmann Ellipse (LLE) by reducing all the four fields in the expression for the four point function as is the standard prescription; also note that the value of $cos\theta(s)$ depends on s.

Important remark. The first point to note is that in the presence of the other states of KK tower, we have to carry out the same analysis as above for each sector. Notice, however, each multiparticle state composed of KK towers has to have the quantum numbers of c (same as d since we consider elastic channels of equal mass scattering). Thus if c carries charge n, then a possible KK state could be $q+l+m = n$ since KK charges can be positive and negative. The second point is when we derive the value of $cos\theta_0$, for each such case, it is rather easy to work out that value will be away from original expression (3.30). Thus the nearest singularity in $cos\theta$ plane is given by the expression (3.31) although there will be Lehmann ellipses associated with higher KK towers.

We expand the scattering amplitude in partial waves (in the Legendre polynomial basis) in the domain of convergence. This domain of analyticity is enlarged (earlier it was only physically permitted values of $cos\theta$) to a region which is an ellipse whose semimajor axis is given by (3.31). Moreover, the absorptive part of the scattering amplitude has a domain of convergence beyond $cos\theta = \pm 1$; it converges inside the large Lehmann ellipse (LLE). Therefore, we are able to write fixed-t dispersion relations as long as t lies in the following domain

$$|t| + |t + 4k^2| < 4k^2 cos\theta_0 \tag{3.32}$$

The absorptive parts A_s and A_u defined on the right hand and left hand cuts respectively, for $s' > s_{thr}$ and $u' > u_{thr}$ are holomorphic in the LLE. Thus, assuming no subtractions

$$F(s,t) = \frac{1}{\pi}\int_{s_{thr}}^{\infty} \frac{ds'\ A_s(s',t)}{s'-s} + \frac{1}{\pi}\int_{u_{thr}}^{\infty} \frac{du'\ A_u(u',t)}{u'+s-4m^2+t} \tag{3.33}$$

We shall settle the important issue of number of required subtraction in the dispersion relations in the next section. Although we have not proved the crossing symmetry explicitly, it will not be hard to provide a proof following the arguments of [28]. Essentially, either one follows the procedures which employed the techniques of Bremmermann, Oehme and Taylor [34] or those of Bross, Epstein and Glaser [35].

4 Asymptotic behavior of the amplitude

We intend to derive the asymptotic behavior of the amplitude in this section. It serves two purposes: (i) To determine the growth properties of the amplitude with energy which is related to the issue of subtractions. (ii) And to derive analog of the Froissart-Martin bound. Let us first resolve the issue of sum over intermediate states in the spectral representations. We shall only indicate the steps followed in I and the interested reader may consults I for the details.

We have not investigated the consequences of unitarity so far. It is a nonlinear relation and imposes strong constraints. Let us define the $\mathbf{T}$-matrix from the $\mathbf{S}$-matrix: $\mathbf{S} = \mathbf{1} - \mathbf{iT}$. The unitarity of $\mathbf{S}$ implies: $(\mathbf{T}^\dagger - \mathbf{T}) = i\mathbf{T}^\dagger\mathbf{T}$. Let us consider the matrix element $< p_d, p_c, in|\mathbf{T}^\dagger - \mathbf{T}|p_b, p_c, in >$ and look at the matrix element $< p_d, p_c, in|\mathbf{T}^\dagger\mathbf{T}|p_b, p_c, in >$. We introduce a complete set of physical states $\sum|\mathcal{N} >< \mathcal{N}| = \mathbf{1}$. Here the set of states $|\mathcal{N} >$ correspond to all admissible physical states consistent with energy-momentum conservation and KK-charge conservation. Thus, at this stage, entire KK tower is to be included. After going to through a series of steps, we arrive the following expression (see I for details)

$$\begin{aligned} T(p_d,p_c;p_b,p_a) - T^*(p_d,p_c;p_b,p_a) = \sum_{\mathcal{N}}\Big[&\delta(p_d+p_c-p_n)\times \\ &T(p_d,p_c;n)T^*(n;p_b,p_a) - \\ &\delta(p_a-p_c-p_n)\times \\ &T(p_d,-p_c;n)T^*(p_d,-p_c;n)\Big] \end{aligned} \tag{4.1}$$

The essential point to note is the presence of the δ-functions in the $r.h.s.$ The presence of the δ-function implies $(p_c + p_d = p_n)$ Thus $(p_c + p_d)^2 = M_n^2$, where M_n^2 is the intermediate physical state mass-squared and $(p_c+p_d)^2 = s$. Therefore, unitarity constrains the number of KK towers that can contribute to the sum; not the entire infinite tower is allowed. Similarly, it is easy to see that the second term corresponds to the cross channel. To recall, *the linear program* is unable to cut off contributions of the entire KK tower; however, unitarity, the nonlinear relation, resolves the issue. In other words, the entire KK tower does

JHEP06(2020)139

not contribute to the spectral representation (3.10) and (3.11). An analogous argument holds for the 'crossed channel' contribution as detailed derivations showed in I.

Let us turn the attention to the partial wave expansion of the amplitude and the power of the positivity property of absorptive part of the amplitude. We recall that the scattering amplitude admits a partial wave expansion

$$F(s,t) = \frac{\sqrt{s}}{k}\sum_{l=0}^{\infty}(2l+1)f_l(s)P_l(cos\theta) \tag{4.2}$$

where $k = |\mathbf{k}|$, the c.m. momentum and θ is the *c.m.* scattering angle. The expansion converges inside the Lehmann ellipse with focii at ± 1 and semimajor axis $1+\frac{const}{2k^2}$. Unitarity leads to the positivity constraints on the partial wave amplitudes

$$0 \le |f_l(s)|^2 \le Im\ f_l(s) \le 1 \tag{4.3}$$

As is well known, the semimajor axis of the Lehmann ellipse shrinks as s grows. Recall that derivation of the Lehmann ellipse is based on the *linear program.* Martin [33] has proved an important theorem. It is known as the procedure for the enlargement of the domain of analyticity. He demonstrated that the scattering amplitude is analytic in the topological product of the domains $D_s \otimes D_t$. This domain is defined by $|t| < \tilde{R}$, $\tilde{R}$ being independent of s and s is outside the cut $s_{thr} + \lambda = 4m_n^2 + \lambda, \lambda > 0$. In order to recognize the importance of this result, we briefly recall the theorem of BEG [36]. It is essentially the study of the analyticity property of the scattering amplitude $F(s,t)$. It was shown that in the neighborhood of any point s_0, t_0; $-T < t_0 \le 0$, s_0 outside the cuts, the amplitude, $F(s,t)$, is analytic in s, and t in a region

$$|s - s_0| < \eta_0(s_0, t_0), \quad |t - t_0| < \eta_0(s_0, t_0) \tag{4.4}$$

Note the following features of BEG theorem: it identifies the domain of analyticity; however, the size of this domain may vary as s_0 and t_0 vary. Furthermore, the size of this domain might shrink to zero; in other words, as $s \to 0$, $\eta(s)$ may tend to zero. The importance Martin's theorem lies in his proof that $\eta(s)$ is bounded from below i.e. $\eta(s) \ge \tilde{R}$ and $\tilde{R}$ is s-independent. It is unnecessary to repeat the proof of Martin's theorem here. The interested reader may consult [28]. Instead, we shall summarize the conditions to be satisfied by the amplitude as stated by Martin [33].

Statement of Martin's Theorem. If following requirements are satisfied by the elastic amplitude

I. $F(s,t)$ satisfies fixed-t dispersion relation in s with finite number of subtractions ($-T_0 \le t \le 0$).

II. $F(s,t)$ is an analytic function of the two Mandelstam variables, s and t, in a neighborhood of $\bar{s}$ in an interval below the threshold, $4m_n^2 - \rho < \bar{s} < 4m_n^2$ and also in some neighborhood of $t = 0$, $|t| < R(\bar{s})$. This statement hold due to the work of Bros, Epstein and Glaser [35, 36].

JHEP06(2020)139

III. Holomorphicity of $A_s(s',t)$ and $A_u(u',t)$: the absorptive parts of $F(s,t)$ on the right hand and left hand cuts with $s' > 4m_n^2$ and $u' > 4m_n^2$ are holomorphic in the LLE.

IV. The absorptive parts $A_s(s',t)$ and $A_u(u',t)$, for $s' > 4m_n^2$ and $u' > 4m_n^2$ satisfy the following positivity properties

$$\left|\left(\frac{\partial}{\partial t}A_s(s',t)\right)^n\right| \leq \left(\frac{\partial}{\partial t}\right)^n A_s(s',t)\bigg|_{t=0}, \quad -4k^2 \leq t \leq 0 \tag{4.5}$$

and

$$\left|\left(\frac{\partial}{\partial t}A_u(u',t)\right)^n\right| \leq \left(\frac{\partial}{\partial t}\right)^n A_u(u',t)\bigg|_{t=0}, \quad -4k^2 \leq t \leq 0 \tag{4.6}$$

where $\mathbf{k}$ is the *c.m.* momentum. Then $F(s,t)$ is analytic in the quasi topological product of the domains $D_s \otimes D_t$. (i) $s \in$ cut-plane: $s \neq 4m_n^2 + \rho, \rho > 0$ and (ii) $|t| < \tilde{R}$, there exists some $\tilde{R}$ such that dispersion relations are valid for $|t| < \tilde{R}$, independent of s. We may follow the standard method to determine $\tilde{R}$. The polynomial boundedness, in s, can be asserted by invoking the simple arguments presented earlier. Consequently, a dispersion relation can be written down for $F(s,t)$ in the domain $D_s \otimes D_t$. The importance of Martin's theorem is appreciated from the fact that it implies that the η of BEG is bounded from below by an s-independent $\tilde{R}$. Moreover, value of $\tilde{R}$ can be determined by the procedure of Martin (see [8] for the derivations).

We outline proof of a few more results as corollaries without providing detailed computations:

(I) The amplitude, $F(s,t)$, satisfies following properties: (i) Polynomial boundedness i.e. $|F(s,t)| < s^N$; N is a finite integer. This follows from the fact that the LSZ reduced amplitudes are tempered distributions. It is necessary that $|t|$ lies within the Lehmann-Martin ellipse. (ii) The partial wave expansion converges inside the Lehmann-Martin ellipse and the positivity conditions are satisfied by partial wave amplitudes (4.3). Then the Froissart-Martin bound is proved. We sketch a pedagogical proof of the Froissart-Martin bound on the total cross section, σ_t, of our interest. Let us consider the absorptive part of the scattering amplitude, $A_s(s,t) = Im\ F(s,t)$. It admits a partial wave expansion which converges inside the large Lehmann ellipse.

$$A_s(s,t) = \frac{\sqrt{s}}{k}\sum_{l=0}^{l=\infty}(2l+1)Im\ f_l(s)P_l\left(1+\frac{t}{2k^2}\right) \tag{4.7}$$

Consider $A_s(s,t)$ at the right extremity value of t, i.e. $t = t_0$, on the ellipse of convergence. Notice the properties: (i) $A_s(s,t) < s^N$; the polynomial boundedness for t inside the ellipse and $P_l(x) > 1$ for $x > 1$. (iii) $0 \leq Imf_l(s) \leq 1$ from partial wave unitarity. Furthermore, since each term in the partial wave expansion (4.7) is positive it is also bounded as

$$(2l+1)Im\ f_l(s)P_l\left(1+\frac{t_0}{2k^2}\right) < s^N\frac{k}{\sqrt{s}} \tag{4.8}$$

We recall that for $x > 1$, $P_l(x) > \frac{\tilde{c}}{(2l+1)}(1 + (2x-2)^{1/2})^l$. Since $P_l(x)$ grows exponentially with x for $x > 1$, $Im\ f_l(s)$ would damp exponentially beyond some cut off value of l

$$Im\ f_l(s) < C' e^{Nlogs - l\sqrt{\frac{t_0}{2k^2}}} \tag{4.9}$$

if the polynomial boundedness (4.8) is to be respected. Thus the effective cut off value is $L_0 = \sqrt{s}(log\ s)$. We can split the partial wave expansion into two parts: a sum from $0\ to\ L_0$ and the rest is $L_0 + 1\ to\ \infty$. Now consider the imaginary part of forward amplitude, $F(s, t = 0)$; we have been considering, so far, the absorptive part at $t = t_0$. It is bounded as

$$Im\ F(s, t = 0) \leq \sum_0^{L_0} (2l + 1) + O(e^{-Nlog\ s}) \tag{4.10}$$

We have used the unitarity bound on $Im\ f_l(s)$ and set $Im\ f_l(s) = 1$; $P_l(1) = 1$ any way. The last term on the $r.h.s$ is the sum of the terms from $L_0 + 1\ to\ \infty$ and it is negligible for large s. Therefore,

$$Im\ F(s, t = 0) \leq L_0^2 = Cslog^2 s \tag{4.11}$$

$C = \frac{4\pi}{t_0}$ fixed from the first principles. Therefore, from the optical theorem

$$\sigma_t \leq Clog^2 s \tag{4.12}$$

This is a quick derivation of the Froissart bound. In our case, t_0 is the lowest threshold for t-channel process and it is $t_0 = 4m_0^2$. Thus the constant, C, also gets fixed.

(II) We have proved the analog of the Jin-Martin bound [37]. The arguments are as follows: the scattering amplitude, $F(s, t)$ is polynomially bounded for $|t|$ lying inside the ellipse of convergence. $F(s, t)$ admits the partial wave expansion (4.2). Note that

$$|F(s, t)| \leq \frac{\sqrt{s}}{k} \sum_{l=0}^{\infty} (2l + 1)|f_l(s)| P_l \left(1 + \frac{|t|}{2k^2}\right) \tag{4.13}$$

since $P_l(1 + x) \leq P_l(1 + |x|)$ and the Legendre polynomial are positive for arguments greater than 1. Utilizing partial wave unitarity and positivity (i.e. $0 \leq |f_l(s)|^2 \leq Im\ f_l(s) \leq 1$), it follows that $|F(s, t)| \leq slog^2 s$ The above inequality is for the $r.h.s$ cut and it also holds for the $l.h.s$ cut. Therefore, by invoking Phragman-Lindelof theorem [39], one arrives at the conclusion that $|F(s, t)|$ is bounded as *constant* $slog^2 s$ in the complex s-plane. Therefore, the fixed-t dispersion relations do not need more than two subtractions as long as $|t|$ lies inside the Lehmann ellipse.

We would like to draw the attention of the reader to the fact that a field theory defined on the manifold $R^{3,1} \otimes S^1$ whose spectrum consists of a massive scalar field and a tower of Kaluza-Klein states satisfies nonforward dispersion relations. This statements begs certain clarifications. The theory satisfies LSZ axioms. The analyticity properties can be derived in the *linear program* of axiomatic field theory which leads to the proof of the existence of the Lehmann ellipses. The role of the KK tower is to be assessed in this program. Once we invoke unitarity constraint stronger results follow and the enlargement of the domain of analyticity in s and t variables can be established.

JHEP06(2020)139

5 Summary and discussions

We summarize our results in this section and discuss their implications. The objective of the present work is to investigate the analyticity property of the scattering amplitude in a field theory with a compactified spatial dimension on a circle i.e. the so called S^1 compactification. We were motivated to undertake this investigation from work of Khuri [27] who considered potential scattering with a compact spatial coordinate. He showed the lack of analyticity of the forward scattering amplitude under certain circumstances. Naturally, it is important to examine what is the situation in relativistic field theories. As has been emphasized by us before, lack of analyticity of scattering amplitude in a QFT will be a matter of concern since analyticity is derived under very general axioms of QFT. Thus a compactified spatial coordinate in a theory with flat Minkowski spacetime coordinates should not lead to unexpected drastic violations of fundamental principles of QFT. In this paper, initially, a five dimensional neutral massive scalar theory of mass, m_0, was considered in a flat Minkowski spacetime. Subsequently, we compactified a spatial coordinate on S^1 leading to a spacetime manifold $R^{3,1} \otimes S^1$. The particles of the resulting theory are a scalar of mass m_0 and the Kaluza-Klein towers. In this work, we have focused on elastic scattering of states carrying nonzero equal KK charges, $n \neq 0$, to prove fixed-t dispersion relations. We have left out the elastic scattering of $n = 0$ states as well as elastic scattering of an $n = 0$ state with an $n \neq 0$ state for nonforward directions. These two cases can be dealt with without much problem from our present work. Moreover, our principal task is to prove analyticity for scattering of $n \neq 0$ states and thus complete the project we started with in order to settle the issue related to analyticity as was raised by Khuri [27] in the context of potential scattering. We showed in I that forward amplitude satisfies dispersion relations. However, it is not enough to prove only the dispersion relations for the forward amplitude but a fixed-t dispersion relation is desirable. We have adopted the LSZ axiomatic formulation, as was the case in I, for this purpose. Our results, consequently, do not rely on perturbation theory whereas, Khuri [27] arrived at his conclusions in the perturbative Greens function techniques as suitable for a nonrelativistic potential model. Thus the work presented here, in some sense, has explored more than what Khuri had investigated in the potential scattering.

We have gone through several steps, as mentioned in the discussion section of I, in order to accomplish our goal. The principal results of this investigations are as follows. First we obtain a spectral representation for the Fourier transform of the causal commutator, $F_c(q)$. We discussed the coincidence region which is important for what followed. In order to identify the singularity free domain, we derived analog of the Jost-Lehmann-Dyson theorem. A departure from the known theorem is that there are several massive states, appearing in the spectral representation, and their presence has to be taken into considerations. Thus, we identified the singularity free region i.e. the boundary of the domain of analyticity. Next, we derived the existence of the Lehmann ellipse. We were able to write down fixed-t dispersion relations for $|t|$ lying within the Lehmann ellipse.

We have proceeded further. It is not enough to obtain the Lehmann ellipse since the semimajor axis of the ellipse shrinks as s increases. Thus it is desirable to derive the analog of Martin's theorem [33]. We appealed to unitarity constraints following Martin and utilized his arguments on the attributes of the absorptive amplitude and showed that indeed Martin's theorem can be proved for the case at hand. As a consequence, the analog of Froissart-Martin upper bound on total cross sections, for the present case, is obtained. The convergence of partial wave expansions within the Lehmann-Martin ellipse and polynomial boundedness for the amplitude, $F(s,t)$ for $|t|$ within Lehmann-Martin ellipse, lead to the Jin-Martin upper bound [37] for the problem we have addressed here. In other words, the amplitude, $F(s,t)$, does not need more than two subtractions to write fixed t dispersion relations in the domain $D_s \otimes D_t$.

We have made two assumptions: (i) existence of stable particles in the entire spectrum of the theory defined on $R^{3,1} \otimes S^1$ geometry. Our arguments are based on the conservation of KK discrete charge $q_n = \frac{n}{R}$; it is the momentum along the compatified direction. (ii) The absence of bound states. We have presented some detailed arguments in support of (ii). To put is very concisely, we conveyed that this flat space $D = 4$ theory with an extra compact S^1 geometry results from toroidal compactification of five dimensional defined in flat Minkowski space. In absence of gravity in $D = 5$, the lower dimensional theory would not have massless gauge field and consequently, BPS type states are absent. It is unlikely that the massive scalars (even with KK charge) would provide bound states. This is our judicious conjecture.

Another interesting aspect needs further careful investigation. Let us start with a five dimensional Einstein theory minimally coupled to a massive neutral scalar field of mass m_0. We are unable to fulfill requirements of LSZ axioms in the case of the five dimensional theory in curved spacetime. Furthermore, let us compactify this theory to a geometry $R^{3,1} \otimes S^1$. Thus the resulting scalar field lives in flat Minkowski space with a compact dimension. We have an Abelian gauge field in $D = 4$, which arises from S^1 compactification of the 5-dimensional Einstein metric. The spectrum of the theory can be identified: (i) There is a massive scalar of mass m_0 descending of $D = 5$ theory accompanied by KK tower of states. (ii) A massless gauge boson and its massive KK partners. (iii) If we expand the five dimensional metric around four dimensional Minkowski metric when we compactify on S^1, we are likely to have massive spin 2 states (analog of KK towers). We may construct a Hilbert space in $D = 4$ i.e with geometry $R^{3,1} \otimes S^1$. It will be interesting to investigate the analyticity properties of the scattering amplitudes and examine the high energy behaviors. Since only a massless spin 1 particle with Abelian gauge symmetry appears in the spectrum, it looks as if the analyticity of amplitudes will not be affected. However, there might be surprises since a massive spin 2 particle is present in the spectrum. Khuri [27] was motivated by the large extra dimension scenario to undertake the problem. He had raised the question what will be the consequences of his conclusions (in the potential scattering model) if indeed the dispersion relation is not valid at LHC energies. However, the field theory we have considered here, the dispersion relations are proved for fixed t. It will be worthwhile to undertake phenomenological analysis to check if there are Froissart-Martin bound violation at extremely high energies. We have noticed that, so far, the issue of the validity of Froissart-Martin bound has not received adequate attention. The data on σ_t is accumulating from the LHC experiments. If the experiments unambigously confirm that energy dependence of the total cross sections show a clear deviation from the $(ln\ s)^2$ behavior then we have to resolve an important problem. The important question would be whether the Froissart bound violation is a challenge to question the axioms of local quantum field theories. Alternatively, one might propose that the violation of the bound is an indication that, at the LHC energies, the extra dimensions are decompactified as envisaged in the large radius extra compact dimension scenario. If there would be evidences in favor of the latter scenario we would witness emergence of new physical phenomena.

Acknowledgments

It is my great pleasure to thank André Martin who inspired me to undertake this investigation. I am grateful to him for numerous valuable discussions. I wish to thanks Stefan Theisen for his encouragements and for discussions. The gracious hospitality of Hermann Nicolai and the Max-Planck Institüt für Gravitational Physik and the Albert Einstein Institüt is gratefully acknowledged.

References

[1] H. Lehmann, K. Symanzik and W. Zimmermann, *Zur Formulierung quantisierter Feldtheorien*, Nuovo Cim. **1** (1955) 205.

[2] J. Maharana, *Analyticity properties of scattering amplitude in theories with compactified space dimensions*, Nucl. Phys. B **943** (2019) 114619 [arXiv:1810.11275] [INSPIRE].

[3] A. Martin, *Scattering Theory: unitarity, analyticity and crossing*, Springer-Verlag, Berlin-Heidelberg-New York, (1969).

[4] A. Martin and F. Cheung, *Analyticity properties and bounds of the scattering amplitudes*, Gordon and Breach, New York (1970).

[5] C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, Dover Publications, Mineola, New York (2008).

[6] M. Froissart, *The proof of dispersion relations*, in *Dispersion Relations and their Connection with Causality*, Academic Press, New York, Varrena Summer School Lectures (1964).

[7] H. Lehmann, *Scattering matrix and field operators*, Nuovo Cim **14** (1959) 153.

[8] G. Sommer, *Present state of rigorous analytic properties of scattering amplitudes*, Fortsch. Phys. **18** (1970) 577 [INSPIRE].

[9] R.J. Eden, *Theorems on high energy collisions of elementary particles*, Rev. Mod. Phys. **43** (1971) 15 [INSPIRE].

[10] S.M. Roy, *High energy theorems for strong interactions and their comparison with experimental data*, Phys. Rept. **5** (1972) 125 [INSPIRE].

[11] A.S. Wightman, *Quantum Field Theory in Terms of Vacuum Expectation Values*, Phys. Rev. **101** (1956) 860 [INSPIRE].

[12] R. Jost, *The General Theory of Quantized Fields*, American Mathematical Society, Providence, Rhodes Island (1965).

[13] R.F. Streater, *Outline of Axiomatic Relativistic Quantum Field Theory*, Rept. Prog. Phys. **38** (1975) 771 [INSPIRE].

[14] L. Klein, *Dispersion Relations and Abstract Approach to Field Theory Field Theory*, Gordon and Breach, Publisher Inc., New York (1961).

[15] S.S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, Raw, Peterson and Company, Evaston, Illinois (1961).

[16] N.N. Bogolibov, A.A. Logunov, A.I. Oksak and I.T. Todorov, *General Principles of Quantum Field Theory*, Klwer Academic Publisher, Dordrecht/Boston/London (1990).

[17] M. Froissart, *Asymptotic behavior and subtractions in the Mandelstam representation*, Phys. Rev. **123** (1961) 1053 [INSPIRE].

[18] A. Martin, *Unitarity and high-energy behavior of scattering amplitudes*, Phys. Rev. **129** (1963) 1432 [INSPIRE].

[19] A. Martin, *Extension of the axiomatic analyticity domain of scattering amplitudes by unitarity-I*, Nuovo Cim. A **42** (1966) 930.

[20] M.L. Goldgerber and K.M. Watson, *Collision Theory*, J. Weyl and Son Inc. (1964).

[21] N. Khuri, *Analyticity of the Schrödinger scattering amplitude and nonrelativistic dispersion relations*, Phys. Rev. **107** (1957) 1148.

[22] D. Wong, *Dispersion relation for nonrelativistic particles*, Phys. Rev. **107** (1957) 302.

[23] I. Antoniadis and K. Benakli, *Extra dimensions at LHC*, Mod. Phys. Lett. A **30** (2015) 1502002 [INSPIRE].

[24] D. Luest and T.R. Taylor, *Limits on stringy signals at the LHC*, Mod. Phys. Lett. A **30** (2015) 15040015.

[25] ATLAS and CMS collaborations, *Searches for extra dimensions with the ATLAS and CMS detectors*, Nucl. Part. Phys. Proc. **273-275** (2016) 541 [INSPIRE].

[26] S. Rappoccio, *The experimental status of direct searches for exotic physics beyond the standard model at the large hadron collider*, Rev. Phys. **4** (2019) 100027.

[27] N.N. Khuri, *Potential scattering on $R^3 \otimes S^1$*, Annals Phys. **242** (1995) 332.

[28] J. Maharana, *Analyticity Properties and Asymptotic Behavior of Scattering Amplitude in Higher Dimensional Theories*, J. Math. Phys. **58** (2017) 012302 [arXiv:1608.06402] [INSPIRE].

[29] R. Jost and H. Lehmann, *Integral-Darstellung kausaler Kommutatoren*, Nuovo Cim. **5** (1957) 1598.

[30] F.J. Dyson, *Integral representations of causal commutators*, Phys. Rev. **110** (1958) 1460 [INSPIRE].

[31] J. Maharana, *Jost-Lehmann-Dyson representation in higher dimensional field theories*, Phys. Lett. B **764** (2017) 212 [INSPIRE].

[32] H. Lehmann, *Analytic properties of scattering amplitudes as functions of momentum transfer*, Nuovo Cim. **10** (1958) 579.

JHEP06(2020)139

[33] A. Martin, *Extension of the axiomatic analyticity domain of scattering amplitudes by unitarity-I*, Nuovo Cim. **42** (1966) 930.

[34] H.J. Bremermann, R. Oehme and J.G. Taylor, *Proof of Dispersion Relations in Quantized Field Theories*, Phys. Rev. **109** (1958) 2178 [INSPIRE].

[35] J. Bros, H. Epstein and V. Glaser, *Some rigorous analyticity properties of the four-point function in momentum space*, Nuovo Cim. **31** (1964) 1265.

[36] J. Bros, H. Epstein and V. Glaser, *A proof of the crossing property for two-particle amplitudes in general quantum field theory*, Commun. Math. Phys. **1** (1965) 240 [INSPIRE].

[37] Y.S. Jin and A. Martin, *Connection Between the Asymptotic Behavior and the Sign of the Discontinuity in One-Dimensional Dispersion Relations*, Phys. Rev. **135** (1964) B1369 [INSPIRE].

[38] K. Symanzik, *Derivation of Dispersion Relations for Forward Scattering*, Phys. Rev. **105** (1957) 743 [INSPIRE].

[39] E.C. Titchmarsh, *The theory of functions*, Oxford University Press, London (1939).