

FREE SUBGROUPS OF 3-MANIFOLD GROUPS

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ABSTRACT. We show that any closed hyperbolic 3-manifold M has a co-final tower of covers $M_i \rightarrow M$ of degrees n_i such that any subgroup of $\pi_1(M_i)$ generated by k_i elements is free, where $k_i \geq n_i^C$ and $C = C(M) > 0$. Together with this result we prove that $\log k_i \geq C_1 \text{sys}_1(M_i)$, where $\text{sys}_1(M_i)$ denotes the systole of M_i , thus providing a large set of new examples for a conjecture of Gromov. In the second theorem $C_1 > 0$ is an absolute constant. We also consider a generalization of these results to non-compact finite volume hyperbolic 3-manifolds.

1. INTRODUCTION

Let $\Gamma < \text{PSL}_2(\mathbb{C})$ be a cocompact Kleinian group and $M = \mathbb{H}^3/\Gamma$ be the associated quotient space. It is a closed orientable hyperbolic 3-orbifold, it is a manifold if Γ is torsion-free. We will call a group Γ *k-free* if any subgroup of Γ generated by k elements is free. We denote the maximal k for which Γ is k -free by $\mathcal{N}_{fr}(\Gamma)$ and we call it the *free rank* of Γ . For example, if S_g is a closed Riemann surface of genus g , then its fundamental group satisfies $\mathcal{N}_{fr}(\pi_1(S_g)) = 2g - 1$. In this note we prove that for any Kleinian group as above there exists an exhaustive filtration of normal subgroups Γ_i of Γ such that $\mathcal{N}_{fr}(\Gamma_i) \geq [\Gamma : \Gamma_i]^C$, where $C = C(\Gamma) > 0$ is a constant. In geometric terms the result can be stated as follows.

Theorem 1. *Let M be a closed hyperbolic 3-orbifold. Then there exists a co-final tower of regular finite-sheeted covers $M_i \rightarrow M$ such that*

$$\mathcal{N}_{fr}(\pi_1(M_i)) \geq \text{vol}(M_i)^C,$$

where $C = C(M)$ is a positive constant which depends only on M .

The proof of the theorem is based on the previous results of Baumslag, Shalen and Wagreich [BS89, SW92], Belolipetsky [B13], and Calegari–Emerton [CE11]. Let us emphasize that although some of the results use arithmetic techniques, our theorem applies to *all* closed hyperbolic 3-orbifolds. A result of similar flavor but for another property of 3-manifold groups was obtained by Long, Lubotzky and Reid in [LLR08]. Indeed, in some parts our construction comes close to their argument.

Together with Theorem 1 we obtain the following theorem of independent interest:

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Theorem 2. *Any closed hyperbolic 3-orbifold admits a sequence of regular manifold covers $M_i \rightarrow M$ such that*

$$\mathcal{N}_{fr}(\pi_1(M_i)) \geq (1 + \varepsilon)^{\text{sys}_1(M_i)},$$

where $\varepsilon > 0$ is an absolute constant and $\text{sys}_1(M_i)$ is the length of a shortest closed geodesic in M_i .

This type of bound was stated by Gromov [G87, Section 5.3.A] for hyperbolic groups in general, but later turned into a conjecture (see [G09, Section 2.4]). We refer to the introduction of [B13] for a related discussion and some other references. In [G09], Gromov particularly mentioned that the conjecture is open even for hyperbolic 3-manifold groups. The first set of examples of hyperbolic 3-manifolds for which the conjecture is true was presented in [B13]. These examples were all arithmetic. Our theorem significantly enlarges this set.

We review the construction of covers $M_i \rightarrow M$ and prove a lower bound for their systoles in Section 2. Theorems 1 and 2 are proved in Section 3. In Section 4 we consider a generalization of the results to non-compact finite volume 3-manifolds. Their groups always contain a copy of $\mathbb{Z} \times \mathbb{Z}$, so have $\mathcal{N}_{fr} = 1$, however, we can modify the definition of the free rank so that it becomes non-trivial for the non-compact manifolds: we define $\mathcal{N}'_{fr}(\Gamma)$ to be the maximal k for which the group Γ is k -semifree, where Γ is called k -semifree if any subgroup generated by k elements is a free product of free abelian groups. With this definition at hand we can extend Gromov's conjecture to the groups of finite volume non-compact manifolds. In Section 4 we prove:

Theorem 3. *Any finite volume hyperbolic 3-orbifold admits a sequence of regular manifold covers $M_i \rightarrow M$ such that*

$$\mathcal{N}'_{fr}(\pi_1(M_i)) \geq (1 + \varepsilon)^{\text{sys}_1(M_i)},$$

where $\varepsilon > 0$ is an absolute constant.

To conclude the introduction we would like to point out one important detail. While in Theorems 2 and 3 we have an absolute constant $\varepsilon > 0$, the constant in Theorem 1 depends on the base manifold. In [B13] it was shown that in arithmetic case $C(M)$ is also bounded below by a universal positive constant. Existence of a bound of this type in general remains an open problem.

Question 1. *Do there exist an absolute constant $C_0 > 0$ such that for any M in Theorem 1 we have $C(M) \geq C_0$.*

2. PRELIMINARIES

Let $\Gamma < \text{PSL}_2(\mathbb{C})$ be a *lattice*, i.e. a finite covolume discrete subgroup. By Mostow–Prasad rigidity, Γ admits a discrete faithful representation into $\text{SL}_2(\mathbb{C})$ with the entries in some (minimal) number field E . Since Γ is finitely generated, there is a finite set

of primes S in E such that $\Gamma < \mathrm{SL}_2(\mathcal{O}_{E,S})$, where $\mathcal{O}_{E,S}$ denotes the ring of S -integers in E .

Following Calegari–Emerton [CE11], we can consider an exhaustive filtration of normal subgroups Γ_i of Γ which gives rise to a co-final tower of hyperbolic 3-manifolds covering \mathbb{H}^3/Γ . The subgroups Γ_i are defined as follows. From the description of Γ given above it follows that it is residually finite and for all but finitely many primes $\mathfrak{p} \in \mathcal{O}_E$ there is an injective map $\phi_{\mathfrak{p}} : \Gamma \rightarrow \mathrm{SL}_2(\hat{\mathcal{O}}_{E,\mathfrak{p}})$ (where $\hat{\mathcal{O}}_{E,\mathfrak{p}}$ denotes the \mathfrak{p} -adic completion of the ring of integers of E). Let p be a rational prime such that for any prime \mathfrak{p} in \mathcal{O}_E which divides p , the correspondent map $\phi_{\mathfrak{p}}$ is injective (this holds for almost all primes p). We can write $p\mathcal{O}_E = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_m^{e_m}$.

For any $j = 1, \dots, m$, the ring $\hat{\mathcal{O}}_{E,\mathfrak{p}_j}$ contains \mathbb{Z}_p as a subring and is a \mathbb{Z}_p -module of dimension $d_j = e_j f_j$, where f_j is the degree of the extension of residual fields $[\mathcal{O}_E/\mathfrak{p}_j : \mathbb{Z}/p\mathbb{Z}]$. If we fix j and a basis $b_1^j, \dots, b_{d_j}^j \in \hat{\mathcal{O}}_{E,\mathfrak{p}_j}$ as \mathbb{Z}_p -module, we have a natural ring homomorphism $\psi_j : \hat{\mathcal{O}}_{E,\mathfrak{p}_j} \rightarrow M_{d_j \times d_j}(\mathbb{Z}_p)$ given by $\psi_j(x) = (x_{rs})$ if $xb_s^j = \sum_{r=1}^{d_j} x_{rs} b_r^j$.

Let $\psi : \prod_{j=1}^m \mathrm{SL}_2(\hat{\mathcal{O}}_{E,\mathfrak{p}_j}) \rightarrow \mathrm{GL}_N(\mathbb{Z}_p)$ be given diagonally by the blocks ψ_1, \dots, ψ_m , where $N = 2 \sum_j d_j$. Let $\phi = \psi \circ \prod_{j=1}^m \phi_{\mathfrak{p}_j} : \mathrm{SL}_2(\mathcal{O}_{E,S}) \rightarrow \mathrm{GL}_N(\mathbb{Z}_p)$. The Zariski closure of the image of ϕ is a group $G < \mathrm{GL}_N(\mathbb{Z}_p)$ of dimension $d \geq 6$ (cf. [CE11, Example 5.7]). It is a p -adic analytic group which admits a normal exhaustive filtration

$$G_i = G \cap \ker(\mathrm{GL}_N(\mathbb{Z}_p) \rightarrow \mathrm{GL}_N(\mathbb{Z}_p/p^i\mathbb{Z}_p)).$$

This filtration gives rise to a filtration of Γ via the normal subgroups $\Gamma_i = \phi^{-1}(G_i)$. The filtration (Γ_i) is exhaustive because ϕ is injective.

Associated to each of the subgroups Γ_i of Γ is a finite-sheeted cover M_i of $M = \mathbb{H}^3/\Gamma$, and by the construction the sequence (M_i) is a co-final tower of covers of M . By Minkowski's lemma, almost all groups G_i are torsion-free, hence associated M_i are smooth hyperbolic 3-manifolds. Therefore, when it is needed we can assume that M is a manifold itself.

We will require a lower bound for the systole of M_i . Such a bound is essentially provided by Proposition 10 of [GL14], which can be seen as a generalization of a result of Margulis [M82] (see also [LLR08]). The main difference is that we do not restrict to arithmetic manifolds. The main technical difference is that while in [op. cit.] the authors consider matrices with real entries we do it for p -adic numbers, which requires replacing norm of a matrix by the height of a matrix. This technical part is more intricate, however, as it is shown below, it does not affect the main argument.

Lemma 2.1. *Suppose M is a compact manifold. Then there is a constant $c_1 = c_1(M) > 0$ such that $\mathrm{sys}_1(M_i) \geq c_1 \log n_i$, where $n_i = [\Gamma : \Gamma_i]$.*

Proof. Since M is compact, we can apply the Milnor–Schwarz lemma. Therefore, if we fix a point $o \in \mathbb{H}^3$, then Γ has a finite symmetric set of generators X such that

the map $(\Gamma, X) \rightarrow \mathbb{H}^3$ given by $\gamma \mapsto \gamma(o)$ is a (C_1, C_2) quasi-isometry. This means that for any pair $\gamma_1, \gamma_2 \in \Gamma$ we have

$$C_1 d_X(\gamma_1, \gamma_2) - C_2 \leq d(\gamma_1(o), \gamma_2(o)) \leq \frac{1}{C_1} d_X(\gamma_1, \gamma_2) + C_2,$$

where $d(\cdot, \cdot)$ denotes the distance function in \mathbb{H}^3 , $d_X(\gamma_1, \gamma_2) = |\gamma_1^{-1}\gamma_2|_X$ and $|\gamma|_X$ is the minimal length of a word in X which represents γ . For any $i \geq 1$, we define $\text{sys}(\Gamma_i, X) = \min\{d_X(1, \gamma) \mid \gamma \in \Gamma_i \setminus \{1\}\}$.

Claim 1: Let $\delta_M > 0$ be the diameter of M . For any $i \geq 1$, we have

$$\text{sys}_1(M_i) \geq C_1 \text{sys}(\Gamma_i, X) - C_2 - 2\delta_M.$$

To prove the claim, consider the Dirichlet fundamental domain $D(o)$ of Γ in \mathbb{H}^3 centered in o . It is easy to see that any point $x \in D(o)$ satisfies $d(x, o) \leq \delta_M$. Now let $\alpha_i \subset M_i$ be a closed geodesic realizing the systole of M_i . As $M_i \rightarrow M$ is a local isometry, the image of α_i in M has the same length (counted with multiplicity). Denote the image by α_i again. We can suppose that $x_i \in D(o)$ is a lift of $\alpha_i(0)$. Thus, there exists a unique nontrivial $\gamma_i \in \Gamma_i$ such that $\text{sys}_1(M_i) = d(x_i, \gamma_i(x_i))$. Note that $d(x_i, o) = d(\gamma_i(x_i), \gamma_i(o)) \leq \delta_M$, therefore, by the triangle inequality we have

$$\text{sys}_1(M_i) \geq d(o, \gamma_i(o)) - 2\delta_M \geq C_1 d_X(1, \gamma_i) - C_2 - 2\delta_M \geq C_1 \text{sys}(\Gamma_i, X) - C_2 - 2\delta_M.$$

Now our problem is reduced to proving that $\text{sys}(\Gamma_i, X)$ grows logarithmically as a function of $[\Gamma : \Gamma_i]$. In order to do so we use arithmetic of the field E in an essential way.

Let $S(E)$ be the set of all places of E , S_∞ be the set of archimedean places, and S_p be the set of places corresponding to the prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_m$, which appear in the definition of M_i . For any $x \in E$, we define the *height* of x by $H(x) = \prod_{v \in S(E)} \max\{1, |x|_v\}$. Recall that for any $x, y \in E$ and an archimedean place v , we have $|x + y|_v \leq 4 \max\{|x|_v, |y|_v\}$, and for any non-archimedean place u , we have $|x + y|_u \leq \max\{|x|_u, |y|_u\}$. Therefore, the height function satisfies $H(x + y) \leq 4^{\#S_\infty} H(x)H(y)$.

We can generalize the definition of height for matrices with entries in E . Thus, for any $M = (m_{ij}) \in \text{SL}_2(E)$, we define $H(M) = \prod_{v \in S(E)} \max\{1, |m_{ij}|_v\}$. We note that $H(M) \geq \max\{H(m_{ij})\}$.

Claim 2: For any $M, N \in \text{SL}_2(E)$, we have $H(MN) \leq 4^{\#S_\infty} H(M)H(N)$.

Indeed, any entry x of MN can be written as $x = au + bt$ with a, b entries of M and u, t entries of N . Therefore, for any $v \in S_\infty$,

$$\max\{1, |x|_v\} \leq 4 \max\{1, |a|_v, |b|_v\} \max\{1, |u|_v, |t|_v\} \leq 4 \max\{1, |m_{ij}|_v\} \max\{1, |n_{ij}|_v\}.$$

For non-archimedean places we have the same inequality without the factor 4. Now if $MN = (x_{ij})$, then these inequalities show that

$$H(MN) = \prod_{v \in S(E)} \max\{1, |x_{ij}|_v\} \leq 4^{\#S_\infty} H(M)H(N).$$

Next we want to estimate from below the height of γ for any nontrivial $\gamma \in \Gamma_i$.

Claim 3: There exists a constant $C_3 > 0$ such that for any $\gamma \in \Gamma_i \setminus \{1\}$ we have $H(\gamma) \geq C_3 p^{ni}$, where $n = [E : \mathbb{Q}]$.

Indeed, let $\gamma = \gamma_{r_1} \cdots \gamma_{r_w(\gamma)} \in \Gamma_i$ be a nontrivial element with $\gamma_{r_j} \in X$ and $w(\gamma) = d_X(1, \gamma)$. We now recall the definition of the group Γ_i . If we write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then for any $l = 1, \dots, m$ we have

$$\begin{pmatrix} \psi_l(a) & \psi_l(b) \\ \psi_l(c) & \psi_l(d) \end{pmatrix} \equiv \begin{pmatrix} I_{d_l} & 0 \\ 0 & I_{d_l} \end{pmatrix} \pmod{(p^i \mathbb{Z}_p)}.$$

By the definition of ψ_l we have that $(a-1)b_j^l, bb_j^l, cb_j^l, (d-1)b_j^l \in p^i \hat{\mathcal{O}}_{E, \mathfrak{p}_l}$ for any $1 \leq j \leq d_l$. Taking $C^* = \min_{l,j} \{|b_j^l|_{\mathfrak{p}_l}\} > 0$, we obtain

$$C^* \max\{|a-1|_{\mathfrak{p}_l}, |b|_{\mathfrak{p}_l}, |c|_{\mathfrak{p}_l}, |d-1|_{\mathfrak{p}_l}\} \leq \text{Norm}(\mathfrak{p}_l)^{-ie_l},$$

for any $l = 1, \dots, m$. This is because $|p|_{\mathfrak{p}_l} = \text{Norm}(\mathfrak{p}_l)^{-e_l}$ by definition, where for an ideal $I \subset \mathcal{O}_E$ the *norm* of I is equal to $\text{Norm}(I) = \#(\mathcal{O}_E/I)$.

Recall that the Product Formula says that for any nonzero $x \in E$ we have $\prod_v |x|_v = 1$. Since γ is nontrivial, at least one of the numbers $\{a-1, b, c, d-1\}$ is not zero. Therefore, if we apply the Product Formula for any nonzero element in this set, we obtain

$$\max\{H(a-1), H(b), H(c), H(d-1)\} \geq \prod_{l=1}^m C^* \text{Norm}(\mathfrak{p}_l)^{ie_l} = (C^*)^m p^{ni}.$$

Moreover, by the estimate of the height of a sum we have

$$\max\{H(a-1), H(b), H(c), H(d-1)\} \leq 4^{\#S_\infty} \max\{H(a), H(b), H(c), H(d)\},$$

therefore,

$$H(\gamma) \geq \max\{H(a), H(b), H(c), H(d)\} \geq \frac{(C^*)^m p^{ni}}{4^{\#S_\infty}} = C_3 p^{ni}.$$

This proves Claim 3.

We can now finish the proof of the lemma. If we take $C_4 = 4^{\#S_\infty} \max\{H(M) \mid M \in X\}$, we have

$$C_3 p^{ni} \leq H(\gamma) \leq (4^{\#S_\infty})^{d_X(1, \gamma) - 1} (\max\{H(M) \mid M \in X\})^{d_X(1, \gamma)} \leq C_4^{d_X(1, \gamma)}.$$

This estimate holds for any nontrivial $\gamma \in \Gamma_i$, hence $C_3 p^{ni} \leq C_4^{\text{sys}(\Gamma_i, X)}$ for any i . On the other hand, there exists a constant $C_5 > 0$ such that $[\Gamma : \Gamma_i] \leq C_5 p^{i \dim(G)}$. These inequalities together imply that

$$\text{sys}(\Gamma_i, X) \geq \frac{n}{\dim(G) \log(C_4)} \log([\Gamma : \Gamma_i]) + \frac{\log(C_3 C_5^{\frac{-n}{\dim(G)}})}{\log(C_4)}.$$

Since $[\Gamma : \Gamma_i] \rightarrow \infty$ and $\text{sys}_1(M_i)$ is bounded below by a positive constant, we conclude that there exists a constant $c_1 = c_1(o, \delta_M, p, \psi_1, \dots, \psi_m) = c_1(M) > 0$ such that $\text{sys}_1(M_i) \geq c_1 \log([\Gamma : \Gamma_i])$ for any $i \geq 1$. \square

Note that the constant c_1 depends on M (cf. Question 1). If M is arithmetic, then by [KSV07] we can take $c_1 = \frac{2}{3} - \epsilon$ for a small $\epsilon > 0$ assuming n_i is sufficiently large. In general case the argument of [KSV07] does not apply, while the proof of Lemma 2.1 does not provide a sufficient level of control over the constants.

3. PROOFS OF THEOREMS 1 AND 2

Following [B13], we define the *systolic genus* of a manifold M by

$$\text{sysg}(M) = \min\{g \mid \text{the fundamental group } \pi_1(M) \text{ contains } \pi_1(S_g)\},$$

where S_g denotes a closed Riemann surface of genus $g > 0$.

Let M be a closed hyperbolic 3-manifold with sufficiently large systole $\text{sys}_1(M)$. By [B13, Theorem 2.1], we have

$$(1) \quad \log \text{sysg}(M) \geq c_2 \cdot \text{sys}_1(M),$$

where $c_2 > 0$ is an absolute constant (for any $\delta > 0$, assuming $\text{sys}_1(M)$ is sufficiently large, we can take $c_2 = \frac{1}{2} - \delta$).

The second ingredient of the proof is a theorem of Calegry–Emerton [CE11], which implies that for the sequences of covers defined in Section 2 we have

$$(2) \quad \dim H_1(M_i, \mathbb{F}_p) \geq \lambda \cdot p^{(d-1)i} + O(p^{(d-2)i})$$

for some rational constant $\lambda \neq 0$. Recall that we have dimension $d = \dim(G) \geq 6$ and the degree of the covers $M_i \rightarrow M$ grows like p^{di} . Hence we can rewrite (2) in the form

$$(3) \quad \dim H_1(M_i, \mathbb{F}_p) \geq c_3 \text{vol}(M_i)^{5/6},$$

where $c_3 > 0$ is a constant depending on M and we assume that $\text{vol}(M_i)$ is sufficiently large.

We note that in contrast with the previous related work, the theorem of [CE11] applies to non-arithmetic manifolds as well as to the arithmetic ones.

Now recall a result of Baumslag–Shalen [BS89, Appendix]. They show that if $\text{sysg}(M) \geq k$ and $\dim H_1(M, \mathbb{Q}) \geq k + 1$, then $\pi_1(M)$ is k -free. In a subsequent paper [SW92], Shalen and Wagreich proved that the same conclusion holds if $\text{sysg}(M) \geq k$ and $\dim H_1(M, \mathbb{F}_p) \geq k + 2$ [loc. cit., Proposition 1.8].

We now bring all the ingredients together. Given a closed hyperbolic 3-orbifold M , for the sequence (M_i) of its manifold covers defined in Section 2 we have:

$$\begin{aligned} \text{sysg}(M_i) &\geq e^{c_2 \cdot \text{sys}_1(M_i)} \text{ (by (1))} \\ &\geq \text{vol}(M_i)^c \text{ (by Lemma 2.1);} \end{aligned}$$

and

$$\dim H_1(M_i, \mathbb{F}_p) \geq c_3 \cdot \text{vol}(M_i)^{5/6} \text{ (by (3)).}$$

Hence by the theorem from [SW92] cited above we obtain

$$\mathcal{N}_{fr}(\pi_1(M_i)) \geq \text{vol}(M_i)^C,$$

where $C = C(M) > 0$ and we assume that $\text{vol}(M_i)$ is sufficiently large. This proves Theorem 1.

For the second theorem recall that the systole of a hyperbolic 3-manifold is bounded above by the logarithm of its volume. Indeed, a manifold M with a systole $\text{sys}_1(M)$ contains a ball of radius $r = \text{sys}_1(M)/2$. The volume of a ball in \mathbb{H}^3 is given by $\text{vol}(B(r)) = \pi(\sinh(2r) - 2r)$, hence we get

$$\begin{aligned} \text{vol}(M) &\geq \pi(\sinh(\text{sys}_1(M)) - \text{sys}_1(M)) \sim \frac{\pi}{2} e^{\text{sys}_1(M)}; \\ \text{vol}(M) &\geq e^{c \cdot \text{sys}_1(M)}, \text{ as } \text{sys}_1(M) \rightarrow \infty. \end{aligned}$$

By Lemma 2.1, the systole of the covers $M_i \rightarrow M$ grows as $i \rightarrow \infty$. Therefore, we can bound both $\text{sysg}(M_i)$ and $\dim H_1(M_i, \mathbb{F}_p)$ below by an exponential function of $\text{sys}_1(M_i)$ with an absolute constant in exponent. Theorem 2 now follows immediately from the theorem of [SW92]. \square

Remark 3.1. It follows from the proof that for any $\delta > 0$, assuming $\text{sys}_1(M_i)$ is large enough, we can take ε in Theorem 2 equal to $e^{\frac{1}{2}-\delta} - 1$. The same bound applies for the constant in Theorem 3, which we prove in the next section.

4. GENERALIZATION TO FINITE VOLUME HYPERBOLIC 3-MANIFOLDS

Let $\Gamma < \text{PSL}_2(\mathbb{C})$ be a finite covolume Kleinian group. The quotient $M = \mathbb{H}^3/\Gamma$ is a finite volume orientable hyperbolic 3-orbifold, which can be either closed or non-compact with a finite number of cusps. The group Γ is a relatively hyperbolic group with respect to the cusp subgroups. In this section we discuss a generalization of Gromov's conjecture and our results to this class of groups.

We call Γ a *k-semifree group* if any subgroup of Γ generated by k elements is a free product of free abelian groups. The maximal k for which Γ is k -semifree is denoted by $\mathcal{N}'_{fr}(\Gamma)$. With this definition, we can generalize Gromov's conjecture to relatively hyperbolic groups. Although the injectivity radius of manifolds with cusps vanish, their systole is still bounded away from zero. Therefore, a natural generalization of Gromov's conjecture would be that $\mathcal{N}'_{fr}(\Gamma)$ is bounded below by an exponential

function of the systole of the associated quotient space M . Theorem 3, which we prove in this section, can be considered as an evidence for this conjecture.

We need to modify the definition of the *systolic genus* of a manifold M in the following way:

$$\text{sysg}(M) = \min\{g > 1 \mid \text{the fundamental group } \pi_1(M) \text{ contains } \pi_1(S_g)\},$$

where S_g denotes a closed Riemann surface of genus g . We excluded the genus $g = 1$ in order to adapt the definition to the non-compact finite volume 3-manifolds which otherwise would all have $\text{sysg} = 1$.

Let M be a finite volume hyperbolic 3-manifold with sufficiently large systole $\text{sys}_1(M)$. By [B13, Theorem 2.1], if M is closed, we have

$$(4) \quad \log \text{sysg}(M) \geq c_2 \cdot \text{sys}_1(M),$$

where $c_2 > 0$ is an absolute constant. We now discuss a generalization of this result to non-compact finite volume 3-manifolds. The first step in the proof of the theorem in [B13] is an application of the theorem of Schoen–Yau and Sacks–Uhlenbeck, which allows to homotop a π_1 -injective map of a surface of genus $g > 1$ into M to a minimal immersion. This result was recently generalized to the finite volume hyperbolic 3-manifolds in the work of Collin–Hauswirth–Mazet–Rosenberg [CHMR17] and Huang–Wang [HW17] (see in particular [HW17, Theorem 1.1]). So let S_g be a closed immersed least area minimal surface in M . In order to establish (4) for M we can suppose that S_g is embedded. Indeed, since $\pi_1(M)$ is *LERF* [A13, Corollary 9.4] there exists a finite covering \tilde{M} of M such that S_g is embedded and π_1 -injective in \tilde{M} . Moreover, $g \geq \text{sysg}(\tilde{M})$ and $\text{sys}_1(\tilde{M}) \geq \text{sys}_1(M)$. If S_g has no accidental parabolic curves, then the systole of S_g with respect to the induced metric satisfies $\text{sys}_1(S_g) \geq \text{sys}_1(M)$ and the rest of the proof in [B13] applies without any changes.

In the presence of accidental parabolics, we can apply the following lemma.

Lemma 4.1 (Compression Lemma). *Let M be a non-compact hyperbolic 3-manifold of finite volume. Suppose that there exists a π_1 -injective embedded closed surface $S_g \subset M$, for some genus $g \geq 2$, such that S_g has an accidental parabolic simple curve α . Then there exist disjoint tori $T_1, \dots, T_n \subset M$, one for each cusp $\mathcal{C}(T_i)$ of M , such that the compact 3-manifold $M' = M \setminus \cup_{i=1}^n \mathcal{C}(T_i)$ has a properly incompressible and boundary-incompressible surface $S_{g',p}$ with $g' \geq \frac{g}{2}$ and $1 \leq p \leq 2$.*

Proof. Suppose that α is associated to a parabolic isometry corresponding to a cusp $\mathcal{C} = T_0 \times [0, \infty)$ of M , where T_0 is a maximal torus. Since S_g is compact we can consider a torus $T = T_0 \times \{t_0\} \subset \mathcal{C}$ for some $t_0 > 0$ sufficiently large such that $S_g \subset M \setminus T_0 \times [t_0, \infty)$. We denote by $\beta \subset T$ the corresponding simple curve homotopic to α .

We first show that there exists an embedding $f : S_g \rightarrow M$ homotopic to the embedding $\iota : S_g \rightarrow M$ such that f is transversal to some torus $T_1 \subset \mathcal{C}$ and $f(S_g) \cap$

$T_1 \times [0, \infty) \subset \mathcal{C}$ is an annulus with boundary curves $f(\alpha_0), f(\alpha_1)$, where α_0, α_1 are the boundary curves of a collar neighborhood of α in S_g .

As an application of the Jaco–Shalen Annulus Theorem [Jaco80, Theorem VIII.13], there exists an embedding $H_0 : \mathbb{S}^1 \times [0, 1] \rightarrow M$ with $H_0(\theta, 0) = \alpha(\theta)$ and $H_0(\theta, 1) = \beta(\theta)$ (see [OT03, Lemma 2.1]). We can suppose that H_0 is transversal to S_g and T and is such that if we denote by \mathcal{A} the image $H_0(\mathbb{S}^1 \times [0, 1])$, then $\mathcal{A} \cap S_g = \alpha$ and $\mathcal{A} \cap M \setminus T \times [0, \infty) = \beta$.

Let \mathcal{D} be a collar neighborhood of α in S_g contained in a tubular neighborhood $\pi : E \subset M \rightarrow \mathcal{A}$ such that $\mathcal{D} \cap \mathcal{A} = \alpha$. Since $\pi : E \rightarrow \mathcal{A}$ is trivial, we can deform \mathcal{D} into E preserving the boundary and moving α along \mathcal{A} . We get a new annulus $\mathcal{D}' \subset M$ with $\partial\mathcal{D}' = \alpha_0 \cup \alpha_1$ and $\mathcal{D}' \cap T = \beta$.

Let ψ be the diffeomorphism between \mathcal{D} and \mathcal{D}' given by the deformation. We can suppose that ψ is the identity in a small neighborhood of the boundary. We now define the map $f : S_g \rightarrow M$ by $f(x) = x$ if $x \notin \mathcal{D}$ and $f(y) = \psi(y)$ if $y \in \mathcal{D}$. It is a smooth embedding homotopic to the inclusion.

By transversality, for some $0 < t_1 < t_0$ we have a torus $T_1 = T_0 \times \{t_1\}$ and a subannulus $\hat{\mathcal{D}} \subset \mathcal{D}$ such that f is transversal to T_1 and

$$f(S_g) \cap M \setminus T_1 \times [0, \infty) = f(S_g \setminus \text{int}(\hat{\mathcal{D}})) \quad \text{and} \quad f(\partial(S_g \setminus \text{int}(\hat{\mathcal{D}}))) = f(\partial\hat{\mathcal{D}}) \subset T_1.$$

This shows that embedding f has the desired properties.

Now, for the torus T_1 constructed above, there exist disjoint tori T_2, \dots, T_n in the cusps of M such that the corresponding cusps $\mathcal{C}(T_j) \cap \mathcal{C}(T_1) = \emptyset$ for all $j = 2, \dots, n$ and $f(S_g \setminus \text{int}(\hat{\mathcal{D}})) \subset M' = M \setminus \cup_{i=1}^n \mathcal{C}(T_i)$, and we have that $f(S_g \setminus \text{int}(\hat{\mathcal{D}})) \subset M'$ is a proper submanifold of M' .

Note that $f(S_g \setminus \text{int}(\hat{\mathcal{D}}))$ is connected with two boundary curves if α does not separate and has two components with a boundary curve if α separates it. In the latter case we consider the component with the maximal genus. Hence in both cases we have a surface $S_{g',p}$ with $g' \geq \frac{g}{2}$ and $1 \leq p \leq 2$ and a proper embedding $f : S_{g',p} \rightarrow M'$.

Recall that a properly embedded surface F in a compact 3-manifold N with boundary is called *boundary-compressible* if either F is a disk and F is parallel to a disk in ∂N , or F is not a disk and there exists a disk $D \subset N$ such that $D \cap F = c$ is an arc in ∂D , $D \cap \partial N = c'$ is an arc in ∂D , with $c \cap c' = \partial c = \partial c'$ and $c \cup c' = \partial D$, and either c does not separate F or c separates F into two components and the closure of neither is a disk. Otherwise, F is *boundary-incompressible* (see [Jaco80, Chapter III]).

Since $S_g \subset M$ is π_1 -injective, it follows from the definition and our construction that $S_{g',p} \subset M'$ is incompressible and boundary-incompressible. \square

We now apply to $S_{g',p}$ a result of Adams and Reid [AR00, Theorem 5.2]. Since $\text{sys}_1(M) = \text{sys}_1(M')$, it immediately implies inequality (4).

The theorem of Calegary–Emerton applies to non-cocompact groups as well as to the cocompact ones.

We finally recall a result of Anderson–Canary–Culler–Shalen [ACCS96]. They show that if $\text{sysg}(M) \geq k$ and $\dim H_1(M, \mathbb{F}_p) \geq k + 2$ for some prime p , then $\pi_1(M)$ is k -semifree [loc. cit., Corollary 7.4]. This theorem generalizes the previous results in [BS89, SW92] to non-compact hyperbolic 3-manifolds. Its proof also makes an essential use of topology of 3-manifolds.

Similar to Section 3, we bring together all the ingredients considered above.

Given a finite volume hyperbolic 3-orbifold M , for the sequence (M_i) of its manifold covers defined in Section 2 we have:

$$\text{sysg}(M_i) \geq e^{c_2 \cdot \text{sys}_1(M_i)} \text{ (by (4))},$$

and

$$\dim H_1(M_i, \mathbb{F}_p) \geq c_3 \cdot \text{vol}(M_i)^{5/6} \text{ (by Calegari–Emerton)}.$$

The fact that a manifold M with systole $\text{sys}_1(M)$ contains a ball of radius $r = \text{sys}_1(M)/2$ is not necessarily true for non-compact finite volume hyperbolic 3-manifolds but it is still possible to bound the volume by an exponential function of the systole. By Lakeland–Leininger [LL14, Theorem 1.3], we have

$$\text{vol}(M) \geq e^{c \cdot \text{sys}_1(M)}, \text{ as } \text{sys}_1(M) \rightarrow \infty$$

(with $c = \frac{3}{4} - \delta$ for any $\delta > 0$, assuming $\text{sys}_1(M)$ is sufficiently large).

Although we do not have a generalization of Lemma 2.1, we do know that $\text{sys}_1(M_i) \rightarrow \infty$ with i because the sequence of covers $M_i \rightarrow M$ is co-final. Therefore, we can bound both $\text{sysg}(M_i)$ and $\dim H_1(M_i, \mathbb{F}_p)$ below by an exponential function of $\text{sys}_1(M_i)$ with an absolute constant in exponent and Theorem 3 now follows from the theorem of [ACCS96]. \square

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