

Invariant Perfect Tensors

Youning Li,^{1,2} Muxin Han,^{3,4} Markus Grassl,^{5,6} and Bei Zeng^{7,8}

¹*Department of Physics, Tsinghua University, Beijing, People's Republic of China*

²*Collaborative Innovation Center of Quantum Matter,
Beijing 100190, People's Republic of China*

³*Department of Physics, Florida Atlantic University, FL 33431, USA*

⁴*Institut für Quantengravitation, Universität Erlangen-Nürnberg,
Staudtstr. 7/B2, 91058 Erlangen, Germany*

⁵*Institut für Optik, Information und Photonik,
Universität Erlangen-Nürnberg, 91058 Erlangen, Germany*

⁶*Max-Planck-Institut für die Physik des Lichts,
Leuchs Division, 91058 Erlangen, Germany*

⁷*Department of Mathematics & Statistics,
University of Guelph, Guelph, Ontario, Canada*

⁸*Institute for Quantum Computing, University of Waterloo, Waterloo, Ontario, Canada*

Invariant tensors are states in the $SU(2)$ tensor product representation that are invariant under the $SU(2)$ action. They play an important role in the study of loop quantum gravity. On the other hand, perfect tensors are highly entangled many-body quantum states with local density matrices maximally mixed. Recently, the notion of perfect tensors recently has attracted a lot of attention in the fields of quantum information theory, condensed matter theory, and quantum gravity. In this work, we introduce the concept of an invariant perfect tensor (IPT), which is a n -valent tensor that is both invariant and perfect. We discuss the existence and construction of IPT. For bivalent tensors, the invariant perfect tensor is the unique singlet state for each local dimension. The trivalent invariant perfect tensor also exists and is uniquely given by Wigner's $3j$ symbol. However, we show that, surprisingly, there does not exist four-valent invariant perfect tensors for any dimension. On the contrary, when the dimension is large, almost all invariant tensors are perfect asymptotically, which is a consequence of the phenomenon of concentration of measure for multipartite quantum states.

I. INTRODUCTION

An invariant n -valent tensor ψ is a state in the $SU(2)$ tensor product representation, and it is invariant under the $SU(2)$ action. Invariant tensors play a central role in the theory of Loop Quantum Gravity (LQG) [1–4], and particularly the structure of Spin-Networks [5–7]. The spin-network state, as a quantum state of gravity, represents the quantization of geometry. Classically an arbitrary three-dimensional geometry can be discretized and built piece by piece by

gluing polyhedral geometries.¹ The spin-network state quantizes the geometry made by polyhedra. As the building block of spin-network, the n -valent invariant tensor represents the quantum geometry of a polyhedron with n faces (explained in Appendix A).

Briefly, the invariant tensor ψ satisfies a quantum constraint equation $\sum_{i=1}^n \mathbf{J}_i \psi = 0$ where \mathbf{J}_i denotes the three $SU(2)$ Lie algebra generators acting at the i -th tensor component. This equation is a quantum analog

¹ The three-dimensional spatial geometry, quantized by spin-networks, serves as the initial data of four-dimensional gravity.

of the (flat) polyhedron closure condition $\sum_{i=1}^n \vec{A}_i = 0$ in three-dimensional space,² where each \vec{A}_i is the oriented area vector of the i -th polyhedron face. The one-to-one correspondence between a flat geometrical polyhedron and n vectors $\vec{A}_1, \dots, \vec{A}_n$ satisfying the closure condition is known as the Minkowski Theorem [11]. Comparing the classical and the quantum closure equations identifies \mathbf{J}_i to be the quantization of \vec{A}_i . The invariant tensor ψ thus represents a quantized geometrical polyhedron. In particular, each polyhedron face area is quantized by $(\vec{A}_i \cdot \vec{A}_i)^{1/2} \sim (\mathbf{J}_i \cdot \mathbf{J}_i)^{1/2} = \sqrt{j_i(j_i + 1)}$, which is the famous quantum area spectrum in LQG [12, 13]. The invariant tensor ψ with fixed j_i for each component is a quantum parameterization of the shapes of polyhedra with fixed face areas [14]. The three-dimensional quantum geometry is constructed by collecting a large number of invariant tensors representing different quantum geometrical polyhedra. It corresponds to the kinematics of four-dimensional quantum gravity.

On the other hand, another notion of special tensors, known as *perfect tensors*, has recently attracted a lot of attention from researchers in quantum information theory, condensed matter theory, and quantum gravity [15–17]. In this work, we consider n -valent tensors with dimension d for each component (i.e., n -qudit quantum states). In this setting, the perfect tensor is a highly entangled many-body quantum state, where its reduced density matrix of any part of the system, involving up to half of the total number of particles of the system, is maximally mixed.

In terms of quantum error-correcting codes, a perfect tensor is a code with large

code distance that is half of the system size. Intimate connections between quantum error-correcting codes, perfect tensors, information scrambling in chaotic many-body quantum systems, and systems with holographic duals, have recently been revealed.

Perfect tensors have been employed to construct the *Tensor Network* as a Conformal Field Theory (CFT) ground state, which realizes the AdS/CFT correspondence [16, 18]. In particular, the perfect tensor network provides an interesting illustration of how the Ryu-Takayanagi formula of Holographic Entanglement Entropy (HEE)³ emerges from many body quantum system.

Furthermore, recently it has been shown that perfect tensors represent quantum channels which are of strongest quantum chaos [15]. The quantum transition defined by perfect tensors turns out to maximally scramble the quantum information such that the initial state cannot be recovered by local measurements. In [15] it was also suggested that a perfect tensor should represent the holographic quantum system dual to the bulk quantum gravity with a black hole.⁴

Given that invariant tensors and perfect tensors relate to quantum gravity from different perspectives, it is then highly desired to incorporate the idea of perfect tensors with that of invariant tensors, a new concept that we call it Invariant Perfect Tensor (IPT). This work is also motivated by

² The closure equation generalized to constant curvature polyhedron has been proposed in [8–10].

³ In the context of bulk-boundary duality, Ryu-Takayanagi Formula conjectures that the CFT entanglement entropy of a spatial region A is proportional the minimal area of the bulk codim-2 surface attached to ∂A [19].

⁴ The recent AdS/CFT computation reveals that a black hole should be dual to a quantum system of fastest scrambling [20], which is consistent with the scrambling feature of perfect tensors.

the recent result in [21], in which the tensor network and HEE Ryu-Takayanagi formula emerge from LQG spin-network with invariant tensors.

The existence and construction of invariant tensors or perfect tensors are to some extent well understood. Moreover, among bivalent tensors, i.e., bipartite quantum states, the existence of IPT is also understood, which is nothing but the spin singlet state. We will show that, among trivalent tensors, invariant perfect tensors can also be constructed uniquely from Wigner's $3j$ symbol. However, the existence and constructions of IPT have been unknown for $n > 3$. As a surprising result, we show that there does not exist any IPT for $n = 4$, for any local dimension d . On the other hand, however, a random 4-valent invariant tensor is nearly perfect for large d . In other words, random invariant tensors also demonstrate a similar behavior of concentration of measure of generic quantum states, although the entropy convergence rate to the maximum possible value turns out to be slower. Our method and results also shed light on more general structure of IPT of $n > 4$.

We organize our paper as follows: in Section II, we introduce basic notations and preliminaries on $SU(2)$ representations. In Section III, we discuss the construction of 3-valent IPT using Wigner's $3j$ symbol. In Section IV, we prove a no-go theorem that there does not exist 4-valent IPT tensor. In Section V, we discuss random 4-valent invariant tensors and show that they are nearly perfect in the large dimension d limit. Finally, a brief discussion will be given in Section VI.

II. NOTATIONS AND PRELIMINARIES

A multipartite quantum system of n -particles has a Hilbert space $\mathcal{H}_n = \otimes_{i=1}^n V_{j_i}$, where each V_{j_i} is a spin- j_i with dimension

$d_i = 2j_i + 1$. The spin angular momentum operators have commutation relations given by $[J^a, J^b] = i\epsilon^{abc}J^c$. An n -valent tensor is a vector $|\psi_n\rangle$ in \mathcal{H}_n .

Let the total spin operator be

$$\mathbf{J} = \sum_{i=1}^n \mathbf{J}^i. \quad (1)$$

An n -valent tensor $|\psi_n\rangle$ is invariant if it satisfies

$$\mathbf{J}|\psi_n\rangle = 0. \quad (2)$$

In the tensor product $\otimes_{i=1}^n V_{j_i}$ of n $SU(2)$ irreducible representations V_{j_i} labeled by spins j_1, \dots, j_n , the dimension of the subspace $\text{Inv}(\otimes_{i=1}^n V_{j_i})$ spanned by the invariant states is given by the following formula [22]

$$\begin{aligned} \dim[\text{Inv}(\otimes_{i=1}^n V_{j_i})] \\ = \frac{2}{\pi} \int_0^\pi d\theta \sin^2(\theta/2) \prod_{i=1}^n \frac{\sin((j_i + \frac{1}{2})\theta)}{\sin(\theta/2)}. \end{aligned}$$

For invariant n -qudit states, take $j_1 = \dots = j_n = j$ with $d = 2j + 1$.

For adding angular momentums, we use the standard Clebsch-Gordan coefficients (CGCs) that are written as

$$C_{m_1 m_2 J M}^{j_1 j_2} = \langle j_1 m_1; j_2 m_2 | J M \rangle. \quad (3)$$

We also use Wigner's $3j$ symbol that is given in terms of CGCs as

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \\ = \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_3 + 1}} \langle j_1 m_1; j_2 m_2 | j_3 m_3 \rangle. \end{aligned} \quad (4)$$

Note that in order the $3j$ symbol is nonzero, the spins j_1, j_2, j_3 have to satisfy the triangle inequality:

$$|j_1 - j_2| \leq j_3 \leq j_1 + j_2.$$

It leads to the geometrical interpretation of $3j$ symbol as a triangle in two-dimensional Euclidean space, whose three edge lengths are j_1, j_2, j_3 .

The $3j$ symbol can be chosen to be purely real, and it is invariant under an even permutation of its columns:

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} \\ &= \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix}. \end{aligned} \quad (5)$$

Moreover, the $3j$ symbol obeys the following orthogonality relation

$$\begin{aligned} \sum_{m_1 m_2} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{pmatrix} \\ = \frac{1}{(2j+1)} \delta_{jj'} \delta_{mm'}. \end{aligned} \quad (6)$$

An n -qudit invariant state is an n -valent tensor with $d_i = 2j_i + 1 = d$. An n -qudit state/tensor $|\psi_n\rangle$ is perfect if for any bipartition, whose number of particles k in the smaller part satisfies $1 \leq k \leq \lfloor n/2 \rfloor$, the entropy of the reduced density matrix is maximal. An n -qudit state $|\psi_n\rangle$ is an invariant perfect tensor (IPT) if it is both invariant and perfect. Our goal is to study the existence and construction of IPTs for n -qudit states.

III. THREE-VALENT IPT: WIGNER'S $3j$ SYMBOLS

We consider three-valent IPTs in this section, and we find that for $n = 3$ there is a unique invariant tensor of $SU(2)$ (up to a rescaling), which is also a perfect tensor.

Consider a tensor product $\otimes_{i=1}^3 V_{j_i}$ of three $SU(2)$ irreducible representations labeled by spins j_1, j_2, j_3 . It is well-known that the subspace $\text{Inv}(\otimes_{i=1}^3 V_{j_i})$ of invariant tensors is one-dimensional in the case of

rank three. The normalized invariant tensor is given by Wigner's $3j$ symbol:

$$(\psi_3)_{m_1, m_2, m_3}^{j_1, j_2, j_3} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix},$$

where the indices m_1, m_2, m_3 transform under $SU(2)$ in j_1, j_2, j_3 representations, respectively.

Now consider the state $|\psi_3\rangle$ given by

$$\sum_{m_1, m_2, m_3} (\psi_3)_{m_1, m_2, m_3}^{j_1, j_2, j_3} |j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle. \quad (7)$$

To get an invariant tensor satisfying $\mathbf{J}|\psi_3\rangle = 0$, we need to have the $3j$ symbols given by the coefficients with which three angular momenta added to zero. We now show that in this case, $|\psi_3\rangle$ is also perfect.

For any choice of two spins j_1, j_2 out of j_1, j_2, j_3 , we define the reduced density matrix $\rho_3 = \text{Tr}_{j_1, j_2} |\psi_3\rangle \langle \psi_3|$. The orthogonality relation implies that

$$\langle j_3, m_3 | \rho_3 | j_3 m'_3 \rangle = \frac{1}{(2j_3 + 1)} \delta_{m_3, m'_3}, \quad (8)$$

and hence the entanglement entropy is maximal $S_3 = \ln(2j_3 + 1)$. So we proved that ψ_3 constructed from Wigner's $3j$ symbols is an invariant perfect tensor.

For a three-qudit state, we have $j_1 = j_2 = j_3$, and $d = 2j_1 + 1$. Notice that for even d , $(\psi_3)_{m_1, m_2, m_3}^{j_1, j_2, j_3}$ is always zero, and for odd d , the invariant tensor is unique.

As the simplest example, we take $j_1 = j_2 = j_3 = 1$. The $3j$ symbol simply give the ϵ symbol (anti-symmetric tensor)

$$\begin{aligned} (\psi_3)_{m_1, m_2, m_3}^{1, 1, 1} &= \begin{pmatrix} 1 & 1 & 1 \\ m_1 & m_2 & m_3 \end{pmatrix} \\ &= \frac{1}{\sqrt{6}} \epsilon_{m_1, m_2, m_3}. \end{aligned} \quad (9)$$

It is readily checked that the reduced density matrix of any single particle is maximally mixed.

IV. FOUR-VALENT IPT: A NO-GO THEOREM

In this section, we discuss the existence of 4-valent IPT. In this case, the dimension of the invariant subspace equals the qudit dimension d . On the other hand, the perfect tensors exist for any $d > 2$, possibly with the exception of $d = 6$ [23]. Since the dimension of invariant subspace grows linearly with d , one might expect that it should be possible to at least find an IPT when d is large. However, surprisingly, it

turns out that there does not exist IPT for any d .

Theorem 1. *There does not exist 4-valent IPTs, for any d .*

To prove this theorem, we start from writing down a general form of invariant tensors. For the 4-qudit invariant tensor, by choosing a coupling scheme, we can formulate the state in terms of the Clebsch-Gordan coefficients as follows:

$$|\psi_4\rangle = \sum_{J=0}^{2j} \alpha(J) \sum_{\text{all } m_i s, M} C_{m_1 m_2 J M}^{j j} C_{m_3 m_4 J - M}^{j j} C_{M - M 0 0}^{J J} |m_1, m_2, m_3, m_4\rangle, \quad (10)$$

where all $m_i s$ run from $-j$ to j and M runs from $-J$ to J .

The perfect state condition requires that

$$\rho_{34} = \rho_{24} = \rho_{23} = \frac{1}{d^2} \mathbb{I}_{d^2}, \quad (11)$$

where $\rho_{ij} = \text{Tr}_{\overline{ij}} |\psi\rangle\langle\psi|$ is the reduced density matrix of the i, j particles, and \mathbb{I}_{d^2} is the identity matrix of size $d^2 \times d^2$, where $d = 2j + 1$.

We will show that Eq. (11) cannot be satisfied for any d . We first examine the consequence of $\rho_{34} = \frac{1}{d^2} \mathbb{I}_{d^2}$, which is given by the following lemma.

Lemma 2. *If $\rho_{34} = \frac{1}{d^2} \mathbb{I}_{d^2}$, then*

$$|\alpha(J)| = \frac{\sqrt{2J+1}}{2j+1}.$$

Proof. According to Eq. (10), the matrix element of ρ_{34} labeled by $m_3 m_4, m'_3 m'_4$ is given by

$$\sum_{J, M} \frac{|\alpha(J)|^2}{2J+1} C_{m_3 m_4 J - M}^{j j} C_{m'_3 m'_4 J - M}^{j j}. \quad (12)$$

Substituting the definition of CGCs in Eq. (3) into Eq. (12), we get

$$\begin{aligned} & \sum_{J, M} \frac{|\alpha(J)|^2}{2J+1} C_{m_3 m_4 J - M}^{j j} C_{m'_3 m'_4 J - M}^{j j} \\ &= \langle j m_3; j m_4 | \hat{O} | j m'_3; j m'_4 \rangle \\ &= \frac{1}{d^2} \delta_{m_3 m'_3} \delta_{m_4 m'_4}, \end{aligned}$$

where $\hat{O} = \sum_{J, M} \frac{|\alpha(J)|^2}{2J+1} |J, -M\rangle\langle J, -M|$. The second equality is true for any element, which means that

$$\sum_{J, M} \frac{|\alpha(J)|^2}{2J+1} |J, -M\rangle\langle J, -M| = \frac{1}{d^2} \mathbb{I}_{d^2}. \quad (13)$$

The completeness of the basis $\{|J, M\rangle\}$ implies that the identity operator has the unique decomposition as

$$\mathbb{I}_{d^2} = \sum_{J, M} |J, -M\rangle\langle J, -M|,$$

so we conclude that

$$|\alpha(J)| = \frac{\sqrt{2J+1}}{2j+1}.$$

□

By Lemma 2, we can rewrite

$$\alpha(J) = \frac{\sqrt{2J+1}}{2j+1} \omega(J),$$

where $\omega(J)$ is a phase factor.

Now we further examine the consequence of $\rho_{24} = \rho_{23} = \frac{1}{d^2} \mathbb{I}_{d^2}$, and show that no choice of $\omega(J)$ can satisfy both conditions. The key idea is the following: ρ_{24} is obtained by tracing out the particles 1, 3 from $|\psi_4\rangle$; on the other hand, ρ_{23} can be obtained by first swapping particles 3 and 4 in $|\psi_4\rangle$, then tracing out the particles 1, 3. Due to the form of $|\psi_4\rangle$ that involves Clebsch-Gordan coefficients, the permutation will result in various $(-1)^J$ factors. Consequently, we will end up with two equations for $\omega(J)$ that contradict each other, which then proves that no choice of $\omega(J)$ can lead to $\rho_{24} = \rho_{23} = \frac{1}{d^2} \mathbb{I}_{d^2}$.

To be more concrete, $\rho_{24} = \frac{1}{d^2} \mathbb{I}_{d^2}$ leads to the equation

$$\sum_J (-1)^J \omega(J) C_{-j j J 0}^{j j} C_{-j j J 0}^{j j} = e^{i\theta}, \quad (14)$$

and $\rho_{24} = \frac{1}{d^2} \mathbb{I}_{d^2}$ leads to an equation

$$\sum_J \omega(J) C_{-j j J 0}^{j j} C_{-j j J 0}^{j j} = e^{i\theta'}. \quad (15)$$

(See Appendix C for details concerning the derivation of Eqs. (14) and (15).) Now we show that these two equations cannot be satisfied simultaneously, which is given by the following lemma.

Lemma 3. $\omega(J) = 1$ is the only solution of $\omega(J)$ to the equation

$$\sum_J \omega(J) C_{-j j J 0}^{j j} C_{-j j J 0}^{j j} = 1, \quad (16)$$

when $\omega(J)$ is a phase factor.

Proof. Firstly, it can be easily checked that, $\omega(J) = 1$ is a solution. Suppose we have another phase factor $\omega_1(J)$, which satisfies Eq. (16), so

$$\sum_J [1 - \text{Re}(\omega_1(J))] C_{-j j J 0}^{j j} C_{-j j J 0}^{j j} = 0,$$

however, $C_{-j j J 0}^{j j} C_{-j j J 0}^{j j} > 0$, and $\text{Re}(\omega_1(J)) \leq 1$, for all $J = 0, 1, \dots, 2j$. This directly leads to the fact that $\omega_1(J) = 1$. □

Using Lemma 3, one sees the intrinsic contradiction between Eq. (14) and Eq. (15), so the conditions $\rho_{34} = \rho_{24} = \rho_{23} = \frac{1}{d^2} \mathbb{I}_{d^2}$ cannot be satisfied simultaneously for any d . One may easily verify that $\alpha(J) = (-1)^J \frac{\sqrt{2J+1}}{2j+1}$ does satisfy $\rho_{34} = \rho_{24} = \frac{1}{d^2} \mathbb{I}_{d^2}$, and is also the unique solution after neglecting an unimportant global phase. In other words, if $\rho_{34} = \rho_{24} = \frac{1}{d^2} \mathbb{I}_{d^2}$ for any 4-valent invariant tensor $|\psi_4\rangle$, then we cannot have $\rho_{23} = \frac{1}{d^2} \mathbb{I}_{d^2}$ at the same time. This hence proves Theorem 1.

V. RANDOM INVARIANT TENSOR AND ASYMPTOTICAL PERFECTNESS

In the last section, we have shown that there does not exist any 4-valent IPT. Then a natural question is whether there exists an invariant tensor that is ‘nearly perfect’. To examine this question, we would like to consider the limit $j \rightarrow \infty$ ($d = 2j + 1$). We know that, in this case, a random tensor exhibits the phenomenon of ‘concentration of measure’, where for any bipartition, the entanglement entropy of the reduced state is near the maximally possible, asymptotically as $j \rightarrow \infty$. Now the question becomes whether this ‘concentration of measure’ phenomenon will also show up in the space of invariant tensors. We give an affirmative answer in this section for the case of 4-valent invariant tensor.

Given an invariant tensor $|I\rangle \in \text{Inv}_{SU(2)}(V_{j_1} \otimes \cdots \otimes V_{j_4})$, we define the density matrix $\rho = |I\rangle\langle I|$. We consider an arbitrary bipartition into two pairs. Without loss of generality, we consider the reduced density matrix $\rho_{34} = \text{Tr}_{12}\rho$ by tracing out the degrees of freedom in $V_{j_1} \otimes V_{j_2}$. The second Renyi entropy S_2 of ρ_{34} is given by

$$e^{-S_2} = \frac{\text{Tr}\rho_{34}^2}{(\text{Tr}\rho_{34})^2}. \quad (17)$$

It is not hard to check that the numerator

$$Z_1 \equiv \text{Tr}\rho_{34}^2 = \text{Tr}[(\rho \otimes \rho)\mathcal{F}_{34}], \quad (18)$$

where the last trace is over the space $(V_{j_1} \otimes \cdots \otimes V_{j_4})^{\otimes 2}$. \mathcal{F}_{34} is a swap operator that swaps particles 3 and 4. The denominator of Eq. (17) can be written similarly as

$$Z_0 \equiv (\text{Tr}\rho_{34})^2 = \text{Tr}[\rho \otimes \rho]. \quad (19)$$

We randomly sample the invariant tensors $|I\rangle$ in the invariant subspace $\mathcal{H}_{\text{inv}} = \text{Inv}_{SU(2)}(V_{j_1} \otimes \cdots \otimes V_{j_4})$, and consider the average

$$\overline{Z_1} = \text{Tr}[\overline{(\rho \otimes \rho)}\mathcal{F}_{34}]. \quad (20)$$

Direct calculation by using Schur's Lemma of Haar random average [24] shows that (see Appendix D)

$$\overline{Z_1} = \frac{2 \sum_I (2I+1)^{-1}}{\dim(\mathcal{H}_{\text{inv}})^2 + \dim(\mathcal{H}_{\text{inv}})} \quad (21)$$

and $\overline{Z_0} = 1$. Therefore, the averaged second Renyi entropy is given by

$$\begin{aligned} \overline{S_2} &= -\ln \frac{\overline{Z_1}}{\overline{Z_0}} \\ &= \ln \left[\dim(\mathcal{H}_{\text{inv}})^2 + \dim(\mathcal{H}_{\text{inv}}) \right] \\ &\quad - \ln \left(2 \sum_I (2I+1)^{-1} \right). \end{aligned} \quad (22)$$

When all spins $j_1 = j_2 = j_3 = j_4 = j$ are equal, we have $\dim(\mathcal{H}_{\text{inv}}) = 2j+1$ and

$$\begin{aligned} \overline{S_2} &= \ln \left[(2j+1)^2 + (2j+1) \right] \\ &\quad - \ln \left(2 \sum_{I=0}^{2j} (2I+1)^{-1} \right). \end{aligned} \quad (23)$$

Let $j \rightarrow \infty$ asymptotically, the leading behavior of $\overline{S_2}$ is

$$\overline{S_2} \sim \ln \left[(2j+1)^2 \right]. \quad (24)$$

Although $\ln \left(2 \sum_{I=0}^{2j} (2I+1)^{-1} \right)$ is also divergent as $j \rightarrow \infty$, the divergence is much slower than $\ln \left[(2j+1)^2 \right]$, indeed,

$$\lim_{j \rightarrow \infty} \frac{\ln \left(2 \sum_{I=0}^{2j} (2I+1)^{-1} \right)}{\ln \left[(2j+1)^2 \right]} = 0. \quad (25)$$

We also estimate the fluctuation of $\overline{S_2}$. In fact, for any small $\delta > 0$, there is a large probability

$$P_\delta = 1 - \frac{3\pi^2}{\delta^2 \Lambda^2}, \quad (26)$$

which is close to 1 as $j \rightarrow \infty$ (since $\Lambda = \sum_{I=0}^{2j} \frac{1}{2I+1} \rightarrow \infty$), such that $|S_2 - \overline{S_2}| \leq \delta$, i.e., the second Renyi entropy is close to the average value $\overline{S_2} \sim \ln \left[(2j+1)^2 \right]$. The derivation of the above result is presented in Appendix E⁵).

Because the von Neumann entropy is lower bounded by the second Renyi, i.e., $S \geq S_2$, and $\ln \left[(2j+1)^2 \right]$ is the maximal value of the entanglement entropy of the 4-valent tensor state, we have for the Von Neumann entropy

$$S \sim \ln \left[(2j+1)^2 \right]. \quad (27)$$

The state is maximally entangled for any partition into two pairs, asymptotically as $j \rightarrow \infty$. Therefore, the random invariant tensor is asymptotically a perfect tensor.

⁵ The idea of the proof is similar to [25]

VI. DISCUSSION

We have introduced the concept of Invariant Perfect Tensors (IPT) and discussed their existence and construction. For 3-valent tensor, IPT exist for integer spin j and is given by the unique spin zero state whose coefficient is Wigner's $3j$ symbol. We showed that there does not exist 4-valent IPT for any single particle spin j . On the other hand, a random 4-valent invariant tensor is asymptotically perfect.

It is natural to ask about the case of $n > 4$. It turns out that the situation is more complicated, and the method used to prove Theorem 1 does not directly apply for $n > 4$. However, one may expect the permutation of particles in the invariant subspace may still cause certain contradictions such that some of the reduced density matrices cannot simultaneously be identity. Numerical results for small local dimensions and $n = 5$ and $n = 6$ indicate such contradictions. One may guess that there might be some fundamental structural reason that IPT might not exist for $n > 4$, although IPT might appear asymptotically when j is large. We leave this for future research.

Acknowledgements

We thank Jianxin Chen, Shawn Cui, Cheng Guo, Guilu Long, Dong Ruan, Nengkun Yu, Ling-Yan Hung and Yidun Wan for helpful discussions. MH acknowledges Yidun Wan and Ling-Yan Hung at Fudan University, Wei Song at Yau's Institute of Mathematics (Tsinghua University), for their hospitality during his visit. MH acknowledges support from the US National Science Foundation through grant PHY-1602867, and the Start-up Grant at Florida Atlantic University, USA. BZ is supported by NSERC and CIFAR.

Appendix A: Geometrical interpretation of invariant tensor

The origin of geometrical interpretation traces back to a classic theorem by Minkowski, which states the following: Given a set of vectors $\vec{A}_1, \dots, \vec{A}_n \in \mathbb{R}^3$ satisfying a closure condition $\sum_{i=1}^n \vec{A}_i = 0$, then there is a unique polyhedron in \mathbb{R}^3 with n faces, whose face areas is given by $|\vec{A}_i|$ and the normal of each face is given by $\vec{A}_i/|\vec{A}_i|$. Therefore, a classical polyhedron geometry can be parameterized by the oriented face area vectors $\vec{A}_1, \dots, \vec{A}_n$ subject to the closure condition.

Loop Quantum Gravity (LQG) provides the result that the polyhedron geometry can be quantized. The quantum polyhedron geometry is obtained by promoting the vectors $\vec{A}_1, \dots, \vec{A}_n$ to vector-valued operators $\hat{A}_1, \dots, \hat{A}_n$. LQG derives the commutation relation between the operators [26, 27]:

$$[\hat{A}_i^a, \hat{A}_j^b] = 8\pi i \gamma \ell_P^2 \delta_{ij} \epsilon^{abc} \hat{A}_i^c, \quad (\text{A1})$$

where $a, b, c = 1, 2, 3$ are the indices of vector components, $i, j = 1, \dots, n$ label the faces of polyhedron; $\ell_P = G_N \hbar$ is the Planck length; γ is called the Barbero-Immirzi parameter in LQG. It is easy to see that different faces correspond to different degrees of freedom, which are commutative. For a given face i , the vector components of \hat{A}_i are non-commutative. The commutation relation is the same as the commutation relation of angular momentum operator $[J^a, J^b] = i\epsilon^{abc} J^c$, or equivalently the commutation relation of the Lie algebra $\mathfrak{su}(2)$.

The Hilbert space may be chosen as a tensor product of $\text{SU}(2)$ irreps $\otimes_{i=1}^n V_{j_i}$, to represent the above operator algebra Eq. (A1). Each \hat{A}_i^a ($a = 1, 2, 3$) is represented as the $\mathfrak{su}(2)$ generator J^a acting on the i -th copy of irrep V_{j_i}

$$\hat{A}_i^a = 8\pi \gamma \ell_P^2 J^a.$$

However, recall that classically there is the closure condition constraining the data $\vec{A}_1, \dots, \vec{A}_n \in \mathbb{R}^3$. The closure condition has to be promoted to an operator constraint

$$\sum_{i=1}^n \hat{A}_i \psi = 8\pi\gamma\ell_P^2 \sum_{i=1}^n \mathbf{J}_i \psi = 0, \quad (\text{A2})$$

which is precisely Eq. (2). Solving the quantum constraint equation reduces $\otimes_{i=1}^n V_{j_i}$ to the invariant tensor subspaces $\text{Inv}(\otimes_{i=1}^n V_{j_i})$.

The invariant tensors parameterizes the quantum geometry of a polyhedron with n faces. A number of geometrical operators can be defined on the Hilbert space $\text{Inv}(\otimes_{i=1}^n V_{j_i})$. For instance, from the classical face area $|\vec{A}_i|$, we have the area operator

$$\begin{aligned} \hat{A}_i \psi &= \sqrt{\vec{A}_i \cdot \vec{A}_i} \psi = 8\pi\gamma\ell_P^2 \sqrt{\mathbf{J}^2} \psi \\ &= 8\pi\gamma\ell_P^2 \sqrt{j_i(j_i + 1)} \psi. \end{aligned} \quad (\text{A3})$$

Here we see that the spin j_i is the quantum number of the i -th face area. The discreteness of j implies that the area spectrum is discrete at the quantum level (at Planck scale). The quantum volume operator can also be defined by quantizing the classical expression of volume, e.g., for a tetrahedron $n = 4$

$$\hat{\text{Vol}}_{\text{tetrahedron}} = \frac{\sqrt{2}}{3} \sqrt{|\hat{A}_1 \cdot (\hat{A}_2 \times \hat{A}_3)|}.$$

The volume operator always commutes with the area operator. The eigenvalue problem of the volume operator can be solved in the Hilbert space $\otimes_{i=1}^n V_{j_i}$ of invariant tensors. The operator spectrum (eigenvalues) is again discrete (the volume spectrum is discrete at Planck scale). The eigenstates corresponding to different volume eigenvalues form a complete orthonormal basis of $\otimes_{i=1}^n V_{j_i}$. Here we see that at the quantum level, the invariant tensors in $\otimes_{i=1}^n V_{j_i}$ actually parameterize the different quantum

shapes of the polyhedron with the same face areas j_i . The different (quantum) shapes of polyhedron correspond to the different (quantum) volumes.

As an example, the trivalent invariant tensor $|\psi_3\rangle$ given in Eq. (7) as a perfect tensor also has an geometrical interpretation. Namely, given a pair of triangles with the same edge lengths j_1, j_2, j_3 , we pick two pairs of edges from two triangles of the same length and glue each pair. The gluing corresponds to taking the inner product \sum_{m_1, m_2} in the orthogonality relation. Because the triangle with fixed edge lengths is rigid, gluing two pairs of edges makes the last pair of edges congruent. This congruence corresponds to δ_{m_3, m'_3} in Eq. (8). From this example, we see that given the geometrical interpretation of an invariant tensor as polygon or polyhedron, the perfectness of the invariant tensor relates to certain rigidity of the polygon or polyhedron.

Appendix B: Perfect tensors as quantum error-correcting codes

An n -qudit state/tensor $|\psi_n\rangle$ is perfect if for any bipartition, whose number of particles k in the smaller part satisfies $1 \leq k \leq \lfloor n/2 \rfloor$, the entropy of the reduced density matrix is maximal. An n -qubit perfect tensor can be equivalently viewed as an $[[n, 0, \delta]]_d$ quantum error-correcting code with the code distance $\delta = \lfloor n/2 \rfloor + 1$.

For the case of n -qubits (i.e., $d = 2$), it is known that there exist $[[2, 0, 2]]_2$, $[[3, 0, 2]]_2$, $[[5, 0, 3]]_2$ and $[[6, 0, 4]]_2$ quantum codes. However the $[[n, 0, \lfloor n/2 \rfloor + 1]]_2$ code does not exist for $n = 4$ and $n > 6$. Recently, it has been shown that a code $[[7, 0, 4]]_2$ does not exist either [28].

For $d > 2$ (possibly with the exception of $d = 6$), $[[4, 0, 3]]_d$ exist, i.e., there exist perfect 4-valent tensors. When $d > 2$ is a prime power, we can use a CSS-type code derived from a classical MDS code with generator

matrix

$$G = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \alpha \end{pmatrix}, \quad (\text{B1})$$

where α is an arbitrary element of the field different from 0 and 1. When $d = d_1 d_2$ is a composite odd number or divisible by four, we can take the tensor product of codes $[[4, 0, 3]]_{d_1}$ and $[[4, 0, 3]]_{d_2}$, considered as code of length four over dimension d . Finally, when d is divisible by two, but not by four, one can use the construction given in [23] using a pair of mutually orthogonal latin squares (MOLS) of order d .

Appendix C: The derivation of Eqs. (14) and (15)

By Lemma 2, we can simply set $\alpha(J) = \frac{\sqrt{2J+1}}{2j+1} \omega(J)$, where $\omega(J)$ is a phase factor. Substituting $\alpha(J) = \frac{\sqrt{2J+1}}{2j+1} \omega(J)$, simplify $\text{Tr}_{13} |\psi_4\rangle\langle\psi_4|$ and $\text{Tr}_{14} |\psi_4\rangle\langle\psi_4|$, then we have

$$\begin{aligned} \rho_{24}^{m_2 m_4, m_2' m_4'} &= \frac{1}{d^2} \sum_{m_1, m_3} F_{m_1 m_2 m_3 m_4} F_{m_1 m_2' m_3 m_4'}^*, \\ \rho_{23}^{m_2 m_3, m_2' m_3'} &= \frac{1}{d^2} \sum_{m_1, m_4} F_{m_1 m_2 m_3 m_4} F_{m_1 m_2' m_3' m_4}^*, \end{aligned}$$

where

$$\begin{aligned} F_{m_1 m_2 m_3 m_4} &= \sum_{J, M} \omega(J) (-1)^{J-M} \\ &\times C_{m_1 m_2}^{j j}{}_{JM} C_{m_3 m_4}^{j j}{}_{J-M}. \end{aligned}$$

Consider the special case $\rho_{24}^{jj, jj}$ and $\rho_{23}^{jj, jj}$,

Eq. (11) leads to

$$\rho_{24}^{jj, jj} = \frac{1}{d^2} |F_{-j, j, -j, j}|^2 = \frac{1}{d^2}, \quad (\text{C1})$$

$$\rho_{23}^{jj, jj} = \frac{1}{d^2} |F_{-j, j, j, -j}|^2 = \frac{1}{d^2}, \quad (\text{C2})$$

where

$$\begin{aligned} F_{-j, j, -j, j} &= \sum_J \omega(J) (-1)^J C_{-j j}^{j j}{}_{J0} C_{-j j}^{j j}{}_{J0}, \\ F_{-j, j, j, -j} &= \sum_J \omega(J) C_{-j j}^{j j}{}_{J0} C_{-j j}^{j j}{}_{J0}. \end{aligned}$$

By Lemma 3, we know that Eq. (C1) and Eq. (C2) contradict each other.

Appendix D: The calculation of \bar{Z}_1 and \bar{Z}_0

In this section, we calculate the average of Z_1 and Z_0 .

By Schur's Lemma [24, 25]

$$\begin{aligned} \overline{\rho \otimes \rho} &= \int dU (U \otimes U) |0\rangle\langle 0| \otimes |0\rangle\langle 0| (U^\dagger \otimes U^\dagger) \\ &= \frac{1}{\dim(\mathcal{H}_{\text{inv}})^2 + \dim(\mathcal{H}_{\text{inv}})} (\mathcal{I} + \mathcal{F}), \end{aligned} \quad (\text{D1})$$

where $|0\rangle$ is an arbitrary reference state in \mathcal{H}_{inv} . The average is over all unitary operators U on \mathcal{H}_{inv} ; \mathcal{I} is the identity operator on $\mathcal{H}_{\text{inv}} \otimes \mathcal{H}_{\text{inv}}$, and \mathcal{F} is the swap operator

$$\begin{aligned} \mathcal{I}|I\rangle \otimes |I'\rangle &= |I\rangle \otimes |I'\rangle, \\ \mathcal{F}|I\rangle \otimes |I'\rangle &= |I'\rangle \otimes |I\rangle. \end{aligned} \quad (\text{D2})$$

The average \bar{Z}_1 is computed as follows

$$\begin{aligned}
& [\dim(\mathcal{H}_{\text{inv}})^2 + \dim(\mathcal{H}_{\text{inv}})] \overline{Z}_1 \\
&= \sum_{\vec{m}, \vec{m}'} \langle m_1; m_2; m_3; m_4 | \otimes \langle m'_1; m'_2; m'_3; m'_4 | (\mathcal{I} + \mathcal{F}) \mathcal{F}_{34} | m_1; m_2; m_3; m_4 \rangle \otimes | m'_1; m'_2; m'_3; m'_4 \rangle \\
&= \sum_{\vec{m}, \vec{m}'} \langle m_1; m_2; m_3; m_4 | \otimes \langle m'_1; m'_2; m'_3; m'_4 | (\mathcal{I} + \mathcal{F}) | m_1; m_2; m'_3; m'_4 \rangle \otimes | m'_1; m'_2; m_3; m_4 \rangle.
\end{aligned}$$

\mathcal{I} and \mathcal{F} act on the invariant tensors in $\mathcal{H}_{\text{inv}} \otimes \mathcal{H}_{\text{inv}}$. So when they acting on

$| m_1; m_2; m'_3; m'_4 \rangle \otimes | m'_1; m'_2; m_3; m_4 \rangle$, they give

$$\begin{aligned}
& (\mathcal{I} + \mathcal{F}) | m_1; m_2; m'_3; m'_4 \rangle \otimes | m'_1; m'_2; m_3; m_4 \rangle \\
&= (\mathcal{I} + \mathcal{F}) P_{\text{inv}} \otimes P_{\text{inv}} | m_1; m_2; m'_3; m'_4 \rangle \otimes | m'_1; m'_2; m_3; m_4 \rangle \\
&= \sum_{I, I'} | I \rangle I_{m_1, m_2, m'_3, m'_4}^{j_1, j_2, j_3, j_4} \otimes | I' \rangle I'_{m'_1, m'_2, m_3, m_4}{}^{j_1, j_2, j_3, j_4} + \sum_{I, I'} | I \rangle I_{m'_1, m'_2, m_3, m_4}^{j_1, j_2, j_3, j_4} \otimes | I' \rangle I'_{m_1, m_2, m'_3, m'_4}{}^{j_1, j_2, j_3, j_4},
\end{aligned}$$

where we have used I to label an orthonormal basis in \mathcal{H}_{inv} . $P_{\text{inv}} = \sum_I | I \rangle \langle I |$ is the projector onto the invariant subspace \mathcal{H}_{inv} . $I_{m_1, m_2, m'_3, m'_4}^{j_1, j_2, j_3, j_4} = \langle I | m_1, m_2, m_3, m_4 \rangle$ is the

invariant tensor component. \overline{Z}_1 is thus expressed as

$$\begin{aligned}
& [\dim(\mathcal{H}_{\text{inv}})^2 + \dim(\mathcal{H}_{\text{inv}})] \overline{Z}_1 \\
&= \sum_{\vec{m}, \vec{m}'} \left(\sum_{I, I'} (I_{m_1, m_2, m_3, m_4}^{j_1, j_2, j_3, j_4})^* I_{m_1, m_2, m'_3, m'_4}^{j_1, j_2, j_3, j_4} (I'_{m'_1, m'_2, m'_3, m'_4}{}^{j_1, j_2, j_3, j_4})^* I'_{m'_1, m'_2, m_3, m_4}{}^{j_1, j_2, j_3, j_4} \right. \\
&\quad \left. + \sum_{I, I'} (I'_{m_1, m_2, m_3, m_4}{}^{j_1, j_2, j_3, j_4})^* I'_{m'_1, m'_2, m_3, m_4}{}^{j_1, j_2, j_3, j_4} (I_{m'_1, m'_2, m'_3, m'_4}{}^{j_1, j_2, j_3, j_4})^* I_{m_1, m_2, m'_3, m'_4}^{j_1, j_2, j_3, j_4} \right). \quad (\text{D3})
\end{aligned}$$

We choose the orthonormal basis $| I \rangle$ to be

such that (as we did in Eq. (10))

$$\begin{aligned}
& I_{m_1, m_2, m_3, m_4}^{j_1, j_2, j_3, j_4} \\
&= \sum_M \frac{(-1)^{I-M}}{\sqrt{2I+1}} C_{m_1, m_2, M}^{j_1, j_2, I} C_{m_3, m_4, -M}^{j_3, j_4, I}.
\end{aligned}$$

It is straightforward to check the orthonormality $\sum_{\vec{m}} (I^{j_1, j_2, j_3, j_4}_{m_1, m_2, m_3, m_4})^* \vec{I}^{j_1, j_2, j_3, j_4}_{m_1, m_2, m_3, m_4} = \delta_{I, \vec{I}}$. Inserting into \overline{Z}_1 , we find the first term

$$\begin{aligned}
& \sum_{I, I'} \sum_{\vec{m}, \vec{m}'} (I^{j_1, j_2, j_3, j_4}_{m_1, m_2, m_3, m_4})^* I^{j_1, j_2, j_3, j_4}_{m_1, m_2, m_3, m_4} (I'^{j_1, j_2, j_3, j_4}_{m'_1, m'_2, m'_3, m'_4})^* I'^{j_1, j_2, j_3, j_4}_{m'_1, m'_2, m'_3, m'_4} \\
&= \sum_{I, I'} \sum_{\vec{m}, \vec{m}'} \sum_{M, \tilde{M}} \frac{(-1)^{2I-M-\tilde{M}}}{2I+1} C_{m_1, m_2, M}^{j_1, j_2, I} C_{m_3, m_4, -M}^{j_3, j_4, I} C_{m_1, m_2, \tilde{M}}^{j_1, j_2, I} C_{m'_3, m'_4, -\tilde{M}}^{j_3, j_4, I} \\
&\times \sum_{N, \tilde{N}} \frac{(-1)^{2I'-N-\tilde{N}}}{2I'+1} C_{m'_1, m'_2, N}^{j_1, j_2, I'} C_{m'_3, m'_4, -N}^{j_3, j_4, I'} C_{m'_1, m'_2, \tilde{N}}^{j_1, j_2, I'} C_{m'_3, m'_4, -\tilde{N}}^{j_3, j_4, I'} \\
&= \sum_{I, I'} \sum_{M, \tilde{M}} \sum_{N, \tilde{N}} \delta_{M, \tilde{M}} \delta_{N, \tilde{N}} \delta^{I, I'} \delta_{M, \tilde{N}} \delta^{I, I'} \delta_{N, \tilde{M}} (2I+1)^{-2} \\
&= \sum_I (2I+1)^{-1}. \tag{D4}
\end{aligned}$$

The second term in Eq. (D3) gives the same result. Therefore

$$\overline{Z}_1 = \frac{2 \sum_I (2I+1)^{-1}}{\dim(\mathcal{H}_{\text{inv}})^2 + \dim(\mathcal{H}_{\text{inv}})}. \tag{D5}$$

in Eq. (D3) gives

The average of Z_0 can be computed in a similar way, by removing the swap operator \mathcal{F}_{34}

$$\begin{aligned}
\overline{Z}_0 &= \frac{1}{\dim(\mathcal{H}_{\text{inv}})^2 + \dim(\mathcal{H}_{\text{inv}})} \sum_{\vec{m}, \vec{m}'} \langle \vec{m} | \otimes \langle \vec{m}' | (\mathcal{I} + \mathcal{F}) | \vec{m} \rangle \otimes | \vec{m}' \rangle \\
&= \frac{1}{\dim(\mathcal{H}_{\text{inv}})^2 + \dim(\mathcal{H}_{\text{inv}})} \sum_{\vec{m}, \vec{m}'} \left(\sum_{I, I'} (I^{j_1, j_2, j_3, j_4}_{m_1, m_2, m_3, m_4})^* I^{j_1, j_2, j_3, j_4}_{m_1, m_2, m_3, m_4} (I'^{j_1, j_2, j_3, j_4}_{m'_1, m'_2, m'_3, m'_4})^* I'^{j_1, j_2, j_3, j_4}_{m'_1, m'_2, m'_3, m'_4} \right. \\
&\quad \left. + \sum_{I, I'} (I^{j_1, j_2, j_3, j_4}_{m_1, m_2, m_3, m_4})^* I^{j_1, j_2, j_3, j_4}_{m'_1, m'_2, m'_3, m'_4} (I'^{j_1, j_2, j_3, j_4}_{m_1, m_2, m_3, m_4})^* I'^{j_1, j_2, j_3, j_4}_{m_1, m_2, m_3, m_4} \right) \\
&= 1. \tag{D6}
\end{aligned}$$

Appendix E: Bound on Fluctuations of S_2

In this section, we estimate the bound on fluctuation of the second Renyi entropy S_2

around the average \overline{S}_2 , under the asymptotical limit $j \rightarrow \infty$. Using the bound, we also show that S_2 concentrates at \overline{S}_2 with a high

probability, which is close to 1 as $j \rightarrow \infty$. The idea of derivation is similar to [25].

We consider the fluctuation:

$$\frac{(\overline{Z_1} - \overline{Z_1})^2}{\overline{Z_1}^2} = \frac{\overline{Z_1^2}}{\overline{Z_1}^2} - 1. \quad (\text{E1})$$

We compute the general average $\overline{Z_1^2} = \frac{\text{Tr}[(\rho \otimes \rho)\mathcal{F}_{34}]^2}{\text{Tr}[(\rho \otimes \rho)\mathcal{F}_{34}]^2}$ by using the following for-

mula [24, 25]

$$\overline{\rho^{\otimes 4}} = \frac{1}{\mathcal{C}_4} \sum_{\sigma \in \text{Sym}_4} \rho \in \mathcal{H}_{\text{inv}}^{\otimes 4} \otimes \mathcal{H}_{\text{inv}}^{*\otimes 4}, \quad (\text{E2})$$

where $\mathcal{C}_m = (\dim \mathcal{H}_{\text{inv}} + m - 1)! / (\dim \mathcal{H}_{\text{inv}} - 1)!$. The sum is over all permutations σ acting on $\mathcal{H}_{\text{inv}}^4$. Inserting the above formula, we have

$$\overline{Z_1^2} = \frac{1}{\mathcal{C}_4} \sum_{\sigma \in \text{Sym}_4} \sum_{\vec{m}^{(i)}} \langle \vec{m}^{(1)} | \dots \langle \vec{m}^{(4)} | \sigma \mathcal{F}_{34}^{\otimes 2} | \vec{m}^{(1)} \rangle \dots | \vec{m}^{(4)} \rangle. \quad (\text{E3})$$

The operation of $\sigma \mathcal{F}_{34}^{\otimes 2}$ gives

$$\begin{aligned} & \sigma \bigotimes_{i=1,3} \mathcal{F}_{34} |m_1^{(i)}; m_2^{(i)}; m_3^{(i)}; m_4^{(i)}\rangle |m_1^{(i+1)}; m_2^{(i+1)}; m_3^{(i+1)}; m_4^{(i+1)}\rangle \\ &= \sum_{I^{(i)}} \sigma \bigotimes_{i=1,3} |I^{(i)}\rangle \langle I^{(i)} | m_1^{(i)}; m_2^{(i)}; m_3^{(i+1)}; m_4^{(i+1)}\rangle \otimes |I^{(i+1)}\rangle \langle I^{(i+1)} | m_1^{(i+1)}; m_2^{(i+1)}; m_3^{(i)}; m_4^{(i)}\rangle \\ &= \sum_{I^{(i)}} \sigma \bigotimes_{i=1,3} |I^{(i)}\rangle I_{m_1^{(i)} m_2^{(i)} m_3^{(i+1)} m_4^{(i+1)}}^{(i)} \otimes |I^{(i+1)}\rangle I_{m_1^{(i+1)} m_2^{(i+1)} m_3^{(i)} m_4^{(i)}}^{(i+1)} \\ &= \sum_{I^{(i)}} \bigotimes_{i=1,3} |I^{\sigma(i)}\rangle \otimes |I^{\sigma(i+1)}\rangle \prod_{i \text{ even}} I_{m_1^{(i)} m_2^{(i)} m_3^{(i+1)} m_4^{(i+1)}}^{(i)} I_{m_1^{(i+1)} m_2^{(i+1)} m_3^{(i)} m_4^{(i)}}^{(i+1)}. \end{aligned} \quad (\text{E4})$$

Taking the inner product, we obtain

$$\begin{aligned} \overline{Z_1^2} &= \frac{1}{\mathcal{C}_4} \sum_{\sigma \in \text{Sym}_4} \sum_{\vec{m}^{(i)}} \sum_{I^{(i)}} \\ & \prod_{i=1,3} I_{m_1^{(i)} m_2^{(i)} m_3^{(i)} m_4^{(i)}}^* \sigma(i) I_{m_1^{(i+1)} m_2^{(i+1)} m_3^{(i+1)} m_4^{(i+1)}}^* \sigma(i+1) I_{m_1^{(i)} m_2^{(i)} m_3^{(i+1)} m_4^{(i+1)}}^{(i)} I_{m_1^{(i+1)} m_2^{(i+1)} m_3^{(i)} m_4^{(i)}}^{(i+1)}. \end{aligned} \quad (\text{E5})$$

Summation over \vec{m}^i yields

$$\begin{aligned}
& \sum_{\vec{m}^{(i)}, \vec{m}^{(i+1)}} I_{m_1^{(i)} m_2^{(i)} m_3^{(i)} m_4^{(i)}}^* \sigma^{(i)} I_{m_1^{(i+1)} m_2^{(i+1)} m_3^{(i+1)} m_4^{(i+1)}}^* \sigma^{(i+1)} I_{m_1^{(i)} m_2^{(i)} m_3^{(i)} m_4^{(i)}}^{(i)} I_{m_1^{(i+1)} m_2^{(i+1)} m_3^{(i+1)} m_4^{(i+1)}}^{(i+1)} \\
&= \sum_{\vec{m}^{(i)}, \vec{m}^{(i+1)}} \sum_{M^{\sigma^{(i)}}} \frac{(-1)^{I^{\sigma^{(i)}} - M^{\sigma^{(i)}}}}{\sqrt{2I^{\sigma^{(i)}} + 1}} C_{m_1^{(i)} m_2^{(i)} M^{\sigma^{(i)}}}^{j_1, j_2, I^{\sigma^{(i)}}} C_{m_3^{(i)} m_4^{(i)} - M^{\sigma^{(i)}}}^{j_3, j_4, I^{\sigma^{(i)}}} \\
& \quad \sum_{M^{(i)}} \frac{(-1)^{I^{(i)} - M^{(i)}}}{\sqrt{2I^{(i)} + 1}} C_{m_1^{(i)} m_2^{(i)} M^{(i)}}^{j_1, j_2, I^{(i)}} C_{m_3^{(i+1)} m_4^{(i+1)} - M^{(i)}}^{j_3, j_4, I^{(i)}} \\
& \quad \sum_{M^{\sigma^{(i+1)}}} \frac{(-1)^{I^{\sigma^{(i+1)}} - M^{\sigma^{(i+1)}}}}{\sqrt{2I^{\sigma^{(i+1)}} + 1}} C_{m_1^{(i+1)} m_2^{(i+1)} M^{\sigma^{(i+1)}}}^{j_1, j_2, I^{\sigma^{(i+1)}}} C_{m_3^{(i+1)} m_4^{(i+1)} - M^{\sigma^{(i+1)}}}^{j_3, j_4, I^{\sigma^{(i+1)}}} \\
& \quad \sum_{M^{(i+1)}} \frac{(-1)^{I^{(i+1)} - M^{(i+1)}}}{\sqrt{2I^{(i+1)} + 1}} C_{m_1^{(i+1)} m_2^{(i+1)} M^{(i+1)}}^{j_1, j_2, I^{(i+1)}} C_{m_3^{(i)} m_4^{(i)} - M^{(i+1)}}^{j_3, j_4, I^{(i+1)}} \\
&= \sum_{M^{\sigma^{(i)}}, M^{(i)}, M^{\sigma^{(i+1)}}, M^{(i+1)}} \frac{1}{(2I^{(i)} + 1)(2I^{(i+1)} + 1)} \delta^{I^{\sigma^{(i)}} I^{(i)}} \delta^{I^{\sigma^{(i+1)}} I^{(i+1)}} \delta^{I^{\sigma^{(i)}} I^{(i+1)}} \delta^{I^{(i)} I^{\sigma^{(i+1)}}} \\
& \quad \times \delta_{M^{\sigma^{(i)}} M^{(i)}} \delta_{M^{\sigma^{(i+1)}} M^{(i+1)}} \delta_{M^{\sigma^{(i)}} M^{(i+1)}} \delta_{M^{(i)} M^{\sigma^{(i+1)}}} \\
&= \frac{1}{(2I^{(i)} + 1)^2} \delta^{I^{\sigma^{(i)}} I^{(i)}} \delta^{I^{\sigma^{(i+1)}} I^{(i+1)}} \delta^{I^{(i)} I^{(i+1)}} \delta^{I^{(i)} I^{(i+1)}} \sum_{M^{(i)}, M^{(i+1)}} \delta_{M^{(i)} M^{(i+1)}} \delta_{M^{(i)} M^{(i+1)}} \\
&= \frac{1}{(2I^{(i)} + 1)} \delta^{I^{\sigma^{(i)}} I^{(i)}} \delta^{I^{\sigma^{(i+1)}} I^{(i+1)}} \delta^{I^{(i)} I^{(i+1)}} \delta^{I^{(i)} I^{(i+1)}}. \tag{E6}
\end{aligned}$$

Inserting the result into \overline{Z}_1^2 gives

$$\begin{aligned}
\overline{Z}_1^2 &= \frac{1}{\mathcal{C}_4} \sum_{\sigma \in \text{Sym}_4} \sum_{I^{(i)}} \prod_{i=1,3} \frac{1}{(2I^{(i)} + 1)} \delta^{I^{\sigma^{(i)}} I^{(i)}} \delta^{I^{\sigma^{(i+1)}} I^{(i+1)}} \delta^{I^{(i)} I^{(i+1)}} \\
&= \frac{1}{\mathcal{C}_4} \left[4 \left(\sum_{I=0}^{2j} \frac{1}{2I+1} \right)^2 + 3! \sum_{I=0}^{2j} \frac{1}{(2I+1)^2} \right]. \tag{E7}
\end{aligned}$$

We have set $j_1 = j_2 = j_3 = j_4 = j$. As $j \rightarrow \infty$, $\sum_{I=0}^{2j} \frac{1}{2I+1}$ is divergent, while $\sum_{I=0}^{\infty} \frac{1}{(2I+1)^2} = \frac{\pi^2}{8}$, and $\sum_{I=0}^{2j} \frac{1}{(2I+1)^{m \geq 2}}$ are all convergent. Therefore if we denote by $\Lambda \equiv \sum_{I=0}^{2j} \frac{1}{2I+1}$,

Given that $\frac{(\mathcal{C}_2)^2}{\mathcal{C}_4} < 1$,

$$\frac{\overline{Z}_1^2}{Z_1^2} = \frac{(\mathcal{C}_2)^2}{\mathcal{C}_4} \left[1 + \frac{3\pi^2}{16} \left(\frac{1}{\Lambda} \right)^2 \right]. \tag{E8}$$

$$\overline{\left(\frac{Z_1}{Z_1} - 1 \right)^2} = \frac{\overline{Z}_1^2}{Z_1^2} - 1 < \frac{3\pi^2}{16} \left(\frac{1}{\Lambda} \right)^2 \tag{E9}$$

By Markov's inequality,

$$\begin{aligned} & \text{Prob} \left(\left| \frac{Z_1}{\overline{Z}_1} - 1 \right| \geq \frac{\delta}{4} \right) \\ & \leq \frac{\overline{\left(\frac{Z_1}{\overline{Z}_1} - 1 \right)^2}}{\left(\frac{\delta}{4} \right)^2} < \frac{3\pi^2}{\delta^2 \Lambda^2}. \end{aligned} \quad (\text{E10})$$

On the other hand, we can also show that

$$\begin{aligned} \overline{Z}_0^2 &= \frac{1}{\mathcal{C}_4} \sum_{\sigma \in \text{Sym}_4} \sum_{I^{(i)}} \delta^{I^{\sigma(1)}, I^{(1)}} \delta^{I^{\sigma(2)}, I^{(2)}} \delta^{I^{\sigma(3)}, I^{(3)}} \delta^{I^{\sigma(4)}, I^{(4)}} \\ &\simeq \frac{1}{\mathcal{C}_4} [(2j+1)^4 + 6(2j+1)^3 + 10(2j+1)^2 + 7(2j+1)]. \end{aligned} \quad (\text{E11})$$

Therefore as $j \rightarrow \infty$

$$\overline{\left(\frac{Z_0}{\overline{Z}_0} - 1 \right)^2} < \frac{3}{j} + O(j^{-2}) \quad (\text{E12})$$

and

$$\begin{aligned} & \text{Prob} \left(\left| \frac{Z_0}{\overline{Z}_0} - 1 \right| \geq \frac{\delta}{4} \right) \\ & < \frac{16}{\delta^2} \left[\frac{3}{j} + O(j^{-2}) \right] < \frac{3\pi^2}{\delta^2 \Lambda^2}. \end{aligned} \quad (\text{E13})$$

The bounds Eq. (E10) and (E13) imply that with the probability of at least $1 - \frac{3\pi^2}{\delta^2 \Lambda^2}$, we have $\left| \frac{Z_{0,1}}{\overline{Z}_{0,1}} - 1 \right| \leq \frac{\delta}{4}$. Then we have

$$\begin{aligned} |S_2 - \overline{S}_2| &= \left| \ln \frac{Z_1}{Z_0} - \ln \frac{\overline{Z}_1}{\overline{Z}_0} \right| \\ &= \left| \ln \frac{Z_1}{\overline{Z}_1} - \ln \frac{Z_0}{\overline{Z}_0} \right| \\ &\leq \left| \ln \frac{Z_1}{\overline{Z}_1} \right| + \left| \ln \frac{Z_0}{\overline{Z}_0} \right| \\ &\leq \frac{\delta}{2} + \frac{\delta}{2} = \delta, \end{aligned} \quad (\text{E14})$$

where we have used that for $\delta \leq 2$, $|\ln(1 \pm \delta/4)| \leq \delta/2$.

Therefore we have shown that for any small $\delta > 0$, there is a large probability

$$P_\delta = 1 - \frac{3\pi^2}{\delta^2 \Lambda^2}, \quad (\text{E15})$$

which is close to 1 as $j \rightarrow \infty$ (since $\Lambda = \sum_{I=0}^{2j} \frac{1}{2I+1} \rightarrow \infty$), such that $|S_2 - \overline{S}_2| \leq \delta$, i.e., the second Renyi entropy is close to the average value $\overline{S}_2 \sim \ln[(2j+1)^2]$.

-
- [1] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, 2007).
- [2] M. Han, W. Huang, and Y. Ma, *Int. J. Mod. Phys. D* **16**, 1397 (2007).
- [3] A. Ashtekar and J. Lewandowski, *Class. Quant. Grav.* **21**, R53 (2004).
- [4] C. Rovelli and F. Vidotto, *Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spin-foam Theory* (Cambridge University Press, 2014).
- [5] R. Penrose, in *Quantum Theory and Beyond*, edited by T. Bastin (Cambridge University Press, Cambridge, 1971).
- [6] C. Rovelli and L. Smolin, *Phys. Rev. D* **52**, 5743 (1995).
- [7] J. C. Baez, *Adv. Math.* **117**, 253 (1996).
- [8] H. M. Haggard, M. Han, and A. Riello, *Annales Henri Poincaré* **17**, 2001 (2016).
- [9] H. M. Haggard, M. Han, W. Kaminski, and A. Riello, *Nucl. Phys. B* **900**, 1 (2015).
- [10] C. Charles and E. R. Livine, *Class. Quant. Grav.* **32**, 135003 (2015).
- [11] H. Minkowski, *Ausgewählte Arbeiten zur Zahlentheorie und zur Geometrie* (Springer, Wien, 1989).
- [12] C. Rovelli and L. Smolin, *Nuclear Physics B*, pp. 593–619.
- [13] A. Ashtekar and J. Lewandowski, *Class. Quant. Grav.* **14**, A55 (1997).
- [14] E. Bianchi, P. Dona, and S. Speziale, *Phys. Rev. D* **83**, 044035 (2011).
- [15] P. Hosur, X.-L. Qi, D. A. Roberts, and B. Yoshida, *JHEP* **02**, 004 (2016).
- [16] F. Pastawski, B. Yoshida, D. Harlow, and J. Preskill, *JHEP* **06**, 149 (2015).
- [17] A. Almheiri, X. Dong, and D. Harlow, *JHEP* **04**, 163 (2015).
- [18] A. Bhattacharyya, Z.-S. Gao, L.-Y. Hung, and S.-N. Liu, *JHEP* **08**, 086 (2016).
- [19] S. Ryu and T. Takayanagi, *Phys. Rev. Lett.* **96**, 181602 (2006).
- [20] J. Maldacena, S. H. Shenker, and D. Stanford, *JHEP* **08**, 106 (2016).
- [21] M. Han and L.-Y. Hung (2016), arXiv:1610.02134.
- [22] E. P. Verlinde, *Nucl. Phys. B* **300**, 360 (1988).
- [23] D. Goyeneche, D. Alsina, J. I. Latorre, A. Riera, and K. Życzkowski, *Phys. Rev. A* **92**, 032316 (2015).
- [24] A. W. Harrow (2013), arXiv:1308.6595.
- [25] P. Hayden, S. Nezami, X.-L. Qi, N. Thomas, M. Walter, and Z. Yang (2016), arXiv:1601.01694.
- [26] A. Ashtekar, A. Corichi, and J. A. Zapata, *Class. Quant. Grav.* **15**, 2955 (1998).
- [27] T. Thiemann, *Class. Quant. Grav.* **18**, 3293 (2001).
- [28] F. Huber, O. Gühne, and J. Siewert (2016), arXiv:1608.06228.