

A NOTE ON THE WEAK SPLITTING NUMBER

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ABSTRACT. The weak splitting number $\text{wsp}(L)$ of a link L is the minimal number of crossing changes needed to turn L into a split union of knots. We describe conditions under which certain \mathbb{R} -valued link invariants give lower bounds on $\text{wsp}(L)$. This result is used both to obtain new bounds on $\text{wsp}(L)$ in terms of the multivariable signature and to recover known lower bounds in terms of the τ and s -invariants. We also establish new obstructions using link Floer homology and apply all these methods to compute wsp for all but two of the 130 prime links with 9 or fewer crossings.

1. INTRODUCTION

Given a link L , the *weak splitting number* $\text{wsp}(L)$ is the minimal number of crossing changes needed to convert L into a *completely split link*, i.e. into a disjoint union of knots contained in pairwise disjoint balls. This paper studies $\text{wsp}(L)$ using a variety of link invariants, including signatures and the J -function from link Floer homology.

The weak splitting number, which was first introduced by Adams [1], must not be confused with the similarly defined *splitting number* $\text{sp}(L)$; the definition of the latter only allows crossing changes between distinct components, which we call *mixed crossing changes*. While the splitting number has been intensively studied [2, 6, 7, 10, 12, 24, 19], the weak splitting number has so far attracted less attention [1, 6, 9, 32]. Indeed, $\text{wsp}(L)$ is harder to compute than $\text{sp}(L)$; one of the main reasons being that the isotopy type of the components of L is not fixed under arbitrary crossing changes. We now review known methods to study wsp and describe new ones, using prime links with 9 or fewer crossings to gauge their efficiency.

A first estimate on wsp is provided by the linking numbers: the sum of their absolute values is a lower bound. Apart from this *linking bound*, the multivariable Alexander polynomial Δ_L also gives rise to obstructions. Indeed, Borodzik, Friedl and Powell [6] proved that if L is an ℓ -component link with $\Delta_L \neq 0$, then $\ell - 1 \leq \text{wsp}(L)$, and if equality is achieved, then Δ_L must factor as

$$(1.1) \quad \Delta_L(t_1, \dots, t_\ell) = f\bar{f} \cdot \prod_{i=1}^{\ell} p_i(t_i) \cdot \prod_{i=1}^{\ell} (t_i - 1)^{m_i}.$$

Another method to compute wsp relies on *slice-torus link invariants* [23, 21, 9]. These are numerical concordance invariants that include Ozsváth and Szabó's τ -invariant [29, 8], and a normalisation of Rasmussen's s -invariant [31, 3]. More precisely, the first two authors

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observed in [9] that if ν is a slice-torus link invariant and $L = K_1 \cup \dots \cup K_\ell$, then

$$(1.2) \quad \left| \nu(L) - \sum_{i=1}^{\ell} \nu(K_i) \right| \leq \text{wsp}(L).$$

Together with the linking bound and Borodzik, Friedl and Powell's *Alexander obstruction*, the *slice-torus bound* in (1.2) allow us to determine the weak splitting number of 114 out of the 130 prime links with 9 or fewer crossings; see Table 2.

In order to determine the remaining cases, we develop novel lower bounds and obstructions. Firstly, we observe that the multivariable signature σ_L and nullity η_L of Cimasoni-Florens [13] can be leveraged to provide lower bounds on the weak splitting number.

Theorem 1.1. *If $L = K_1 \cup \dots \cup K_\ell$ is an oriented link and $\underline{\omega} = (\omega_1, \dots, \omega_\ell) \in (\mathbb{S}^1)^\ell$, then the following inequality holds:*

$$\left| \sigma_L(\underline{\omega}) - \sum_{i=1}^{\ell} \sigma_{K_i}(\omega_i) \right| + \left| \eta_L(\underline{\omega}) - \sum_{i=1}^{\ell} \eta_{K_i}(\omega_i) - \ell + 1 \right| + 3 \sum_{i < j} |\ell k(K_i, K_j)| \leq 4 \text{wsp}(L).$$

While this *signature bound* has appeared in the (unpublished) PhD thesis of the third named author [14], here we provide an alternative proof. To state the theorem on which this alternative proof is based, and which is one of the main results of this paper, we introduce some terminology. A *self crossing change* is a crossing change that involves only one component. If a link is oriented, a *positive crossing change* is one that changes a negative crossing into a positive crossing.

Theorem 1.2. *Let I be an \mathbb{R} -valued oriented link invariant. Suppose there exists $a, b, b' \in \mathbb{R}$ such that*

$$(1.3) \quad I(L) - I(L') \in [a, b] \quad \text{or} \quad I(L) - I(L') \in [-b', b']$$

depending on whether L and L' are related by a positive self crossing change or a mixed crossing change. If $\delta := (b - a) - b' \geq 0$, then for each oriented link $L = K_1 \cup \dots \cup K_\ell$

$$(1.4) \quad \left| I(L) - I\left(\bigsqcup_{i=1}^{\ell} K_i\right) \right| + \delta \sum_{i=1}^{\ell} |\ell k(K_i, K_j)| \leq (b - a) \text{wsp}(L).$$

Theorem 1.2 thus provides a template to produce lower bounds on wsp . As applications, we recover the slice-torus and signature bounds; see Corollary 3.3 for a third application.

Before returning to links with 9 or fewer crossings, we pause and compare the lower bounds that we have obtained so far. One might expect the obstructions from link homology theories to be more powerful than classical invariants. The next proposition shows that this is not always the case (cf. Propositions 2.8 and 2.7, and Remark 2.9), answering a question posed in [9, Remark 1.5].

Proposition 1.3. *The linking bound, the slice-torus bound, and the signature bound are independent. More precisely, for each of the above bounds there are infinitely many links for which the given bound is sharper than the other two. Moreover, the difference between the values of any two among the above-mentioned bounds can be arbitrarily high.*

We now return to weak splitting numbers of links with 9 or fewer crossings. Using the signature bound from Theorem 1.1, we are able to determine 6 of the missing values in Table 2. The remainder of this article develops methods to investigate the 10 remaining cases.

Inspired by [7, Theorem 7.7], we first develop new obstructions based on the J -function from link Floer homology [17, 7]. The definition and properties of the J -function are reviewed in Section 3. For the moment we only note that the J -function

$$J_L: \mathbb{Z}^\ell \longrightarrow \mathbb{Z}_{\geq 0}$$

is an invariant of the ℓ -component oriented link L . To state our result we also need the integer-valued knot concordance invariant ν^+ introduced by Hom and Z. Wu [18]. In fact, $\nu^+(K)$ can be defined as the minimal $m \in \mathbb{Z}_{\geq 0}$ such that $J_K(m) = 0$; see Section 3 as well as [28, Definition 2.12 and Proposition 2.13].

Finally, given a splitting sequence for an oriented link $L = K_1 \cup \dots \cup K_\ell$, we use s_i (resp. $m_{i,j}^+$) to denote the number of self crossing changes performed on K_i (resp. the number of positive mixed crossing changes involving both K_i and K_j).

Theorem 1.4. *Let $L = K_1 \cup \dots \cup K_\ell$ be an oriented link, and let $\{\varepsilon_{i,j}\}_{i \neq j} \subset \{0, 1\}^{\ell^2 - \ell}$ be a sequence of $\ell^2 - \ell$ integers with $\varepsilon_{i,j} + \varepsilon_{j,i} = 1$. If $J_L(v_1, \dots, v_\ell) \neq 0$ then, for some i ,*

$$(1.5) \quad v_i < \nu^+(K_i) + s_i + \sum_{j \neq i} \varepsilon_{i,j} m_{i,j}^+.$$

This new obstruction still does not allow us to determine the missing values of Table 2. Nevertheless, in Example 3.5 we describe an infinite family of links for which the J -function determines wsp , whereas the linking and signature bounds are ineffective.

To conclude the computation of wsp for the links in Table 2, Section 4 uses homotopical considerations as well as covering link calculus. Here, recall that for an ℓ -component link $L = K_1 \cup \dots \cup K_\ell$ with K_i unknotted, one can form the 2-fold cover $p: \mathbb{S}^3 \rightarrow \mathbb{S}^3$ branched along K_i . The link $\tilde{L} = p^{-1}(L \setminus K_i) \subset \mathbb{S}^3$ is called the *covering link* of L .

Proposition 1.5. *Let $L = K_1 \cup \dots \cup K_\ell$ be an ℓ -component link.*

- (1) *If L can be split via k crossing changes that do not involve an unknotted component K_i , then the corresponding covering link satisfies $\text{wsp}(\tilde{L}) \leq 2k$.*
- (2) *If L can be split via self crossing changes that do not involve K_1 , then $L \setminus K_1$ is null-homotopic in the exterior of K_1 .*
- (3) *If L has pairwise vanishing linking numbers and is not null-homotopic, then either $\text{wsp}(L) = \text{sp}(L)$ or $3 \leq \text{wsp}(L) \leq \text{sp}(L) - 1$.*

Combining Proposition 1.5 with the previously described methods, we are able to determine the weak splitting numbers of all but two of the prime links with 9 or fewer crossings. All our lower bounds and obstructions failed to determine the weak splitting number for the links $L9a29$ and $L9a30$ in Thistlethwaite's link table.

Organisation. In Section 2, we establish Theorem 1.2 and Theorem 1.1. In Section 3, we review the J -function and prove (a generalisation of) Theorem 1.4. In Section 4, we prove the homotopical obstructions of Proposition 1.5, while Section 5 lists the weak splitting numbers of all but two of the 130 links with 9 or fewer crossings.

2. LOWER BOUNDS

2.1. Linking numbers. This subsection shows that the linking numbers as well as the number of ‘‘obstructive sublinks’’ provide a lower bound on the weak splitting number.

Given a link L , a non-split sublink $J \subseteq L$ is called *obstructive* if it has vanishing linking matrix. A collection of sublinks $J_1, \dots, J_k \subseteq L$ is called an *obstructive collection* if each J_i is obstructive, and if the J_i 's are pairwise disjoint (i.e. they do not have common components). We use n_{oc} to denote the maximal number of elements among all obstructive collections of sublinks of L .

The next result shows that linking numbers and the number of obstructive links provide lower bounds on the weak splitting number. The proof is identical to that of [10, Lemma 2.1], with a small *caveat*: there exists links with an arbitrary number of components, which have pairwise linking number 0, and wsp equal to 1.

Lemma 2.1. *If a splitting sequence for an ℓ -component link $L = K_1 \cup \dots \cup K_\ell$ has s self crossing changes and m mixed crossing changes, then*

$$\sum_{i < j} |\ell k(K_i, K_j)| \leq m \quad \text{and} \quad m \equiv \sum_{i < j} \ell k(K_i, K_j) \pmod{2}.$$

Additionally, the linking numbers give a lower bound on the weak splitting number:

$$(2.1) \quad \sum_{i < j} |\ell k(K_i, K_j)| + n_{oc}(L) \leq \text{wsp}(L).$$

The following example shows how Lemma 2.1 can be applied in practice.

Example 2.2. The 3-component link $L9a47$ has $\text{wsp}(L9a47) = 3$. Indeed, the inspection of a diagram shows that $\text{wsp}(L9a47) \leq 3$. The equality follows from Lemma 2.1, since we have $\sum_{i < j} |\ell k(K_i, K_j)| = 2$, and $L9a47$ contains the Whitehead link as a sublink.

We observe that linking numbers provide a condition for the equality $\text{sp}(L) = \text{wsp}(L)$.

Remark 2.3. *We argue that if $N := \sum_{i < j} \ell k(K_i, K_j) = \text{wsp}(L)$, then $\text{wsp}(L) = \text{sp}(L)$. We need only show that $\text{sp}(L) \leq \text{wsp}(L)$. Choose a minimal splitting sequence with s self crossing changes and m mixed crossing changes. By Lemma 2.1 we have $N \leq m \leq s + m = N$. It follows that $s = 0$ and $\text{sp}(L) \leq m = \text{wsp}(L)$.*

2.2. General Bounds. We prove Theorem 1.2: any \mathbb{R} -valued oriented link invariant that has a bounded behaviour with respect to crossing changes (recall (1.3) in the statement of Theorem 1.2) provides a lower bound on the weak splitting number.

Proof of Theorem 1.2. Fix a minimal splitting sequence $L = L^{(0)}, L^{(1)}, \dots, L^{(k)}$. Denote by s the number of self crossing changes in said splitting sequence, and denote by m the number of mixed crossing changes. In particular, we have $\text{wsp}(L) = k = s + m$. Given an oriented link $J = J_1 \cup \dots \cup J_\ell$, we consider the difference

$$i(J) := I(J) - I\left(\bigsqcup J_i\right),$$

and we wish to study the behaviour of $i(L^{(j)}) - i(L^{(j-1)})$. First, when $L^{(j)}$ is obtained from $L^{(j-1)}$ by a self crossing change, we can apply (1.3) to deduce that

$$(2.2) \quad \begin{cases} -(b-a) \leq I(L^{(j)}) - I(L^{(j-1)}) - b \leq 0 & \text{if the crossing is positive,} \\ 0 \leq I(L^{(j)}) - I(L^{(j-1)}) + b \leq (b-a) & \text{if the crossing is negative.} \end{cases}$$

For each r , use $K_1^{(r)}, \dots, K_\ell^{(r)}$ to denote the components of $L^{(r)}$. Consider the links $\sqcup_r K_r^{(j)}$ and $\sqcup_r K_r^{(j-1)}$ obtained as the split unions of the components of $L^{(j)}$ and $L^{(j-1)}$, respectively. These links differ by a self crossing change, which is of the same type as the crossing

change performed to pass from $L^{(j-1)}$ to $L^{(j)}$. Thus, a second application of (1.3) gives the following inequalities:

$$(2.3) \quad \begin{cases} 0 \leq -I\left(\bigsqcup_r K_r^{(j)}\right) + I\left(\bigsqcup_r K_r^{(j-1)}\right) + b \leq (b-a) & \text{if the crossing is positive,} \\ -(b-a) \leq -I\left(\bigsqcup_r K_r^{(j)}\right) + I\left(\bigsqcup_r K_r^{(j-1)}\right) - b \leq 0 & \text{if the crossing is negative.} \end{cases}$$

Adding the inequalities in (2.2) to those in (2.3), we obtain (regardless of the type of the crossing) the inequality

$$(2.4) \quad -(b-a) \leq i(L^{(j)}) - i(L^{(j-1)}) \leq (b-a).$$

Now, assume that the crossing change between $L^{(j-1)}$ and $L^{(j)}$ involves two different components. A similar reasoning to the one above yields the following inequality:

$$(2.5) \quad -b' \leq i(L^{(j)}) - i(L^{(j-1)}) \leq b'.$$

Recall that m (resp. s) denotes the number of mixed (resp. self) crossing changes in our fixed splitting sequence for L . We have that for s indices j_1, \dots, j_s Equation (2.5) holds, while for the remaining m indices Equation (2.4) holds. Adding all these equations, and taking into account that $i(L^{(s+m)}) = 0$, we get

$$|i(L)| \leq (b-a)s + b'm = (b-a)(s+m) + (b' - (b-a))m.$$

Now, since $0 \leq (b-a) - b' = \delta$, Lemma 2.1 implies that

$$-\delta m \leq -\delta \sum_{i < j} |\ell k(K_i, K_j)|$$

and the result is an immediate consequence of the following computation:

$$|i(L)| \leq (b-a)(s+m) - \delta m \leq (b-a) \text{wsp}(L) - \delta \sum_{i < j} |\ell k(K_i, K_j)|. \quad \square$$

We can now recover [9, Theorem 1.4] from Theorem 1.2. In particular, it follows from [9, Examples 2.1, 2.2, and 2.3] that the s , τ and s_n -invariants (i.e. the \mathfrak{sl}_n -analogues of s [25, 33]) all give rise to lower bounds for wsp . The reader is referred to [9] for the definition and general properties of slice-torus link invariants.

Corollary 2.4 (Slice-torus bound). *Let ν be a slice-torus link invariant. If L is an ℓ -component oriented link, then*

$$\left| \nu(L) - \sum_{i=1}^{\ell} \nu(K_i) \right| \leq \text{wsp}(L).$$

Proof. Slice-torus link invariants are known to satisfy the hypotheses of Theorem 1.2 with $a = 0$ and $b = b' = 1$; see [9, Proposition 2.9]. Since slice-torus link invariants are, by definition, additive under disjoint unions, the corollary follows. \square

Theorem 1.2 can be used to obtain a lower bound for wsp from (finite) families of invariants which are uniformly bounded with respect to crossing changes, in the sense of (2.6). We note that the bound obtained in the following proposition is stronger than the bound obtained by applying naively Theorem 1.2 to the sum of the invariants.

Proposition 2.5. *Let $\{I_1, \dots, I_k\}$ be a family of \mathbb{R} -valued oriented link invariants. Suppose there exists $\Delta, \beta \in \mathbb{R}$ such that*

$$(2.6) \quad \sum_{j=1}^k |I_j(L) - I_j(L')| \leq \Delta \quad \text{or} \quad \sum_{j=1}^k |I_j(L) - I_j(L')| \leq \beta$$

depending on whether L and L' are related by a self crossing change or a mixed crossing change. If $\delta := 2\Delta - \beta \geq 0$, then for each oriented link $L = K_1 \cup \dots \cup K_\ell$, we have

$$(2.7) \quad \sum_{j=1}^k \left| I_j(L) - I_j \left(\bigsqcup_{i=1}^{\ell} K_i \right) \right| + \delta \sum_{i < j} |\ell k(K_i, K_j)| \leq 2\Delta \text{wsp}(L).$$

Proof. Fix $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_k) \in \{\pm 1\}^k$, and consider the sum $I(\underline{\varepsilon}) = \sum_{j=1}^k \varepsilon_j I_j$. Using (2.6), a quick verification shows that $I(\underline{\varepsilon})$ satisfies the hypothesis of Theorem 1.2 with $-a = b = \Delta$ and $b' = \beta$, for each choice of $\underline{\varepsilon}$. Applying Theorem 1.2, we deduce that the following inequality holds for every $\underline{\varepsilon}$ and every L :

$$(2.8) \quad \left| I(\underline{\varepsilon})(L) - I(\underline{\varepsilon}) \left(\bigsqcup_{i=1}^{\ell} K_i \right) \right| + \delta \sum_{i < j} |\ell k(K_i, K_j)| \leq 2\Delta \text{wsp}(L).$$

To conclude, it remains to arrange the position of the absolute values; compare (2.8) with (2.7). To achieve this, fix an arbitrary link $L = K_1 \cup \dots \cup K_\ell$, and choose any sequence $\underline{\varepsilon}$ of signs so that

$$\varepsilon_j \left(I_j(L) - I_j \left(\bigsqcup_{i=1}^{\ell} K_i \right) \right) = \left| I_j(L) - I_j \left(\bigsqcup_{i=1}^{\ell} K_i \right) \right|, \quad \text{for all } j \in \{1, \dots, k\}.$$

Since such a choice can be performed for each L , the proof of the proposition is concluded. \square

As an application of Proposition 2.5 we (re-)obtain the lower bounds on wsp that appeared in the third author's (unpublished) PhD thesis [14, Proposition 4.4.5].

We briefly recall the definition of the multivariable signature and nullity, referring to [13] for details. A C -complex for an ordered link $L = K_1 \cup \dots \cup K_\ell$ consists of a collection F of Seifert surfaces F_1, \dots, F_ℓ for the components K_1, \dots, K_ℓ that intersect only along clasps. Given a C -complex and a sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_\ell)$ of ± 1 's, there are 2^ℓ generalized Seifert matrices A^ε , which extend the usual Seifert matrix. Note that for all ε , we have $A^{-\varepsilon} = (A^\varepsilon)^T$. Using this fact, one can check that for any $\omega = (\omega_1, \dots, \omega_\ell) \in (\mathbb{S}^1)^\ell$, the following matrix is Hermitian:

$$H(\omega) = \sum_{\varepsilon} \prod_{i=1}^{\ell} (1 - \bar{\omega}_i^{\varepsilon_i}) A^\varepsilon.$$

Since $H(\omega)$ vanishes as soon as one of the coordinates of ω is equal to 1, it is convenient to restrict our attention to $\omega \in \mathbb{T}_*^\ell := (\mathbb{S}^1 \setminus \{1\})^\ell$. We use $\beta_0(F)$ to denote the number of connected components of a C -complex F .

Definition 2.6. The *multivariable signature and nullity* of an ordered link L at $\omega \in \mathbb{T}_*^\ell$ are

$$\sigma_L(\omega) := \text{sign } H(\omega); \quad \eta_L(\omega) := \text{null } H(\omega) + \beta_0(F) - 1.$$

The multivariable signature and nullity are known not to depend on the choice of the C -complex [13, Theorem 2.1]. Note that the signature is *not* a slice-torus invariant: even though it satisfies the first three axioms of [9, Definition 2], it fails to satisfy the fourth. Nonetheless, we can use Proposition 2.5 to sidestep this issue and to establish that σ_L and η_L provide lower bounds on the weak splitting number.

Proof of Theorem 1.1. The invariants $I_1(L) = \sigma_L(\omega)$ and $I_2(L) = \eta_L(\omega)$ satisfy the hypotheses of Proposition 2.5 with $\Delta = 2$ and $b' = 1$; see [12, proof of Theorem 3.1], [4,

Lemma 6.2], as well as [13, Section 5]. Furthermore, the multivariable signature and nullity behave as follows under disjoint unions (cf. [13, Proposition 2.13]):

$$\sigma_{\sqcup_{i=1}^{\ell} K_i}(\omega) = \sum_i \sigma_{K_i}(\omega_i), \quad \eta_{\sqcup_{i=1}^{\ell} K_i}(\omega) = \sum_i \eta_{K_i}(\omega_i) + \ell - 1.$$

By Proposition 2.5, the announced inequality is established. \square

2.3. Comparing the slice-torus and the signature bounds. In this subsection we compare the slice-torus bound and the signature bounds, and prove their independence.

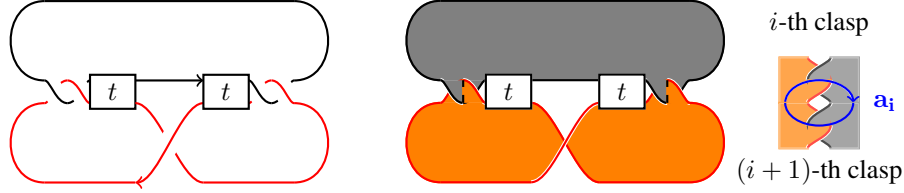


FIGURE 1. A diagram for the link L_{t+1} (left), a C-complex bounding it (centre), and the generator associated to the i -th clasp for $i \neq 2t$ (right). The boxes marked with t indicate the presence of $|t|$ full twists equal to those illustrated if $t > 0$, and their mirror if $t < 0$.

Proposition 2.7. *There is an infinite family $\{L_t\}_{t \geq 1}$ of links for which the slice-torus bound is sharp, but for which the signature and the linking bounds are not. Furthermore, the difference between the values provided for the links $\{L_t\}_{t \geq 1}$ by any two among these bounds increases linearly in t .*

Proof. Consider the diagram representing the 2-bridge link L_t illustrated in Figure 1. It can easily be seen that a self crossing change (on the unique crossing involving only one component) turns L_{t+1} into L_t . Thus, we deduce that $\text{wsp}(L_t) \leq t$. Notice that the linking number of L_t is zero, and therefore the linking bound from Lemma 2.1 is 1 (and is independent of t).

We now use slice-torus link invariants to establish the equality $\text{wsp}(L_t) = t$. A quick computation using [9, Theorem 1.3] shows that $\nu(L_t) \geq t$, for any slice-torus link invariant ν . Since the components of L_t are unknots, it follows from Corollary 2.4 that

$$\text{wsp}(L_t) = \nu(L_t) = t.$$

It remains to show that the signature bound cannot be used to determine $\text{wsp}(L_t)$. The generalised Seifert matrices corresponding to the C-complex F_t shown in Figure 1 are of size $\text{rank } {}_{\mathbb{Z}}H_1(F_t) = 2t - 1$. We deduce that

$$|\sigma_{L_t}(\omega)| + |\eta_{L_t}(\omega) - 1| \leq |\sigma_{L_t}(\omega)| + |\eta_{L_t}(\omega)| + 1 \leq (2t - 1) + 1 = 2t.$$

Thus, the lower bound in Theorem 1.1 does not exceed $\lceil t/2 \rceil$, and therefore cannot be sharp for $t \geq 2$. This concludes the proof of the proposition. \square

Next, we construct an infinite family of links for which the signature bound is stronger than the slice-torus and linking bounds.

Proposition 2.8. *There is an infinite family $\{L'_n\}$ of links for which the signature bound is sharper than the linking and the slice-torus bounds, and their difference grows linearly with n .*

Proof. A brief inspection of the diagram representing the 3-component link $L = L10a129$, see [11], shows that $\text{wsp}(L) \leq 3$. Since L contains the Whitehead link as a sublink, Lemma 2.1 implies that $\text{wsp}(L) = 3$. Consider the link L'_n obtained by connect-summing¹ n copies of L . This connected sum can be taken so that the linking number bound fails; if we take the connected sum along two components each of which is part of a Whitehead sublink in the corresponding copy of the $L10a129$, then the number of obstructive sublinks does not increase (thus the linking bound is $2n + 1$). Note however that the choice of the component where the connected sum is performed is immaterial for the remainder of the argument. For instance, regardless of this choice, we have $\text{wsp}(L'_n) \leq 3n$.

By [13, Proposition 2.12], the multivariable signature and nullity (as well as linking numbers) are additive with respect to the connected sum. Therefore, it suffices to compute the signature bound for L to obtain the bound for L'_n . Denote by $\sigma^{LT}(\omega)$ the Levine-Tristram signature, then [13, Proposition 2.5] asserts that for any link J we have

$$(2.9) \quad \sigma_J(\omega, \dots, \omega) = \sigma_J^{LT}(\omega) + \sum_{i < k} \ell k(J_i, J_k).$$

Using the Seifert matrices for $L10a29$ provided by LinkInfo [11], we see that the signature bound for L at $\omega = e^{\pi i/4}$ is $10/4$. Using the aforementioned additivity argument, and since the number of components increases by 2 at each connected sum, we get

$$\frac{10}{4}n \leq \text{wsp}(L'_n) \leq 3n.$$

It remains to argue that the slice-torus bound is not greater than $10n/4$. While slice-torus invariants are not additive under connected sums, they are known to satisfy the following sub-additivity property [9, Remark 2.8]:

$$\nu(L_1) + \nu(L_2) - 1 \leq \nu(L_1 \#_{K_1, K_2} L_2) \leq \nu(L_1) + \nu(L_2).$$

On the other hand, since L is non-split and alternating, we have²

$$(2.10) \quad \nu(L) = \frac{-\sigma(L) + \ell - 1}{2}.$$

Therefore, we obtain $\nu(L) \in \{0, 2\}$ by [11]. Consequently, regardless of this choice, we obtain $\nu(L'_n) \leq n\nu(L) \leq 2n$. This concludes the proof of the proposition. \square

We conclude this section by using the examples of Proposition 2.8 to provide examples where the linking bound is stronger than the signature bound and the slice-torus bound.

Remark 2.9. *As in Proposition 2.8, we consider the 3-component link $L := L10a129$. If, instead of performing the connected sums along one of the components of the Whitehead sublink of L (as we did in Proposition 2.8), one performs the connected sum along the third component, then the resulting link L'_n contains as many disjoint obstructive sublinks as connected summands. Thus, the linking bound gives $3n \leq \text{wsp}(L'_n)$, and the equality follows. The fact that the other two bounds cannot be sharp in this case (with an arbitrarily high difference) follows from the proof of Proposition 2.8.*

¹For the definition of connected sums of links, see for instance [16, Section 4.6].

²Alternating diagrams are homogeneous [15], and thus the bound in [9, Theorem 1.3] is sharp for these diagrams [20, Section 5 and Remark 6.3]. In particular, the value of all slice-torus invariants coincides for non-split alternating links. As $s(L) = -\sigma(L)$ for non-split alternating links, (2.10) follows from [9, Example 2.2].

3. BOUNDS FROM HEEGAARD-FLOER HOMOLOGY

We prove Theorem 1.4, which provides lower bounds for wsp via link Floer homology. First however, we briefly review the H -function of a link [17, 7], an invariant that is extracted from the minus flavor $\text{CFL}^-(L)$ of link Floer homology [29, 31, 30].

Let \mathbb{F}_2 be the field with two elements. Given an ℓ -component link L , the complex $\text{CFL}^-(L)$ is a complex of free $\mathbb{F}_2[U_1, \dots, U_\ell]$ -modules, endowed with an absolute \mathbb{Z} -grading d and a filtration for each component of L . The action of the variable U_i drops the d -grading by 2, and each filtration level by 1. If we use $\underline{\ell k}(L) \in \mathbb{Q}^\ell$ to denote the vector with $\ell k(K_i, L \setminus K_i)/2$ as its i -th entry, then the ℓ filtrations of $\text{CFL}^-(L)$ can be re-interpreted as a unique filtration \mathcal{F} indexed by an element of the lattice

$$\mathbb{H}(L) = \mathbb{Z}^\ell + \underline{\ell k}(L).$$

In fact, there is a filtered complex $\text{CFL}^-(D)$ for each Heegaard diagram D of L , and $\text{CFL}^-(L)$ is the filtered homotopy type, as a complex of $\mathbb{F}_2[U_1, \dots, U_\ell]$ -modules, of any $\text{CFL}^-(D)$ [30].

As the actions of the U_i 's on $\text{CFL}^-(L)$ are all homotopic [30], the homology of $\text{CFL}^-(L)$ can be seen as an $\mathbb{F}_2[U]$ -module, where U acts as any of the U_i . It is also known that, for each $\underline{m} \in \mathbb{H}(L)$, the homology $H_*(\mathcal{F}_{\underline{m}}\text{CFL}^-(L))$ of the \underline{m} -th filtration level decomposes into an $\mathbb{F}_2[U]$ -summand and an $\mathbb{F}_2[U]$ -torsion summand [26]. The H -function of L at $\underline{m} \in \mathbb{H}(L)$ is then defined as

$$H_L(\underline{m}) = \min \{d \mid \text{rank}_{\mathbb{F}[U]}(H_{-2d}(\mathcal{F}_{\underline{m}}\text{CFL}^-(L))) \neq 0\}.$$

This function was first introduced by Gorsky and Nemethi [17], see also [7]. It is known that H_L takes non-negative values [17, Proposition 3.10] and, as in [7], we work with the following shifted version of H_L .

Definition 3.1. The J -function of an ℓ -component link L is the function

$$J_L: \mathbb{Z}^\ell \longrightarrow \mathbb{Z}_{\geq 0} \quad \underline{m} \mapsto H_L(\underline{m} + \underline{\ell k}(L)).$$

We use $\underline{e}_i \in \mathbb{Z}^\ell$ to denote the i -th vector of the canonical basis. We collect the properties of the J -function in the following proposition; proofs can be found in [7, Propositions 3.10 and 3.11, and Theorem 6.20].

Proposition 3.2. For an oriented link L , the J -function satisfies the following properties.

- (1) For $i = 1, \dots, \ell$, and $\underline{v} \in \mathbb{Z}^\ell$, the function J_L satisfies

$$J_L(\underline{v}) \leq J_L(\underline{v} - \underline{e}_i) \leq J_L(\underline{v}) + 1.$$

- (2) Let L' be obtained from L via a positive crossing change, and let $\underline{v} \in \mathbb{Z}^\ell$.

- (a) if the crossing change is a self crossing change on the i -th component, then

$$J_{L'}(\underline{v} + \underline{e}_i) \leq J_L(\underline{v}) \leq J_{L'}(\underline{v});$$

- (b) if the crossing change is mixed and involves the i -th and the j -th components of L then, for each $* \in \{i, j\}$,

$$J_{L'}(\underline{v}) \leq J_L(\underline{v}) \leq J_{L'}(\underline{v} - \underline{e}_*).$$

- (3) If $L = K_1 \cup \dots \cup K_n$ is a completely split link, then

$$J_L(v_1, \dots, v_\ell) = \sum_{i=1}^{\ell} J_{K_i}(v_i).$$

Using all three items of Proposition 3.2 and Theorem 1.2 (with $a = -1, b = 0$ and $b' = 1$), we obtain the following result.

Corollary 3.3. *For each oriented link L , we have the following:*

$$\left| J_L(v_1, \dots, v_\ell) - \sum_{i=1}^{\ell} J_{K_i}(v_i) \right| \leq \text{wsp}(L).$$

In order to prove Theorem 1.4 however, we need one more lemma.

Lemma 3.4. *Assume an n -component link L can be split using $\text{wsp}(L) = s + m$ crossing changes with s self crossing changes and m mixed crossing changes. Then L can be converted into the split union of its components in $2s + m$ crossing changes. Furthermore, if the link L is oriented, then the $2s$ self crossing changes can be taken to be s positive and s negative crossing changes.*

Proof. Using $\text{wsp}(L)$ crossing changes, one can turn $L = K_1 \cup \dots \cup K_n$ into an n component split link $K'_1 \sqcup \dots \sqcup K'_n$ for some knots K'_1, \dots, K'_n . Let s_i be the number of crossing changes needed to pass from K_i to K'_i while splitting L . As $s = s_1 + \dots + s_n$ and s_i is greater or equal to the Gordian distance³ between K_i and K'_i , the link $K'_1 \sqcup \dots \sqcup K'_n$ can be converted into $K_1 \sqcup \dots \sqcup K_n$ using s additional crossing changes. In the case the links are oriented, then these last s self crossing changes can be taken to be of the opposite sign with respect to the s self crossing changes performed on L . \square

We now prove Theorem 1.4 from the introduction. First however, we recall some notation. Given a splitting sequence for an oriented link $L = K_1 \cup \dots \cup K_\ell$, we use s_i (resp. $m_{i,j}^+$) to denote the number of self crossing changes performed on K_i (resp. the number of positive mixed crossing changes involving both K_i and K_j).

Proof of Theorem 1.4. We prove the contrapositive. Use Lemma 3.4 to convert L into the split union of its components via $2s + m$ crossing changes, where exactly s of these $2s$ self crossing changes are negative. Applying the second item of Proposition 3.2, we deduce that

$$(3.1) \quad J_L(\underline{v}) \leq J_{\sqcup_{i=1}^{\ell} K_i} \left(\underline{v} - \sum_{i=1}^{\ell} \left[s_i + \sum_{j \neq i} \varepsilon_{i,j} m_{i,j}^+ \right] \underline{e}_i \right).$$

Recall from the introduction that if K is a knot and $m \geq \nu^+(K)$, then $J_K(m) = 0$ [28, Definition 2.12 and Proposition 2.13]. Combining this with the third item of Proposition 3.2, we see that the right hand side of (3.1) vanishes if no v_i satisfies (1.5). The assertion now follows since the J -function is non-negative. \square

Example 3.5. We use Theorem 1.4 to show that the family $L_t = K_t^1 \cup K_t^2$ of 2-bridge links from Figure 1 has $\text{wsp}(L_t) = t$. Since the L_t are L-space links, their J -functions can be recovered from the potential function [7, Corollary 3.32].⁴ Applying [7, Section 7.4], the potential function of L_t is

$$\nabla_{L_t}(t_1, t_2) = (-1)^t \sum_{|i+\frac{1}{2}|+|j+\frac{1}{2}|\leq t} (-1)^{i+j} t_1^{i+\frac{1}{2}} t_2^{j+\frac{1}{2}}.$$

³This is the minimal number of crossing changes needed to pass from one given knot to another.

⁴Borodzik and Gorsky state this in terms of a symmetrized version $\Delta_L(t_1, \dots, t_n) \in \mathbb{Z}[t_1^{\pm\frac{1}{2}}, \dots, t_n^{\pm\frac{1}{2}}]$ of the Alexander polynomial for which they additionally fix a sign [7, Subsection 2.1]. In other words, they are working with the potential function $\nabla_L(t_1, \dots, t_n)$. Furthermore [7, Equation (3.3)] implicitly makes use of [5, Theorem 1.1] to obtain an equality, instead of an equality up to signs.

If we set $\tilde{J}_{L_t}(i, j) := J_{L_t}(i, j) - J_{K_t^1}(i) - J_{K_t^2}(j)$, then applying [7, Corollary 3.32] and rearranging the sums of the corresponding generating function yields

$$\tilde{\mathbf{J}}_{L_t}(t_1, t_2) := \sum_{i,j} \tilde{J}_{L_t}(i, j) t_1^i t_2^j = \sum_{j=0}^{t-1} \sum_{i=0}^{t-j-1} (-1)^{i+j+t+1} \left(\sum_{k=0}^{2i} t_1^{k-i} \right) \left(\sum_{h=0}^{2j} t_2^{h-j} \right).$$

It follows from the above equalities and a tedious computation that the bound provided by Corollary 3.3 is at most $\lceil t/2 \rceil$; see also [22, Figure 4]. Using successively that L_t has unknotted components (as well as $J_{\bigcirc}(v) = 0$ for $v \geq 0$, equivalently $\nu^+(\bigcirc) = 0$), and the above computations, we obtain that for $r = 0, \dots, t-1$

$$(3.2) \quad \tilde{J}_{L_t}(r, t-1-r) = J_{L_t}(r, t-1-r) = 1.$$

We already showed in Proposition 2.7 that $\text{wsp}(L) \leq t$. By way of contradiction, assume that $\text{wsp}(L) \leq t-1$, so that $s_1 + s_2 + m_{1,2}^+ \leq t-1$. Since $s_1 \leq t-1$, we apply (3.2) with $r = s_1$ to obtain $J_{L_t}(s_1, t-1-s_1) = 1$. Since J_{L_t} is non-increasing (by the first item of Proposition 3.2), we deduce that $J_{L_t}(s_1, s_2 + m_{1,2}^+) \geq 1$. As $J_{L_t}(s_1, s_2 + m_{1,2}^+) \neq 0$, Theorem 1.4 applied to the sequence $(\varepsilon_{1,2}, \varepsilon_{2,1}) = (0, 1)$ implies that either $s_1 < s_1$ or $s_2 + m_{1,2}^+ < s_2 + m_{1,2}^+$. This is a contradiction in both cases and thus $\text{wsp}(L_t) = t$.

4. HOMOTOPICAL OBSTRUCTIONS

4.1. Link homotopy. We show how the homotopy type of a link provides restrictions on its weak splitting number. Here, recall that two links L and L' are link-homotopic if and only if they are related by a sequence of ambient isotopies and self crossing changes. Furthermore, a link is nullhomotopic if it is link-homotopic to an unlink.

Proposition 4.1. *If a link L has pairwise vanishing linking numbers and is not nullhomotopic, then either $\text{wsp}(L) = \text{sp}(L)$ or $3 \leq \text{wsp}(L) \leq \text{sp}(L) - 1$.*

Proof. Write $\text{wsp}(L) = s + m$, so that L can be split using s self crossing changes and m mixed crossing changes. If one can find such a sequence with $s = 0$, then there is a minimal splitting sequence without self crossing changes. Thus $\text{sp}(L) \leq m = \text{wsp}(L)$, which implies $\text{wsp}(L) = \text{sp}(L)$. Otherwise, every weak splitting sequence must have $s > 0$ and m even: indeed L has pairwise vanishing linking numbers. Since $\text{sp}(L)$ must be even and since m cannot be 0 (because L is not nullhomotopic), we immediately deduce that $\text{wsp}(L) = s + m \in \{3, \dots, \text{sp}(L) - 1\}$, concluding the proof of the proposition. \square

Since the Milnor invariants with non-repeating indices are invariant under link homotopy [27], Proposition 4.1 can easily be applied in practice.

Example 4.2. We show that the 3-component link $L = L9a54$ has $\text{wsp}(L) = 3$. Since L has vanishing pairwise linking numbers, the linking obstruction is ineffective, and in fact both the slice-torus bound and the signature bound give $2 \leq \text{wsp}(L) \leq 3$.

Since L is known to have $\text{sp}(L) = 4$ [10], if we manage to show that L is not nullhomotopic, then Proposition 4.1 will imply that $\text{wsp}(L) = 3$. As L can be obtained from the Borromean rings J via a single self crossing change, we obtain $\mu_{123}(L) = \mu_{123}(J) = 1$ and thus L is not nullhomotopic. We conclude that $\text{wsp}(L) = 3$, as claimed.

The following lemma can be used to obstruct the existence of minimal weak splitting sequences without mixed crossing changes.

Proposition 4.3. *Assume that the link L can be completely split with only self crossing changes not involving a fixed component, say K_1 . Then, $L \setminus K_1$ is null-homotopic in the complement of K_1 . In particular, each component of $L \setminus K_1$ is null-homotopic in $\mathbb{S}^3 \setminus K_1$.*

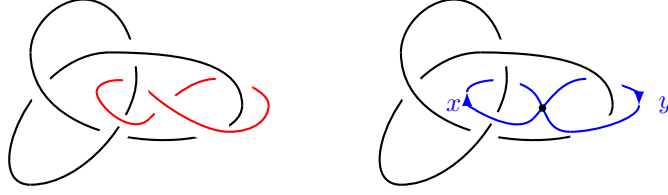


FIGURE 2. A diagram of the link $L7a3$ (left), and a pair of generators of the fundamental group of its trefoil component K_1 (right).

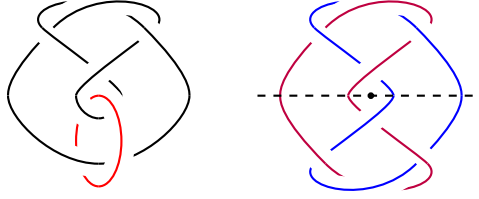


FIGURE 3. A diagram of the link $L7a3$ (left), and a diagram for \tilde{L} (right).

Proof. A self crossing change in $L \setminus K_1$ does not change its homotopy type in $\mathbb{S}^3 \setminus K_1$. As any knot in a 3-manifold that sits inside a 3-ball is null-homotopic, the result follows. \square

4.2. Covering link calculus. We use covering link calculus to study wsp . Given an n -component link $L = K_1 \cup \dots \cup K_n$ with K_i unknotted, one can form the 2-fold cover $p: S^3 \rightarrow S^3$ branched along K_i . The link $\tilde{L} = p^{-1}(L \setminus K_i)$ is called the *covering link* of L with respect to K_i . For a proof of the next result, we refer to [10, Section 3].

Proposition 4.4. *Let $L = K_1 \cup \dots \cup K_n$ be an n -component link with K_i unknotted. A crossing change not involving K_i results in two crossing changes in the covering link. In particular, if L can be split via k crossing changes not involving K_i , then $\text{wsp}(\tilde{L}) \leq 2k$.*

We show how Proposition 4.4 can be used in conjunction with Proposition 4.3: the former obstructs the existence of self crossing sequences in knotted components, while the latter obstructs the existence of self crossing changes in unknotted components.

Example 4.5. We claim that the link $L = K_1 \cup K_2 = L7a3$ in Figure 2 has $\text{wsp}(L) = 2$. First, an inspection of the diagram shows that $\text{wsp}(L) \leq 2$, and that $\ell k(K_1, K_2) = 0$. Furthermore, all the techniques illustrated in Sections 2 and 3 imply that $1 \leq \text{wsp}(L)$.

Assume, by contradiction, that $\text{wsp}(L) = 1$. Since L has vanishing linking numbers, any minimal splitting sequence is realised by a single self crossing change. First, we show that the self crossing change cannot occur within the trefoil component K_1 of L . Denote by \tilde{L} the lift of K_1 to the double cover of \mathbb{S}^3 branched along K_2 , see Figure 3. Since \tilde{L} has linking number ± 4 , Lemma 2.1 gives $\text{wsp}(\tilde{L}) \geq 4$, contradicting Proposition 4.4.

It remains to show that L cannot be split by a self crossing change within its unknotted component K_2 . By Proposition 4.3, K_2 must be trivial in $\pi_1(\mathbb{S}^3 \setminus K_1)$, which

Link as in [11]	θ	Link as in [11]	θ	Link as in [11]	θ
$L9a52\{1, 0\}$	$3\pi/97$	$L9n14\{0\}$	$5\pi/19$	$L9n17\{0\}$	$59\pi/61$
$L9n24\{1, 0\}$	$2\pi/13$	$L9n28\{0, 0\}$	$2\pi/17$		

TABLE 1. The roots of unity $\omega = e^{2i\theta}$ used to compute the signature bound of Theorem 1.1 for the entries marked with \star in Table 2.

admits $\langle x, y \mid yxy = xyx \rangle$ as a Wirtinger presentation. Here, x and y are the generators depicted in Figure 2. With respect to these generators, K_2 can be written as xy^{-1} (or $x^{-1}y$ depending on the orientation). If K_2 were nullhomotopic then $x = y$, and thus $\pi_1(\mathbb{S}^3 \setminus K_1) = \mathbb{Z}$ which is absurd since K_1 is a trefoil. Therefore, $\text{wsp}(L) = 2$.

5. THE WEAK SPLITTING NUMBER OF SMALL LINKS

Table 2 below lists $\text{wsp}(L)$ for prime links with 9 or fewer crossings. Its second column indicates which of the previously described methods we use among the following:

- (0) non-splitness: the link is non-split and has $\text{wsp}(L) \leq 1$;
- (1) the linking number bound from Lemma 2.1;
- (2) the slice-torus or signature bound, for the values of $\omega \neq 1$ used see Table 1;⁵
- (3) the Alexander polynomial obstructions from [6];
- (4) the covering link calculus obstruction from Proposition 4.4;
- (5) the homotopical considerations of Lemmas 4.1 and 4.3;
- (6) the combination of covering link calculus and the fundamental group obstruction.

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⁵In Table 1, we only list *one* variable $\omega \in S^1$ for the multivariable signature σ_L : for these links, we are using $\sigma_L(\omega, \dots, \omega)$ and its relation to the Levine-Tristram signature at ω ; recall (2.9).

name	wsp	method	name	wsp	method	name	wsp	method
L2a1	1	(0)	L9a1	1	(0)	L9a48	4	(1)
L4a1	2	(1)	L9a2	1	(0)	L9a49	4	(1)
L5a1	1	(0)	L9a3	1	(0)	L9a50	3	(2)
L6a1	2	(1)	L9a4	2	(6)	L9a51	4	(1)
L6a2	3	(1)	L9a5	2	(1)	L9a52	3	(2)*
L6a3	3	(1)	L9a6	2	(1)	L9a53	2	(4)
L6a4	2	(3)	L9a7	2	(1)	L9a54	3	(2)&(5)
L6a5	3	(1)	L9a8	2	(6)	L9a55	4	(1)
L6n1	3	(1)	L9a9	2	(4)	L9n1	2	(1)
L7a1	1	(0)	L9a10	2	(6)	L9n2	2	(6)
L7a2	2	(1)	L9a11	2	(1)	L9n3	1	(0)
L7a3	2	(6)	L9a12	2	(1)	L9n4	2	(1)
L7a4	2	(4)	L9a13	2	(1)	L9n5	2	(2)
L7a5	1	(0)	L9a14	2	(6)	L9n6	1	(0)
L7a6	2	(2)	L9a15	2	(1)	L9n7	2	(1)
L7a7	3	(1)	L9a16	2	(1)	L9n8	1	(0)
L7n1	2	(1)	L9a17	2	(6)	L9n9	2	(1)
L7n2	1	(0)	L9a18	2	(6)	L9n10	2	(1)
L8a1	1	(0)	L9a19	2	(1)	L9n11	2	(1)
L8a2	1	(0)	L9a20	2	(3)	L9n12	2	(1)
L8a3	2	(1)	L9a21	1	(0)	L9n13	1	(0)
L8a4	1	(0)	L9a22	2	(3)	L9n14	2	(2)*
L8a5	2	(1)	L9a23	3	(1)	L9n15	3	(1)
L8a6	2	(1)	L9a24	2	(3)	L9n16	3	(1)
L8a7	2	(1)	L9a25	2	(3)	L9n17	2	(2)*
L8a8	2	(2)	L9a26	2	(2)	L9n18	4	(1)
L8a9	2	(3)	L9a27	1	(0)	L9n19	4	(1)
L8a10	3	(1)	L9a28	3	(1)	L9n20	3	(1)
L8a11	3	(1)	L9a29	2/3	(2)	L9n21	3	(1)
L8a12	4	(1)	L9a30	2/3	(2)	L9n22	3	(1)
L8a13	4	(1)	L9a31	1	(0)	L9n23	3	(2)
L8a14	4	(1)	L9a32	3	(1)	L9n24	3	(2)*
L8a15	3	(1)	L9a33	3	(1)	L9n25	2	(3)
L8a16	3	(3)	L9a34	2	(1)	L9n26	3	(2)
L8a17	4	(1)	L9a35	2	(3)	L9n27	1	(0)
L8a18	4	(1)	L9a36	3	(2)	L9n28	3	(2)*
L8a19	2	(1)	L9a37	2	(1)			
L8a20	4	(1)	L9a38	1	(0)			
L8a21	4	(1)	L9a39	2	(1)			
L8n1	2	(1)	L9a40	2	(2)			
L8n2	1	(0)	L9a41	2	(1)			
L8n3	4	(1)	L9a42	2	(3)			
L8n4	4	(1)	L9a43	3	(1)			
L8n5	2	(1)	L9a44	3	(1)			
L8n6	4	(1)	L9a45	3	(1)			
L8n7	4	(1)	L9a46	2	(2)			
L8n8	4	(1)	L9a47	3	(3)			

TABLE 2. Weak splitting numbers of prime links with 9 or fewer crossings. For the entries with a \star , the values of ω used to compute the signature bound are listed in Table 1.

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