TOROIDAL CONFINEMENT (THEORY)

Plasma Equilibria of Tokamak Type

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We present a general formalism for the description of an axisymmetric plasma equilibrium. This is a model for the steady operation of a Tokamak device. We use the hydromagnetic equations taking into account effects such as tensorial resistivity and finite thermal conductivity. The reformulation of this set leads to an equivalent set, including the generalisation to toroidal geometry of the Bennett-Pinch relation, and an expression for the resistive plasma loss which shows explicitly the effect of the discharge current. This mathematically concise presentation of the full resistive equilibrium problem is appropriate to practical calculations. As an example we consider a steady state on the resistive time scale and for the case of small inverse aspect-ratio calculate the plasma displacement, and the radial distributions of all equilibrium quantities.

To describe the stationary state of a Tokamak plasma, we use the following MHD equations in M.K.S. units and standard notation

$$\mathrm{rot}\,\underline{B} \times \underline{B} = \mu_0 \nabla p \tag{1}$$

$$\mathrm{div}\,\underline{B} = 0 \tag{2}$$

$$\underline{E} + \underline{v} \times \underline{B} = \boldsymbol{\eta} \cdot \underline{j} \quad ; \quad \underline{E} = -\nabla\varphi - \frac{U}{2\pi}\nabla\xi \tag{3}$$

$$\mathrm{div}\,\rho\underline{v} = Q \tag{4}$$

$$\mathrm{div}\,e\rho\underline{v} = \boldsymbol{\eta} : \underline{j}\,\underline{j} - p\,\mathrm{div}\,\underline{v} + \mathrm{div}(\kappa\nabla T) + Q_E \tag{5}$$

$$\boldsymbol{\eta} = \eta_\perp \mathbf{1} - (\eta_\perp - \eta_\parallel)\underline{B}\,\underline{B}/B^2 \tag{6}$$

The expression for $\underline{E}$ follows from the assumption that there is no time variation of $\underline{B}$ in the plasma region. U is the ring voltage, ξ the angle about the axis of symmetry and e the specific internal energy. To complete the system we add the ideal gas equation of state. We assume that the temperature T is constant along field lines. We imagine that the plasma is enclosed in an ideally conducting container.

The following observations should be made:

(1) We consider a stationary state, by which is meant that the time variation is slow enough to neglect all partial derivatives with respect to time.

(2) The plasma losses due to ion-electron collisions, as described by the resistivity $\boldsymbol{\eta}$ are balanced by a plasma source Q.

(3) In the stationary state under investigation, plasma flows are retained but are such that inertia effects are negligible.

The usual description of an axisymmetric situation is carried out with the use of cylindrical coordinates. The introduction of flux functions leads to a more elegant and convenient description [1]. We employ for $\underline{B}$ and $\underline{j}$ the fluxes the "long" and "short" way, where each flux is evaluated in the appropriate direction between the magnetic axis and a magnetic surface. We call the magnetic fluxes the long and the short way F and G respectively. The corresponding fluxes of $\underline{j}$, the currents, we call I and J. By their definition, the equations used and the assumption on T, F, G, I, J, T, p and ρ are surface quantities. The labeling of magnetic surfaces can be done in terms of any one of these, we usually emplɒy the poloidal magnetic flux G and denote derivatives with respect to this variable by a dot.

A straightforward analysis reveals that the basic equations can be rewritten in the following form

$$\mathrm{div}\,\frac{\nabla G}{R^2} + \frac{\Lambda\dot{\Lambda}}{R^2} + 4\pi^2\mu_0\dot{p} = 0 \tag{7}$$

$$\dot{\Lambda} I - \Lambda\dot{I} = \frac{U}{\eta_\parallel \iota} \tag{8}$$

$$\dot{I} + \frac{1}{\mu_0 \iota}\dot{\Lambda} + \dot{p}\dot{V} = 0 \tag{9}$$

$$\int\limits_{F(x)=F} Q\,d^3\tau = M = \left\{\eta_\perp \int\limits_{F(x)=F}\frac{|\nabla p|}{B^2}\,dS - \eta_\parallel \frac{p'\Lambda^2}{\iota^2}\left(\int\limits_{F(x)=F}\frac{1}{B^2}\frac{dS}{|\nabla F|} - \frac{V'^2}{\Lambda+\mu_0\iota I}\right) + \frac{\mu_0 I U V'}{\Lambda+\mu_0\iota I}\right\}\rho \tag{10}$$

$$p = \rho kT/m \tag{11}$$

$$\frac{k\gamma}{m(\gamma-1)}(TM)^{\cdot} + U\dot{I} = \frac{d}{dG}\left\{\dot{T}\frac{d}{dG}\left(\int \kappa|\nabla G|^2 d^3\tau\right) + \int Q_E\,d^3\tau\right\} \tag{12}$$

(7) is the fundamental equilibrium equation. (8) is Ohm's law in the flux formulation and (9) is the differential form of the Bennett-Pinch relation in the toroidal case. The last two equations express mass and energy balance. The quantities needed to explain this system we list in the following lines

$$\underline{B} = \frac{1}{2\pi}(\nabla\xi \times \nabla G + \Lambda\nabla\xi) \quad , \quad \Lambda = \mu_0(J_A - J) \tag{13}$$

$$\frac{1}{\iota} \equiv \dot{F} = \Lambda\frac{d}{dG}\int\frac{d^3\tau}{(2\pi R)^2} \quad , \quad V \equiv \int d^3\tau \tag{14}$$

$$M \equiv \int\limits_{F(x)=F} \rho\underline{v}\cdot d\underline{S} \quad , \quad {}' = \frac{d}{dF} \tag{15}$$

R is the distance from the axis of symmetry. We note that the determination of the six quantities G, Λ, I, p, T and ρ from (7) to (12) determines the flows, so that the velocity $\underline{v}$ consists of two parts

$$\underline{v} = \hat{\underline{v}} + \left(\psi_s - \frac{\Lambda}{B^2}\dot{\varphi}_s\right)\underline{B} + \frac{\dot{\varphi}_s}{B^2}\underline{B}\times\nabla G \tag{16}$$

(1) $\hat{\underline{v}}$, which is calculable from G, Λ and ρ, and (2) the remaining part, which is divergencefree and everywhere tangent to the magnetic surfaces. This part depends on arbitrary surface quantities ψ_s and φ_s, which come from the integration of two magnetic differential equations.

(10) is the expression for the resistive plasma loss and is made up of three different terms:

(1) the so called "Classical Diffusion" term involving $j_\perp$,

(2) the correction due to toroidicity, first derived by Pfirsch and Schlüter [2]. By Schwarz's inequality this term can quite generally be shown to be always positive,

(3) a new term, involving the ring voltage and always negative.

The structure of the derived set of equations suggests the following procedure for an approximate determination of all equilibrium quantities. Leaving for a moment the equilibrium equation (7) out of discussion, closer investigation of the remaining relations shows that the prescription of any family of nested toroidal surfaces G=const. allows the calculation of the surface quantities I, J, p, ρ and T by ordinary differential equations. However the question now is do the solutions so found, satisfy the equilibrium condition? Certainly they do not on average violate the equilibrium condition, because (9) is (7) averaged over a magnetic surface. But this does not avoid local violation of equation (7).

What is possible and what we propose, is to choose a family of surfaces, in other words a coordinate system, which anticipates the expected geometry of the magnetic surfaces and which is provided with largely arbitrary built-in functions. After solving for the surface quantities we can use the built-in functions to modify shape and position of the magnetic surfaces in such a manner that the approximation with respect to the equilibrium equation is as good as possible.

As an example we have done this for a large aspect-ratio torus and for the special case of no mass and energy sources.

The results will be presented.

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[1] M.D. Kruskal, R.M. Kulsrud, Phys.Fluids 1 , 265 (1958)

[2] D. Pfirsch, A. Schlüter, MPI/PA/7/62 (1962).