No-boundary solutions are robust to quantum gravity corrections

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The no-boundary proposal is a theory of the initial conditions of the universe formulated in semi-classical gravity, and relying on the existence of regular (complex) solutions of the equations of motion. We show by explicit computation that regular no-boundary solutions are modified, but not destroyed, upon inclusion of expected quantum gravity corrections that involve higher powers of the Riemann tensor as well as covariant derivatives thereof. We illustrate our results with examples drawn from string theory. Our findings provide a crucial self-consistency test of the no-boundary framework.

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CONTENTS

I. INTRODUCTION

The Hartle-Hawking no-boundary proposal [1, 2] provides a theory of the quantum state of the universe. As such it is a theory of the initial conditions of the universe, meaning that it provides (relative) probabilities for different evolutions of the universe [3]. The proposal is formulated in semi-classical gravity and relies on the existence of solutions of the Einstein equations that replace the big bang singularity with a smooth geometry. In Lorentzian signature it is however not possible to find a regular solution that starts out at zero size. The insight of Hartle and Hawking was that in Euclidean signature regular solutions can exist, the prototype being a 4-sphere of constant positive curvature. In the simplest case of a cosmological constant one may then think of a no-boundary geometry as a gluing of a Euclidean onto a Lorentzian solution. Once a scalar field is added the solutions are necessarily complex, and they smoothly interpolate between Euclidean and Lorentzian signature [4].

There are two crucial features of no-boundary solutions, namely that they are compact and that they are regular (i.e. Euclidean) near the big bang. Both features are necessary in order to obtain a consistent semi-classical description. However, from a quantum point of view, these two features do not commute: compactness requires specifying a vanishing initial size while regularity corresponds to specifying an initial Euclidean expansion rate. Since size and expansion rate are conjugate variables that must satisfy the uncertainty principle, both conditions cannot be imposed simultaneously. Recent work has shown that fixing a zero initial size leads to trouble [5], while one can obtain a consistent path integral definition of the no-boundary proposal when one specifies the initial expansion rate to be Euclidean [6, 7]. This construction is also supported by the analogous calculation in anti de-Sitter space, where one may use well known results in black hole thermodynamics as guidance [8]. Thus the latest understanding of the no-boundary proposal is that it should not be thought of as a sum over compact metrics, but rather as a sum over geometries of all sizes that start out as purely spatial (Euclidean) metrics. Then, as the universe grows, the signature changes to Lorentzian – time is not present at the "beginning", where one only has space. The no-boundary geometry, which is both Euclidean *and* compact, then arises as the dominant (saddle point) contribution to the path integral.

The regularity of no-boundary geometries is crucial to the proposal since otherwise there is no chance that one may trust the results of semi-classical gravity. After all, gravity is non-renormalisable and one expects an eventual full theory of quantum gravity to have an effective description as general relativity augmented by a series of quantum corrections of higher order in the Riemann tensor. A singularity in the solution would imply an infinite sensitivity to such curvature corrections. But then one must wonder whether a solution with the required characteristics (regularity, finite action) still exists in the presence of the expected quantum gravity corrections. This is the topic of the present paper.

If we were looking for solutions with constant 4-curvature, the answer would be almost trivial since terms of higher order in the Riemann tensor (even with covariant derivatives included) would have a simple structure and such corrections would be suppressed with powers of the 4-curvature (assumed to be well below the Planck scale). But realistic no-boundary solutions have varying curvature, and can be quite different from the toy model (half-sphere + de Sitter) geometry. Moreover there exist ekpyrotic no-boundary solutions which have a geometrical shape that is very different from that of inflationary instantons [9, 10]. Technically, the problem may be formulated as follows: in a universe with scale factor $a(t)$, the Riemann tensor contains terms of the form

$$Riem \sim \frac{1}{a^2}\,, \frac{\dot{a}^2}{a^2}\,, \frac{\ddot{a}}{a}\,, \tag{1}$$

and thus it is not at all clear that there will be a smooth solution when $a \to 0$. In fact, it seems that the problem will get worse when considering higher powers of the Riemann tensor[1]. Nevertheless, as we will show in this paper, there exist conspiracies between the various terms in the Riemann tensor such that for a large class of theories, including all the known corrections stemming from string theory, smooth solutions continue to exist. Even when covariant derivatives are included in the correction terms, no-boundary solutions are robust to these corrections in the sense that the solutions will be modified somewhat, but their smoothness property is not endangered. This result represents an important self-consistency check of the no-boundary proposal, as it implies that the results obtained using only the setting of semi-classical gravity will continue to hold without drastic modification in more complete theories of quantum gravity.

The plan of this article is as follows. We will begin in section II by reviewing the salient features of the no-boundary proposal that we will require. In section III we will consider all actions composed solely of Riemann terms, i.e. terms that are scalar contractions of Riemann tensors, for metrics of closed Friedmann-Lemaître-Robertson-Walker (FLRW) form. Then in section IV we will focus on specific extensions of general relativity and quantum gravity

[1] Very few works have looked into this question in the past, in particular see Hawking and Luttrell [11] and Vilenkin [12] on quadratic gravity, and van Elst et al. on including a cubic Ricci scalar term [13].

corrections, and see if they admit a consistent and regular no-boundary solution. Section V will be devoted to the study of covariant derivatives of Riemann terms, that appear in some quantum gravity corrections. Our conclusions are in section VI. We employ the convention that the Riemann tensor is defined as $R^{\lambda}{}_{\mu\alpha\nu} = \partial_\alpha \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\alpha} + \Gamma^{\beta}_{\mu\nu}\Gamma^{\lambda}_{\beta\alpha} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\lambda}_{\beta\nu}$ and the Ricci tensor as $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$.

II. THE NO-BOUNDARY ANSATZ

The no-boundary wavefunction is a function of the (e.g. current) spatial metric of the universe h_{ij} and matter configuration $\tilde{\phi}$, defined as the path integral

$$\Psi(h_{ij}, \tilde{\phi}) = \int^{h_{ij},\tilde{\phi}} D\phi D g_{\mu\nu} e^{\frac{i}{\hbar}S} \,, \tag{2}$$

$$S = \frac{1}{8\pi G}\int d^4x\sqrt{-g}\left[\frac{R}{2} - \Lambda + \cdots\right] + \frac{1}{8\pi G}\int_{h_{ij}} d^3y\sqrt{h}K \,, \tag{3}$$

where in the action the dots stand for matter contributions ϕ and eventual additional curvature terms. The cosmological constant is denoted by Λ. A Gibbons-Hawking-York surface term (involving the trace of the extrinsic curvature K) is added on the final boundary, allowing one to fix the spatial metric there, but no such term is added at the "no-boundary hypersurface" so as to allow for the imposition of a momentum condition there, forcing metrics to be Euclidean near the nucleation of the universe – for full details see [7, 8]. This path integral can then be evaluated in the saddle point approximation, with a no-boundary geometry providing the dominant contribution. In the present work we will not consider the difficult problem of defining the path integral in the presence of higher derivative terms in the action, rather we will assume that the saddle point approximation will remain valid. More to the point, we will investigate whether suitable candidates for a no-boundary saddle point geometry exist.

It is useful to first look at the case of a closed FLRW metric in the presence of perfect fluid matter. The metric is given by

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2\left[\mathrm{d}\psi^2 + \sin^2\psi\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] , \tag{4}$$

where ψ and θ range from 0 to π and ϕ ranges from 0 to 2π. The lapse function $N(t)$ and the scale factor $a(t)$ both only depend on time. For the fluid, we will assume a stress tensor of perfect fluid form $T^{\mu\nu} = p(t)g^{\mu\nu} + (\rho(t) + p(t))\, u^\mu u^\nu$ where $\rho(t)$ is the energy density, $p(t)$ the pressure and u^μ the 4-velocity. Then the constraint and equations of motion are

$$\frac{\dot{a}^2}{N^2} + 1 = \frac{a^2}{3}(\Lambda + 8\pi G\rho) \,, \tag{5}$$

$$\frac{2\ddot{a}}{aN^2} + \frac{\dot{a}^2}{a^2N^2} + \frac{1}{a^2} - \Lambda = -8\pi G p \,, \tag{6}$$

$$a\dot{\rho} + 3\dot{a}(\rho + p) = 0 \,. \tag{7}$$

We are now looking for a solution that is regular as $a(t) \to 0$ (we will choose the origin of the time coordinate such that this coincides with $t \to 0$). From the equations above one can see that this can only be achieved if

$$\dot{a}^2(t \to 0) = -N^2 \quad ; \quad \ddot{a}(t \to 0) = 0 \quad ; \quad (\rho + p)(t \to 0) = 0 \,. \tag{8}$$

This is precisely the no-boundary solution. The condition on $\dot{a}$ immediately implies that the metric is Euclidean near $t = 0$. Meanwhile, the condition on the energy density and pressure implies that near $t = 0$ the only form of matter that is allowed is one which has the equation of state of a cosmological constant there. An example is a scalar field that approaches a constant value at $t = 0$, i.e. for which $\dot{\phi}(t = 0) = 0$. No other form of matter is allowed near the "big bang" (also sometimes called the South Pole of the instanton), as this would destroy the regularity of the solution. This means that for our purposes we can actually ignore matter contributions and focus only on gravitational terms.

Given that we need to focus on gravitational terms, do we need to worry mainly about anisotropies near the South Pole? To see that this is not the case, consider a Bianchi IX metric,

$$\mathrm{d}s^2_{IX} = -N^2\mathrm{d}t^2 + \frac{a^2}{4}\Big[e^{\beta_+ + \sqrt{3}\beta_-}(\sin\psi \mathrm{d}\theta - \cos\psi \sin\theta \mathrm{d}\phi)^2 + e^{\beta_+ - \sqrt{3}\beta_-}(\cos\psi \mathrm{d}\theta + \sin\psi \sin\theta \mathrm{d}\phi)^2 + e^{-2\beta_+}(\mathrm{d}\psi + \cos\theta \mathrm{d}\phi)^2\Big] \,; \tag{9}$$

in (t, ψ, θ, ϕ) coordinates, with $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ and $\psi \in [0, 4\pi]$. Neglecting matter, the constraint and equations of motion for the Einstein-Hilbert action are

$$\frac{3\dot{a}^2}{a^2} - \frac{3}{4}(\dot{\beta}_+^2 + \dot{\beta}_-^2) - \frac{N^2}{a^2}U(\beta_+, \beta_-) - N^2\Lambda \;=\; 0 \,; \tag{10}$$

$$\frac{\dot{a}^2}{a^2N^2} + \frac{2\ddot{a}}{aN^2} + \frac{3}{4N^2}(\dot{\beta}_+^2 + \dot{\beta}_-^2) - \frac{1}{3a^2}U(\beta_+, \beta_-) - \Lambda = 0 \,; \tag{11}$$

where

$$U(\beta_+,\beta_-)=\left(e^{-4\beta_+}+e^{2\beta_+-2\sqrt{3}\beta_-}+e^{2\beta_++2\sqrt{3}\beta_-}-2e^{2\beta_+}-2e^{-\beta_+-\sqrt{3}\beta_-}-2e^{-\beta_++\sqrt{3}\beta_-}\right)\,. \tag{12}$$

Close to $t=0$ the no-boundary ansatz (8) again leads to a solution, provided that in addition $(\dot\beta_+^2+\dot\beta_-^2)(t\to 0)=0$ and $U(\beta_+,\beta_-)(t\to 0)=-3$. This implies that the anisotropies β_+ and β_- are necessarily going to zero when $t\to 0$. Similar arguments apply to inhomogeneities.

We conclude that close to the no-boundary point, we can focus on the isotropic and homogeneous part of the metric, i.e. on the scale factor. To determine the existence of no-boundary solutions we will therefore focus on a Taylor series ansatz of the form

$$\begin{cases} a(t)=a_1 t+\dfrac{a_3}{6}t^3+\dfrac{a_4}{24}t^4+\dfrac{a_5}{120}t^5+\mathcal{O}(t^6)\,; \\ a_1^2=-N^2\,. \end{cases} \tag{13}$$

Our aim will be to see if such a series solution exists in the presence of quantum gravity corrections. Before embarking on this task, a few remarks:

1. The regularity condition $\dot a^2(0)=-N^2$ leads to two complex conjugated solutions, $\dot a(0)=a_1=\pm iN$. These actually correspond to the Vilenkin [14] and Hartle-Hawking [2] choices. Our present work will not distinguish between the two, but for discussions of the differences see e.g. [5, 15–18].

2. The coefficient $a_1=\pm iN$ on its own just describes flat space. Therefore, $a(t)=a_1t$ will always be a solution of any action constructed purely from Riemann tensors. However it is not clear whether for arbitrary actions we can have non-vanishing $a_3, a_5, \ldots$ coefficients that will define a no-boundary solution regular in time.

3. The coefficient a_3 is related to how fast the universe is expanding. This can be seen from the no-boundary solution for general relativity in the presence of a cosmological constant $\Lambda\equiv 3H^2$, which in Euclidean time $\tau=-iNt$ is given by

$$a(\tau)=\frac{1}{H}\sin(H\tau)=\tau-\frac{1}{6}H^2\tau^3+\cdots\,. \tag{14}$$

We recover $a_1^2=-N^2$, independently of H, and moreover we can see that a_3 is proportional to H^2. Therefore, for generic theories that allow solutions with different expansion rates, we should expect a_3 to remain a free parameter, labelling the various solutions. These solutions with different expansion rates will have different actions, and thus obtain different probabilities. In fact it is in this sense that the no-boundary proposal provides a quantum theory of initial conditions.

III. RIEMANN TERMS

In this section we will investigate the impact of adding terms of higher order in the Riemann tensor, without the inclusion of covariant derivatives. As explained in the previous section, we can reduce our investigation to that of the scale factor in a closed FLRW universe, with metric (4). In this spacetime, the only non-vanishing components of the Riemann tensor ${R^{\mu\nu}}_{\rho\sigma}$ are of the form ${R^{ab}}_{ab}$ and ${R^{ab}}_{ba}$ with $a, b = 0, \ldots, 3$, $a \neq b$ and no summation on a and b implied. Therefore all scalar contractions composed of n Riemann tensors $R_{\mu\nu\rho\sigma}$ and $2n$ inverse metrics $g^{\mu\nu}$ can in this FLRW background be written as contractions of n ${R^{ab}}_{ab}$ or ${R^{ab}}_{ba}$ (where n can be any integer). Moreover, these 24 non-zero components have simple expressions in terms of the lapse and scale-factor functions: $\forall\ i, j = 1, 2, 3$ with $i \neq j$ and no summation on the indices implied,

$$ {R^{ij}}_{ij} = \frac{\dot{a}^2 + N^2}{a^2 N^2} \equiv A_1 \quad \text{and} \quad {R^{0i}}_{0i} = \frac{\ddot{a}N - \dot{a}\dot{N}}{aN^3} \equiv A_2 \,. \tag{15} $$

We define a Riemann term to be any scalar combination of Riemann tensors and metric terms. As a consequence of (15), any Riemann term can be written as a polynomial in A_1 and A_2 on a closed FLRW background. Basic examples are the Ricci scalar $R = 6(A_1 + A_2)$, the Ricci tensor squared $R^{\mu\nu}R_{\mu\nu} = 12(A_1^2 + A_1 A_2 + A_2^2)$ and the Riemann tensor squared $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 12(A_1^2 + A_2^2)$.

A. General action and constraint

Since all Riemann terms are polynomials in A_1 and A_2, the most general action containing only such terms will take the form

$$ S = \int \mathrm{d}^4x \sqrt{-g} \cdot f\left(R_{\mu\nu\rho\sigma}, g^{\alpha\beta}\right) = 2\pi^2 \int \mathrm{d}t\ a^3 N \sum_{p_1, p_2 \in \mathbb{N}^2} c_{p_1,p_2} A_1^{p_1} A_2^{p_2} \,, \tag{16} $$

where c_{p_1,p_2} is a constant depending on the precise form of f for each couple $\{p_1, p_2\}$.

In order to later find the equations of motions, we slightly manipulate this action. The lapse N is a non-dynamical variable whose equation of motion is a constraint on the system.

Therefore, given that we will work in a gauge where N is constant, any term containing more than one power of $\dot{N}$ will later disappear at the level of the equations of motion. Decomposing $A_2^{p_2}$ with the Newton formula,

$$A_2^{p_2} = \left(\frac{\ddot{a}}{aN^2} - \frac{\dot{a}\dot{N}}{aN^3}\right)^{p_2} = \sum_{l=0}^{p_2} \binom{p_2}{l} \left(-\frac{\dot{a}\dot{N}}{aN^3}\right)^l \left(\frac{\ddot{a}}{aN^2}\right)^{p_2-l}, \tag{17}$$

the relevant part is given by the terms $l = 0$ and $l = 1$, so we replace

$$\left(\frac{\ddot{a}N - \dot{a}\dot{N}}{aN^3}\right)^{p_2} \to \left(\frac{\ddot{a}}{aN^2}\right)^{p_2-1} \left(\frac{\ddot{a}}{aN^2} - p_2\frac{\dot{a}\dot{N}}{aN^3}\right). \tag{18}$$

We also rewrite

$$A_1^{p_1} = \left(\frac{\dot{a}^2 + N^2}{a^2N^2}\right)^{p_1} = \frac{1}{a^{2p_1}} \sum_{j=0}^{p_1} \binom{p_1}{j} \frac{\dot{a}^{2j}}{N^{2j}}. \tag{19}$$

The action (16) then reduces to

$$S = 2\pi^2 \sum_{p_1,p_2\in\mathbb{N}^2} c_{p_1,p_2} \sum_{j=0}^{p_1} \binom{p_1}{j} \int \mathrm{d}t \left[\frac{1}{N^{2p_2-1+2j}} \frac{\dot{a}^{2j}\ddot{a}^{p_2}}{a^{2p_1+p_2-3}} - p_2 \frac{\dot{N}}{N^{2p_2+2j}} \frac{\dot{a}^{2j+1}\ddot{a}^{p_2-1}}{a^{2p_1+p_2-3}}\right]. \tag{20}$$

We can now calculate the constraint equation by variating the general action (20) with respect to the lapse function $N(t)$. Using

$$\frac{\dot{N}}{N^{2p_2+2j}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{1}{2p_2+2j-1} \cdot \frac{1}{N^{2p_2+2j-1}}\right), \tag{21}$$

we can rewrite (20) as

$$\begin{aligned} S = 2\pi^2 \sum_{p_1,p_2} c_{p_1,p_2} \sum_{j=0}^{p_1} \binom{p_1}{j} \int \frac{\mathrm{d}t}{N^{2p_2+2j-1}} \Bigg[\frac{\dot{a}^{2j}\ddot{a}^{p_2}}{a^{2p_1+p_2-3}} - \frac{p_2}{2p_2+2j-1} \Bigg\{ (2j+1)\frac{\dot{a}^{2j}\ddot{a}^{p_2}}{a^{2p_1+p_2-3}} \\ + (p_2-1)\frac{\dot{a}^{2j+1}\ddot{a}^{p_2-2}\dddot{a}}{a^{2p_1+p_2-3}} - (2p_1+p_2-3)\frac{\dot{a}^{2j+2}\ddot{a}^{p_2-1}}{a^{2p_1+p_2-2}} \Bigg\} \Bigg] \end{aligned} \tag{22}$$

$$\equiv 2\pi^2 \sum_{p_1,p_2} c_{p_1,p_2} \sum_{j=0}^{p_1} \binom{p_1}{j} \int \frac{\mathrm{d}t}{N^{2p_2+2j-1}} \cdot \mathcal{L}_{p_1,p_2,j}(a, \dot{a}, \ddot{a}, \dddot{a}). \tag{23}$$

Varying w.r.t. the lapse then yields

$$\delta_N S = 2\pi^2 \sum_{p_1,p_2} c_{p_1,p_2} \sum_{j=0}^{p_1} \binom{p_1}{j} \int \mathrm{d}t \left(\frac{-(2p_2+2j-1)\delta N}{N^{2p_2+2k}}\right) \cdot \mathcal{L}_{p_1,p_2,j}\ ; \tag{24}$$

so that the constraint equation of this system is

$$0 = \frac{\delta S}{\delta N} = -2\pi^2 \sum_{p_1,p_2} c_{p_1,p_2} \sum_{j=0}^{p_1} \binom{p_1}{j} \frac{2p_2+2j-1}{N^{2p_2+2j}} \cdot \mathcal{L}_{p_1,p_2,j}\ ; \tag{25}$$

$$\Leftrightarrow \; 0 = \frac{\delta S}{\delta N} = -2\pi^2 \sum_{p_1,p_2} \frac{c_{p_1,p_2}}{N^{2p_2}} \frac{\ddot{a}^{p_2-1}}{a^{2p_1+p_2-2}} \sum_{j=0}^{p_1} \binom{p_1}{j} \frac{\dot{a}^{2j}}{N^{2j}} \Bigg[(2j-1)(1-p_2)a\ddot{a} - p_2(p_2-1)\frac{a\dot{a}a^{(3)}}{\ddot{a}} + p_2(2p_1+p_2-3)\dot{a}^2 \Bigg] . \tag{26}$$

Using Newton's binomial formula,

$$\begin{cases} \displaystyle\sum_{k=0}^{p_1} \binom{p_1}{l} \frac{\dot{a}^{2l}}{N^{2l}} = \left(\frac{\dot{a}^2}{N^2}+1\right)^{p_1} = a^{2p_1} A_1^{p_1} \, ; \\ \displaystyle\sum_{j=0}^{p_1} \binom{p_1}{j} j \frac{\dot{a}^{2j}}{N^{2j}} = p_1 \frac{\dot{a}^2}{N^2} \left(\frac{\dot{a}^2}{N^2}+1\right)^{p_1-1} = p_1 \frac{\dot{a}^2}{a^2 N^2} a^{2p_1} A_1^{p_1} \, ; \end{cases} \tag{27}$$

the constraint equation (26) reduces to

$$0 = \frac{\delta S}{\delta N} = 2\pi^2 \sum_{p_1,p_2} c_{p_1,p_2} \Bigg[2p_1(p_2-1)\frac{a\dot{a}^2}{N^2} A_2^{p_2} A_1^{p_1-1} + (1-p_2)a^3 A_2^{p_2} A_1^{p_1} + p_2(p_2-1)\frac{a\dot{a}a^{(3)}}{N^4} A_2^{p_2-2} A_1^{p_1} - p_2(2p_1+p_2-3)\frac{a\dot{a}^2}{N^2} A^{p_2-1} A_1^{p_1} \Bigg] . \tag{28}$$

We have verified that the equation of motion for the scale factor, obtained by varying the action with respect to a, is implied by the constraint equation in the sense that it can be obtained by deriving the constraint with respect to time. From now on we shall therefore work exclusively with the constraint equation (28).

B. Order by order equations with the no-boundary ansatz

Now we are ready to insert the no-boundary ansatz into the Friedmann constraint equation (28) for the general action (20). We will then analyse the resulting equations order by order in t. This will provide conditions the action must obey so as to admit a no-boundary solution.

We first make the observation that the constraint equation (28) (hence also the equation of motion), and the no-boundary conditions (8), are all invariant under the transformation

$$\begin{cases} t \to -t, \\ a \to -a, \end{cases} \quad \Rightarrow \quad a(-t) = -a(t) \; ; \tag{29}$$

so the function a must be odd in t. Thus all coefficients of even powers of t in the Taylor expansion are zero, and the no-boundary ansatz (13) can in fact be simplified to

$$\begin{cases} a(t) = a_1 t + \dfrac{a_3}{6} t^3 + \dfrac{a_5}{120} t^5 + O(t^7) \, ; \\ a_1^2 = -N^2 \, . \end{cases} \tag{30}$$

The fact that a is an odd function of t implies that for any solution $a(t)$, there will always exist a time-reversed solution, but both will have the same signature as the metric only depends on $a(t)^2$. For this second solution, the proper time runs in the opposite coordinate time direction t. Since there is also always a complex conjugate solution for each solution (see 1), this makes for four solutions in total.

We start by plugging (30) into the expressions for A_1 and A_2, obtaining the expansions

$$\begin{cases} A_1 = \dfrac{\dot{a}^2(t)+N^2}{a^2(t)N^2} = -\dfrac{a_3}{a_1^3} + \dfrac{\left(a_3^2 - a_1 a_5\right)}{12 a_1^4}\,t^2 + \dfrac{\left(a_3 a_5 - a_1 a_7\right)}{360 a_1^4}\,t^4 + O(t^6)\,; \\ A_2 = \dfrac{\ddot{a}(t)}{a(t)N^2} = -\dfrac{a_3}{a_1^3} + \dfrac{\left(a_3^2 - a_1 a_5\right)}{6 a_1^4}\,t^2 - \dfrac{\left(10 a_3^3 - 13 a_1 a_3 a_5 + 3 a_1^2 a_7\right)}{360 a_1^5}\,t^4 + O(t^6)\,. \end{cases} \tag{31}$$

The fact that these expansions start at order t^0 is non-trivial since A_1 and A_2 both contain powers of $a(t)$ in their denominators, so they could in principle have been singular as $t \to 0$, but this is precisely what the no-boundary solution prevents. The combination $A_2 - A_1$ only starts at order t^2.

Then we plug the no-boundary ansatz (30) into the Friedmann constraint equation (28) (see appendix A). The surprise is that even though we allow terms of arbitrary order in the Riemann tensor, all coefficients of negative powers of t vanish automatically and the first non-trivial condition arises at order t. In fact, at the two lowest non-trivial orders (t and t^3) we obtain two conditions on the coefficients c_{p_1,p_2}:

$$\text{Order } t: \quad \sum_{p_1,p_2} \frac{c_{p_1,p_2}}{N^{2P}} a_1^{4-P} a_3^{P-1} \left(p_2 - p_1\right) = 0\,; \tag{32}$$

$$\text{Order } t^3: \quad \sum_{p_1,p_2} \frac{c_{p_1,p_2}}{N^{2P}}\, a_1^{3-P} a_3^{P-2} \left(a_3^2 \cdot G_3[p_1,p_2] + a_1 a_5 \cdot G_5[p_1,p_2]\right) = 0\,; \tag{33}$$

where $P \equiv p_1 + p_2$ and

$$G_3[p_1,p_2] = \frac{1}{6}\left(p_1^2 - 15 p_1 + 6 - 4 p_2^2 + 12 p_2\right) \quad ; \quad G_5[p_1,p_2] = \frac{p_1}{6}\left(1 - p_1\right) - \frac{2 p_2}{3}\left(1 - p_2\right)\,. \tag{34}$$

One way of easily satisfying the first condition (32) is by requiring that

$$\forall \{p_1, p_2\} \in \mathbb{N}^2\,, \quad c_{p_1,p_2} = c_{p_2,p_1}\,. \tag{35}$$

This special case in fact covers most known examples:

- any term of the form R^n, $\forall\, n \in \mathbb{N}$, satisfies (35) since $R = 6(A_1 + A_2)$. In particular this implies that $f(R)$ theory, and hence gravity plus a scalar field, will admit a no-boundary solution.

- quadratic terms and all their powers since $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 6\,(A_1^2 + A_2^2)$ and $R_{\mu\nu}R^{\mu\nu} = 12\,(A_1^2 + A_1 A_2 + A_2^2)$.

We then turn to the second condition (33). Provided the expression factoring a_5 is not zero, this condition in fact determines the value of a_5 in terms of a_1 and a_3:

$$a_5 \cdot \sum_{p_1,p_2} \frac{c_{p_1,p_2}}{N^{2P}}\, a_1^{3-P} a_3^{P-2} G_5[p_1,p_2] = -\frac{a_3^2}{a_1} \cdot \sum_{p_1,p_2} \frac{c_{p_1,p_2}}{N^{2P}}\, a_1^{3-P} a_3^{P-2} G_3[p_1,p_2]\,. \tag{36}$$

When we are in the special case where (35) is satisfied, we can simplify (36) by symmetrising the expressions G_3 and G_5 in the exchange of p_1 and p_2, and we find

$$a_5 = -\frac{a_3^2}{a_1} \cdot \frac{\sum_{p_1,p_2} \frac{c_{p_1,p_2}}{N^{2P}} a_1^{3-P} a_3^{P-2} \Big[4 - p_1(p_1+1) - p_2(p_2+1)\Big]}{\sum_{p_1,p_2} \frac{c_{p_1,p_2}}{N^{2P}} a_1^{3-P} a_3^{P-2} \Big[p_1(p_1-1) + p_2(p_2-1)\Big]}\,. \tag{37}$$

At higher orders in t the additionally appearing coefficients $a_7, a_9, \ldots$ will be fixed in terms of the lower ones. Thus all theories of this form admit no-boundary solutions as $a \to 0$, with a_3 remaining a free parameter effectively corresponding to solutions with different expansion rates.

The single exception to this statement is the case where the left-hand side of (36) vanishes, with the consequence that a_3 is fixed in terms of a_1. This corresponds to ordinary general relativity in the presence of a cosmological constant. Expanding (5) one straightforwardly finds

$$a_3 = -\frac{a_1^3\Lambda}{3}\ ;\ a_5 = -\frac{5a_3^2}{a_1} - 2a_1^2 a_3\Lambda = \frac{a_1^5\Lambda^2}{9}\ ; \quad etc. \tag{38}$$

For this theory the no-boundary solution corresponds to complexified de Sitter space with fixed expansion rate determined by the cosmological constant.

What we have done so far is to find general conditions that Riemann terms need to satisfy if they are to preserve the existence of no-boundary solutions. In the next section we will examine specific examples of extensions of general relativity to see whether or not they fulfil these requirements. But before doing so it may be helpful, for the sake of illustration, to see what goes wrong if the condition (32) is not satisfied. Even though we do not have a covariant expression for them, let us consider actions like

$$\int \mathrm{d}t a^3 N A_1 \; ; \quad \text{or} \quad \int \mathrm{d}t a^3 N A_1 A_2^2 \; ; \quad \text{etc,} \tag{39}$$

that are in violation of (32). The constraint equation for the action $\int \mathrm{d}t a^3 N A_1$ gives

$$(a_1^2 - N^2)t + a_1 a_3 t^3 + O(t^5) = 0\,; \tag{40}$$

so even in the presence of matter (only appearing at order t^3), this would imply $a_1 = \pm N$, corresponding to Minkowski spacetime rather than Euclidean space near $a = 0$. This is inconsistent with the no-boundary ansatz. Here we see that it is not enough for an approximately flat solution to exist near $a = 0$, it must be flat and Euclidean at the same time. Even this is not enough, as the next example will show: if we turn to $\int \mathrm{d}t a^3 N A_1 A_2^2$ for instance, the constraint equation is

$$\frac{2a_1 a_3^2}{N^6} t + \Big(\frac{4a_1 a_3 a_5}{3N^6} - 2a_1^3 \Lambda\Big) t^3 + O(t^5) = 0 \; ; \tag{41}$$

where we have included a cosmological constant Λ and assumed the no-boundary relation $a_1^2 = -N^2$. At order t one is forced to set a_3 to zero, but then at the next order the constraint cannot be satisfied. Hence this action does not admit a no-boundary solution.

Having gained a better appreciation for the non-triviality of the no-boundary regularity condition we now turn our attention to specific examples of theories containing higher orders of the Riemann tensor in the action.

IV. NO-BOUNDARY SOLUTIONS FOR EXTENSIONS OF GENERAL RELATIVITY

A. Quadratic gravity

The most straightforward extension of Einstein gravity is quadratic gravity, analysed in this context in [11, 12]. It has the advantage of being a renormalisable theory of gravity [19], but it suffers from the presence of a ghost. Lots of efforts are being made in order to make sense of this ghost, see e.g. [20]. Quadratic gravity has many uses, such as in Starobinsky's inflation [21], in asymptotic safety [22, 23], and it has interesting general implications near the big bang, where it automatically enforces the suppression of anisotropies and inhomogeneities [24].

We will first consider pure quadratic gravity, where the action only contains R^2 terms. This theory is scale invariant and has the action

$$S_{\text{pure quad}} = \int \mathrm{d}^4x\sqrt{-g}\left(\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right). \tag{42}$$

On closed FLRW background, we recall that

$$R^2 = 36\left(A_1 + A_2\right)^2 \;\; ; \;\; R_{\mu\nu}R^{\mu\nu} = 12\left(A_1^2 + A_1A_2 + A_2^2\right) \text{ and } R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 12\left(A_1^2 + A_2^2\right).$$

In four dimensions, the Gauss-Bonnet term $\int \mathrm{d}^4x\sqrt{-g}\,\mathcal{G}$ is a topological invariant and does not contribute to the dynamics. On a closed FLRW background,

$$\mathcal{G} \equiv R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta}R_{\alpha\beta} + R^2 = 24A_1A_2\,; \tag{43}$$

and the associated constraint equation obtained by inserting $\{p_1 = 1, p_2 = 1\}$ in (28) is automatically null. To study the dynamics the action can therefore effectively be reduced to

$$S_{\text{pure quad}}\big|_{reduced} = 2\pi^2 \int \mathrm{d}t\, a^3 N\, \epsilon(A_1^2 + A_2^2) \quad ; \quad \text{with } \epsilon = 36\alpha + 12\beta + 12\gamma\,. \tag{44}$$

This time even at order t^1 the constraint equation is automatically satisfied because the action (44) is symmetric in A_1 and A_2, and therefore satisfies the condition (35). At next order in t, the constraint equation yields

$$\frac{(a_1a_3^2 - a_1^2a_5)}{N^4}\cdot t^3 + O\left(t^5\right) = 0\,, \tag{45}$$

solved by $a_5 = a_3^2/a_1$. The coefficient a_3 is left undetermined, as expected from the scale invariance of the theory.

Next we can consider coupling quadratic gravity to ordinary general relativity,

$$S_{\text{quad}} = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{16\pi G} - \frac{\Lambda}{8\pi G} + \frac{\omega}{3\sigma}R^2 - \frac{1}{2\sigma}C^2 + \epsilon\mathcal{G}\right), \tag{46}$$

where we wrote the action in terms of the Weyl tensor C, which vanishes for a FLRW metric:

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 = 0\,;$$

and the Gauss-Bonnet combination $\mathcal{G}$, which does not contribute to the dynamics as we just saw. Therefore the relevant part of the quadratic action to compute the dynamics on a FLRW background is

$$S_{\text{quad,reduced}} = 2\pi^2 \int \mathrm{d}t a^3 N\Big[\frac{1}{8\pi G}\left(3A_1 + 3A_2 - \Lambda\right) + \frac{12\omega}{\sigma}(A_1^2 + A_2^2)\Big]. \tag{47}$$

The constraint equation for this action is

$$\left(\alpha a_1^3\Lambda + \frac{a_3^2\beta}{a_1^3} - \frac{a_5\beta}{a_1^2} + 3\alpha a_3\right)\cdot t^3 + O\left(t^5\right) = 0\,; \tag{48}$$

where $\alpha = \frac{1}{8\pi G}$ and $\beta = \frac{12\omega}{\sigma}$. The no-boundary solution is

$$a_5 = \frac{a_3^2}{a_1} + \frac{\alpha}{\beta}\big(a_1^5\Lambda + 3a_1^2 a_3\big)\,; \tag{49}$$

valid when $\alpha \sim \beta$ or $\alpha \ll \beta$. Then a_3 is left undetermined.

When $\alpha \gg \beta$, the solution is instead

$$a_3 \;=\; \frac{-3a_1^3 \pm \sqrt{9a_1^6 - 4\frac{\beta}{\alpha}a_1^6\Lambda + 4\frac{\beta^2}{\alpha^2}a_1 a_5}}{2\beta/\alpha} \xrightarrow{\alpha\gg\beta} \begin{cases} -\dfrac{a_1^3\Lambda}{3} + O\big(\beta/\alpha\big)\;; \\ -3a_1^3\dfrac{\alpha}{\beta} + \dfrac{a_1^3\Lambda}{3} + O\big(\beta/\alpha\big)\;. \end{cases} \tag{50}$$

The first branch corresponds to the Einstein-Hilbert solution, while the second branch is not physical as it gives a solution with curvature a_3 bigger than the Planck scale (α), and a non smooth limit $\beta \to 0$. The second branch arises due to the presence of higher derivatives in the action, and is associated with the ghost.

B. Heterotic string theory

The low-energy effective theory from heterotic string theory is the *Einstein – Maxwell – axion – dilaton* gravity containing a dilaton field ϕ, gauge fields F (Maxwell) and a 3-form H (axion), see e.g. [25, 26]. At first order in the inverse string tension α', an S-matrix calculation in heterotic string theory leads to the effective Einstein frame action [26]

$$S_{\text{heterotic}} = \frac{1}{2\kappa_D^2}\int \mathrm{d}^D x\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 + \frac{\alpha'}{8}e^{-\phi/2}\Big(\mathcal{G} + \frac{3}{16}(\partial\phi)^4\Big) - V(\phi) + \cdots\right); \tag{51}$$

where we have assumed that the compactification has led to a potential $V(\phi)$ for the dilaton (in general we may expect additional terms). Note that, as discussed in section II, the axion H and the gauge fields F have been consistently set to zero. If additional scalar fields arise due to the compactification, then these will behave analogously to the dilaton, so that we may use the dilaton as a stand-in for all of the scalars. In the gravitational sector, the first correction in α' is given by the Gauss-Bonnet combination. Because of the dilaton dependent prefactor, it is not a topological invariant this time, and we must include its effects. The constraint reads

$$\frac{\delta}{\delta N}\Big(\mathcal{L}_{\text{heterotic}}\Big) = \frac{\delta}{\delta N}\Big(6a^3 N(A_1 + A_2)\Big) - a^3\Big[\frac{\dot\phi^2}{2} - \frac{\alpha'}{128}e^{-\phi/2}\dot\phi^4 + V(\phi)\Big]$$

$$+\,3\alpha' e^{-\phi/2}\frac{\delta}{\delta N}\Big(a^3 N A_1 A_2\Big) - \frac{3\alpha'}{2}\dot\phi\, e^{-\phi/2}\frac{\dot a a^2}{N^2}A_1 = 0\,, \tag{52}$$

where the second line follows from

$$\frac{\delta}{\delta N}\Big(a^3 N\, A(N,\dot N,t)\, B(t)\Big) = B\,\frac{\delta\big(a^3 N A\big)}{\delta N} - \dot B\,\frac{\partial\big(a^3 N A\big)}{\partial \dot N} \quad \text{for} \quad A \equiv \mathcal{G} \text{ and } B \equiv e^{-\phi/2}\,. \tag{53}$$

Equation (52) is odd under the transformation $t \to -t$, $a \to -a$ and $\phi \to \phi$. We will also need the equation of motion for the scalar ϕ, which is given by

$$\nabla^2\phi - \frac{\alpha'}{16}e^{-\phi/2}\Big(\mathcal{G} + 3(\nabla_\mu\phi)(\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + \frac{3}{2}\nabla^2\phi(\partial\phi)^2 - \frac{9}{16}\big(\partial\phi\big)^4\Big) - V_{,\phi} = 0\,. \tag{54}$$

On a closed FLRW background and for a homogeneous field $\phi(t)$ this translates into

$$\ddot\phi - \frac{3\alpha'}{16}e^{-\phi/2}\Big(8A_1A_2 + \frac{3}{2}\dot\phi^2\ddot\phi - \frac{3}{16}\dot\phi^4\Big) - V_{,\phi} = 0\,. \tag{55}$$

This equation (55) is even under the transformation $t \to -t$, $a \to -a$ and $\phi \to \phi$.

Now we look for Taylor series solutions to equations (52) and (55) around $t = 0$. From the transformation rules of the equations of motion (52) and (55) under $t \to -t$, $a \to -a$ and $\phi \to \phi$, we know that $a(t)$ must be an odd function of time, while $\phi(t)$ must be even:

$$\begin{cases} a(t) = a_1 t + \dfrac{a_3}{6}t^3 + \dfrac{a_5}{120}t^5 + \dots \\ \phi(t) = \phi_0 + \dfrac{\phi_2}{2}t^2 + \dfrac{\phi_4}{24}t^4 + \dots \end{cases} \tag{56}$$

This is already enough to realise that ϕ will be constant at first order in time close to the no-boundary point $t \to 0$. When plugging (56) in the constraint equation (52) and expanding in orders of t, the leading order gives

$$-\frac{3a_1 e^{-\phi_0/2}\alpha'}{2N^4}(a_1^2 + N^2)\phi_2 t + O(t^3) = 0\,; \tag{57}$$

that is solved by the usual no-boundary solution $a_1^2 = -N^2$. Then we turn to the equation of motion for ϕ (55) where at leading order we find

$$\phi_2 - \frac{3a_3^2}{2a_1^6}\alpha' e^{-\phi_0/2} - V_{,\phi}(\phi_0) + O(t^2) = 0\,. \tag{58}$$

This equation fixes ϕ_2 as a function of ϕ_0, a_1 and a_3. Implementing this solution for ϕ_2, the next order of the constraint equation gives us a cubic equation for a_3 in terms of a_1 and ϕ_0:

$$\Big[-\frac{9a_3^3}{4a_1^8}\alpha'^2 e^{-\phi_0} + 6a_3\Big(1 - e^{-\phi_0/2}\frac{\alpha' V_{,\phi}(\phi_0)}{4a_1^2}\Big) - a_1^3 V(\phi_0)\Big]t^3 + O(t^5) = 0\,. \tag{59}$$

We conclude that the heterotic string action (51) possesses a family of no-boundary solutions, this time usefully labelled by ϕ_0, the dilaton value at the South Pole.

C. Type II string theory in D=10 spacetime dimensions

The low-energy effective action, obtained by looking at quantum corrected amplitudes from type II string theory in $D = 10$ dimensions order by order in α', reads [27, 28]

$$S = \int \mathrm{d}^D x \sqrt{-G} \left(R + (\alpha')^3 \mathcal{E}^{(D)}_{(0,0)} \mathcal{R}^4 + (\alpha')^5 \mathcal{E}^{(D)}_{(1,0)} \nabla^4 \mathcal{R}^4 + (\alpha')^6 \mathcal{E}^{(D)}_{(0,1)} \nabla^6 \mathcal{R}^4 + \dots \right); \quad (60)$$

where G is the determinant of the metric in D dimensions, while $\mathcal{E}^{(D)}_{(p,q)}$ are coefficient functions that depend on the compactification. General compactifications imply the presence of additional curvature terms (along the lines discussed above) and scalars (discussed in section V D) as well as numerous gauge fields which we can set to zero (cf. the discussion in section II). Here we will focus on the α'^3 type II correction to Einstein gravity (60) which is given by the $\mathcal{R}^4$ term, a special combination of four Riemann tensors defined as

$$\mathcal{R}^4 = t_8^{ijklmnpq} t_8^{abcdefgh} R_{ijab} R_{klcd} R_{mnef} R_{pqgh} \,. \quad (61)$$

t_8 is a special 8-rank tensor whose explicit expression can be found in [29] (chapter 9, Appendix A) to be:

$$\begin{aligned} t^{ijklmnpq} = & -\frac{1}{2}\epsilon^{ijklmnpq} \\ & -\frac{1}{2}\Big[\left(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}\right)\left(\delta^{mp}\delta^{nq} - \delta^{mq}\delta^{np}\right) + \left(\delta^{km}\delta^{ln} - \delta^{kn}\delta^{lm}\right)\left(\delta^{pi}\delta^{qj} - \delta^{pj}\delta^{qi}\right) \\ & \quad + \left(\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm}\right)\left(\delta^{kp}\delta^{lq} - \delta^{kq}\delta^{lp}\right) \Big] \\ & +\frac{1}{2}\Big[\delta^{jk}\delta^{lm}\delta^{np}\delta^{qi} + \delta^{jm}\delta^{nk}\delta^{lp}\delta^{qi} + \delta^{jm}\delta^{np}\delta^{qk}\delta^{li} + \text{ 45 more terms} \\ & \quad \text{obtained by antisymmetrizing on the pairs } ij,\ kl,\ mn \text{ and } pq \Big]. \end{aligned} \quad (62)$$

The quantity $\mathcal{R}^4$ is therefore a Riemann term, so we can determine if it will admit a no-boundary solution by simply looking at its structure in terms of A_1 and A_2 and see if it meets condition (32). We start by computing the explicit structure of $\mathcal{R}^4$ in terms of Riemann tensors with the xAct package [30]:

$$\begin{aligned} \mathcal{R}^4 = & 12(R_{abcd}R^{abcd})^2 + 6R^{abcd}R_{ab}{}^{ij}(4R_{ij}{}^{kl}R_{cdkl} - R_{ic}{}^{kl}R_{jdkl}) - 12R_{abij}R_{cdkl}R^{abci}R^{djkl} \\ & + \frac{3}{2}R_{abij}R^{acid}R^{jl}{}_{ck}R^{bk}{}_{dl} + \frac{3}{4}R_{abij}R^{acid}R_{ckdl}R^{bkjl} \\ & + \epsilon^{ijklmnpq}R_{ij}{}^{ab}\Big[2R_{klef}R_{mn}{}^{ef}R_{pqab} - \frac{1}{2}R_{kl}{}^{ef}R_{mnae}R_{pqbf} - \frac{1}{2}R_{klae}R_{mn}{}^{fe}R_{pqbf} \end{aligned}$$

$$+2R_{kl}{}^{ef}R_{mnab}R_{pqef} - \frac{1}{2}R_{klae}R_{mnbf}R_{pq}{}^{ef} + 2R_{klab}R_{mn}{}^{ef}R_{pqef}\Big]$$

$$+\frac{1}{4}\epsilon^{ijklmnpq}\epsilon^{efghabcd}R_{ijab}R_{klcd}R_{efmn}R_{ghpq}. \tag{63}$$

We must be aware that these expressions are originally valid only in 10 dimensions. When going down to 4 dimensions, there will be new fields (and different associated terms) appearing through the compactification, when indices point in the internal dimensions. These gauge fields and scalars will depend on the details of the compactification. However, as discussed in section II, we expect gauge field to be zero and scalar fields constant at the no-boundary point. Therefore the only part of (63) that we are really interested in is the one where all indices point in the (four) external spacetime dimensions. But then all the terms containing an 8 rank tensor ϵ are set to zero, and we are left with

$$\begin{aligned}\mathcal{R}^4\big|_{4d,\text{truncated}} = {} & 12(R_{\mu\nu}{}^{\rho\sigma}R^{\mu\nu}{}_{\rho\sigma})^2 + 6R^{\mu\nu}{}_{\rho\sigma}R_{\mu\nu}{}^{\xi\eta}(4R_{\xi\eta}{}^{\kappa\lambda}R^{\rho\sigma}{}_{\kappa\lambda} - R_{\xi}{}^{\rho\kappa}{}_{\lambda}R_{\eta}{}^{\sigma}{}_{\kappa}{}^{\lambda}) \\ & - 12R_{\mu\nu}{}^{\xi\eta}R^{\rho\sigma}{}_{\kappa\lambda}R^{\mu\nu}{}_{\rho\xi}R_{\sigma\eta}{}^{\kappa\lambda} + \frac{3}{2}R^{\mu\nu}{}_{\xi\eta}R_{\mu\rho}{}^{\xi\sigma}R^{\eta}{}_{\lambda}{}^{\rho}{}_{\kappa}R_{\nu}{}^{\kappa}{}_{\sigma}{}^{\lambda} \\ & + \frac{3}{4}R_{\mu\nu}{}^{\xi\eta}R^{\mu\rho}{}_{\xi\sigma}R_{\rho\kappa}{}^{\sigma\lambda}R^{\nu\kappa}{}_{\eta\lambda}\,;\end{aligned} \tag{64}$$

where $\mu,\nu,\rho,\sigma,\xi,\eta,\kappa,\lambda$ are now spacetime indices running from $\{0,\ldots,3\}$. This expression (64) is now ready to be expressed in terms of A_1 and A_2. Using (15), we compute that on this background all the terms of expression (64) can be written in terms of two quantities that we denote $\mathcal{R}_1$ and $\mathcal{R}_2$:

$$12(R_{\mu\nu}{}^{\rho\sigma}R^{\mu\nu}{}_{\rho\sigma})^2 = 12^3\left(A_1^4 + 2A_1^2A_2^2 + A_2^4\right) \equiv 12\mathcal{R}_1\,; \tag{65}$$

$$24R^{\mu\nu}{}_{\rho\sigma}R_{\mu\nu}{}^{\xi\eta}R_{\xi\eta}{}^{\kappa\lambda}R^{\rho\sigma}{}_{\kappa\lambda} = 8\cdot 12^2\left(A_1^4 + A_2^4\right) \equiv 24\mathcal{R}_2\,; \tag{66}$$

$$-6R^{\mu\nu}{}_{\rho\sigma}R_{\mu\nu}{}^{\xi\eta}R_{\xi}{}^{\rho\kappa}{}_{\lambda}R_{\eta}{}^{\sigma}{}_{\kappa}{}^{\lambda} = -12^2(A_1^4 + A_2^4) = -3\mathcal{R}_2\,; \tag{67}$$

$$-12R_{\mu\nu}{}^{\xi\eta}R^{\rho\sigma}{}_{\kappa\lambda}R^{\mu\nu}{}_{\rho\xi}R_{\sigma\eta}{}^{\kappa\lambda} = -12^2\cdot 4\left(A_1^4 + A_1^2A_1^2 + A_2^4\right) = -2\mathcal{R}_1 - 6\mathcal{R}_2\,; \tag{68}$$

$$\frac{3}{2}R^{\mu\nu}{}_{\xi\eta}R_{\mu\rho}{}^{\xi\sigma}R^{\eta}{}_{\lambda}{}^{\rho}{}_{\kappa}R_{\nu}{}^{\kappa}{}_{\sigma}{}^{\lambda} = 9\left(3A_1^4 + 2A_1^2A_2^2 + 3A_2^4\right) = \frac{1}{16}\mathcal{R}_1 + \frac{3}{8}\mathcal{R}_2\,; \tag{69}$$

and finally

$$\frac{3}{4}R_{\mu\nu}{}^{\xi\eta}R^{\mu\rho}{}_{\xi\sigma}R_{\rho\kappa}{}^{\sigma\lambda}R^{\nu\kappa}{}_{\eta\lambda} = 18\left(A_1^4 + 2A_1^2A_2^2 + A_2^4\right) = \frac{1}{8}\mathcal{R}_1\,. \tag{70}$$

Therefore the expression (64) reads

$$\mathcal{R}^4\big|_{4d,\text{truncated}} = \frac{163}{16}\mathcal{R}_1 + \frac{123}{8}\mathcal{R}_2 = 1467(A_1^2 + A_2^2)^2 + 738(A_1^4 + A_2^4)\,. \tag{71}$$

The quantities $\mathcal{R}_1$ and $\mathcal{R}_2$ are both symmetric under the exchange of A_1 and A_2, so they satisfy the condition (35). Therefore, the $\mathcal{R}^4$ term satisfies the leading order condition (32), and will admit a no-boundary solution.

It might look a bit astonishing that this very complicated scalar combination of four Riemann tensors has such a simple expression in terms of A_1 and A_2, that is moreover symmetric in the exchange of A_1 and A_2. This might lead us to think that this could be a general property of any scalar combination of Riemann tensors, but if we look at the two following combinations:

$$R_{\mu\nu}{}^{\rho\sigma}R^{\mu\xi}{}_{\rho\sigma}R_{\xi\kappa}{}^{\nu\lambda}R^{\alpha\kappa}{}_{\alpha\lambda} = 48A_1^4 + 36A_2^4 + 48A_1A_2^3 + 24A_1^3A_2 + 60A_1^2A_2^2\,; \tag{72}$$

and

$$R^{\mu\nu}{}_{\xi\eta}R_{\mu\rho}{}^{\xi\sigma}R^{\rho\kappa}{}_{\nu\lambda}R_{\sigma\kappa}{}^{\eta\lambda} = 12(A_1^4 + A_1^2A_2^2 + A_2^4) + 12A_1(A_1^3 + A_1A_2^2 + 2A_2^3)\,; \tag{73}$$

we see that they are both not symmetric under the exchange of A_1 and A_2. However, they still satisfy the leading order condition (32), and therefore admit a no-boundary solution.

We may conclude that known Riemann terms stemming from string theory have a structure that allows for no-boundary solutions. What is more, all of the covariant Riemann terms that we have investigated allow for no-boundary solutions. It would of course be very interesting if one could prove a general result in this direction. The next orders in α' of the type II string theory (60) are not Riemann terms anymore, but rather involve covariant derivatives acting on Riemann tensors. Unfortunately, it is not possible to treat covariant derivative terms as systematically as we treated Riemann terms, because they depend on higher and higher time derivatives of the scale factor $a(t)$. We will therefore study them on a case by case basis, starting with the easiest expressions and ending with the first string theory covariant derivative term, written schematically as $\nabla^4\mathcal{R}^4$ in (60).

V. COVARIANT DERIVATIVES OF RIEMANN TERMS

When covariant derivatives enter the game, it is even less trivial that their contributions to the constraint equation will still admit consistent and regular solutions. Indeed we have seen that Riemann terms are linear combinations of A_1 and A_2, and these quantities only start at order t^0. Therefore, when acting on them with time derivatives, there is no risk of ending up with negative powers of t, that could bring singularities. But the covariant derivative is also composed of the Christoffel symbol part: $\nabla\cdot \sim \partial \cdot + \Gamma\cdot$. The non zero Christoffel symbols are schematically

$$g^{ki}\Gamma^0_{ij} \sim \frac{\dot{a}}{aN^2} \quad ; \quad \Gamma^i_{j0} \sim \frac{\dot{a}}{a} \quad \text{and} \quad \Gamma^i_{jk} \sim 1 \; ; \tag{74}$$

(by $\sim$ we indicate only the time dependence, not the angular dependence). The quantity $\dot{a}/a \sim t^{-1}$ is singular, and we can fear that covariant derivatives introduce singularities into the constraint equations. Therefore we need to check term by term the existence of regular solutions in the covariant derivative terms that we need.

First consider again the transformation

$$\begin{cases} t \to -t \,, \\ a \to -a \,. \end{cases} \tag{75}$$

On a closed FLRW background, if we consider the action

$$S = \int \mathrm{d}t a^3 N \; \mathcal{L} \; ; \tag{76}$$

then the constraint equation of this action will be

$$\frac{\delta}{\delta N}\Big(a^3 N \mathcal{L}\Big) \equiv \frac{\partial\big(a^3 N\mathcal{L}\big)}{\partial N} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\big(a^3 N\mathcal{L}\big)}{\partial \dot{N}}\right] + \ldots \; = 0 \; . \tag{77}$$

This constraint equation will be odd under the transformation (75) only if $\mathcal{L}$ is even under this same transformation. Now A_1 and A_2 are even under this transformation, hence such are all Riemann terms. Because the FLRW metric doesn't contain any mixed term g_{0i}, time derivatives will always come in pairs. The Christoffel symbols (74) with one 0 index are odd under (75) and will also always come in pairs or with one time derivative. Therefore all covariant derivatives of Riemann terms will be even under this transformation, and their constraint equation odd. Thus we may keep using the reduced no-boundary ansatz (30) instead of the full ansatz (13).

By studying terms with up to four covariant derivatives acting on Riemann terms, we will encounter expressions with up to four derivatives acting on $a(t)$. To ease the upcoming expressions, we therefore define

$$A_3 \equiv \frac{a^{(3)}}{aN^3} - \frac{\dot{a}\ddot{N}}{aN^4} - \Big(\frac{3\dot{N}}{N^2} + \frac{\dot{a}}{aN}\Big)A_2 \; ; \tag{78}$$

$$A_4 \equiv \frac{a^{(4)}}{aN^4} - \frac{\dot{a}N^{(3)}}{aN^5} - \frac{6\dot{N}}{N^2}A_3 - \Big(\frac{6\dot{a}\dot{N}}{aN^3} + \frac{3\dot{N}^2}{N^4} + \frac{4\ddot{N}}{N^3}\Big)A_2 - A_2^2 \; . \tag{79}$$

The calculations involving covariant derivatives are rather lengthy, so we are not going to display them entirely here. Rather, we will explicitly show the simplest example that arises when two covariant derivatives act on one Riemann tensor, and relegate the results of lengthier calculations to the appendix. Our focus will be on terms of the form $\nabla^4 R^4$.

A. An explicit example: two covariant derivatives acting on one Riemann tensor

The following quantity is a scalar term where two covariant derivatives act on one Riemann tensor:

$$\mathcal{A} \equiv \nabla^2 R = -6\bigg(A_4 + \frac{3\dot{a}}{aN} A_3 + 2A_2(A_2 - A_1) \bigg) \,. \tag{80}$$

We can directly observe that $\mathcal{A}$ is a total derivative, so its constraint equation will be null. We will however derive this result explicitly for illustrative purposes.

To compute the constraint equation of $\mathcal{A}$ we need to compute those of the terms A_4 and $\frac{\dot{a}}{aN} A_3$, or more precisely, of the actions

$$S_{A_4} = \int \mathrm{d}t a^3 N A_4 \quad \text{and} \quad S_{\dot{a}A_3} = \int \mathrm{d}t a^3 N \frac{\dot{a}}{aN} A_3 \,. \tag{81}$$

In a closed FLRW background, the constraint equation for the action S_{A_4} is

$$0 = \frac{\partial (a^3 N A_4)}{\partial N} - \frac{\mathrm{d}}{\mathrm{d}t}\bigg[\frac{\partial (a^3 N A_4)}{\partial \dot{N}}\bigg] + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg[\frac{\partial (a^3 N A_4)}{\partial \ddot{N}}\bigg] - \frac{\mathrm{d}^3}{\mathrm{d}t^3}\bigg[\frac{\partial (a^3 N\, A_4)}{\partial N^{(3)}}\bigg] \equiv \frac{\delta}{\delta N}\Big[a^3 N A_4\Big] \,. \tag{82}$$

We make the whole derivation explicitly for this first case:[2]

$$a^3 N\, A_4 = \frac{a^2 a^{(4)}}{N^3} - \frac{a\ddot{a}^2}{N^3} - \frac{6a^2 a^{(3)} \dot{N}}{N^4} + \frac{2a\dot{a}\ddot{a}\dot{N}}{N^4} - \frac{4a^2\ddot{a}\ddot{N}}{N^4} - \frac{a^2\dot{a}N^{(3)}}{N^4}; \tag{83}$$

$$\Rightarrow \begin{cases} \dfrac{\partial (a^3 N\, A_4)}{\partial N} = -\dfrac{3}{N^4}\Big(a^2 a^{(4)} - a\ddot{a}^2\Big)\,; \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\bigg[\dfrac{\partial (a^3 N\, A_4)}{\partial \dot{N}}\bigg] = \dfrac{1}{N^4}\Big(- 6a^2 a^{(4)} + 2\dot{a}^2\ddot{a} + 2a\ddot{a}^2 - 10a\dot{a}a^{(3)}\Big)\,; \\ \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg[\dfrac{\partial (a^3 N\, A_4)}{\partial \ddot{N}}\bigg] = \dfrac{1}{N^4}\Big(- 4a^2 a^{(4)} - 8\dot{a}^2\ddot{a} - 8a\ddot{a}^2 - 16a\dot{a}a^{(3)}\Big)\,; \\ \dfrac{\mathrm{d}^3}{\mathrm{d}t^3}\bigg[\dfrac{\partial (a^3 N\, A_4)}{\partial N^{(3)}}\bigg] = \dfrac{1}{N^4}\Big(- a^2 a^{(4)} - 12\dot{a}^2\ddot{a} - 6a\ddot{a}^2 - 8a\dot{a}a^{(3)}\Big)\,. \end{cases} \tag{84}$$

[2] In this paper, it is always implicitly understood that the following expressions are evaluated at constant lapse N, so that we can drop all terms containing more than one power of a derivative of N.

So using (82) we find that the constraint equation for the action S_{A_4} is

$$\frac{\delta}{\delta N}\Big[a^3 N A_4\Big] = \frac{1}{N^4}\Big(2\dot{a}^2\ddot{a} - a\ddot{a}^2 + 2a\dot{a}a^{(3)}\Big)\,. \tag{85}$$

We use exactly the same procedure for all coming terms, but only display the final results. For the action $S_{\dot{a}A_3}$, we find the constraint equation to be

$$\frac{\delta}{\delta N}\Big[a^3 N \frac{\dot{a}}{aN} A_3\Big] = \frac{1}{N^4}\Big(- 2a\dot{a}a^{(3)} + a\ddot{a}^2 - 2\dot{a}^2\ddot{a}\Big)\,. \tag{86}$$

The only missing piece to get the constraint equation for $\nabla^2 R$ (80) is the $A_2(A_2 - A_1)$ term. This one is a simple $A_1^{p_1} A_2^{p_2}$ term, so we read off its contribution from (28):

$$\frac{\delta}{\delta N}\Big[a^3 N A_2\big(A_2 - A_1\big)\Big] = \frac{1}{N^4}\Big(2a\dot{a}a^{(3)} - a\ddot{a}^2 + 2\dot{a}^2\ddot{a}\Big)\,. \tag{87}$$

The constraint equation for $\mathcal{A}$ is therefore

$$\begin{aligned}\delta\mathcal{A} \equiv \frac{\delta}{\delta N}\Big[a^3 N \mathcal{A}\Big] &= -6\bigg(\frac{\delta}{\delta N}\Big[a^3 N A_4\Big] + 3\frac{\delta}{\delta N}\Big[a^3 N \frac{\dot{a}}{aN} A_3\Big] + 2\frac{\delta}{\delta N}\Big[a^3 N A_2\big(A_2 - A_1\big)\Big]\bigg) \\ &= \;0\,. \end{aligned} \tag{88}$$

which is the expected result since this term is a total derivative.

B. General recipe

Using the straightforward method presented in the previous subsection, we can compute all possible covariant derivatives terms. However, we can ease our life even more by decomposing the calculations further. Assume we know the constraint equations for the two actions

$$S_A = \int \mathrm{d}t\, a^3 N\, A \quad \text{and} \quad S_B = \int \mathrm{d}t\, a^3 N\, B\,, \tag{89}$$

where A and B are functions of a, N and their time derivatives. Then the constraint equation for the action

$$S_{A\cdot B} = \int \mathrm{d}t\, a^3 N\, A\cdot B\,, \tag{90}$$

will be given by

$$\begin{aligned}\frac{\delta}{\delta N}\Big[a^3 N\, A\cdot B\Big] &= A\cdot\frac{\delta}{\delta N}\Big[a^3 N\, B\Big] + B\cdot\frac{\delta}{\delta N}\Big[a^3 N\, A\Big] - a^3 A\cdot B \\ &\quad - \dot{A}\cdot\bigg[\frac{\partial(a^3 N\, B)}{\partial\dot{N}} - 2\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial(a^3 N\, B)}{\partial\ddot{N}}\Big) + 3\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(\frac{\partial(a^3 N B)}{\partial N^{(3)}}\Big)\bigg]\end{aligned}$$

$$+\ddot{A}\Big[\frac{\partial(a^3N\,B)}{\partial\ddot{N}} - 3\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial(a^3NB)}{\partial N^{(3)}}\Big)\Big] - A^{(3)}\frac{\partial(a^3NB)}{\partial N^{(3)}}$$
$$-\dot{B}\cdot\Big[\frac{\partial(a^3N\,A)}{\partial\dot{N}} - 2\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial(a^3N\,A)}{\partial\ddot{N}}\Big) + 3\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(\frac{\partial(a^3NA)}{\partial N^{(3)}}\Big)\Big]$$
$$+\ddot{B}\Big[\frac{\partial(a^3N\,A)}{\partial\ddot{N}} - 3\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial(a^3NA)}{\partial N^{(3)}}\Big)\Big] - B^{(3)}\frac{\partial(a^3NA)}{\partial N^{(3)}}\,. \tag{91}$$

This assumes that the highest derivative of N on which A and B depend is of third order, as it will be the case in this work. It is however trivial to extend (91) to include higher orders.

Using equation (91) enables us to build iteratively the constraint equations of more and more involved expressions of A_1, A_2, A_3 and A_4. To illustrate this a bit more, suppose we want to compute the constraint equations of the four following covariant expressions:

$$\begin{aligned}&\mathcal{B}_1 \equiv (\nabla_\mu R_{\alpha\beta\gamma\delta})(\nabla^\mu R^{\alpha\beta\gamma\delta})\,; \quad \mathcal{B}_2 \equiv (\nabla_\mu R_{\alpha\beta})(\nabla^\mu R^{\alpha\beta})\,;\\ &\mathcal{B}_3 \equiv (\nabla_\mu R)(\nabla^\mu R) \qquad \text{and} \quad \mathcal{B}_4 \equiv (\nabla_\mu R^\mu{}_{\alpha\beta\gamma})(\nabla_\nu R^{\nu\alpha\beta\gamma})\,;\end{aligned} \tag{92}$$

that are expressed in terms of the quantities A_1, A_2 and A_3 as

$$\mathcal{B}_1 = -12\Big[A_3^2 + \frac{8\dot{a}^2}{a^2N^2}(A_2-A_1)^2\Big]\,; \tag{93}$$
$$\mathcal{B}_2 = -12\Big[A_3^2 + \frac{2\dot{a}}{aN}(A_2-A_1)A_3 + \frac{6\dot{a}^2}{a^2N^2}(A_2-A_1)^2\Big]\,; \tag{94}$$
$$\mathcal{B}_3 = -36\Big[A_3^2 + \frac{4\dot{a}}{aN}(A_2-A_1)A_3 + \frac{4\dot{a}^2}{a^2N^2}(A_2-A_1)^2\Big]\,; \tag{95}$$
$$\mathcal{B}_4 = -6\Big[2(A_2-A_1)\frac{\dot{a}}{aN} + A_3\Big]^2\,. \tag{96}$$

Then we just need to compute the constraint equations for the two actions

$$S_{A_3} = \int \mathrm{d}t a^3 N A_3 \quad \text{and} \quad S_{\dot{a}(A_2-A_1)} = \int \mathrm{d}t a^3 N\frac{\dot{a}}{aN}\big(A_2-A_1\big)\,; \tag{97}$$

and then combine them using (91).[3]

The general recipe we apply to compute the constraint equations of all covariant derivative terms is therefore

1. Decompose the expression in terms of A_1, A_2, A_3 and A_4.[4]

2. Find the basic blocks needed to build each terms in this expression (e.g. (97) in the previous example), and compute their constraint equation.

[3] Notice also that other terms like $R^{\mu\nu}(\nabla^2 R_{\mu\nu})$ or $R(\nabla^2 R)$ can be obtained from these $\mathcal{B}$ terms (92) by integrating by parts, since two terms differing by a total derivative lead to the same constraint equation.

[4] This is only valid for terms where at most four covariant derivatives are acting on Riemann tensors.

3. Use the formula (91) (iteratively if needed) to combine the basic blocks and get the complete constraint equation for the initial covariant expression.

4. Plug in the no-boundary ansatz (30). This step is commutative with the previous one.

Using this method, we computed the constraint equations of all the $\mathcal{B}$ terms (92) as well as those of the following terms where four covariant derivatives act on two Riemann tensors (see Appendix B):

$$\begin{aligned} &\mathcal{C}_1 \equiv \nabla^2 R_{\alpha\beta\gamma\delta}\, \nabla^2 R^{\alpha\beta\gamma\delta}\ ; && \mathcal{C}_2 \equiv \nabla^2 R_{\alpha\beta}\, \nabla^2 R^{\alpha\beta}\ ; && \mathcal{C}_3 \equiv \nabla^2 R\, \nabla^2 R\ ; \\ &\mathcal{C}_4 \equiv \nabla_\mu \nabla_\nu R_{\alpha\beta\gamma\delta} \nabla^\mu \nabla^\nu R^{\alpha\beta\gamma\delta}\ ; && \mathcal{C}_5 \equiv \nabla_\mu \nabla_\nu R_{\alpha\beta} \nabla^\mu \nabla^\nu R^{\alpha\beta}\ ; && \mathcal{C}_6 \equiv \nabla_\mu \nabla_\nu R\, \nabla^\mu \nabla^\nu R\ . \end{aligned} \tag{98}$$

Remarkably, all the constraint equations of these expressions only start at order t^3, although we could expect them to start at order t^{-1}, and are therefore not singular. This peculiar feature will continue to hold for the cases of four derivatives acting on four Riemann tensor that we are now going to address.

C. Four covariant derivatives acting on four Riemann tensors

We are now ready to evaluate the contributions to the constraint equation stemming from the $\nabla^4\mathcal{R}^4$ terms (these terms are discussed in more detail in [31], see also [32]). We once again consider the truncated part of $\mathcal{R}^4$, expressed in terms of the two quantities $\mathcal{R}_1$ and $\mathcal{R}_2$,

$$\mathcal{R}^4\big|_{4d,\text{truncated}} = \frac{163}{16}\mathcal{R}_1 + \frac{123}{8}\mathcal{R}_2\,; \tag{99}$$

with

$$\mathcal{R}_1 = (R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})^2 \quad \text{and} \quad \mathcal{R}_2 = R^{\alpha\beta}{}_{\gamma\delta}R_{\alpha\beta}{}^{\epsilon\zeta}R_{\epsilon\zeta}{}^{\eta\theta}R^{\gamma\delta}{}_{\eta\theta}\,. \tag{100}$$

There are three types of terms that one can write and that are inequivalent using integration by parts when four covariant derivatives act on four Riemann tensors:[5]

$$(\nabla R)^4 \quad ; \quad (\nabla^2 R)^2 R^2 \quad \text{and} \quad (\nabla^2 R)(\nabla R)^2 R\,. \tag{101}$$

For these three types, we will construct all possible independent terms where the four Riemann tensors are either $\mathcal{R}_1$ or $\mathcal{R}_2$.

[5] The R here does not refer to the Ricci scalar but is a schematic way of writing the Riemann tensor without bothering about the indices.

a. Type 1: $(\nabla R)^4$ *terms.* These terms can all be written as linear combinations of the four following terms:

$$\begin{aligned} &\mathcal{D}_1 \equiv \left(\nabla_\mu R_{\alpha\beta\gamma\delta}\, \nabla^\mu R^{\alpha\beta\gamma\delta}\right)^2 \quad ; \quad &&\mathcal{D}_2 \equiv \left(\nabla_\mu R_{\alpha\beta\gamma\delta} \nabla_\nu R^{\alpha\beta\gamma\delta} \nabla^\mu R_{\epsilon\zeta\eta\theta} \nabla^\nu R^{\epsilon\zeta\eta\theta}\right) \quad ; \\ &\mathcal{D}_3 \equiv \left(\nabla_\mu R^{\alpha\beta}{}_{\gamma\delta} \nabla^\mu R_{\alpha\beta}{}^{\epsilon\zeta} \nabla_\nu R_{\epsilon\zeta}{}^{\eta\theta} \nabla^\nu R_{\eta\theta}{}^{\gamma\delta}\right) \quad ; \quad &&\mathcal{D}_4 \equiv \left(\nabla_\mu R^{\alpha\beta}{}_{\gamma\delta} \nabla_\nu R_{\alpha\beta}{}^{\epsilon\zeta} \nabla^\mu R_{\epsilon\zeta}{}^{\eta\theta} \nabla^\nu R_{\eta\theta}{}^{\gamma\delta}\right) \quad ; \end{aligned} \tag{102}$$

that can be expressed in terms of A_1, A_2 and A_3 (see Appendix C). Computing their contributions to the constraint equation requires the computation of the following constraint equations:

$$\zeta_1 \equiv \frac{\delta}{\delta N}\left[a^3 N\, A_3^4\right] = \frac{32(a_3^2 - a_1 a_5)^3}{9a_1^{15}} t^3 + O\left(t^5\right) \; ; \tag{103}$$

$$\zeta_2 \equiv \frac{\delta}{\delta N}\left[a^3 N \frac{\dot{a}^4}{a^4 N^4}(A_2 - A_1)^4\right] = \frac{(a_1 a_5 - a_3^2)^3}{72 a_1^{15}} \cdot t^3 + O\left(t^5\right) \; ; \tag{104}$$

$$\zeta_3 \equiv \frac{\delta}{\delta N}\left[a^3 N\, \frac{\dot{a}^2}{a^2 N^2} A_3^2 (A_2 - A_1)^2\right] = O\left(t^5\right) \; ; \tag{105}$$

$$\zeta_4 \equiv \frac{\delta}{\delta N}\left[a^3 N \frac{\dot{a}^3}{a^3 N^3} A_3 \big(A_2 - A_1\big)^3\right] = \frac{(a_1 a_5 - a_3^2)^3}{36 a_1^{15}} \cdot t^3 + O\left(t^5\right) \; . \tag{106}$$

Combining these, we get the contributions to the constraint coming from the four $\mathcal{D}$ terms that are displayed in Appendix C. Let us stress here that up to order t^3, these four terms have the same structure involving the combination $a_3^2 - a_1 a_5$,

$$\delta \mathcal{D}_i = \alpha_i \frac{(a_3^2 - a_1 a_5)^3}{a_1^{15}} t^3 + O(t^5)\, ; \tag{107}$$

where α_i are numerical factors.

b. Type 2: $(\nabla^2 R)^2 R^2$ *terms* In this case we can construct 8 different independent expressions:

$$\begin{aligned} &\mathcal{E}_1 \equiv \nabla^2 R_{\alpha\beta\gamma\delta} \big(\nabla^2 R^{\alpha\beta\gamma\delta}\big) R_{\epsilon\zeta\eta\theta} R^{\epsilon\zeta\eta\theta} \quad ; \quad &&\mathcal{E}_2 \equiv \nabla_\mu \nabla_\nu (R_{\alpha\beta\gamma\delta}) \nabla^\mu \nabla^\nu (R^{\alpha\beta\gamma\delta}) R_{\epsilon\zeta\eta\theta} R^{\epsilon\zeta\eta\theta} \; ; \\ &\mathcal{E}_3 \equiv \big((\nabla^2 R_{\alpha\beta\gamma\delta}) R^{\alpha\beta\gamma\delta}\big)^2 \quad ; \quad &&\mathcal{E}_4 \equiv \nabla_\mu \nabla_\nu (R_{\alpha\beta\gamma\delta}) \nabla^\mu \nabla^\nu (R_{\epsilon\zeta\eta\theta}) R^{\alpha\beta\gamma\delta} R^{\epsilon\zeta\eta\theta} \; ; \\ &\mathcal{E}_5 \equiv \nabla^2 R^{\alpha\beta}{}_{\gamma\delta} \big(\nabla^2 R_{\alpha\beta}{}^{\epsilon\zeta}\big) R_{\epsilon\zeta}{}^{\eta\theta} R^{\gamma\delta}{}_{\eta\theta} \quad ; \quad &&\mathcal{E}_6 \equiv \nabla_\mu \nabla_\nu (R^{\alpha\beta}{}_{\gamma\delta}) \nabla^\mu \nabla^\nu (R_{\alpha\beta}{}^{\epsilon\zeta}) R_{\epsilon\zeta}{}^{\eta\theta} R^{\gamma\delta}{}_{\eta\theta} \; ; \\ &\mathcal{E}_7 \equiv \nabla^2 R^{\alpha\beta}{}_{\gamma\delta} \big(\nabla^2 R_{\epsilon\zeta}{}^{\eta\theta}\big) R_{\alpha\beta}{}^{\epsilon\zeta} R^{\gamma\delta}{}_{\eta\theta} \quad ; \quad &&\mathcal{E}_8 \equiv \nabla_\mu \nabla_\nu (R^{\alpha\beta}{}_{\gamma\delta}) \nabla^\mu \nabla^\nu (R_{\epsilon\zeta}{}^{\eta\theta}) R_{\alpha\beta}{}^{\epsilon\zeta} R^{\gamma\delta}{}_{\eta\theta} \; . \end{aligned} \tag{108}$$

These are expressed in terms of the quantities A_1, A_2, A_3, A_4 and are displayed in Appendix C.

c. Type 3: $(\nabla^2 R)(\nabla R)^2 R$ terms The possible terms constructed from $\mathcal{R}_1$ and $\mathcal{R}_2$ are:

$$\begin{aligned}
\mathcal{F}_1 &\equiv (\nabla^2 R_{\alpha\beta\gamma\delta}) R^{\alpha\beta\gamma\delta} \nabla_\mu R_{\epsilon\zeta\eta\theta} \nabla^\mu R^{\epsilon\zeta\eta\theta} \quad ; \quad \mathcal{F}_2 \equiv (\nabla_\mu \nabla_\nu R_{\alpha\beta\gamma\delta}) R^{\alpha\beta\gamma\delta} \nabla^\mu R_{\epsilon\zeta\eta\theta} \nabla^\nu R^{\epsilon\zeta\eta\theta} \; ; \\
\mathcal{F}_3 &\equiv (\nabla^2 R_{\alpha\beta\gamma\delta}) R_{\epsilon\zeta\eta\theta} \nabla_\mu R^{\alpha\beta\gamma\delta} \nabla^\mu R^{\epsilon\zeta\eta\theta} \quad ; \quad \mathcal{F}_4 \equiv (\nabla_\mu \nabla_\nu R_{\alpha\beta\gamma\delta}) R_{\epsilon\zeta\eta\theta} \nabla^\mu R^{\alpha\beta\gamma\delta} \nabla^\nu R^{\epsilon\zeta\eta\theta} \; ; \\
\mathcal{F}_5 &\equiv \left(\nabla^2 R^{\alpha\beta}{}_{\gamma\delta}\right) R_{\alpha\beta}{}^{\epsilon\zeta} \nabla_\mu R_{\epsilon\zeta}{}^{\eta\theta} \nabla^\mu R^{\gamma\delta}{}_{\eta\theta} \quad ; \quad \mathcal{F}_6 \equiv \nabla_\mu \nabla_\nu (R^{\alpha\beta}{}_{\gamma\delta}) R_{\alpha\beta}{}^{\epsilon\zeta} \nabla^\mu R_{\epsilon\zeta}{}^{\eta\theta} \nabla^\nu R^{\gamma\delta}{}_{\eta\theta} \; ; \\
\mathcal{F}_7 &\equiv \left(\nabla^2 R^{\alpha\beta}{}_{\gamma\delta}\right) R_{\epsilon\zeta}{}^{\eta\theta} \nabla_\mu R_{\alpha\beta}{}^{\epsilon\zeta} \nabla^\mu R^{\gamma\delta}{}_{\eta\theta} \quad ; \quad \mathcal{F}_8 \equiv \nabla_\mu \nabla_\nu (R^{\alpha\beta}{}_{\gamma\delta}) R_{\epsilon\zeta}{}^{\eta\theta} \nabla^\mu R_{\alpha\beta}{}^{\epsilon\zeta} \nabla^\nu R^{\gamma\delta}{}_{\eta\theta} \; .
\end{aligned} \tag{109}$$

Again they can be expressed in terms of A_1, A_2, A_3 and A_4, see Appendix C.

To compute the contribution to the constraint equation stemming from $\mathcal{E}$ and $\mathcal{F}$ terms, we will need to compute those of the following basic expressions:

$$\begin{aligned}
&\gamma_1 = A_1^2 A_2^2 (A_2 - A_1)^2 \; ; &&\gamma_2 = A_2^4 (A_2 - A_1)^2 \; ; &&\gamma_3 = A_1 A_2^3 (A_2 - A_1)^2 \; ; \\
&\gamma_4 = A_1^2 A_4^2 \; ; &&\gamma_5 = A_2^2 A_4^2 \; ; &&\gamma_6 = A_1 A_2^2 (A_2 - A_1) A_4 \; ; \\
&\gamma_7 = A_2 A_3^2 A_4 \; ; &&\gamma_8 = A_1 A_2 (A_2 - A_1) A_3^2 \; ; &&\gamma_9 = \frac{\dot a}{aN} A_1^2 A_2 (A_2 - A_1) A_3 \; ; \\
&\gamma_{10} = \frac{\dot a}{aN} A_2^3 (A_2 - A_1) A_3 \; ; &&\gamma_{11} = \frac{\dot a}{aN} A_1 A_2^2 (A_2 - A_1) A_3 \; ; &&\gamma_{12} = \frac{\dot a}{aN} A_1^2 A_3 A_4 \; ; \\
&\gamma_{13} = \frac{\dot a}{aN} A_1 A_2 A_3 A_4 \; ; &&\gamma_{14} = \frac{\dot a}{aN} A_2^2 A_3 A_4 \; ; &&\gamma_{15} = \frac{\dot a}{aN} A_1 A_3^3 \; ; \\
&\gamma_{16} = \frac{\dot a}{aN} A_2 A_3^3 \; ; &&\gamma_{17} = \frac{\dot a^2}{a^2 N^2} A_1^2 A_2 (A_2 - A_1)^2 \; ; &&\gamma_{18} = \frac{\dot a^2}{a^2 N^2} A_1 A_2^2 (A_2 - A_1)^2 \; ; \\
&\gamma_{19} = \frac{\dot a^2}{a^2 N^2} A_2^3 (A_2 - A_1)^2 \; ; &&\gamma_{20} = \frac{\dot a^2}{a^2 N^2} A_1^2 A_3^2 \; ; &&\gamma_{21} = \frac{\dot a^2}{a^2 N^2} A_1 A_2 A_3^2 \quad ; \\
&\gamma_{22} = \frac{\dot a^2}{a^2 N^2} A_2^2 A_3^2 \; ; &&\gamma_{23} = \frac{\dot a^2}{a^2 N^2} A_1^2 (A_2 - A_1) A_4 \; ; &&\gamma_{24} = \frac{\dot a^2}{a^2 N^2} A_1 A_2 (A_2 - A_1) A_4 \; ; \\
&\gamma_{25} = \frac{\dot a^2}{a^2 N^2} A_2^2 (A_2 - A_1) A_4 \; ; &&\gamma_{26} = \frac{\dot a^3}{a^3 N^3} A_1^2 (A_2 - A_1) A_3 \; ; &&\gamma_{27} = \frac{\dot a^3}{a^3 N^3} A_1 A_2 (A_2 - A_1) A_3 \; ; \\
&\gamma_{28} = \frac{\dot a^3}{a^3 N^3} A_2^2 (A_2 - A_1) A_3 \; ; &&\gamma_{29} = \frac{\dot a^4}{a^4 N^4} A_1^2 (A_2 - A_1)^2 \; ; &&\gamma_{30} = \frac{\dot a^4}{a^4 N^4} A_1 A_2 (A_2 - A_1)^2 \; ; \\
&\gamma_{31} = \frac{\dot a^4}{a^4 N^4} A_2^2 (A_2 - A_1)^2 \, .
\end{aligned}$$

We denote $\Gamma_i \equiv \frac{\delta}{\delta N}\left[a^3 N \, \gamma_i\right]$ the constraint contributions from these basic expressions. All $\mathcal{E}$ and $\mathcal{F}$ terms can be expressed as linear combinations of the γ terms, so their constraint equations will be equal to the same linear combination of the corresponding Γ terms.

First we compute the contributions from all the γ terms, and plug in them the no-boundary ansatz (30). Then we expand all Γs to third order in t. Only nine out of these 31 terms actually start at order t^{-1} (as we expected of terms where four covariant derivatives act on Riemann terms). They are, to leading order,[6]

$$\Gamma_{23} = \Gamma_{24} = \Gamma_{25} = -\frac{2a_3^2\big(a_3^2 - a_1 a_5\big)}{a_1^{13} t} \quad ; \quad \Gamma_{26} = \Gamma_{27} = \Gamma_{28} = -\frac{2a_3^2\big(a_3^2 - a_1 a_5\big)}{3a_1^{13} t} \; ; \tag{110}$$

$$\text{and} \quad \Gamma_{29} = \Gamma_{30} = \Gamma_{31} = -\frac{a_3^2\big(a_3^2 - a_1 a_5\big)}{3a_1^{13} t} \; . \tag{111}$$

In the $\mathcal{E}$ and $\mathcal{F}$ terms, these nine terms appear in the eleven following combinations, which all give contributions that start at least at order t:

$$\frac{\delta}{\delta N}\left[\frac{\dot a^2}{a^2 N^2}(A_2 - A_1)^3 A_4\right] \equiv \Gamma_{23} - 2\Gamma_{24} + \Gamma_{25} = O(t^3) \; ; \tag{112}$$

$$\frac{\delta}{\delta N}\left[\frac{\dot a^2}{a^2 N^2}A_1(A_2 - A_1)^2 A_4\right] \equiv \Gamma_{24} - \Gamma_{23} = O(t) \; ; \tag{113}$$

$$\frac{\delta}{\delta N}\left[\frac{\dot a^2}{a^2 N^2}A_2(A_2 - A_1)^2 A_4\right] \equiv \Gamma_{25} - \Gamma_{24} = O(t) \; ; \tag{114}$$

$$\frac{\delta}{\delta N}\left[\frac{\dot a^3}{a^3 N^3}A_3(A_2 - A_1)^3\right] = \zeta_4 \equiv \Gamma_{26} - 2\Gamma_{27} + \Gamma_{28} = O(t^3) \; ; \tag{115}$$

$$\frac{\delta}{\delta N}\left[\frac{\dot a^3}{a^3 N^3}A_3 A_1(A_2 - A_1)^2\right] \equiv \Gamma_{27} - \Gamma_{26} = O(t) \; ; \tag{116}$$

$$\frac{\delta}{\delta N}\left[\frac{\dot a^3}{a^3 N^3}A_3 A_2(A_2 - A_1)^2\right] \equiv \Gamma_{28} - \Gamma_{27} = O(t) \; ; \tag{117}$$

$$\frac{\delta}{\delta N}\left[\frac{\dot a^4}{a^4 N^4}(A_2 - A_1)^4\right] = \zeta_2 \equiv \Gamma_{29} - 2\Gamma_{30} + \Gamma_{31} = O(t^3) \; ; \tag{118}$$

$$\frac{\delta}{\delta N}\left[\frac{\dot a^4}{a^4 N^4}A_1(A_2 - A_1)^3\right] \equiv \Gamma_{30} - \Gamma_{29} = O(t) \; ; \tag{119}$$

$$\frac{\delta}{\delta N}\left[\frac{\dot a^4}{a^4 N^4}A_2(A_2 - A_1)^3\right] \equiv \Gamma_{31} - \Gamma_{30} = O(t) \; ; \tag{120}$$

$$\frac{\delta}{\delta N}\left[\frac{\dot a^3}{a^3 N^3}A_1^2(A_2 - A_1)\Big(A_3 - \frac{\dot a}{aN}(A_2 - A_1)\Big)\right] \equiv \Gamma_{26} - 2\Gamma_{29} = O(t) \; ; \tag{121}$$

$$\frac{\delta}{\delta N}\left[\frac{\dot a^2}{a^2 N^2}A_1^2(A_2 - A_1)\Big(A_4 - \frac{\dot a}{aN}A_3 - \frac{4\dot a^2}{a^2 N^2}(A_2 - A_1)\Big)\right] \equiv \Gamma_{23} - \Gamma_{26} - 4\Gamma_{29} = O(t) \; . \tag{122}$$

In fact, astonishingly, the cancellations go even further and the contribution at order t also vanishes identically. The full expressions, which start at order t^3, are listed in Appendix C. Schematically, the order t^3 contribution of all $\delta\mathcal{F}$ terms can be written as

$$\delta\mathcal{F} = \frac{(a_3^2 - a_1 a_5)t^3}{a_1^{15}}\Big[\lambda_1\big(a_3^2 - a_1 a_5\big)^2 + \lambda_2 a_1 a_5\big(a_3^2 - a_1 a_5\big) + \lambda_3 a_1 a_3\big(a_3 a_5 - a_1 a_7\big)\Big] + O(t^5) \; ; \tag{123}$$

[6] Beware that these equalities are only valid at order t^{-1}.

where λ_1, λ_2 and λ_3 take different numerical values for each combination of derivatives. As for the $\mathcal{E}$ terms, their contribution to the constraint is of the form

$$\delta\mathcal{E} = \frac{t^3}{a_1^{15}}\Big[\mu_1\big(a_3^2 - a_1a_5\big)^3 + \mu_2 a_1a_5\big(a_3^2 - a_1a_5\big)^2 + a_1a_3\big(\mu_3a_5a_3 + \mu_4a_1a_7\big)\big(a_3^2 - a_1a_5\big) + \mu_5a_1^2a_3a_5\big(a_3a_5 - a_1a_7\big) + \mu_6a_1^2a_3^2\big(a_1a_9 - a_3a_7\big)\Big] + O(t^5)\,; \tag{124}$$

where μ_i are numerical factors varying for each case.

We are now in position to compute the type II string theory constraint equation up to fifth order in α', and see whether this action admits a no-boundary solution.

D. Constraint equation for type II string theory

When compactified down to four dimensions, the type II action is of the form

$$S^{4d}_{\text{type II}} = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\Big[R-(\partial\phi)^2-2V(\phi)+(\alpha')^3\mathcal{E}_{(0,0)}\mathcal{R}^4+(\alpha')^5\mathcal{E}_{(1,0)}\nabla^4\mathcal{R}^4+O(\alpha'^6)+\cdots\Big], \tag{125}$$

where we included a single scalar field with a potential $V(\phi)$, but where the ellipsis stands for many additional scalars and gauge fields, with the precise form of the action depending on the details of the compactification. In looking for no-boundary solutions we may once again neglect the contribution due to the gauge fields. In the same vein, the contributions in higher powers of α' should be thought of as containing compactification dependent coefficient functions θ, δ_i, ϵ_i and η_i, in front of the specific combinations $\mathcal{D}, \mathcal{E}, \mathcal{F}$ that we introduced in section V C:

$$\mathcal{E}_{(0,0)}\mathcal{R}^4 = \theta\left(\mathcal{R}_1 + \frac{246}{163}\mathcal{R}_2\right) \quad \text{and} \quad \mathcal{E}_{(1,0)}\nabla^4\mathcal{R}^4 = \sum_{i=1}^{4}\delta_i\mathcal{D}_i + \sum_{i=1}^{8}\epsilon_i\mathcal{E}_i + \sum_{i=1}^{8}\eta_i\mathcal{F}_i\,. \tag{126}$$

Does this theory now admit no-boundary solutions? As we demonstrated in the last section, the constraint equation, which provides the litmus test for the existence of regular solutions, does not receive α' corrections at order t^{-1} nor at order t when the no-boundary ansatz (30) is plugged in, due to the specific form of the $\mathcal{D}, \mathcal{E}, \mathcal{F}$ terms. This rather astonishing result may have an underlying explanation in the fact that no-boundary solutions approach Euclidean flat space smoothly near the South Pole, and hence covariant derivatives acting on the corresponding Riemann tensors are suppressed. In fact, the first non-trivial contributions to the constraint equation arise at order t^3, where the constraint takes the form

$$\begin{aligned}
&-6a_3t^3 - 2V(\phi_0)a_1^3t^3 + (\alpha')^3\mathcal{E}_{(0,0)}\left[2205\cdot\frac{2a_3^2}{a_1^9}\left(3a_1a_5-4a_3^2\right)t^3 - 2934\cdot\frac{a_3^2}{a_1^9}\left(2a_3^2-a_1a_5\right)t^3\right]\\
&+(\alpha')^5\mathcal{E}_{(0,1)}\left[\#_1\cdot\frac{\left(a_3^2-a_1a_5\right)^3}{a_1^{15}}t^3 + \#_2\cdot\frac{a_5\left(a_3^2-a_1a_5\right)^2}{a_1^{14}} + \left(\#_3\cdot a_3^2a_5 + \#_4\cdot a_1a_3a_7\right)\frac{a_3^2-a_1a_5}{a_1^{14}}\right.\\
&\qquad\left.+\#_5\cdot\frac{a_3a_5\left(a_3a_5-a_1a_7\right)}{a_1^{13}} + \#_6\cdot\frac{a_3^2\left(a_1a_9-a_3a_7\right)}{a_1^{13}}\right] = 0\,. \qquad (127)
\end{aligned}$$

Here we denoted $\phi(0)=\phi_0$ and the numerical coefficients at order α'^5 by $\#_i$. In the absence of higher order corrections we would have learned that $a_3 = -\frac{V(\phi_0)}{3}a_1^3$, i.e. that the initial expansion rate depends on the location of the scalar field on the potential. Once the higher order terms are added, new families of solutions arise, and depending on the coefficient functions, a_5, a_7 and even a_9 can enter the constraint equation. At higher orders in t, higher order terms in the series expansion for $a(t)$ will of course also appear, and in this manner higher coefficients will continue to be given in terms of the lower order ones. Also, for terms with more derivatives, such as terms of the form $\nabla^6\mathcal{R}^4$, we expect higher $a(t)$ Taylor series coefficients to appear, in analogy with the results for $\mathcal{C}$ terms (see Appendix B). For perturbative solutions, a self-consistency check will be that the solutions should have a smooth limit as $\alpha'\to 0$, very much like the limit $\beta\to 0$ encountered in section IV A on quadratic gravity. What is clear however is that, given the current knowledge about α' corrections, perturbative no-boundary solutions exist in type II string theory.

VI. CONCLUSIONS

The general expectation in cosmology is that as we approach the big bang, quantum gravity corrections will become more and more important, to the extent that we might remain ignorant about the initial stages of the universe until we will have fully uncovered quantum gravity. The no-boundary proposal, which is arguably the best understood theory for the initial conditions of the universe, goes somewhat against the grain by being formulated merely in semi-classical gravity. The question that concerned us in the present paper was whether the no-boundary proposal stands a chance of providing reliable answers given our current, partial, knowledge of quantum gravity.

The very lack of a complete theory of quantum gravity means that we are not able to answer this question fully, yet the problem is still tractable to the extent that the general structure of perturbative quantum gravity corrections is known. Such corrections are expected to involve higher powers of the Riemann tensor as well as covariant derivatives acting on these tensors. The question thus becomes whether no-boundary solutions continue to exist in the presence of such correction terms. We have been able to derive explicit conditions, in particular Eq. (32), that terms composed solely of Riemann tensors have to satisfy in order for no-boundary solutions to exist. This requirement is met for $f(R)$ gravity, quadratic gravity, Gauss-Bonnet gravity, heterotic string theory as well as type II string theory including the first non-trivial order in α'. What is more, by considering specific examples, we have been able to show that terms involving covariant derivatives acting on Riemann tensors may also coexist with no-boundary solutions. Here we studied the specific example provided by type II string theory up to order α'^5. An interesting open question is whether the structure of string theory is such that it allows for no-boundary solutions in general.

Our results provide an important consistency check of the no-boundary proposal, as they show that for large classes of theories the results obtained in semi-classical gravity are robust. This in no way precludes the existence of qualitatively new solutions in full quantum gravity, but it does imply that no-boundary solutions will continue to exist in perturbative quantum gravity. Combined with the recent progress in constructing a consistent path integral implementation [6–8], our results put the no-boundary proposal on a rather firm theoretical footing.

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Appendix A: Constraint equation of Riemann terms in the no-boundary ansatz

We plug the no-boundary ansatz (30) into the Friedmann constraint equation (28), and expand it at lowest orders in t. From (31) we know that at lowest order $A_1 = A_2 = \frac{a_3}{a_1 N^2}$. Therefore we get

$$0 = 2\pi^2 \sum_{p_1,p_2} c_{p_1,p_2}\Bigg[2p_1(p_2-1)\frac{a_1 t\cdot a_1^2}{N^2}\Big(\frac{a_3}{a_1N^2}\Big)^{P-1} + p_2(p_2-1)\frac{a_1 t\cdot a_1\cdot a_3}{N^4}\Big(\frac{a_3}{a_1N^2}\Big)^{P-2} - p_2(2p_1+p_2-3)\frac{a_1 t\cdot a_1^2}{N^2}\Big(\frac{a_3}{a_1N^2}\Big)^{P-1}\Bigg] + O(t^3)\,; \tag{A1}$$

where we defined $P = p_1 + p_2$ for simplicity.

This leading order equation can be further simplified to

$$2\pi^2 \sum_{p_1,p_2} \frac{c_{p_1,p_2}}{N^{2P}} a_1^{4-P} a_3^{P-1}\left[2p_2 - 2p_1\right] t + O(t^3) = 0\,. \tag{A2}$$

Let us now look at the next order. Because $a(t)$ is an odd function of t, and hence A_1 and A_2 are even functions of t (see (31)), the t^2 order of the Friedmann constraint will vanish. We directly consider the t^3 order of the Friedmann constraint:

$$\begin{aligned}2\pi^2 \sum_{p_1,p_2} c_{p_1,p_2}\Bigg[&2p_1(p_2-1)\frac{\big(a_1t+\frac{a_3t^3}{6}\big)\big(a_1+\frac{a_3t^2}{2}\big)^2}{N^2}\Big(\frac{a_3}{a_1N^2}+\frac{a_3^2-a_1a_5}{12N^4}t^2\Big)^{p_1-1}\Big(\frac{a_3}{a_1N^2}+\frac{a_3^2-a_1a_5}{6N^4}t^2\Big)^{p_2}\\ &+p_2(p_2-1)\frac{\big(a_1t+\frac{a_3t^3}{6}\big)\big(a_1+\frac{a_3t^2}{2}\big)\big(a_3+\frac{a_5t^2}{2}\big)}{N^4}\Big(\frac{a_3}{a_1N^2}+\frac{a_3^2-a_1a_5}{12N^4}t^2\Big)^{p_1}\Big(\frac{a_3}{a_1N^2}+\frac{a_3^2-a_1a_5}{6N^4}t^2\Big)^{p_2-2}\\ &-p_2(2p_1+p_2-3)\frac{\big(a_1t+\frac{a_3t^3}{6}\big)\cdot\big(a_1+\frac{a_3t^2}{2}\big)^2}{N^2}\Big(\frac{a_3}{a_1N^2}+\frac{a_3^2-a_1a_5}{12N^4}t^2\Big)^{p_1}\Big(\frac{a_3}{a_1N^2}+\frac{a_3^2-a_1a_5}{6N^4}t^2\Big)^{p_2-1}\\ &+(1-p_2)\big(a_1t\big)^3\Big(\frac{a_3}{a_1N^2}\Big)^P\Bigg] + O(t^5) = 0\,.\end{aligned} \tag{A3}$$

This can then be simplified to

$$2\pi^2 \sum_{p_1,p_2} \frac{c_{p_1,p_2}}{N^{2P}} a_1^{3-P} a_3^{P-2}\Big(a_3^2\cdot G_3[p_1,p_2] + a_1a_5\cdot G_5[p_1,p_2]\Big)t^3 + O(t^5) = 0\,, \tag{A4}$$

with

$$G_3[p_1,p_2] = \frac{1}{6}\left(p_1^2 - 15p_1 + 6 - 4p_2^2 + 12p_2\right) \quad \text{and} \quad G_5[p_1,p_2] = \frac{p_1}{6}\left(1-p_1\right) - \frac{2p_2}{3}\left(1-p_2\right)\,. \tag{A5}$$

Appendix B: Constraint equations for $\mathcal{B}$ and $\mathcal{C}$ terms

Here we display the constraint equations of $\mathcal{B}$ terms where the no-boundary ansatz has been plugged in. Writing $\delta\mathcal{B} \equiv \frac{\delta}{\delta N}\big(a^3 N\mathcal{B}\big)$, we find

$$\delta\mathcal{B}_1 = -12\left[\frac{4}{15a_1^6}\Big(25a_3^3 - 29a_1a_3a_5 + 4a_1^2a_7\Big)\,t^3\right] + O(t^5)\,; \tag{B1}$$

$$\delta\mathcal{B}_2 = -\,12\left[\frac{1}{15a_1^6}\Big(85a_3^3 - 101a_1a_3a_5 + 16a_1^2a_7\Big)\,t^3\right] + O(t^5)\,; \tag{B2}$$

$$\delta\mathcal{B}_3 = -\,36\left[\frac{2}{15a_1^6}\Big(35a_3^3 - 43a_1a_3a_5 + 8a_1^2a_7\Big)\,t^3\right] + O(t^5)\,; \tag{B3}$$

$$\delta\mathcal{B}_4 = -\,\frac{4}{5a_1^6}\Big(35a_3^3 - 43a_1a_3a_5 + 8a_1^2a_7\Big)t^3 + O(t^5)\,. \tag{B4}$$

All those $\nabla^2 R^2$ terms possess a no-boundary solution which specifies a_7 in terms of a_1, a_3, and a_5, but where the latter are not specified by the $\nabla^2 R^2$ terms alone.

We now look at the constraint equations for $\mathcal{C}$ terms. Their expressions in terms of A_1, A_2, A_3 and A_4 are

$$\mathcal{C}_1 = 12\left[4\Big[A_2(A_2-A_1) + \frac{2\dot{a}^2}{a^2N^2}(A_2-A_1) + \frac{\dot{a}}{aN}A_3\Big]^2 + \Big[A_4 - \frac{4\dot{a}^2}{a^2N^2}(A_2-A_1) + \frac{\dot{a}}{aN}A_3\Big]^2\right]; \tag{B5}$$

$$\begin{aligned}\mathcal{C}_2 = \; & 12\Bigg[A_4^2 + \frac{4\dot{a}}{aN}A_3A_4 + \frac{7\dot{a}^2}{a^2N^2}A_3^2 + 2A_4A_2(A_2-A_1) + 4A_2^2(A_2-A_1)^2 - \frac{4\dot{a}^2}{a^2N^2}A_4(A_2-A_1) \\ & + \frac{16\dot{a}^4}{a^4N^4}(A_2-A_1)^2 + \frac{8\dot{a}^2}{a^2N^2}A_2(A_2-A_1)^2 + \frac{10\dot{a}}{aN}A_3A_2(A_2-A_1) + \frac{4\dot{a}^3}{a^3N^3}A_3(A_2-A_1)\Bigg];\end{aligned} \tag{B6}$$

$$\mathcal{C}_3 = 36\left[\frac{3\dot{a}}{aN}A_3 + 2A_2(A_2-A_1) + A_4\right]^2; \tag{B7}$$

$$\begin{aligned}\mathcal{C}_4 = 12\Bigg[&A_4^2 - \frac{4\dot{a}}{aN}A_3A_4 + \frac{19\dot{a}^2}{a^2N^2}A_3^2 + \frac{16\dot{a}}{aN}A_3A_2(A_2-A_1) - \frac{80\dot{a}^3}{a^3N^3}A_3(A_2-A_1) \\ & + \frac{160\dot{a}^4}{a^4N^4}(A_2-A_1)^2 - \frac{48\dot{a}^2}{a^2N^2}A_2(A_2-A_1)^2 + 8A_2^2(A_2-A_1)^2\Bigg];\end{aligned} \tag{B8}$$

$$\begin{aligned}\mathcal{C}_5 = 12\Bigg[&A_4^2 - \frac{2\dot{a}}{aN}A_3A_4 + \frac{11\dot{a}^2}{a^2N^2}A_3^2 - \frac{34\dot{a}^3}{a^3N^3}A_3\big(A_2-A_1\big) + \frac{8\dot{a}}{aN}A_3A_2\big(A_2-A_1\big) + 2A_4A_2\big(A_2-A_1\big) \\ & - \frac{6\dot{a}^2}{a^2N^2}A_4\big(A_2-A_1\big) + \frac{104\dot{a}^4}{a^4N^4}\big(A_2-A_1\big)^2 - \frac{36\dot{a}^2}{a^2N^2}A_2\big(A_2-A_1\big)^2 + 6A_2^2\big(A_2-A_1\big)^2\Bigg];\end{aligned} \tag{B9}$$

$$\begin{aligned}\mathcal{C}_6 = 36\Bigg[&\Big[A_4 + 2A_2(A_2-A_1)\Big]^2 - \frac{12\dot{a}^2}{a^2N^2}(A_2-A_1)A_4 - \frac{24\dot{a}^2}{a^2N^2}A_2(A_2-A_1)^2 + \frac{3\dot{a}^2}{a^2N^2}A_3^2 \\ & + \frac{12\dot{a}^3}{a^3N^3}(A_2-A_1)A_3 + \frac{48\dot{a}^4}{a^4N^4}(A_2-A_1)^2\Bigg].\end{aligned} \tag{B10}$$

Writing $\delta\mathcal{C} \equiv \frac{\delta}{\delta N}\big(a^3N\mathcal{C}\big)$, we find

$$\delta\mathcal{C}_1 = \frac{8t^3\left(3262a_1a_3^2a_5 + 60a_1^3a_9 - 2135a_3^4 - a_1^2\left(592a_3a_7 + 595a_5^2\right)\right)}{35a_1^9} + O\left(t^5\right)\;; \tag{B11}$$

$$\delta\mathcal{C}_2 = \frac{4t^3\left(3528a_1a_3^2a_5 - 5a_1^2\left(161a_5^2 - 24a_1a_9\right) - 1995a_3^4 - 848a_1^2a_3a_7\right)}{35a_1^9} + O\left(t^5\right)\;; \tag{B12}$$

$$\delta\mathcal{C}_3 = \frac{48t^3\left(133a_1a_3^2a_5 - 15a_1^2\left(7a_5^2 - 2a_1a_9\right) + 70a_3^4 + 128a_3a_7N^2\right)}{35a_1^9} + O\left(t^5\right)\;; \tag{B13}$$

$$\delta\mathcal{C}_4 = \frac{8t^3\left(1008a_1a_3^2a_5 + 12a_1^3a_9 - 735a_3^4 - 19a_1^2\left(8a_3a_7 + 7a_5^2\right)\right)}{7a_1N^8} + O\left(t^5\right)\;; \tag{B14}$$

$$\delta\mathcal{C}_5 = \frac{2t^3\left(13048a_1a_3^2a_5 - 5a_1^2\left(413a_5^2 - 48a_1a_9\right) - 8855a_3^4 - 2368a_1^2a_3a_7\right)}{35a_1^9} + O\left(t^5\right)\ ; \tag{B15}$$

$$\delta\mathcal{C}_6 = \frac{12t^3\left(2968a_1a_3^2a_5 - 15a_1^2\left(49a_5^2 - 8a_1a_9\right) - 1505a_3^4 - 848a_1^2a_3a_7\right)}{35a_1^9} + O\left(t^5\right)\ . \tag{B16}$$

These six ∇^4R^2 terms all admit a regular no-boundary solution, for which the coefficient a_9 is fixed in terms of a_1, a_3, a_5 and a_7 at order t^3 of the constraint. This ensures the existence of a solution if these ∇^4R^2 terms are combined with Riemann terms and ∇^2R^2 terms, since a_9 is a new degree of freedom at order t^3.

Appendix C: Constraint equations from $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ terms

Expressions of $\mathcal{D}$ terms as functions of A_1, A_2 and A_3:

$$\mathcal{D}_1 = 144\left[A_3^4 + \frac{16\dot{a}^2}{a^2N^2}A_3^2\left(A_2-A_1\right)^2 + \frac{64\dot{a}^4}{a^4N^4}\left(A_2-A_1\right)^4\right]; \tag{C1}$$

$$\mathcal{D}_2 = 48\left[3A_3^4 + \frac{24\dot{a}^2}{a^2N^2}A_3^2(A_2-A_1)^2 + \frac{64\dot{a}^4}{a^4N^4}\left(A_2-A_1\right)^4\right]; \tag{C2}$$

$$\mathcal{D}_3 = 48\left[A_3^4 + \frac{4\dot{a}^2}{a^2N^2}A_3^2(A_2-A_1)^2 + \frac{40\dot{a}^4}{a^4N^4}\left(A_2-A_1\right)^4\right]; \tag{C3}$$

$$\mathcal{D}_4 = 48\left[A_3^4 + \frac{16\dot{a}^3}{a^3N^3}A_3\left(A_2-A_1\right)^3 + \frac{20\dot{a}^4}{a^4N^4}\left(A_2-A_1\right)^4\right]. \tag{C4}$$

Expressions of $\mathcal{E}$ terms as functions of A_1, A_2, A_3 and A_4:

$$\begin{aligned}\mathcal{E}_1 =& 144\left(A_1^2+A_2^2\right)\Bigg[4\Big(\frac{\dot{a}}{aN}A_3 + \frac{2\dot{a}^2}{a^2N^2}(A_2-A_1) + A_2(A_2-A_1)\Big)^2 \\ &+ \Big(A_4 + \frac{4\dot{a}^2}{a^2N^2}(A_1-A_2) + \frac{\dot{a}}{aN}A_3\Big)^2\Bigg];\end{aligned} \tag{C5}$$

$$\begin{aligned}\mathcal{E}_2 =& 144\left(A_1^2+A_2^2\right)\Bigg[A_4^2 - \frac{4\dot{a}}{aN}A_4A_3 + \frac{19\dot{a}^2}{a^2N^2}A_3^2 + \frac{16\dot{a}}{aN}A_3A_2(A_2-A_1) + \frac{80\dot{a}^3}{a^3N^3}A_3(A_1-A_2) \\ &+ 8A_2^2(A_2-A_1)^2 + \frac{160\dot{a}^4}{a^4N^4}(A_2-A_1)^2 - \frac{48\dot{a}^2}{a^2N^2}A_2(A_2-A_1)^2\Bigg];\end{aligned} \tag{C6}$$

$$\mathcal{E}_3 = 144\left[A_2A_4 + \frac{\dot{a}}{aN}A_3(A_2+2A_1) - 2A_1A_2(A_1-A_2) - \frac{4\dot{a}^2}{a^2N^2}(A_1-A_2)^2\right]^2; \tag{C7}$$

$$\begin{aligned}\mathcal{E}_4 =& 48\Bigg[12A_1^2A_2^2(A_2-A_1)^2 + 3A_2^2A_4^2 + 12A_1A_2^2(A_2-A_1)A_4 + \frac{16\dot{a}^4}{a^4N^4}(A_2-A_1)^2(A_2^2-5A_1A_2+13A_1^2) \\ &- \frac{12\dot{a}^3}{a^3N^3}A_3(A_2-A_1)\Big(2(A_2-A_1)^2 - 4A_1(A_2-A_1) - 3A_1A_2\Big) \\ &- \frac{12\dot{a}^2}{a^2N^2}(A_2-A_1)\Big(3A_1A_2A_4 - A_3^2(A_2-A_1) + 6A_1^2A_2(A_2-A_1)\Big) + \frac{9\dot{a}^2}{a^2N^2}A_3^2A_2^2\end{aligned}$$

$$-\frac{12\dot{a}}{aN}A_3A_2(A_2-A_1)\Big(A_4+2A_1(A_2-A_1)\Big)\Bigg]\ ; \tag{C8}$$

$$\mathcal{E}_5=48\Bigg[4A_1^2\Big[\frac{\dot{a}}{aN}A_3+\frac{2\dot{a}^2}{a^2N^2}(A_2-A_1)+A_2(A_2-A_1)\Big]^2+A_2^2\Big[A_4+\frac{\dot{a}}{aN}A_3+\frac{4\dot{a}^2}{a^2N^2}(A_1-A_2)\Big]^2\Bigg]\ ; \tag{C9}$$

$$\begin{aligned}\mathcal{E}_6=&48\Bigg[16\frac{\dot{a}^4}{a^4N^4}(A_2-A_1)^2(7A_1^2+3A_2^2)-12\frac{\dot{a}^2}{a^2N^2}A_2(A_2-A_1)^2(A_2^2+3A_1^2)\\&+2A_2^2(A_2-A_1)^2(A_2^2+3A_1^2)+\frac{\dot{a}^2}{a^2N^2}A_3^2(8A_1^2+11A_2^2)+16A_3\frac{\dot{a}^3}{a^3N^3}(A_1-A_2)(2A_2^2+3A_1^2)\\&+4\frac{\dot{a}}{aN}A_3A_2(A_2-A_1)(A_2^2+3A_1^2)+A_2^2A_4^2-4\frac{\dot{a}}{aN}A_3A_2^2A_4\Bigg]\ ;\end{aligned} \tag{C10}$$

$$\begin{aligned}\mathcal{E}_7=&48\Bigg[\frac{16\dot{a}^4}{a^4N^4}(A_2-A_1)^2(A_1^2+A_2^2)+4A_1^2A_2^2(A_2-A_1)^2-\frac{8\dot{a}^3}{a^3N^3}A_3(A_2-A_1)(A_2^2-2A_1^2)\\&+\frac{\dot{a}^2}{a^2N^2}(4A_1^2+A_2^2)A_3^2+\frac{8\dot{a}^2}{a^2N^2}A_2^2A_4(A_1-A_2)+\frac{16\dot{a}^2}{a^2N^2}A_1^2A_2(A_2-A_1)^2\\&+\frac{2\dot{a}}{aN}A_2^2A_3A_4+\frac{8\dot{a}}{aN}A_1^2A_2A_3(A_2-A_1)+A_2^2A_4^2\Bigg]=\mathcal{E}_5\ ;\end{aligned} \tag{C11}$$

$$\begin{aligned}\mathcal{E}_8=&48\Bigg[\frac{4\dot{a}^4}{a^4N^4}(A_2-A_1)^2(3A_2^2+18A_2A_1+19A_1^2)-\frac{8\dot{a}^3}{a^3N^3}A_3(A_2-A_1)(A_2^2+6A_2A_1+3A_1^2)\\&+\frac{\dot{a}^2}{a^2N^2}A_3^2(8A_2A_1+7A_2^2+4A_1^2)-\frac{24\dot{a}^2}{a^2N^2}A_1A_2(A_2-A_1)^2(A_2+A_1)\\&+\frac{8\dot{a}}{aN}A_2A_1A_3(A_2^2-A_1^2)-\frac{4\dot{a}}{aN}A_2^2A_3A_4+4A_2^2A_1(A_2-A_1)^2(A_2+A_1)+A_2^2A_4^2\Bigg]\ .\end{aligned} \tag{C12}$$

Expressions of $\mathcal{F}$ terms through A_1, A_2, A_3 and A_4 quantities:

$$\mathcal{F}_1=144\Bigg[\frac{8\dot{a}^2}{a^2N^2}(A_2-A_1)^2+A_3^2\Bigg]\Bigg[2A_2A_1(A_2-A_1)-\frac{4\dot{a}^2}{a^2N^2}(A_2-A_1)^2+\frac{\dot{a}}{aN}A_3(A_2+2A_1)+A_2A_4\Bigg]\ ; \tag{C13}$$

$$\begin{aligned}\mathcal{F}_2=&48\Bigg[-\frac{16\dot{a}^4}{a^4N^4}(A_2-A_1)^3(A_2+2A_1)-\frac{12\dot{a}^3}{a^3N^3}A_3(A_2-A_1)^2(A_2-2A_1)\\&+\frac{24\dot{a}^2}{a^2N^2}A_1A_2(A_2-A_1)^3+\frac{18\dot{a}^2}{a^2N^2}A_3^2A_1(A_1-A_2)+\frac{12\dot{a}^2}{a^2N^2}A_2A_4(A_2-A_1)^2\\&+\frac{6\dot{a}}{aN}A_3^3(A_1-A_2)+6A_3^2A_1A_2(A_2-A_1)+3A_2A_3^2A_4\Bigg]\ ;\end{aligned} \tag{C14}$$

$$\mathcal{F}_3=144\Bigg[\frac{2\dot{a}}{aN}A_1(A_2-A_1)+A_2A_3\Bigg]\Bigg[\frac{\dot{a}}{aN}A_3^2+\frac{4\dot{a}}{aN}A_2(A_2-A_1)^2+\frac{8\dot{a}^3}{a^3N^3}(A_2-A_1)^2+A_3A_4\Bigg]\ ; \tag{C15}$$

$$\mathcal{F}_4=144\Bigg[\frac{2\dot{a}}{aN}A_1(A_2-A_1)+A_2A_3\Bigg]\Bigg[\frac{4\dot{a}}{aN}(A_2-A_1)^2\big(A_2-\frac{6\dot{a}^2}{a^2N^2}\big)-\frac{2\dot{a}}{aN}A_3^2+\frac{8\dot{a}^2}{a^2N^2}A_3(A_2-A_1)+A_3A_4\Bigg]\ ; \tag{C16}$$

$$\mathcal{F}_5 =48\Big[-\frac{8\dot{a}^4}{a^4N^4}(A_2-A_1)^3(A_2-3A_1)+\frac{2\dot{a}^3}{a^3N^3}A_3(A_2-A_1)^2(A_2+6A_1)-\frac{4\dot{a}^2}{a^2N^2}A_2A_3^2(A_2-A_1)$$
$$+\frac{2\dot{a}^2}{a^2N^2}A_2A_4(A_2-A_1)^2+\frac{12\dot{a}^2}{a^2N^2}A_1A_2(A_2-A_1)^3+\frac{\dot{a}}{aN}A_2A_3^3+A_2A_3^2A_4\Big]\,; \quad \text{(C17)}$$

$$\mathcal{F}_6 =48\Big[-\frac{2\dot{a}^4}{a^4N^4}(A_2-A_1)^3(9A_2+13A_1)+\frac{6\dot{a}^3}{a^3N^3}A_1A_3(A_2-A_1)^2+\frac{2\dot{a}^2}{a^2N^2}A_3^2(A_2^2-A_1^2)$$
$$+\frac{12\dot{a}^2}{a^2N^2}A_1A_2(A_2-A_1)^3+\frac{2\dot{a}}{aN}A_2^2A_3(A_2-A_1)^2-\frac{2\dot{a}}{aN}A_2A_3^3+A_2A_3^2A_4\Big]\,; \quad \text{(C18)}$$

$$\mathcal{F}_7 =48\Big[\frac{8\dot{a}^4}{a^4N^4}(A_2-A_1)^3(A_2+A_1)+\frac{2\dot{a}^3}{a^3N^3}A_3(A_2-A_1)^2(2A_2+5A_1)+\frac{2\dot{a}^2}{a^2N^2}A_1A_4(A_2-A_1)^2$$
$$-\frac{4\dot{a}^2}{a^2N^2}A_2A_3^2(A_2-A_1)+\frac{4\dot{a}^2}{a^2N^2}A_2(A_2-A_1)^3(A_2+2A_1)+\frac{\dot{a}}{aN}A_2A_3^3+A_2A_3^2A_4\Big]\,; \quad \text{(C19)}$$

$$\mathcal{F}_8 =48\Big[\frac{2\dot{a}^4}{a^4N^4}(A_2-A_1)^3(5A_2-27A_1)-\frac{6\dot{a}^3}{a^3N^3}A_3(A_2-A_1)^2(2A_2-3A_1)+\frac{12\dot{a}^2}{a^2N^2}A_1A_2(A_2-A_1)^3$$
$$+\frac{4\dot{a}^2}{a^2N^2}A_2A_3^2(A_2-A_1)+\frac{2\dot{a}}{aN}A_2^2A_3(A_2-A_1)^2-\frac{2\dot{a}}{aN}A_2A_3^3+A_2A_3^2A_4\Big]\,. \quad \text{(C20)}$$

We display here the constraint equations obtained for all $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ terms (using again the notation $\delta\mathcal{D}\equiv\frac{\delta}{\delta N}\big(a^3N\mathcal{D}\big)$ and similarly for $\mathcal{E}$ and $\mathcal{F}$):

$$\delta\mathcal{D}_1 = 144\Big[\zeta_1+16\zeta_3+64\zeta_2\Big]=\frac{384(a_3^2-a_1a_5)^3}{a_1^{15}}\,t^3+O\left(t^5\right)\,; \quad \text{(C21)}$$

$$\delta\mathcal{D}_2 = 12\Big[64\zeta_2+12\big[\zeta_1+8\zeta_3+16\zeta_2\big]\Big]=\frac{1408(a_3^2-a_1a_5)^3}{3a_1^{15}}\,t^3+O\left(t^5\right)\,; \quad \text{(C22)}$$

$$\delta\mathcal{D}_3 = 2\Big[864\zeta_2+24\big[\zeta_1+4\zeta_3+4\zeta_2\big]\Big]=\frac{144(a_3^2-a_1a_5)^3}{a_1^{15}}\,t^3+O\left(t^5\right)\,; \quad \text{(C23)}$$

$$\delta\mathcal{D}_4 = 4\Big[240\zeta_2+192\zeta_4+12\zeta_1\Big]=\frac{136(a_3^2-a_1a_5)^3}{a_1^{15}}\,t^3+O\left(t^5\right)\,; \quad \text{(C24)}$$

$$\delta\mathcal{E}_1 = 144\cdot\frac{4t^3}{105a_1^{15}}\Big[60a_1^2a_3^2\big(a_1a_9-a_3a_7\big)+532a_1^2a_3a_5(a_3a_5-a_1a_7)-3675\big(a_3^2-a_1a_5\big)^3$$
$$-3535a_1a_5\big(a_3^2-a_1a_5\big)^2-a_1a_3\big(616a_5a_3+924a_1a_7\big)\big(a_3^2-a_1a_5\big)\Big]+O\left(t^5\right)\,; \quad \text{(C25)}$$

$$\delta\mathcal{E}_2 = 144\cdot\frac{4t^3}{63a_1^{15}}\Big[36a_1^2a_3^2\big(a_1a_9-a_3a_7\big)+420a_1^2a_3a_5(a_3a_5-a_1a_7)-2793\big(a_3^2-a_1a_5\big)^3$$
$$-2765a_1a_5\big(a_3^2-a_1a_5\big)^2-a_1a_3\big(1246a_5a_3+560a_1a_7\big)\big(a_3^2-a_1a_5\big)\Big]+O\left(t^5\right)\,; \quad \text{(C26)}$$

$$\delta\mathcal{E}_3 = 144\cdot\frac{2t^3}{315a_1^{15}}\Big[180a_1^2a_3^2\big(a_1a_9-a_3a_7\big)+588a_1^2a_3a_5(a_3a_5-a_1a_7)-6860\big(a_3^2-a_1a_5\big)^3$$
$$-5915a_1a_5\big(a_3^2-a_1a_5\big)^2-a_1a_3\big(-4046a_5a_3+2996a_1a_7\big)\big(a_3^2-a_1a_5\big)\Big]+O\left(t^5\right)\,; \quad \text{(C27)}$$

$$\delta\mathcal{E}_4 = 48\cdot\frac{2t^3}{315a_1^{15}}\Big[540a_1^2a_3^2\big(a_1a_9-a_3a_7\big)+3276a_1^2a_3a_5(a_3a_5-a_1a_7)-23450\big(a_3^2-a_1a_5\big)^3$$
$$-22575a_1a_5\big(a_3^2-a_1a_5\big)^2-a_1a_3\big(-4179a_5a_3+7644a_1a_7\big)\big(a_3^2-a_1a_5\big)\Big]+O\left(t^5\right)\,; \quad \text{(C28)}$$

$$\delta\mathcal{E}_5 = \delta E_7 = 48\cdot\frac{t^3}{315a_1^{15}}\Big[360a_1^2a_3^2\big(a_1a_9-a_3a_7\big)+3192a_1^2a_3a_5(a_3a_5-a_1a_7)-29470\big(a_3^2-a_1a_5\big)^3 - 28000a_1a_5\big(a_3^2-a_1a_5\big)^2-a_1a_3\big(2240a_5a_3+7000a_1a_7\big)\big(a_3^2-a_1a_5\big)\Big]+O\left(t^5\right)\ ; \tag{C29}$$

$$\delta\mathcal{E}_6 = 48\cdot\frac{t^3}{315a_1^{15}}\Big[360a_1^2a_3^2\big(a_1a_9-a_3a_7\big)+4200a_1^2a_3a_5(a_3a_5-a_1a_7)-32795\big(a_3^2-a_1a_5\big)^3 - 32305a_1a_5\big(a_3^2-a_1a_5\big)^2-a_1a_3\big(11396a_5a_3+6664a_1a_7\big)\big(a_3^2-a_1a_5\big)\Big]+O\left(t^5\right)\ ; \tag{C30}$$

$$\delta\mathcal{E}_8 = 48\cdot\frac{t^3}{630a_1^{15}}\Big[720a_1^2a_3^2\big(a_1a_9-a_3a_7\big)+8400a_1^2a_3a_5(a_3a_5-a_1a_7)-65695\big(a_3^2-a_1a_5\big)^3 - 64610a_1a_5\big(a_3^2-a_1a_5\big)^2-a_1a_3\big(22792a_5a_3+13328a_1a_7\big)\big(a_3^2-a_1a_5\big)\Big]+O\left(t^5\right)\ ; \tag{C31}$$

$$\delta\mathcal{F}_1 = 144\cdot\frac{2\big(a_3^2-a_1a_5\big)t^3}{45a_1^{15}}\Big[235a_3^2(a_3^2-a_1a_5)-64a_1a_3\big(a_3a_5-a_1a_7\big)\Big]+O\left(t^5\right)\ ; \tag{C32}$$

$$\delta\mathcal{F}_2 = 48\cdot\frac{2\big(a_3^2-a_1a_5\big)t^3}{45a_1^{15}}\Big[105a_1a_5\big(a_3^2-a_1a_5\big)+12a_1a_3\big(a_3a_5-a_1a_7\big)-160\big(a_3^2-a_1a_5\big)^2\Big]+O\left(t^5\right)\ ; \tag{C33}$$

$$\delta\mathcal{F}_3 = 144\cdot\frac{4\big(a_3^2-a_1a_5\big)t^3}{45a_1^{15}}\Big[60a_1a_5\big(a_3^2-a_1a_5\big)-18a_1a_3\big(a_3a_5-a_1a_7\big)+5\big(a_3^2-a_1a_5\big)^2\Big]+O\left(t^5\right)\ ; \tag{C34}$$

$$\delta\mathcal{F}_4 = 144\cdot\frac{2\big(a_3^2-a_1a_5\big)t^3}{45a_1^{15}}\Big[-155a_1a_5\big(a_3^2-a_1a_5\big)+32a_1a_3\big(a_3a_5-a_1a_7\big)-30\big(a_3^2-a_1a_5\big)^2\Big]+O\left(t^5\right)\ ; \tag{C35}$$

$$\delta\mathcal{F}_5 = 48\cdot\frac{\big(a_3^2-a_1a_5\big)t^3}{90a_1^{15}}\Big[725a_3^2(a_3^2-a_1a_5)-128a_1a_3\big(a_3a_5-a_1a_7\big)\Big]+O\left(t^5\right)\ ; \tag{C36}$$

$$\delta\mathcal{F}_6 = 48\cdot\frac{\big(a_3^2-a_1a_5\big)t^3}{180a_1^{15}}\Big[-725a_1a_5\big(a_3^2-a_1a_5\big)+128a_1a_3\big(a_3a_5-a_1a_7\big)-265\big(a_3^2-a_1a_5\big)^2\Big]+O\left(t^5\right)\ ; \tag{C37}$$

$$\delta\mathcal{F}_7 = 48\cdot\frac{\big(a_3^2-a_1a_5\big)t^3}{90a_1^{15}}\Big[725a_1a_5\big(a_3^2-a_1a_5\big)-128a_1a_3\big(a_3a_5-a_1a_7\big)+10\big(a_3^2-a_1a_5\big)^2\Big]+O\left(t^5\right)\ ; \tag{C38}$$

$$\delta\mathcal{F}_8 = 48\cdot\frac{\big(a_3^2-a_1a_5\big)t^3}{180a_1^{15}}\Big[-725a_1a_5\big(a_3^2-a_1a_5\big)+128a_1a_3\big(a_3a_5-a_1a_7\big)-275\big(a_3^2-a_1a_5\big)^2\Big]+O\left(t^5\right). \tag{C39}$$

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